Proceedings of the Edinburgh Mathematical Society (2013) **56**, 509–513
DOI:10.1017/S0013091512000272

BOUND FOR THE ORDER FOR P-ELEMENTARY SUBGROUPS IN THE PLANE CREMONA GROUP OVER A PERFECT FIELD

A. L. FOMIN

Department of Higher Algebra, Faculty of Mechanics and Mathematics, Moscow State Lomonosov University, Vorobievy Gory, Moscow 119899, Russia (fin-al@yandex.ru)

(Received 4 August 2010)

Abstract We obtain a sharp bound for p-elementary subgroups in the Cremona group $\mathrm{Cr}_2(k)$ over an arbitrary perfect field k.

Keywords: Cremona group; finite subgroups; p-groups

2010 *Mathematics subject classification:* Primary 14E07

1. Introduction

Let k be a field. The plane Cremona group $\mathrm{Cr}_2(k)$ over k is the group of birational transformations of $\mathbb{P}^2$ that are defined over k, or equivalently the group of k-automorphisms of the field $k(x, y)$. The study of finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ has a history of nearly one and a half centuries. But dealing with fields k, which are not algebraically closed, started only a few years ago, in [**2**].

A finite abelian group A is called a *p-elementary group*, where p is a prime number, if $A \cong (\mathbb{Z}/p)^r$; r is called the rank of A and is denoted by $\operatorname{rank} A$. In [**1**], Beauville classified maximal p-elementary subgroups in $\mathrm{Cr}_2(k)$ over an algebraically closed field k of arbitrary characteristic up to conjugacy. The purpose of the present paper is to find a sharp bound for p-elementary subgroups in the plane Cremona group $\mathrm{Cr}_2(k)$ over an arbitrary perfect field k.

For a perfect field k, denote by $\bar{k}$ its algebraic closure and set $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$. For a prime number p it is always assumed that $p \neq \mathrm{Char}(k)$. Note that in the case $p = \mathrm{Char}(k)$ there exist groups isomorphic to $(\mathbb{Z}/p)^r$ in $\mathrm{Cr}_2(k)$ for any $r > 0$ (for instance the group generated by $(x, y) \mapsto (x, y + x^q)$, $q = 1, \dots, r$). Define $t = [k(\zeta_p) : k]$, where $\zeta_p \in \bar{k}$ is any primitive root of unity of degree p. It is clear that t divides $p - 1$.

Our main result is the following.

Theorem 1.1. *Let $A \subset \mathrm{Cr}_2(k)$ be a p-elementary subgroup, where k is a perfect field. Then*

$$\operatorname{rank} A \leqslant \begin{cases} 4 & \text{if } p = 2, \\ 3 & \text{if } p = 3,\ t = 1, \\ 2 & \text{if } p = 3,\ t = 2 \text{ and } p > 3,\ t = 1, 2, \\ 1 & \text{if } t = 3, 4, 6, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Moreover, this bound is attained for any $p \neq \mathrm{Char}(k)$.

2. Bounds for a p-torsion subgroup of a torus

2.1. Let T be an algebraic torus of dimension d defined over k. In [**3**], Serre obtained a sharp bound for the order of finite p-subgroups in $T(k)$. Below we give a similar bound for p-elementary subgroups.

Theorem 2.1. *In the notation above, $\operatorname{rank} T(k)[p] \leqslant d/\varphi(t)$, where $T(k)[p]$ is a p-torsion subgroup of $T(k)$ and φ is Euler's function. Moreover, this bound is attained for a suitable torus defined over k.*

Proof. Let $\mathrm{X}(T)$ and $\Upsilon(T)$ be the groups of characters and cocharacters of T over $\bar{k}$, where $\rho\colon \Gamma_k \to \mathrm{Aut}(\Upsilon(T))$ is the action of the Galois group and $\rho_p\colon \Gamma_k \to \mathrm{Aut}(\Upsilon(T)/p)$ is its reduction modulo p. In addition, let $\boldsymbol{\mu}_p \subset \bar{k}^*$ be the group of the roots of unity of degree p, and let $\chi\colon \Gamma_k \to \mathrm{Aut}(\boldsymbol{\mu}_p) \cong (\mathbb{Z}/p)^*$ be the action of the Galois group.

It is clear that

$$T(k)[p] = T(\bar{k})[p]^{\Gamma_k} \quad \text{and} \quad T(\bar{k})[p] \cong \mathrm{Hom}(\mathrm{X}(T)/p, \boldsymbol{\mu}_p) \cong \Upsilon(T)/p \otimes \boldsymbol{\mu}_p,$$

with all isomorphisms being compatible with the actions of the Galois group. Obviously,

$$\operatorname{rank}(\Upsilon(T)/p \otimes \boldsymbol{\mu}_p)^{\Gamma_k} \leqslant \operatorname{rank}(\Upsilon(T)/p \otimes \boldsymbol{\mu}_p)^g \quad \text{for any } g \in \Gamma_k$$

and g acts on $\Upsilon(T)/p \otimes \boldsymbol{\mu}_p$ as $\rho_p(g) \otimes \chi(g) = \chi(g)\rho_p(g) \otimes 1$. Using any isomorphism $\boldsymbol{\mu}_p \cong \mathbb{Z}/p$ and $\Upsilon(T)/p \otimes \boldsymbol{\mu}_p \cong \Upsilon(T)/p$, it is possible to identify the set of fixed points of g in $\Upsilon(T)/p \otimes \boldsymbol{\mu}_p$ with the set of fixed points of $\chi(g)\rho_p(g)$ in $\Upsilon(T)/p$, which is merely the eigenspace of $\rho_p(g)$ with eigenvalue $\chi(g)^{-1}$.

We fix $g \in \Gamma_k$ such that $\chi(g)$ is of order t and set $\chi(g)^{-1} = \varepsilon$. Since $\rho(g)$ has finite order, its characteristic polynomial F is the product of cyclotomic polynomials, $F = \prod_i \Phi_{d_i}$, and the characteristic polynomial of $\rho_p(g)$ is $\bar{F} = \prod_i \bar{\Phi}_{d_i}$, where $\bar{\Phi}$ denotes the reduction a polynomial Φ modulo p. To prove the theorem, we need to find an upper bound for the multiplicity of ε as the root of $\bar{\Phi}_{d_i}$.

Lemma 2.2. *In the above notation, the multiplicity of $\varepsilon \in (\mathbb{Z}/p)^*$ as the root of $\bar{\Phi}_n$ is the same for all ε of the fixed order t, and it is positive if and only if $n = tp^f$.*

https://doi.org/10.1017/S0013091512000272 Published online by Cambridge University Press

Proof of Lemma 2.2. First, if $p \mid n$ and $q = p^f$, then $\bar{\Phi}_{nq} \equiv \bar{\Phi}_n^{\varphi(q)} \pmod{p}$, so we can assume that $p \nmid n$.

Let $\mathcal{O}$ be the integral closure of $\mathbb{Z}$ in the field $\mathbb{Q}(\zeta_n)$, where $\zeta_n \in \mathbb{C}$ is any primitive root of unity of degree n, $\boldsymbol{\mu}_n \subset \mathcal{O}^*$ is the group of the roots of unity of degree n and $\mathfrak{p} \subset \mathcal{O}$ is any prime ideal such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. Then

$$\Phi_n(X) = \prod_{\zeta}(X - \zeta) \quad \text{and} \quad \bar{\Phi}_n(X) = \prod_{\zeta}(X - \bar{\zeta})$$

in $\mathcal{O}/\mathfrak{p}$, where ζ runs through all primitive roots of unity of degree n. It is well known that the natural map $\boldsymbol{\mu}_n \to (\mathcal{O}/\mathfrak{p})^*$ is injective, so $\bar{\zeta}$ is of order n in $(\mathcal{O}/\mathfrak{p})^*$ for any ζ. This implies that the set of roots of $\bar{\Phi}_n$ in $\mathcal{O}/\mathfrak{p}$ coincides with the set of all elements of order n in $(\mathcal{O}/\mathfrak{p})^*$.

Suppose that $\bar{\Phi}_n$ has a root $\varepsilon \in (\mathbb{Z}/p)^*$ of order t; then $t = n$ and any element of order t in $(\mathbb{Z}/p)^*$ is a simple root of $\bar{\Phi}_n$. This proves all statements of the lemma. □

Going back to the proof of Theorem 2.1 we see that it follows from the above lemma that the multiplicity of ε as the root of $\bar{\Phi}_{d_i}$ is bounded from above by $\varphi(d_i)/\varphi(t)$, and its multiplicity as the root of $\bar{F}$ is bounded from above by $d/\varphi(t)$, since $\sum_i \varphi(d_i) = d$.

To prove the second statement of Theorem 2.1, it is enough to construct a torus of dimension $d = \varphi(t)$ defined over k such that $\operatorname{rank} T(k)[p] > 0$. This is done in [**3**] (see the proof of Theorem 4′ therein). □

3. Proof of the main theorem

In this section we prove Theorem 1.1.

3.1. Let $A \subset \mathrm{Cr}_2(k)$ be a p-elementary subgroup. It is known [**2**, Theorem 5] that A can be represented as a subgroup of $\mathrm{Aut}_k(S)$, where S is a smooth projective surface defined and rational over k, which is of one of the following two types.

(i) There exists an A-equivariant conic bundle structure $f\colon S \to C$, where C is a smooth curve of genus 0, such that $\operatorname{rank}\operatorname{Pic}(S/C)^A = 1$ (though we do not need this fact, note that if S is rational over k, then $C \cong \mathbb{P}^1$ over k since $S(k) \neq \varnothing$ and thus $C(k) \neq \varnothing$).

(ii) S is a Del Pezzo surface such that $\operatorname{rank}\operatorname{Pic}(S)^A = 1$.

Proposition 3.1. *If $p \nmid n$, any p-elementary subgroup $A \subset G(k)$, where G is a k-form of PGL_n, is contained in a maximal torus defined over k.*

Proof. This statement was proved in [**1**, Lemma 3.1] for $k = \bar{k}$. The centralizer of A in G, which is defined over k as A itself is, contains a maximal torus defined over k, which is the maximal torus in G. Since A consists of semisimple elements, any maximal torus that centralizes A must contain it. □

https://doi.org/10.1017/S0013091512000272 Published online by Cambridge University Press

3.2. In what follows we shall study all possible cases for $\operatorname{rank} A$ in order to find in each case the restrictions on t and then we shall prove that under the restrictions obtained such an A exists. The case $p = 2$ will be dealt with separately, as it does not involve the value of t.

3.3. Suppose that $\operatorname{rank} A \geqslant 1$. It was proved in [**2**, Theorem 2] that in this case $t \in \{1, 2, 3, 4, 6\}$ and, moreover, for these values of t there is an element of order p in $A \subset \operatorname{Cr}_2(k)$.

3.4. Suppose that $\operatorname{rank} A \geqslant 2$. We shall prove that $t \leqslant 2$. We can assume that $p > 3$ (as otherwise there is nothing to prove) and that A is a subgroup of $\operatorname{Aut}_k(S)$ as it is described above. Define $\bar{S} = S \otimes \bar{k}$. We have two possibilities for S specified in § 3.1.

Let $f\colon S \to C$ be an A-equivariant conic bundle. The action of A on the base defines the homomorphism $A \to \operatorname{Aut}_k(C)$. Denote by $\bar{A}$ its image and by A_0 its kernel. Obviously, A_0 is an automorphism group of the generic fibre of f, which is a smooth curve of genus 0 over the field $k(C)$. The automorphism group of the base is a k-form of PGL_2, and the automorphism group of the generic fibre is a $k(C)$-form of PGL_2. It is readily seen that t has the same value for k and $k(C)$. Since p is odd, it follows from Proposition 3.1 that $\bar{A}$ and A_0 are contained in tori of dimension 1 defined over k and $k(C)$, respectively. Theorem 2.1 yields that $\operatorname{rank} A_0 \leqslant 1$ and $\operatorname{rank} \bar{A} \leqslant 1$, with the equality being possible only if $t \leqslant 2$. Finally, we obtain that $\operatorname{rank} A \leqslant 2$, and the equality implies that $t \leqslant 2$.

Let S be a Del Pezzo surface. It follows from [**1**, Proposition 3.9] and [**2**, Theorem 5] that $9 \geqslant K_S^2 \geqslant 6$ and $K_S^2 \neq 7$. We consider the possibilities for K_S^2 case by case.

(i) If $K_S^2 = 9$, then $\bar{S} \cong \mathbb{P}^2$. Therefore, $\operatorname{Aut}(S)$ is a k-form of PGL_3 and Proposition 3.1 gives that A is contained in a torus of dimension 2 defined over k. According to Theorem 2.1 this is possible only if $t \leqslant 2$.

(ii) If $K_S^2 = 8$, then $\bar{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$ (otherwise $\bar{S}$ contains a unique (-1)-curve which must be defined over k; this contradicts $\operatorname{rank} \operatorname{Pic}(S)^A = 1$). Then the connected component $\operatorname{Aut}(S)^\circ$ is a k-form of $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ of index 2 in $\operatorname{Aut}(S)$. It is clear that $A \subset \operatorname{Aut}(S)^\circ$ since $p > 3$, and by Proposition 3.1 A is contained in a torus of dimension 2, and thus $t \leqslant 2$.

(iii) If $K_S^2 = 6$, then the connected component $\operatorname{Aut}(S)^\circ$ is a two-dimensional torus and $\operatorname{Aut}(S)/\operatorname{Aut}(S)^\circ \otimes \bar{k} \cong S_3 \times \mathbb{Z}/2$. As above, $A \subset \operatorname{Aut}(S)^\circ$ since $p > 3$, and we obtain that $t \leqslant 2$.

Now we prove that there exists a p-elementary subgroup of rank 2 in $\operatorname{Cr}_2(k)$ whenever $t \leqslant 2$. Applying Theorem 2.1, we obtain that for such t there exists a two-dimensional torus T defined over k such that $T(k)$ contains a p-elementary subgroup A of rank 2. Thus, the well-known fact that T is rational over k [**4**, § 4.9] yields that $A \subset \operatorname{Cr}_2(k)$.

3.5. Suppose now that $\operatorname{rank} A \geqslant 3$ and p is odd. It is shown in [**1**, Propositions 2.6 and 3.10] that $p = 3$, $\operatorname{rank} A = 3$ and S must be a cubic surface in $\mathbb{P}^3$. We claim that $t = 1$.

It follows from Proposition 3.1 that $A \subset T(k)$, where $T \subset \mathrm{PGL}_4$ is a maximal torus defined over k. We use notation from the proof of Theorem 2.1. Since PGL_4 is a group of inner type, for any $g \in \varGamma_k$, $\rho(g)$ acts on $\varUpsilon(T)$ as an element of the Weyl group. Let $F = \prod_i \varPhi_{d_i}$ be the characteristic polynomial of $\rho(g)$ and let $\bar{F} = \prod_i \bar{\varPhi}_{d_i}$ be its reduction modulo 3. Note that each d_i divides one of the invariant degrees of the Weyl group; therefore, each $d_i \in \{1, 2, 3, 4\}$. Suppose that $t = 2$; then the multiplicity of $-1 \in (\mathbb{Z}/3)^*$ as the root of $\bar{F}$ is equal to 3. It follows easily from Lemma 2.2 that each $d_i = 2$ and $F(X) = (X + 1)^3$. Since $\rho(g)$ has finite order, $\rho(g) = -1$, but it is well known that -1 does not belong to the Weyl group of PGL_4. So we conclude that the case $t = 2$ is impossible. This completes the proof of (1.1) for $p > 2$.

To prove the second statement of Theorem 1.1 in the case $p = 3$ and $t = 1$, i.e. k contains the primitive cubic root of unity, consider the Fermat cubic given by equation $X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$ in $\mathbb{P}^3$. It is rational over k and evidently admits the action of 3-elementary group A with $\operatorname{rank} A = 3$, so $A \subset \mathrm{Cr}_2(k)$.

3.6. Finally, suppose that $p = 2$. It was proved in [**1**, Propositions 2.6 and 3.11] that $\operatorname{rank} A \leqslant 4$. On the other hand, $\mathbb{P}^1$ admits $(\mathbb{Z}/2)^2$ as the automorphism group for every field k; hence, there exists an action of the group $A \cong (\mathbb{Z}/2)^4$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $A \subset \mathrm{Cr}_2(k)$. This completes the proof of the main theorem.

Acknowledgements. The author was supported in part by Leading Scientific Schools Grant 4713.2010.1.

References

1. A. Beauville, *p*-elementary subgroups of the Cremona group, *J. Alg.* **314** (2007) 553–564.
2. I. V. Dolgachev and V. A. Iskovskikh, On elements of prime order in the plane Cremona group over a perfect field, *Int. Math. Res. Not.* **18** (2009), 3467–3485.
3. J.-P. Serre, Bounds for the order of finite subgroup of $G(k)$, in *Group representation theory* (ed. M. Geck, D. Testerman and J. Thévenaz), Fundamental Sciences (EPFL Press, Lausanne, 2006).
4. V. E. Voskresenskii, *Algebraic groups and their birational invariants*, Translations of Mathematical Monographs, Volume 179 (American Mathematical Society, Providence, RI, 1998).

https://doi.org/10.1017/S0013091512000272 Published online by Cambridge University Press