Canad. Math. Bull. Vol. **56** (4), 2013 pp. 769–784
http://dx.doi.org/10.4153/CMB-2011-198-7

A Non-zero Value Shared by an Entire Function and its Linear Differential Polynomials

Indrajit Lahiri and Imrul Kaish

Abstract. In this paper we study uniqueness of entire functions sharing a non-zero finite value with linear differential polynomials and address a result of W. Wang and P. Li.

1 Introduction, Definitions, and Results

Let f be a non-constant entire function in the open complex plane $\mathbb{C}$. We denote by $\overline{E}(a; f)$, $\overline{E}_{1)}(a; f)$, and $\overline{E}_{(2}(a; f)$ the set of all distinct a-points, simple a-points, and distinct multiple a-points of f.

In 1986 G. Jank, E. Mues, and L. Volkmann [2] proved a uniqueness theorem for entire functions sharing a single value with two derivatives. Their result can be stated as follows.

Theorem A ([2]) *Let f be a non-constant entire function and let a be a non-zero finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

Theorem A has been extended to general order derivatives and linear differential polynomials by several authors.

Throughout the paper we denote by L a non-constant linear differential polynomial in f of the form

$$L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}, \tag{1.1}$$

where $a_1, a_2, \ldots, a_n (\neq 0)$ are constants.

Inspired by Theorem A, P. Li [4] proved the following result.

Theorem B ([4]) *Let f be a non-constant entire function and let $L(\not\equiv 0)$ be given by (1.1). If f and $L^{(1)}$ share a finite non-zero value a counting multiplicities, and $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L)$, then $L = L^{(1)}$ and $f = f^{(1)}$ or $f = a + \frac{1}{a} L(L - a)$.*

In 2004 W. Wang and P. Li [5] improved Theorem B and proved the following result.

Theorem C ([5]) *Let f be a non-constant entire function, $a \in \mathbb{C}\backslash\{0, \infty\}$, and let $L(\not\equiv 0)$ be given by (1.1). If $\overline{E}(a; f) = \overline{E}(a; L^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L)$, then one of the following holds:*

Received by the editors August 19, 2011; revised November 24, 2011.
Published electronically February 3, 2012.
AMS subject classification: **30D35**.
Keywords: entire function, linear differential polynomial, value sharing.

(i) $f = f^{(1)} = L$;
(ii) $L = L^{(1)}$ *and* $f = a + \frac{1}{a}L(L - a)$;
(iii) $f = a + c_1 e^{\frac{3}{2}z} + c_2 e^{3z}$ *and* $L = -2c_1 e^{\frac{3}{2}z} - \frac{1}{2}c_2 e^{3z}$, *where* $3c_1^2 = 2ac_2$ *and* c_1, c_2 *are non-zero constants.*

So far as we understand there is a major lacuna in the proof of Theorem B and the same is carried forward to the proof of Theorem C. In fact, in [4, Lemma 4] it is shown that $\phi = (L^{(1)} - f^{(1)})/(f - a)$ is a constant. To do this, Li [4] claimed the following:

$$L^{(2)} = (A^{(1)} + \xi L^{(2)}) + (\xi^{(1)} + \eta\phi)L^{(1)} + (\eta^{(1)} - \eta\phi)(f - a),$$

which is [4, (5) on p. 4]. But calculation reveals that it should be

$$L^{(2)} = (A^{(1)} + \xi L^{(2)}) + (\xi^{(1)} + \eta)L^{(1)} + (\eta^{(1)} - \eta\phi)(f - a).$$

Consequently the identity $a_n^2\phi^{2n+3} + R[\phi] \equiv 0$, as claimed in [4, p. 4], should be $a_n^2\phi^{2n+2} + R[\phi] \equiv 0$, where $R[\phi]$ is a differential polynomial in ϕ with degree not greater than $2n + 2$. Therefore, one cannot use Clunie's lemma to show that ϕ is a constant. In this paper we reconsider Theorem C and prove a modified version of it.

For standard definitions and notations of the value distribution theory we refer the reader to [1]. However, we require the following definitions.

Definition 1.1 Let f and g be two non-constant meromorphic functions defined in $\mathbb{C}$. For $a, b \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid g = b)$ ($\overline{N}(r, a; f \mid g = b)$) the counting function (reduced counting function) of those a-points of f that are the b-points of g.

Definition 1.2 Let f and g be two non-constant meromorphic functions defined in $\mathbb{C}$. For $a, b \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid g \neq b)$ ($\overline{N}(r, a; f \mid g \neq b)$) the counting function (reduced counting function) of those a-points of f that are not the b -points of g.

Definition 1.3 Let f be a non-constant meromorphic function in $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. For $A \subset \mathbb{C}$ we denote by $N_A(r, a; f)$ ($\overline{N}_A(r, a; f)$) the counting function (reduced counting function) of those a-points of f that belong to A.

Definition 1.4 Let f be a non-constant meromorphic function defined in $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer k we denote by $N(r, a; f \mid\geq k)$ ($N(r, a; f \mid\leq k)$) the counting function of those a-points of f whose multiplicities are not less (greater) than k. By $\overline{N}(r, a; f \mid\geq k)$ and $\overline{N}(r, a; f \mid\leq k)$ we denote the corresponding reduced counting functions.

Definition 1.5 Let f be a non-constant meromorphic function in $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. Suppose that $A \subset \mathbb{C}$ and let k be a positive number. We denote by $N_A(r, a; f \mid\geq k)$ ($\overline{N}_A(r, a; f \mid\geq k)$) the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than k and that belong to A.

In a similar manner, we define $N_A(r, a; f \mid\leq k)$ and $\overline{N}_A(r, a; f \mid\leq k)$.

The following definition is well known.

Definition 1.6 Let f be a non-constant meromorphic function in $\mathbb{C}$. Suppose that

$$M_j[f] = a_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}} \cdots (f^{(p_j)})^{n_{p_j j}}$$

is a differential monomial in f, where a_j is a small function of f.

We denote by $\gamma_{M_j} = \sum_{k=0}^{p_j} n_{kj}$ and by $\Gamma_{M_j} = \sum_{k=0}^{p_j}(1+k)n_{kj}$ the degree and weight of $M_j[f]$ respectively.

The numbers $\gamma_P = \max_{1\le j\le n} \gamma_{M_j}$ and $\Gamma_P = \max_{1\le j\le n} \Gamma_{M_j}$ are respectively called the degree and weight of the differential polynomial $P[f] = \sum_{j=1}^{n} M_j[f]$.

We now state the main result of the paper.

Theorem 1.7 *Let f be a non-constant entire function of finite order, let a be a non-zero finite number, and let L given by (1.1) be such that $L^{(1)}$ is non-constant and $|1-a_1|+|a_2| \neq 0$.*

Let $A = \overline{E}(a; f)\backslash\overline{E}(a; L^{(1)})$ and $B = \overline{E}(a; L^{(1)})\backslash\{\overline{E}(a; f^{(1)}) \cap \overline{E}(a; L)\}$. Suppose further that

(i) $N_A(r, a; f) + N_B(r, a; L^{(1)}) = S(r, f)$,
(ii) $\overline{E}_{1)}(a; L^{(1)}) \subset \overline{E}(a; f)$,
(iii) $\overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(1)})$, *and*
(iv) $\overline{E}_{(2}(a; f) \cap \overline{E}(0; L^{(1)}) = \varnothing$.

Then one of the following holds:

(a) $f = L = \alpha e^z$, *where α is a nonzero constant;*
(b) $f = a + (\alpha^2/a)e^{2z} - \alpha e^z$ *and* $L = \alpha e^z$, *where* $\sum_{k=1}^{n} 2^k a_k = 0$, $\sum_{k=1}^{n} a_k = -1$ *and α is a nonzero constant;*
(c) $f = a + c_1 e^{\frac{3}{2}z} + c_2 e^{3z}$ *and* $L = -2c_1 e^{\frac{3}{2}z} - \frac{1}{2}c_2 e^{3z}$, *where* $3c_1^2 = 2ac_2$ *and c_1, c_2 are non-zero constants.*

Putting $A = B = \varnothing$ we get the following corollary.

Corollary 1.8 *Let f be a non-constant entire function of finite order, let a be a non-zero finite number, and let L given by (1.1) be such that $L^{(1)}$ is non-constant and $|1-a_1|+|a_2| \neq 0$. Further suppose that $\overline{E}(a; f) \subset \overline{E}(a; L^{(1)}) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L)$ and $\overline{E}_{1)}(a; L^{(1)}) \subset \overline{E}(a; f)$. Then the conclusion of Theorem 1.7 holds.*

As a consequence of Corollary 1.8 we obtain the following corollary.

Corollary 1.9 *Let f be a non-constant entire function of finite order, let a be a non-zero finite number, and let L given by (1.1) be such that $L^{(1)}$ is non-constant and $|1-a_1|+|a_2| \neq 0$. If $\overline{E}(a; f) = \overline{E}(a; L^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L)$, then the conclusion of Theorem 1.7 holds.*

2 Lemmas

In this section we present some necessary lemmas.

Lemma 2.1 *Let f be a non-constant entire function and let a be a non-zero finite complex number. Then $f = L = \alpha e^z$, where α is a non-zero constant, provided the following hold:*

(i) $m(r, a; f) = S(r, f)$,
(ii) $\overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$,
(iii) $N_A(r, a; f) = S(r, f)$, *where* $A = \overline{E}(a; f)\backslash\{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)}) \cap \overline{E}(a; f^{(1)})\}$.

Proof Let

$$\lambda = \frac{f^{(1)} - a}{f - a}. \tag{2.1}$$

From the hypothesis we see that λ has no simple pole and $T(r, \lambda) = S(r, f)$. From (2.1) we get

$$f^{(1)} = \lambda_1 f + \mu_1, \tag{2.2}$$

where $\lambda_1 = \lambda$ and $\mu_1 = a(1 - \lambda)$.

Differentiating (2.2) we get $f^{(k)} = \lambda_k f + \mu_k$, where λ_k and μ_k are meromorphic functions satisfying $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k$ and $\mu_{k+1} = \mu_k^{(1)} + \mu_1\lambda_k$ for $k = 1, 2, 3, \ldots$. Also, we see that $T(r, \lambda_k) + T(r, \mu_k) = S(r, f)$ for $k = 1, 2, 3, \ldots$.

Now

$$L = \left(\sum_{k=1}^{n} a_k\lambda_k\right) f + \sum_{k=1}^{n} a_k\mu_k = \xi f + \eta, \text{ say.} \tag{2.3}$$

Clearly $T(r, \xi) + T(r, \eta) = S(r, f)$. Differentiating (2.3) we get

$$L^{(1)} = \xi f^{(1)} + \xi^{(1)} f + \eta^{(1)}. \tag{2.4}$$

Let $z_0 \notin A$ be an a-point of f. Then from (2.3) and (2.4) we get $a\xi(z_0) + \eta(z_0) = a$ and $a\xi(z_0) + a\xi^{(1)}(z_0) + \eta^{(1)}(z_0) = a$.

If $a\xi + \eta \not\equiv a$, then

$$N(r, a; f) \le N(r, a; f \mid\le 1) + N_A(r, a; f) \le N(r, a; a\xi + \eta) + S(r, f) = S(r, f),$$

which is impossible because $m(r, a; f) = S(r, f)$. Hence $a\xi + \eta \equiv a$. Similarly $a\xi + a\xi^{(1)} + \eta^{(1)} \equiv a$. This implies that $\xi \equiv 1$ and $\eta \equiv 0$. So from (2.3) we get $L \equiv f$.

By actual calculation we see that $\lambda_2 = \lambda^2 + \lambda^{(1)}$ and $\lambda_3 = \lambda^3 + 3\lambda\lambda^{(1)} + \lambda^{(2)}$. In general, we now verify that

$$\lambda_k = \lambda^k + P_{k-1}[\lambda], \tag{2.5}$$

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

where $P_{k-1}[\lambda]$ is a differential polynomial in λ with constant coefficients such that $\gamma_{P_{k-1}} \leq k-1$ and $\Gamma_{P_{k-1}} \leq k$. Also each term of $P_{k-1}[\lambda]$ contains some derivative of λ.

Let (2.5) be true. Then

$$\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k = (\lambda^k + P_{k-1}[\lambda])^{(1)} + \lambda\big(\lambda^k + P_{k-1}[\lambda]\big) = \lambda^{k+1} + P_k[\lambda],$$

noting that differentiation does not increase the degree of a differential polynomial but increases its weight by 1. So (2.5) is verified by mathematical induction.

Since $\xi \equiv 1$, we get from (2.5)

$$\sum_{k=1}^{n} a_k\lambda^k + \sum_{k=1}^{n} a_k P_{k-1}[\lambda] \equiv 1. \tag{2.6}$$

Let z_0 be a pole of λ with multiplicity $p(\geq 2)$. Then z_0 is a pole of $\sum_{k=1}^{n} a_k\lambda^k$ with multiplicity np, and it is a pole of $\sum_{k=1}^{n} a_k P_{k-1}[\lambda]$ with multiplicity not exceeding $(n-1)p+1$. Since $np > (n-1)p+1$, it follows that z_0 is a pole of the left-hand side of (2.6) with multiplicity np, which is impossible. So λ is an entire function. If λ is transcendental, then by Clunie's lemma we get from (2.6) that $T(r,\lambda) = S(r,\lambda)$, which is a contradiction. If λ is a polynomial of degree $d(\geq 1)$, then the left-hand side of (2.6) is a polynomial of degree nd with leading coefficient $a_n(\neq 0)$, which is also a contradiction. Therefore, λ is a constant, and so from (2.5) we get $\lambda_k = \lambda^k$ for $k = 1, 2, 3, \ldots$.

Since $\xi \equiv 1$, we see that $\sum_{k=1}^{n} a_k\lambda^k = 1$. Also from (2.2), we obtain $f^{(2)} = \lambda f^{(1)}$ and so $f^{(1)} = \alpha\lambda e^{\lambda z}$ and $f = \alpha e^{\lambda z} + \beta$, where $\alpha(\neq 0)$ and β are constants.

Now $L = (\sum_{k=1}^{n} a_k\lambda^k)\alpha e^{\lambda z} = \alpha e^{\lambda z}$. Since $f \equiv L$, we get $\beta = 0$. Since

$$N_A(r,a;f) = S(r,f) \quad \text{and} \quad N(r,a;f) = T(r,f) + S(r,f),$$

we see that $\overline{E}(a;f) \cap \overline{E}(a;f^{(1)}) \neq \varnothing$. So $f^{(1)} = \lambda f$ implies $\lambda = 1$. Hence $f = \alpha e^z$. This proves the lemma. ∎

Lemma 2.2 ([3]) *Let f be a non-constant entire function in $\mathbb{C}$, let a be a finite non-zero complex number, and let L given by (1.1) be such that $L^{(1)}$ is non-constant.*

Further suppose that $E_{1)}(a;f) \subset E(a;f^{(1)})$ and $N_A(r,a;f) + N_B(r,a;L) = S(r,f)$, where $A = E(a;f) \setminus E(a;L)$ and $B = E(a;L) \setminus \{E(a;f^{(1)}) \cap E(a;L^{(1)})\}$. Then one of the following cases holds:

- (i) *$f = a + \alpha e^z$ and $L = \alpha e^z$, where α is a nonzero constant;*
- (ii) *$f = L = \alpha e^z$, where α is a nonzero constant;*
- (iii) *$f = a + \frac{\alpha^2}{a}e^{2z} - \alpha e^z$ and $L = \alpha e^z$, where $\sum_{k=1}^{n} 2^k a_k = 0$, $\sum_{k=1}^{n} a_k = -1$, and α is a nonzero constant.*

Lemma 2.3 *Let f be a non-constant entire function in $\mathbb{C}$, let a be a finite non-zero complex number, and let L given by (1.1) be such that $L^{(1)}$ is non-constant. Let $A = \overline{E}(a;f)\setminus\overline{E}(a;L^{(1)})$ and $B = \overline{E}(a;L^{(1)})\setminus\{\overline{E}(a;f^{(1)}) \cap \overline{E}(a;L)\}$. If $\overline{E}(a;f) \neq \varnothing$ and $N_A(r,a;f) + N_B(r,a;L^{(1)}) = S(r,f)$, then $\overline{N}(r,a;L^{(1)} \mid f \neq a) = S(r,f)$.*

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

Proof We put

$$C=\overline{E}(a;f)\cap\overline{E}(a;f^{(1)})\cap\overline{E}(a;L)\cap\overline{E}(a;L^{(1)}),$$
$$D=\{\overline{E}(a;f)\cap\overline{E}(a;L^{(1)})\}\backslash C,$$

and henceforth we use these notations.

Let $\chi=(L-f^{(1)})/(f-a)$ and $\phi=(L^{(1)}-f^{(1)})/(f-a)$. Then $m(r,\chi)+m(r,\phi)=S(r,f)$ and

$$\begin{aligned}N(r,\chi)+N(r,\phi)&\le 2\{N_A(r,a;f)+N_D(r,a;f)\}\le 2(n+1)\overline{N}_D(r,a;f)+S(r,f)\\&=2(n+1)N_D(r,a;L^{(1)})+S(r,f)\le 2(n+1)N_B(r,a;L^{(1)})+S(r,f)\\&=S(r,f).\end{aligned}$$

So

$$T(r,\chi)+T(r,\phi)=S(r,f). \tag{2.7}$$

First we suppose that $L\not\equiv f^{(1)}$. Then by the hypothesis

$$\begin{aligned}\overline{N}(r,a;L^{(1)})&\le N\Big(r,1;\frac{L}{f^{(1)}}\Big)+N_B(r,a;L^{(1)})\le T\Big(r,\frac{L}{f^{(1)}}\Big)+S(r,f)\\&=N(r,\frac{L}{f^{(1)}})+S(r,f)\le N(r,0;f^{(1)})+S(r,f).\end{aligned}\tag{2.8}$$

Again,

$$\begin{aligned}m(r,a;f)&\le m\Big(r,\frac{f^{(1)}}{f-a}\Big)+m(r,0;f^{(1)})=T(r,f^{(1)})-N(r,0;f^{(1)})+S(r,f)\\&\le T(r,f)-N(r,0;f^{(1)})+S(r,f)\end{aligned}$$

and so

$$N(r,0;f^{(1)})\le N(r,a;f)+S(r,f). \tag{2.9}$$

From (2.8) and (2.9) we get

$$\overline{N}(r,a;L^{(1)})\le N(r,a;f)+S(r,f). \tag{2.10}$$

Also, we see that

$$\begin{aligned}N_D(r,a;f)&\le(n+1)\overline{N}_D(r,a;f)=(n+1)\overline{N}_D(r,a;L^{(1)})\\&\le(n+1)N_B(r,a;L^{(1)})=S(r,f).\end{aligned}$$

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

Now by (2.10) we get

$$\begin{aligned}\overline{N}(r,a;L^{(1)}\mid f\neq a)&=\overline{N}(r,a;L^{(1)})-\overline{N}_C(r,a;L^{(1)})-N_D(r,a;L^{(1)})\\&\leq N(r,a;f)-\overline{N}_C(r,a;f)-\overline{N}_D(r,a;f)\\&=N(r,a;f)-N_C(r,a;f)+S(r,f)\\&=N_A(r,a;f)+N_D(r,a;f)+S(r,f)=S(r,f).\end{aligned}$$

Next we suppose that $f^{(1)}\not\equiv L^{(1)}$. Then by the hypothesis

$$\begin{aligned}\overline{N}(r,a;L^{(1)})&\leq N\Big(r,1;\frac{L^{(1)}}{f^{(1)}}\Big)+N_B(r,a;L^{(1)})\leq T\Big(r,\frac{L^{(1)}}{f^{(1)}}\Big)+S(r,f)\\&=N\Big(r,\frac{L^{(1)}}{f^{(1)}}\Big)+S(r,f)\leq N(r,0;f^{(1)})+S(r,f),\end{aligned}$$

which is (2.8). Now proceeding as above we get $\overline{N}(r,a;L^{(1)}\mid f\neq a)=S(r,f)$.

Finally we suppose that $L\equiv f^{(1)}$ and $L^{(1)}\equiv f^{(1)}$. Then $L=f+c$, where c is a constant. Hence $f^{(1)}=f+c$. Since $\overline{E}(a;f)\neq\varnothing$, we see that $a+c\neq 0$, because on integration we set $f=-c+\alpha e^z$, where α is a non-zero constant. Hence $N(r,a;f)\neq S(r,f)$. Also, we see that $f^{(1)}\equiv L\equiv L^{(1)}=\alpha e^z$, and since $N_A(r,a;f)=S(r,f)$, we get $\overline{E}(a;f)\cap\overline{E}(a;L^{(1)})\neq\varnothing$. So $f+c=L^{(1)}$ implies that $c=0$. Therefore, $f=L^{(1)}$ and so $\overline{N}(r,a;L^{(1)}\mid f\neq a)=S(r,f)$. This proves the lemma. ■

Lemma 2.4 *Let f be a non-constant entire function. Then, for a non-zero finite number a,*

$$T(r,f)\leq N(r,a;f)+\overline{N}(r,a;R)+S(r,f),$$

where R is a non-constant linear differential polynomial in $f^{(1)}$ with constant coefficients.

Proof If f is a non-constant meromorphic function and ψ is a non-constant linear differential polynomial in f, then by [1, Theorem 3.2 on p. 57] we get

$$T(r,f)\leq\overline{N}(r,\infty;f)+N(r,0;f)+\overline{N}(r,1;\psi)+S(r,f).$$

Since f is entire and R is a linear differential polynomial in $f^{(1)}$, the lemma follows from the above inequality replacing f by $f-a$ and putting $\psi=\frac{R}{a}$. This proves the lemma. ■

3 Proof of Theorem 1.7

Proof We put $\psi=(L-L^{(1)})/(f-a)$ and $\phi=(L^{(1)}-f^{(1)})/(f-a)$. Since $\psi=\chi-\phi$, by (2.7) we get $T(r,\psi)+T(r,\phi)=S(r,f)$. We now consider the following cases.

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

Case 1. Let $\psi \equiv 1$. Then

$$L^{(1)} = L - (f - a), \tag{3.1}$$

$$L^{(1)} = f^{(1)} + \phi(f - a). \tag{3.2}$$

Differentiating (3.1) and using (3.2) we get

$$L^{(2)} = L^{(1)} - f^{(1)} = \phi(f - a). \tag{3.3}$$

Differentiating (3.2) we get

$$L^{(2)} = f^{(2)} + \phi f^{(1)} + \phi^{(1)}(f - a). \tag{3.4}$$

Eliminating $L^{(2)}$ from (3.3) and (3.4) we get

$$f^{(2)} = -\phi f^{(1)} + (\phi - \phi^{(1)})(f - a). \tag{3.5}$$

Differentiating (3.5) and using it repeatedly we get

$$f^{(k+1)} = \big\{ (-\phi)^k + P_{k-1}[\phi] \big\} f^{(1)} + \big\{ \phi^k + Q_k[\phi] \big\} (f - a),$$

where $P_{k-1}[\phi]$, $Q_k[\phi]$ are differential polynomials in ϕ with constant coefficients and $\Gamma_{P_{k-1}} \leq k$, $\gamma_{P_{k-1}} \leq k - 1$. Therefore

$$L^{(1)} = \sum_{k=1}^{n} a_k \big\{ (-\phi)^k + P_{k-1}[\phi] \big\} f^{(1)} + \sum_{k=1}^{n} a_k \big\{ \phi^k + Q_k[\phi] \big\} (f - a). \tag{3.6}$$

Let $\overline{E}(a; f) = \varnothing$. Since f is of finite order, we can put $f = a + e^p$, where p is a polynomial with $\deg(p) \geq 1$. Differentiating repeatedly we get $f^{(k)} = T_k e^p$, where T_k is a polynomial with $\deg(T_k) = k(\deg(p) - 1)$. So $L = \sum_{k=1}^{n} a_k f^{(k)} = Pe^p$ and $L^{(1)} = \sum_{k=1}^{n} a_k f^{(k+1)} = Qe^p$, where P, Q are polynomials with $\deg(P) = n(\deg(p) - 1)$ and $\deg(Q) = (n + 1)(\deg(p) - 1)$. From (3.1) we get $P = Q + 1$, which implies $\deg(p) = 1$. So P, Q are constants. Therefore, $\overline{E}_{1)}(a; L^{(1)}) \neq \varnothing$ and this contradicts the hypothesis $\overline{E}_{1)}(a; L^{(1)}) \subset \overline{E}(a; f)$. Therefore $\overline{E}(a; f) \neq \varnothing$.

Let us recall the following sets from the proof of Lemma 2.3 : $C = \overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$ and $D = \{\overline{E}(a; f) \cap \overline{E}(a; L^{(1)})\} \backslash C$. We now verify that

$$\overline{N}_C(r, a; f) \neq S(r, f). \tag{3.7}$$

If $\overline{N}_C(r, a; f) = S(r, f)$, we get

$$\begin{aligned} N(r, a; f) &= N_A(r, a; f) + N_C(r, a; f) + N_D(r, a; f) \\ &\leq \overline{N}_C(r, a; f) + (n + 1)\overline{N}_D(r, a; f) + S(r, f) \\ &= (n + 1)N_D(r, a; L^{(1)}) + S(r, f) \\ &\leq (n + 1)N_B(r, a; L^{(1)}) + S(r, f) = S(r, f). \end{aligned} \tag{3.8}$$

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

Since $\overline{E}(a; f) \neq \varnothing$, by (3.8) and Lemma 2.3 we obtain

$$\overline{N}(r, a; L^{(1)}) \leq \overline{N}(r, a; f) + \overline{N}(r, a; L^{(1)} \mid f \neq a) = S(r, f). \tag{3.9}$$

By Lemma 2.4 we get from (3.8) and (3.9) that $T(r, f) = S(r, f)$, a contradiction. Thus (3.7) is verified. So from (3.6) we get

$$\overline{N}_C(r, a; f) \leq N\left(r, 1; \sum_{k=1}^{n} a_k \big\{ (-\phi)^k + P_{k-1}[\phi] \big\} \right) = S(r, f),$$

which is a contradiction unless

$$\sum_{k=1}^{n} a_k \big\{ (-\phi)^k + P_{k-1}[\phi] \big\} \equiv 1. \tag{3.10}$$

If ϕ is a polynomial of degree $p(\geq 1)$, then the left-hand side of (3.10) is a polynomial of degree np with leading coefficient $(-1)^n a_n (\neq 0)$. This is a contradiction.

If ϕ is transcendental, by Clunie's lemma we get from (3.10) that $m(r, \phi) = S(r, \phi)$. By the hypothesis we see that ϕ has no simple pole. Let z_0 be a pole of ϕ with multiplicity $q(\geq 2)$. Then z_0 is a pole of $a_n(-\phi)^n$ with multiplicity nq. Also, z_0 is a pole of

$$\sum_{k=1}^{n-1} \big\{ a_k (-\phi)^k + P_{k-1}[\phi] \big\} + a_n P_{n-1}[\phi]$$

with multiplicity at most $n + (q-1)(n-1) = q(n-1) + 1$. Since $q \geq 2$, we see that $nq > q(n-1) + 1$, and so z_0 becomes a pole of the left-hand side of (3.10), which is impossible. Therefore, ϕ is entire and so $T(r, \phi) = S(r, \phi)$, a contradiction. Hence ϕ is a constant.

So from (3.5) we get

$$f^{(2)} + \phi f^{(1)} - \phi(f - a) = 0. \tag{3.11}$$

Solving (3.11) we obtain

$$f = \begin{cases} c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + a & \text{if } \lambda_1 \neq \lambda_2, \\ (c_1 + c_2 z) e^{\lambda_1 z} + a & \text{if } \lambda_1 = \lambda_2, \end{cases}$$

where c_1, c_2 are constants and λ_1, λ_2 are roots of the equation $\lambda^2 + \phi\lambda - \phi = 0$.

If $\lambda_1 = \lambda_2$, then $N(r, a; f) = N(r, 0; c_1 + c_2 z) = S(r, f)$, which contradicts (3.7). So $\lambda_1 \neq \lambda_2$, and by (3.7) we get $c_1 c_2 \neq 0$.

Let $\lambda_1 = \alpha + \beta$ and $\lambda_2 = \alpha - \beta$, where $2\alpha = -\phi$ and $2\beta = +\sqrt{\phi^2 + 4\phi} \neq 0$. Then

$$f = a + e^{(\alpha-\beta)z} \big(\sqrt{c_1} e^{\beta z} - i\sqrt{c_2} \big) \big(\sqrt{c_1} e^{\beta z} + i\sqrt{c_2} \big).$$

This shows that all a-points of f are simple. Hence

$$\overline{E}(a; f) = \overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(1)}) \subset \overline{E}(a; L^{(1)}).$$

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

From (3.3) we see that

$$\overline{E}_{(2}(a; L^{(1)}) \subset \overline{E}(0; L^{(2)}) = \overline{E}(a; f).$$

Since by the hypothesis $\overline{E}_{1)}(a; L^{(1)}) \subset \overline{E}(a; f)$, we get $\overline{E}(a; L^{(1)}) = \overline{E}(a; f)$.

Since $\overline{E}(a; L^{(1)}) = \overline{E}(a; f) = \overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$ and from (3.3) we get $L^{(3)} = \phi f^{(1)}$, each a-point of $L^{(1)}$ is a double a-point. Therefore,

$$L^{(1)} - a = (f - a)^2 e^h, \tag{3.12}$$

where h is an entire function.

Since the order of f is 1 and that of $L^{(1)}$ is at most 1, h is a polynomial of degree at most 1. Since $\lambda_1 \neq \lambda_2$, we see that $\phi \neq 0$. Differentiating (3.12) and using (3.3) we get

$$(2\lambda_1 + \gamma)c_1 e^{(\lambda_1+\gamma)z} + (2\lambda_2 + \gamma)c_2 e^{(\lambda_2+\gamma)z} = \phi e^{-\delta}, \tag{3.13}$$

where we put $h = \gamma z + \delta$ and γ, δ are constants.

We now verify that at least one of $\lambda_1 + \gamma$ and $\lambda_2 + \gamma$ is zero. Otherwise, by the second fundamental theorem, we get

$$\begin{aligned} T\big(r, e^{(\lambda_1+\gamma)z}\big) &\leq N\Big(r, \frac{\phi e^{-\delta}}{(2\lambda_1 + \gamma)c_1}; e^{(\lambda_1+\gamma)z}\Big) + S\big(r, e^{(\lambda_1+\gamma)z}\big) \\ &= N\big(r, 0; e^{(\lambda_2+\gamma)z}\big) + S\big(r, e^{(\lambda_1+\gamma)z}\big) = S\big(r, e^{(\lambda_1+\gamma)z}\big), \end{aligned}$$

which is a contradiction unless $2\lambda_1 + \gamma = 0$. So from (3.13) we get $2\lambda_2 + \gamma = 0$, which is impossible, as $\lambda_1 \neq \lambda_2$.

Hence we can suppose that $\lambda_2 + \gamma = 0$. Then from (3.13) we get $2\lambda_1 + \gamma = 0$, and so $2\lambda_1 = \lambda_2$. Since λ_1 and λ_2 are roots of $\lambda^2 + \phi\lambda - \phi = 0$, we get $\lambda_1 = \frac{3}{2}$, $\lambda_2 = 3$ and $\phi = -\frac{9}{2}$.

From (3.3) we get $L^{(2)} = -\frac{9}{2}(c_1 e^{\frac{3}{2}z} + c_2 e^{3z})$, and on integration, $L^{(1)} = -3(f - a) + \frac{3}{2}c_2 e^{3z} + d$, where d is a constant. In view of (3.7), let $z_0 \in C$. Then

$$\frac{3}{2}c_2 e^{3z_0} + d = a, \tag{3.14}$$

$$c_1 e^{\frac{3}{2}z_0} + c_2 e^{3z_0} = 0. \tag{3.15}$$

Since $f^{(1)} = \frac{3}{2}c_1 e^{\frac{3}{2}z} + 3c_2 e^{3z}$, we get

$$\frac{3}{2}c_1 e^{\frac{3}{2}z_0} + 3c_2 e^{3z_0} = a. \tag{3.16}$$

From (3.14), (3.15), and (3.16) we obtain $d = 0$. Now eliminating z_0 from (3.14) and (3.15), we get $3c_1^2 = 2ac_2$.

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

Since $f = a + c_1 e^{\frac{3}{2}z} + c_2 e^{3z}$ and $L^{(1)} = -3(f-a) + \frac{3}{2}c_2 e^{3z}$, we get from (3.1)

$$L = L^{(1)} + (f - a) = -2c_1 e^{\frac{3}{2}z} - \frac{1}{2}c_2 e^{3z}.$$

Case 2. Let $\psi \not\equiv 1$. Then

$$\overline{N}(r, a; L^{(1)} \mid \geq 2) \leq N(r, 1; \psi) = S(r, f). \tag{3.17}$$

We now consider the following subcases.

Subcase 2.1. Let $\phi \equiv 0$. Then $L^{(1)} \equiv f^{(1)}$, and so $L = f + d$, where d is a constant.

Let $\overline{E}(a; f) = \varnothing$. Since f is of finite order, we can put $f = a + e^p$, where p is a non-constant polynomial. Since $\overline{E}_{1)}(a; f^{(1)}) = \overline{E}_{1)}(a; L^{(1)}) \subset \overline{E}(a; f) = \varnothing$, by (3.17) we get $\overline{N}(r, a; f^{(1)}) = S(r, f) = S(r, f^{(1)})$. Also, $\overline{N}(r, 0; f^{(1)}) = \overline{N}(r, 0; p^{(1)}) = S(r, f^{(1)})$, and so by the second fundamental theorem $T(r, f^{(1)}) = S(r, f^{(1)})$, a contradiction. Hence $\overline{E}(a; f) \neq \varnothing$, and so (3.7) is also valid.

Since $f^{(1)} \equiv L^{(1)}$, from (3.17) we get

$$N(r, a; f^{(1)} \mid \geq 2) \leq (n + 1)\overline{N}(r, a; f^{(1)} \mid \geq 2) = S(r, f).$$

Let

$$g_1 = \frac{L^{(2)} - (1 - \psi)L^{(1)}}{f^{(1)} - a} \quad \text{and} \quad g_2 = \frac{L^{(2)} - (1 - \psi)L^{(1)}}{f - a}.$$

If $z_0 \in C$, then clearly $L^{(2)}(z_0) - (1 - \psi(z_0))L^{(1)}(z_0) = 0$. So by Lemma 2.3 we get

$$\begin{aligned} N(r, g_1) &\leq N(r, a; f^{(1)} \mid \geq 2) + N_B(r, a; f^{(1)}) + N(r, a; f^{(1)} \mid f \neq a) \\ &\leq N_B(r, a; L^{(1)}) + (n + 1)\overline{N}(r, a; f^{(1)} \mid f \neq a) + S(r, f) \\ &= (n + 1)\overline{N}(r, a; L^{(1)} \mid f \neq a) + S(r, f) = S(r, f), \end{aligned}$$

and $N(r, g_2) \leq N_A(r, a; f) + (n+1)N_B(r, a; L^{(1)}) = S(r, f)$. Also, $m(r, g_1) + m(r, g_2) = S(r, f)$, and so $T(r, g_1) + T(r, g_2) = S(r, f)$.

Let $L^{(2)} - (1 - \psi)L^{(1)} \not\equiv 0$. Then

$$m\Big(r, \frac{f^{(1)} - a}{f - a}\Big) = m\Big(r, \frac{g_2}{g_1}\Big) = S(r, f),$$

and so $m(r, a; f) = S(r, f)$. Therefore by Lemma 2.1 we get $f = L = \alpha e^z$, where α is a non-zero constant.

Next let

$$L^{(2)} - (1 - \psi)L^{(1)} \equiv 0. \tag{3.18}$$

We suppose that $\psi \not\equiv 0$. Differentiating $L - L^{(1)} = \psi(f - a)$ we get

$$L^{(1)} - L^{(2)} \equiv \psi^{(1)}(f - a) + \psi f^{(1)}. \tag{3.19}$$

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

Eliminating $L^{(2)}$ from (3.18) and (3.19) we obtain $\psi^{(1)}(f-a)\equiv 0$. Since f is non-constant, we obtain $\psi^{(1)}\equiv 0$ and so ψ is a non-zero constant.

Let $a+d=0$. Then

$$\psi=\frac{L-L^{(1)}}{f-a}=1-\frac{f^{(1)}}{f-a},$$

and so $f^{(1)}/(f-a)=1-\psi=c$, say, a non-zero constant. This implies that $f=a+Ke^{cz}$, where $K(\neq 0)$ is a constant. Since $L=f+d=f-a=Ke^{cz}$, we get $L^{(1)}=cKe^{cz}$. Since by (3.7), $C\neq\varnothing$, there exists z_0 such that $L(z_0)=L^{(1)}(z_0)=a$ and so $c=1$. This implies a contradiction, as $\psi\not\equiv 0$. Therefore, $a+d\neq 0$, and so

$$\frac{1}{f-a}=\frac{1}{a+d}\Big(\frac{f+d}{f-a}-1\Big)=\frac{1}{a+d}\Big(\frac{L}{f-a}-1\Big),$$

which implies that $m(r,a;f)=S(r,f)$. So by Lemma 2.1 we get $f=L=\alpha e^{z}$, where α is a non-zero constant. This implies a contradiction as $\psi\not\equiv 0$.

Therefore indeed $\psi\equiv 0$. Then $L^{(1)}\equiv L$ and so $L=\alpha e^{z}$, where α is a non-zero constant. Since by (3.7) there exists $z_0\in C$, we get $f(z_0)=L(z_0)=a$ and so $d=0$. Therefore $f=L=\alpha e^{z}$.

Subcase 2.2. Let $\phi\not\equiv 0$. First we suppose that $\psi\equiv 0$. Then $L\equiv L^{(1)}$, and we can apply Lemma 2.2. If Lemma 2.2(i) or (ii) holds, then $\phi\equiv 0$, which is a contradiction. Therefore Lemma 2.2(iii) holds.

Next we suppose that $\psi\not\equiv 0$. If $1+(\frac{1}{\phi})^{(1)}\equiv 0$, then on integration we get $\phi=\frac{1}{c-z}$, where c is a constant. This is impossible, as the hypothesis implies that ϕ has no simple pole. Hence $1+(\frac{1}{\phi})\not\equiv 0$.

Now

$$m(r,f)=m\Big(r,a+\frac{L^{(1)}-f^{(1)}}{\phi}\Big)\leq m(r,f^{(1)})+S(r,f)\leq m(r,f)+S(r,f),$$

and so

$$T(r,f)=T(r,f^{(1)})+S(r,f). \tag{3.20}$$

Differentiating $f=a+\frac{L^{(1)}-f^{(1)}}{\phi}$ we get

$$\frac{f^{(1)}}{f^{(1)}-a}=\frac{1}{1+\Big(\frac{1}{\phi}\Big)^{(1)}}\Big\{\Big(\frac{1}{\phi}\Big)^{(1)}\frac{L^{(1)}}{f^{(1)}-a}+\Big(\frac{1}{\phi}\Big)\frac{L^{(2)}-f^{(2)}}{f^{(1)}-a}\Big\}.$$

This implies that $m(r,f^{(1)}/(f^{(1)}-a))=S(r,f)$, and so $m(r,a;f^{(1)})=S(r,f)$.

From the definitions of ϕ and ψ we get

$$L-L^{(1)}=\psi(f-a), \tag{3.21}$$

$$L^{(1)}-f^{(1)}=\phi(f-a). \tag{3.22}$$

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

Differentiating (3.21) and using (3.22) we obtain

$$(1-\psi)L^{(1)} - L^{(2)} = (\psi^{(1)} - \phi\psi)(f-a). \tag{3.23}$$

Let $\psi^{(1)} - \phi\psi \equiv 0$. Then

$$\phi = \frac{\psi^{(1)}}{\psi}. \tag{3.24}$$

The hypothesis implies that ϕ has no simple pole, and clearly $\frac{\psi^{(1)}}{\psi}$ has no multiple pole. So from (3.24) we can infer that ϕ and ψ are entire functions.

Since $\psi^{(1)} - \phi\psi \equiv 0$, from (3.23) we get

$$L^{(2)} = (1-\psi)L^{(1)}. \tag{3.25}$$

Since ψ is entire, (3.25) implies that $L^{(1)}$ has no zero, and so $L^{(1)} = e^h$, where h is an entire function. Since f and so $L^{(1)}$ is of finite order, h is a polynomial. From (3.25) we get that $\psi = 1 - h^{(1)}$ is also a polynomial. Since ϕ is entire, (3.24) implies that ψ is a constant and so $\phi \equiv 0$, which is a contradiction. Therefore $\psi^{(1)} - \phi\psi \not\equiv 0$.

From (3.23) we get

$$f = a + \frac{1-\psi}{\psi^{(1)} - \phi\psi} L^{(1)} \Big\{ 1 - \frac{L^{(2)}}{(1-\psi)L^{(1)}} \Big\},$$

and so

$$m(r,f) \le m(r,L^{(1)}) + S(r,f) \le m(r,L) + S(r,f) \le m(r,f) + S(r,f).$$

Therefore,

$$T(r,f) = T(r,L) + S(r,f) = T(r,L^{(1)}) + S(r,f). \tag{3.26}$$

Eliminating $f-a$ from (3.21) and (3.23) we get

$$L = \frac{\psi^{(1)} + \psi - \psi^2 - \phi\psi}{\psi^{(1)} - \phi\psi} L^{(1)} - \frac{\psi}{\psi^{(1)} - \phi\psi} L^{(2)}.$$

Hence $m(r, L/(L-a)) = S(r,f)$ and so $m(r,a;L) = S(r,f)$. Since $m(r,a;f^{(1)}) + m(r,a;L) = S(r,f)$, we get from (3.20) and (3.26)

$$N(r,a;f^{(1)}) = N(r,a;L) + S(r,f). \tag{3.27}$$

We now suppose that $L \not\equiv f^{(1)}$. Then $\chi = \frac{L-f^{(1)}}{f-a} \not\equiv 0$, and by (2.7) we get $T(r,\chi) = S(r,f)$.

First we suppose that $\chi \not\equiv 1$. Then $L^{(1)} - f^{(2)} = \chi f^{(1)} + \chi^{(1)}(f-a)$. Let $z_0 \in C$ be a multiple a-point of $f^{(1)}$ that is not a pole of χ. Then from above we see that

https://doi.org/10.4153/CMB-2011-198-7 Published online by Cambridge University Press

$\chi(z_0) = 1$. So $\overline{N}_C(r, a; f^{(1)} \mid\geq 2) \leq N(r, 1; \chi) + N(r, \infty; \chi) = S(r, f)$. Also in view of (3.27) we note that $N(r, a; f^{(1)} \mid L \neq a) = S(r, f)$.

Let $\overline{E}(a; f) \neq \varnothing$. Then by Lemma 2.3 we get $\overline{N}(r, a; L^{(1)} \mid f \neq a) = S(r, f)$. We put $X = \{\overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(1)})\} \backslash \overline{E}(a; f)$. Then

$$N_X(r, a; f^{(1)}) \leq (n+1)\overline{N}_X(r, a; f^{(1)}) \leq (n+1)N(r, a; L^{(1)} \mid f \neq a) = S(r, f).$$

We put $Y = \{\overline{E}(a; L) \cap \overline{E}(a; f^{(1)})\} \backslash \overline{E}(a; f)$. If $z_0 \in Y$, then clearly $\chi(z_0) = 0$. So

$$N_Y(r, a; f^{(1)}) \leq (n+1)\overline{N}_Y(r, a; f^{(1)}) \leq (n+1)N(r, 0; \chi) = S(r, f).$$

We now put $Z = \{\overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L)\} \backslash \overline{E}(a; L^{(1)})$. Then

$$N_Z(r, a; f^{(1)}) \leq (n+1)\overline{N}_Z(r, a; f^{(1)}) \leq (n+1)N_A(r, a; f) = S(r, f).$$

Therefore,

$$\begin{aligned} N(r, a; f^{(1)} \mid\geq 2) \leq N_C(r, a; f^{(1)} \mid\geq 2) &+ N_X(r, a; f^{(1)} \mid\geq 2) + N_Y(r, a; f^{(1)} \mid\geq 2) \\ &+ N_Z(r, a; f^{(1)} \mid\geq 2) + N(r, a; f^{(1)} \mid L \neq a) \\ &\leq (n+1)\overline{N}_C(r, a; f^{(1)} \mid\geq 2) + S(r, f) = S(r, f). \end{aligned}$$

Let $\overline{E}(a; f) = \varnothing$. Since f is of finite order, we can put $f = a + e^p$, where p is a non-constant polynomial. Then

$$N(r, a; f^{(1)} \mid\geq 2) \leq 2N(r, 0; f^{(2)}) = 2N(r, 0; (p^{(1)})^2 + p^{(2)}) = S(r, f).$$

Now we suppose that $\chi \equiv 1$. Then

$$L \equiv f^{(1)} + f - a. \tag{3.28}$$

Differentiating (3.28) and using (3.22) we get

$$f^{(2)} = \phi(f - a). \tag{3.29}$$

- Let $z_0 \in \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L)$. Then from (3.28) we see that $z_0 \in \overline{E}(a; f)$. Hence $\overline{E}(a; f^{(1)}) \cap \overline{E}(a; L) \subset \overline{E}(a; f)$.
- Let z_0 be a multiple a-point of $f^{(1)}$ and an a-point of L. Then z_0 is a simple a-point of f and so in view of (3.28) z_0 is a simple a-point of L.
- Let z_0 be a simple a-point of $f^{(1)}$ and an a-point of L. Then z_0 is a simple a-point of f, and so by hypothesis z_0 is not a pole of ϕ. Then from (3.29) we get $f^{(2)}(z_0) = 0$, which is a contradiction.
- Let z_0 be a multiple a-point of L and an a-point of $f^{(1)}$. Then z_0 is a simple a-point of f and so by hypothesis z_0 is not a pole of ϕ. So from (3.29) we get $f^{(2)}(z_0) = 0$ and z_0 is a multiple a-point of $f^{(1)}$. Hence (3.28) implies that z_0 is a multiple a-point of f, which is a contradiction.

Now using (3.27) we get

$$N(r,a;f^{(1)}) = N(r,a;f^{(1)} \mid L=a) + S(r,f) \geq 2N(r,a;L \mid f^{(1)}=a) + S(r,f)$$
$$= 2N(r,a;L) + S(r,f) = 2N(r,a;f^{(1)}) + S(r,f),$$

and so $N(r,a;f^{(1)}) = S(r,f)$. This implies that $N(r,a;f^{(1)} \mid\geq 2) = S(r,f)$.

Since $N(r,a;f^{(1)} \mid\geq 2) = S(r,f)$, in view of (3.27) we obtain

$$N(r,a;f^{(1)}) \leq N\Big(r,1;\frac{L}{f^{(1)}}\Big) + S(r,f) \leq T\Big(r,\frac{L}{f^{(1)}}\Big) + S(r,f)$$
$$= N\Big(r,\frac{L}{f^{(1)}}\Big) + S(r,f) \leq N(r,0;f^{(1)}) + S(r,f). \tag{3.30}$$

Using (3.20) we can achieve (2.9). Since $m(r,a;f^{(1)}) = S(r,f)$, by (2.9), (3.20), and (3.30), we get $T(r,f) = T(r,f^{(1)}) + S(r,f) \leq N(r,a;f) + S(r,f)$, and so $m(r,a;f) = S(r,f)$. Hence by Lemma 2.1 we get $f \equiv L$, which is impossible as $\phi \not\equiv 0$. Therefore, $L \equiv f^{(1)}$ and so $L = a_1L + a_2L^{(1)} \cdots + a_nL^{(n-1)}$ and $L^{(1)} = a_1L^{(1)} + a_2L^{(2)} + \cdots + a_nL^{(n)}$. Since $|1-a_1| + |a_2| \neq 0$, we get $m(r, L^{(1)}/(L^{(1)}-a)) = S(r,f)$, which implies $m(r,a;L^{(1)}) = S(r,f)$. Since $m(r,a;L) = S(r,f)$, by (3.26) we get

$$N(r,a;L^{(1)}) = N(r,a;L) + S(r,f). \tag{3.31}$$

In view of (3.31) we get $N(r,a;L \mid\geq 2) \leq N(r,a;L \mid L^{(1)} \neq a) = S(r,f)$, and so

$$N(r,a;L) \leq N\Big(r,1;\frac{L^{(1)}}{L}\Big) + S(r,f) \leq T\Big(r,\frac{L^{(1)}}{L}\Big) + S(r,f)$$
$$= N\Big(r,\frac{L^{(1)}}{L}\Big) + S(r,f) \leq N(r,0;L) + S(r,f). \tag{3.32}$$

Also by (3.26) we get

$$m(r,a;f) \leq m\Big(r,\frac{L}{f-a}\Big) + m(r,0;L)$$
$$= T(r,L) - N(r,0;L) + S(r,f) = T(r,f) - N(r,0;L) + S(r,f)$$

and so

$$N(r,0;L) \leq N(r,a;f) + S(r,f). \tag{3.33}$$

So using (3.26), (3.32), and (3.33) we obtain

$$m(r,a;f) = T(r,f) - N(r,a;f) + S(r,f) = T(r,L) - N(r,a;f) + S(r,f)$$
$$= N(r,a;L) + m(r,a;L) - N(r,a;f) + S(r,f) \leq S(r,f).$$

Hence by Lemma 2.1 we get $f = L = \alpha e^z$, where α is a non-zero constant. This contradicts the fact that $\phi \not\equiv 0$ and proves the theorem. ■

Acknowledgment The authors are thankful to the referee for carefully reading the manuscript and giving valuable suggestions towards improvement of the paper.

References

[1] W. K.Hayman, *Meromorphic functions.* Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[2] G. Jank, E. Mues, and L. Volkmann, *Meromorphe Funktionen, die mit ihrer ersten und zweiten Ableitung einen endlichen wert teilen.* Complex Variables Theory Appl. **6**(1986), no. 1, 51–71.

[3] I. Lahiri and G. K. Ghosh, *Entire functions sharing one value with linear differential polynomials.* Analysis (Munich) **31**(2011), no. 4, 331–340. http://dx.doi.org/10.1524/anly.2011.1125

[4] P. Li, *Value sharing on entire functions and their linear differential polynomials.* J. Univ. Sci. Technol. China **29**(1999), no. 4, 387–393.

[5] W. Wang and P. Li, *Unicity of entire functions and their linear differential polynomials.* Complex Var. Theory Appl. **49**(2004), no. 11, 821–832. http://dx.doi.org/10.1080/02781070412331298552

University of Kalyani, Department of Mathematics, West Bengal 741235, INDIA
e-mail: ilahiri@hotmail.com

Baharampur College, Department of Mathematics, Gupta Para, Baharampur, West Bengal 742101, INDIA
e-mail: imrulksh3@gmail.com