GROUPS OF POSITIVE OPERATORS

H. A. DYE AND R. S. PHILLIPS

1. Introduction. Semi-groups of bounded positive operators on certain function spaces enter the theory of stochastic processes of the diffusion type in an essential way. It is a matter of experience that these semi-groups cannot be imbedded in groups of positive operators, or, in more special terms, that the solution of a diffusion equation does not define a one-parameter group of positive operators on the natural function space. The present work originated with an effort to explain this circumstance by showing, under appropriate conditions, that a group of positive operators will solve only a first order partial differential equation (see §3). Aspects of this problem, however, pointed the way to a general study of group representations by bounded positive operators on $C_0(X)$, the space of real-valued continuous functions vanishing at infinity on a locally compact Hausdorff space X. A typical problem arising here, for example, was that of determining when a bounded positive group representation on $C_0(X)$ is equivalent to a pure flow (or isometric) representation. In the main, then, this paper deals with general questions of this type.

The existence of a certain canonical factorization of elements in a group of positive operators provides the technical basis for our study. Expressly, any representation $\sigma \to U_\sigma$ of a group G by bounded positive operators on $C_0(X)$ splits into a product $U_\sigma = L_{\theta(\cdot,\sigma)} T_\sigma$ of a flow representation $\sigma \to T_\sigma$ of G by isometries of $C_0(X)$ and pointwise multiplications by functions in $P(X)$, the class of all positive continuous functions on X bounded away from 0 and infinity. In particular, the group of all positive operators on $C_0(X)$ belonging to a given flow splits into a semi-direct product of that flow by $P(X)$. While this theorem is not essentially new, its implications have not been studied extensively.

It develops that equivalence properties of the positive representations of G on $C_0(X)$ hinge on the analysis of certain functional identities. The multiplication factors $\theta(\cdot, \sigma)$ arising in the factorization of U_σ satisfy the characteristic identity $\theta(\cdot, \sigma\tau) = \theta(\cdot\sigma, \tau)\, \theta(\cdot, \sigma)$, and the representation $[U_\sigma]$ will be equivalent to a pure flow (in a natural sense) if and only if $\theta(\cdot, \sigma)$ has the form $\theta(\cdot, \sigma) = g(\cdot)/g(\cdot\sigma)$, for some g in $P(X)$. To provide a natural algebraic vehicle for this analysis, certain elementary notions from the cohomology theory of Eilenberg and MacLane **(2)** are discussed in §4. Algebraic techniques suggested by this theory, and involving in particular cohomology group $H^1(G, P(X))$, are employed variously throughout the rest of the paper.

Received July 25, 1955. Portions of this research were supported by the Office of Ordnance Research, U.S. Army, under Contracts DA-36-034-ORD-1292 and DA-04-495-ORD-613. In addition, the investigation was started while the second named author held a John Simon Guggenheim Fellowship.

Our main results in this direction are contained in §§5, 6, and 7. In §5, we prove that a bounded positive representation belonging to an ergodic flow is already equivalent to that flow, and show by example that bounded representations are not in general equivalent to flows. In §6, we study the automorphism group of the group of all positive operators on $C_0(X)$ belonging to a given flow. Theorems here concern the semi-direct product structure of the group of bounded automorphisms, and the characterization of the group of flow-related automorphisms modulo inner automorphisms. In §7, we show that the adjoint representation of a given strongly continuous bounded positive representation of a topological group G on $C_0(X)$ will be equivalent to the adjoint of the flow representation provided only a Borel measurable factorization of the multiplication factor $\theta(\cdot, \sigma)$ exists. Equivalence of the adjoint representations of two positive representations $[U_\sigma]$ and $[V_\sigma]$ of G on $C_0(X)$ implies that the spectra of U_σ and V_σ coincide.

The appendix contains an application of the foregoing theory to semi-groups of operators. Two one-parameter groups of operators are exhibited with the property that the sum of their infinitesimal operators has no extension generating even a semi-group or operators.

2. Factorization of positive operators. Let $L(X)$ [resp., $C_0(X)$] denote the algebra of all real-valued continuous functions with compact support [resp., vanishing at infinity] on the locally compact Hausdorff space X. We take these spaces with the customary norm on continuous functions, namely

$$||f|| = \sup_{x \epsilon X} |f(x)|,$$

so that $L(X)$ is dense in $C_0(X)$. A *positive operator* on $L(X)$ [resp. $C_0(X)$] is by definition an everywhere defined linear transformation of $L(X)$ $[C_0(X)]$ into itself which carries non-negative functions into non-negative functions. Our discussions will center on the class of bounded positive operators which have bounded positive inverses—a decisive restriction—and in this section, we derive the basic factorization theorem cited in the introduction.

LEMMA 2.1. *Let U be a bounded regular operator on $L(X)$ [resp. $C_0(X)$], which together with its inverse is positive. Then there exists a positive continuous function $p(\cdot)$ on X, bounded away from 0 and ∞, and a homeomorphism $x \to x\sigma$ of X such that*

$$U = L_p T_\sigma, \tag{2.1}$$

where L_p denotes pointwise left multiplication by $p(\cdot)$ and T_σ is the automorphism $(T_\sigma f)(x) = f(x\sigma)$ of $L(X)[C_0(X)]$ implemented by σ. Components in this factorization are uniquely determined, and one has $||U|| = ||p||$.

Proof. Suppose that U is a bounded positive operator on $C_0(X)$ having a bounded positive inverse. We prove first the useful fact that U maps $L(X)$ onto itself. For this, it suffices to prove that Uf lies in $L(X)$ for each f in the positive cone of $L(X)$, namely $L^+(X)$. Consider then an $f \in L^+(X)$ and choose an $h \in L^+(X)$ which assumes the value 1 on all of the support of f. We approximate to Uh by a function $g \in L^+(X)$ chosen so that

$$||Uh - g|| < (2||U^{-1}||)^{-1}, \; 0 \leqslant g \leqslant Uh.$$

We have $0 \leqslant U^{-1}g \leqslant h$ and $||U^{-1}g - h|| < \frac{1}{2}$. It follows that $U^{-1}g > \frac{1}{2}$ on the support of f, and therefore, the support of Uf lies in the (compact) support of g, proving our assertion. In view of this fact, it will clearly suffice to prove the theorem for $L(X)$.

We next show that there exists a one-to-one map of X on itself, $x \rightarrow x\sigma$, such that $g(x\sigma) = 0$ if and only if $Ug(x) = 0$ for each $g \in L(X)$. Again it is clear that we can restrict our considerations to $L^+(X)$. We further note that U and its inverse being positive implies that U describes a linear order isomorphism in $L(X)$ so that $U(f \vee g) = Uf \vee Ug$. Now for fixed $x \in X$, we set

$$I \equiv [h; h \in L^+(X), \quad Uh(x) = 0].$$

It follows from the above remarks that I is a closed positive cone, closed with respect to the lattice operation $\vee$, and neither empty nor all of $L^+(X)$. Let $Z(h) \equiv [y; h(y) = 0]$. If $h_1, h_2 \in I$, then $h_1 \vee h_2 \in I$ and

$$Z(h_1 \vee h_2) = Z(h_1) \wedge Z(h_2).$$

Consequently if $F \equiv \cap [Z(h); h \in I]$ is disjoint from a given compact set C, then there is an $h \in I$ with $Z(h) \cap C = \phi$ (the null set). Now if $g \in L^+(X)$ has C as its support, then $0 \leqslant g \leqslant \alpha h$ for α sufficiently large and therefore $0 \leqslant Ug(x) \leqslant \alpha Uh(x) = 0$ so that $g \in I$. Since I is a proper subset of $L^+(X)$, it follows from this that F is necessarily non-empty. On the other hand, F can contain no more than one point. For if $y_1, y_2 \in F$, it is easy to construct functions $k_1, k_2 \in L^+(X)$ such that $k_i(y_i) > 0$, $i = 1, 2$, and $k_1 \wedge k_2 = 0$. Thus

$$0 = U(k_1 \wedge k_2)(x) = [Uk_1(x)] \wedge [Uk_2(x)].$$

Consequently either k_1 or k_2 lies in I, so that F cannot contain both y_1 and y_2. Denote the single point in F by $x\sigma$. We see that $Ug(x) = 0$ implies $g(x\sigma) = 0$, and $x\sigma$ is the only point for which this assertion holds for all $g \in L^+(X)$. On the other hand if $f \in L^+(X)$ and f vanishes identically in a neighborhood of $x\sigma$, then a compact support for f is disjoint from $F = \{x\sigma\}$ and hence as above $f \in I$. Now any $g \in L^+(X)$ with $g(x\sigma) = 0$ can be approximated in norm by functions of the type f and hence any such g belongs to I; that is $g(x\sigma) = 0$ implies $Ug(x) = 0$. Finally to show that σ maps X onto itself we have only to derive the corresponding assertions for U^{-1} and note that these involve σ^{-1} in place of σ.

For each $x \in X$ choose a $g_x \in L(X)$ so that $g_x(x\sigma) = 1$; set $p(x) = Ug_x(x)$. Then for any $f \in L(X)$, $f - f(x\sigma)\, g_x$ vanishes at $x\sigma$ and therefore

$$Uf(x) = f(x\sigma)\, Ug_x(x) = p(x) f(x\sigma).$$

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

This is the desired representation of U. It follows from this representation that $p(x)$ is non-negative and bounded on X and, since

$$U^{-1}f(x) = [p(x\sigma^{-1})]^{-1} f(x\sigma^{-1}),$$

we see that $p(x)$ is also bounded away from 0. To prove that $x \to x\sigma$ is a homeomorphism let a neighborhood $N(x_0\sigma)$ be given and choose $f \in L^+(X)$ so that $N(x_0\sigma)$ is a support for f and $f(x_0\sigma) > 0$. Since $Uf \in L(X)$, the set

$$N(x_0) \equiv [x, p(x) f(x\sigma) > 0]$$

defines a neighborhood of x_0 with the property that $[N(x_0)]\sigma \subset N(x_0\sigma)$. The mapping σ is therefore continuous and a similar argument applied to U^{-1} establishes the continuity of σ^{-1}. It is now easy to prove that p is continuous on X. In fact if $N(x_0\sigma)$ is chosen to have a compact closure, then there exists an $f \in L^+(X)$ with $f(x) = 1$ for all $x \in N(x_0\sigma)$. In this case $p(x)$, which is identical with $Uf(x)$ in $N(x_0) \equiv N(x_0\sigma)\sigma^{-1}$, is seen to be continuous at at x_0. Finally we note that the uniqueness of the factorization (2.1) follows trivially from the fact that $L_pT_\sigma = I$ (identity operator) entails $p = 1$, $\sigma = e$ (the identity homeomorphism), and so all parts of the Lemma are proved.

When X is compact the Lemma implies the following result, due to Kadison **(6)**:

COROLLARY. *If X is compact, then any linear order isomorphism of $L(X)$ which conserves the identity is implemented by a homeomorphism of X.*

With this, we pass to a characterization of groups of positive operators. Some notation is needed. Given a group G and a topological space X, we say that G acts on X if a representation of G in the group of homeomorphisms of X is given. A *flow* of G in $C_0(X)$ (X locally compact) is a representation $\sigma \to T_\sigma$ of G by a group of isometries of $C_0(X)$. Given a flow G in $C_0(X)$, one can find an action $\sigma \to x\sigma$ of G on X such that $(T_\sigma f)(x) = f(x\sigma)$ for all f in $C_0(X)$ (cf. **(1)** and **(10)**).

THEOREM 2.1. *Let $\sigma \to U_\sigma$ be a representation of a group G by bounded positive operators on $C_0(X)$. These operators U_σ have a factorization*

$$U_\sigma = L_{\theta(\cdot,\,\sigma)}T_\sigma \tag{2.2}$$

where, for each σ, $\theta(\cdot, \sigma)$ is a positive continuous function on X, bounded away from 0 and infinity, and $\sigma \to T_\sigma$ is a flow representation of G on $C_0(X)$ implemented by an action $x \to x\sigma$ of G on X. These functions $\theta(x, \sigma)$ satisfy

$$\theta(x, \sigma\tau) = \theta(x\sigma, \tau)\,\theta(x, \sigma) \text{ and } \theta(x, e) \equiv 1. \tag{2.3}$$

If G is a topological group, and if the representation $\sigma \to U_\sigma$ is strongly continuous, then

(2.4) *the mapping $(x, \sigma) \to x\sigma$ is continuous on $X \times G$ to X, and*

(2.5) *the function $\theta(x, \sigma)$ is continuous on $X \times G$.*

Conversely, given an action $x \to x\sigma$ of G on X and a function $\theta(x, \sigma)$ subject to all the above conditions, then $\sigma \to U_\sigma = L_{\theta(\cdot, \sigma)} T_\sigma$ defines a strongly continuous representation of G on $L(X)$.

Proof. The representation identity $(U_\sigma)^{-1} = U_{\sigma^{-1}}$ assures that each U_σ has a bounded positive inverse, and therefore, the existence of the factorization (2.2) follows from the Lemma. Now,

$$L_{\theta(\cdot, \sigma\tau)} T_{\sigma\tau} = U_{\sigma\tau} = U_\sigma U_\tau = L_{\theta(\cdot, \sigma)} T_\sigma L_{\theta(\cdot, \tau)} T_\tau = L_{\theta(\cdot, \sigma)\theta(\cdot\sigma, \tau)} T_\sigma T_\tau.$$

By the uniqueness of factorization, therefore, $T_{\sigma\tau} = T_\sigma T_\tau$ and

$$\theta(\cdot, \sigma\tau) = \theta(\cdot\sigma, \tau)\, \theta(\cdot, \sigma),$$

proving the first part of (2.3). That $\theta(\cdot, e) = 1$ follows trivially from this identity.

We turn to the topological properties. First, given a compact C in X and an open $V \supset C$, we argue, there exists a neighborhood N of the identity e in G so that $CN \subset V$. In fact, choose an h in $L(X)$ which is 1 on C and 0 outside V, and then apply strong continuity to choose N so that

$$|\theta(\cdot, \sigma)\, h(\cdot\sigma) - h(\cdot)| < 1,$$

for all σ in N. Trivially, this entails $CN \subset V$.

With this, we can see that (2.4) holds: given x_0, σ_0, and a neighborhood V of $x_0\sigma_0$, choose a neighborhood U of x_0 with compact closure which satisfies $(U^-)\, \sigma_0 \subset V$. Now, by the preceding paragraph, choose a neighborhood N of e so that $(U^-)\, \sigma_0 N \subset V$. This proves (2.4).

For (2.5), note first that it suffices to prove joint continuity at each pair x, e (e the identity); in fact, given this, suppose $\lim x_\alpha = x_0$ and $\lim \sigma_\beta = e$. Then

$$\lim \theta(x_\alpha, \sigma_0\sigma_\beta) = \lim \theta(x_\alpha\sigma_0, \sigma_\beta) \cdot \lim \theta(x_\alpha, \sigma_0) = \theta(x_0, \sigma_0),$$

so joint continuity will be established in general. To prove joint continuity at x, e, consider any compact set C in X. Choose D compact with C in its interior, choose a symmetric neighborhood N of e in G so that $CN \subset D$, and finally, let f be a function in $L(X)$ which is 1 on D. Since $f(x\sigma) = 1$ on $C \times N$, we then have, for x in C and σ in N,

$$|\theta(x, \sigma) - 1| = |[\theta(x, \sigma) - 1] f(x\sigma)| \leqslant |\theta(x, \sigma) f(x\sigma) - f(x)| + |f(x\sigma) - f(x)|.$$

Shrinking N if necessary, we can arrange that the first term on the right be arbitrarily small for all $x \in C$ by appealing to the strong continuity of U_σ; the second term vanishes on C by our choice of f. Thus (2.5) is established.

The converse follows readily from the fact that, for any f in $L(X)$, we can choose a neighborhood N of e so that $|f(\cdot\sigma) - f(\cdot)| < \epsilon$, for all σ in N, together with the fact that the numbers $\theta(x, \sigma)$ will be uniformly close to 1, for x in the support of f and σ in some neighborhood of e. It may be noted here that the converse will hold with $C_0(X)$ replacing $L(X)$ if it is known that the functions $\theta(\cdot, \sigma)$ are uniformly bounded for σ in some neighborhood of the identity; this will be the case, for example, if G is locally compact.

3. The infinitesimal operator of a group representation. When G is the additive group of real numbers with the usual topology, then one can define an *infinitesimal operator* for a strongly continuous representation $[U_t; -\infty < t < \infty]$ of G by bounded positive operators on $C_0(X)$ as

$$\lim_{\eta\to 0} \frac{U_\eta - I}{\eta} f \equiv Af, \tag{3.1}$$

where the domain of A, in symbols $D(A)$, consists of all $f \in C_0(X)$ for which this limit exists. It can be shown (see (**5**, chap. IX)) that A is a closed linear operator with dense domain and that for $f \in D(A)$

$$\frac{d}{dt} U_t f = AU_t f, \qquad -\infty < t < \infty. \tag{3.2}$$

Thus if A happens to be a differential operator, then $u(t, x) \equiv U_t f(x)$ satisfies the differential equation

$$\frac{\partial}{\partial t} u(t, x) = [Au(t, \cdot)](x). \tag{3.3}$$

We shall now determine the precise form of the infinitesimal operator for the above group representation under the assumption that A is a differential operator. This requires a certain amount of specialization. In the first place the concept of a differential operator does not make sense unless X is a differentiable manifold. In general this is not in itself sufficient and we therefore make the following additional assumption, which has the effect of imposing a degree of local regularity on A.

ASSUMPTION D. *All functions of class $C^{(\infty)}$ with compact supports belong to $D(A)$.*

THEOREM 3.1. *Let $[U_t; -\infty < t < \infty]$ be a strongly continuous group of linear bounded positive operators on $C_0(X)$ to itself where X is an n-dimensional manifold of class $C^{(\infty)}$. If the domain of the infinitesimal operator satisfies Assumption D, then there exists a continuous scalar $\beta(x)$ and a continuous contravariant vector field $\bar{\alpha}(x)$ such that*

$$[Af](x) = \bar{\alpha}(x) \cdot \nabla f(x) + \beta(x) f(x), \qquad f \in D(A) \cap C^{(1)}, \tag{3.4}$$

where ∇ is the gradient operator.

Proof. Since this is a local problem, we may suppose that X is represented in a neighborhood $N(x_0)$ of a given $x_0 \in X$ by the euclidean coordinates $(x^1, x^2, \ldots, x^n)$. It follows from Assumption D that $D(A)$ contains a function $f_0(x)$ which is identically one in some neighborhood, say $N_1(x_0)$, of x_0. Hence making use of the representation (2.2) and the property (2.4), we see that there exists an $N_2(x_0)$ and a $\delta_1 > 0$ such that

$$\eta^{-1}[U_\eta f_0 - f_0](x) = \eta^{-1}[\theta(x, \eta) f_0(x\eta) - f_0(x)] = \eta^{-1}[\theta(x, \eta) - 1]$$

for all $x \in N_2(x_0)$ and $|\eta| < \delta_1$. Since $f_0 \in D(A)$, the incremental ratio $\eta^{-1}[U_\eta f_0 - f_0]$ converges in norm as $\eta \to 0$ and therefore

$$\frac{\partial}{\partial t}\theta(x, t)|_{t=0} \equiv \beta(x)$$

exists uniformly in $N_2(x_0)$. It follows that $\beta(x)$ is continuous in x; it is obvious that $\beta(x)$ does not depend on the local coordinate system.

The domain of A also contains functions $f_i(x) = x^i - x_0^i$ $(i = 1, 2, \ldots, n)$ in some neighborhood $N_3(x_0)$. Again by (2.4) there exists an $N_4(x_0)$ and a $\delta_2 > 0$ such that

$$\begin{aligned}\eta^{-1}[U_\eta f_i - f_i](x) &= \eta^{-1}[\theta(x, \eta) f_i(x\eta) - f_i(x)] \\ &= \eta^{-1}[\theta(x, \eta) - 1][(x\eta)^i - x_0^i] + \eta^{-1}[(x\eta)^i - x^i]\end{aligned}$$

for all $x \in N_4(x_0)$ and $|\eta| < \delta_2$. As before the limit exists uniformly in x as $\eta \to 0$ and since the first term in the right member converges to a limit we see that

$$\frac{\partial}{\partial t}(xt)^i|_{t=0} \equiv \alpha^i(x)$$

exists. The limit being uniform with respect to x in $N_4(x_0)$, it follows that $\alpha^i(x)$ is continuous in $N_4(x_0)$.

Finally suppose $f \in D(A) \cap C^{(1)}$. Then writing

$$\eta^{-1}[U_\eta f - f](x) = \eta^{-1}[\theta(x, \eta) - 1]f(x\eta) + \eta^{-1}[f(x\eta) - f(x)]$$

and passing to the limit as $\eta \to 0$, we obtain

$$[Af](x) = \beta(x)f(x) + \sum_{i=1}^{n} \alpha^i(x)\frac{\partial}{\partial x^i}f(x)$$

for all $x \in N_2(x_0) \cap N_4(x_0)$. We see from this expression that the $\alpha^i(x)$ are the components of a contravariant vector field in the above local coordinate system. This completes the proof.

We note in particular that the infinitesimal operator of $[U_t]$ cannot be a second order differential operator. As a consequence the solution to a diffusion equation can never define a strongly continuous group of linear bounded positive operators on $C_0(X)$.

4. Introducing the cohomology group $H^1(G, P)$. Consider two representations $[U_\sigma^{(1)}]$ and $[U_\sigma^{(2)}]$ of a group G on $L(X)$ by bounded positive operators. Write $P(X)$ (or simply P) for the class of all positive continuous functions on X which are bounded away from 0 and infinity. We call these representations $[U_\sigma^{(i)}]$ *P-equivalent* if, for some p in P,

$$L_p U_\sigma^{(1)} L_p^{-1} = U_\sigma^{(2)}. \tag{4.1}$$

Suppose (4.1) holds, and let $\theta^{(i)}(\cdot, \sigma)$ and $T_\sigma^{(i)}$ denote the corresponding multiplication and flow factors of the $U_\sigma^{(i)}$, in the sense of (2.2). The uniqueness of factorization shows that these constituents are related by

(4.2) $$T_\sigma^{(1)} = T_\sigma^{(2)},$$
and
(4.3) $$\theta^{(1)}(\cdot, \sigma)[p(\cdot)/p(\cdot\sigma)] = \theta^{(2)}(\cdot, \sigma).$$

In particular, if $\theta^{(1)} = 1$, so in other words $U_\sigma^{(1)}$ is a pure flow, and if we write simply U_σ for $U_\sigma^{(2)}$, then (4.3) becomes

(4.4) $$\theta(\cdot, \sigma) = p(\cdot)/p(\cdot\sigma).$$

In other words, a necessary and sufficient condition that a representation of G on $L(X)$ by bounded positive operators be P-equivalent to a pure flow is that its multiplication factor $\theta(\cdot, \sigma)$ have the form (4.4), for some function p in P. (In this connection, one should remark that the notion of P-equivalence is not so restrictive as might first appear; see Corollary 6.1 below.)

We see then that significant properties of positive representations of G on $L(X)$ are connected with functional identities (for example (2.3) and (4.4)) involving their multiplication factors. On the other hand, as the informed reader will note, these identities are cohomology statements in the sense of the Eilenberg-MacLane cohomology theory (**2**, p. 55). While our work here has a very limited contact with this theory (in that we study only $H^1(G, P)$, the first cohomology group of G with coefficients in P), it is none the less advantageous to adopt a few of these notions for our purposes. These we review in the following.

Definition 4.1. Let G be a group, X a locally compact Hausdorff space, and P the multiplicative abelian group of all positive continuous functions on X which are bounded away from 0 and infinity. Assume that G acts on X. When G is a topological group, we say this action is *continuous* if the mapping $(x, \sigma) \to x\sigma$ is jointly continuous. By a *cochain* (more precisely, a 1-cochain) we mean any function $\theta(\cdot, \sigma)$ on G to $P(X)$. If G is topological and acts continuously, we call a cochain continuous if it is jointly continuous on $X \times G$. A *cocycle* is a cochain satisfying the identity (2.3), viz.

$$\theta(x, \sigma\tau) = \theta(x\sigma, \tau)\,\theta(x, \sigma).$$

The multiplicative abelian group of cocycles is denoted $Z^1(G, P)$. A *coboundary* is a cochain $\theta(\cdot, \sigma)$ having the form $\theta(x, \sigma) = p(x)/p(x\sigma)$, for some p in P. $B^1(G, P)$ will denote their group (clearly, a subgroup of $Z^1(G, P)$). By the (first) cohomology group $H^1(G, P)$ (of G with values in P) we mean the quotient group $Z^1(G, P)/B^1(G, P)$.

Returning to the study of positive representations of G on $L(X)$, let us say that a positive representation $[U_\sigma]$ of G on $L(X)$ *belongs to the given flow* $[T_\sigma]$ of G on $L(X)$ if there exists a cocycle $\theta(\cdot, \sigma)$ in $Z^1(G, P)$ so that $U_\sigma = L_{\theta(\cdot, \sigma)}T_\sigma$. By (4.3), any representation of G on $L(X)$ P-equivalent to $[U_\sigma]$ also belongs to the flow $[T_\sigma]$ and has for its multiplication factor a cocycle cohomologous with $\theta(\cdot, \sigma)$. These remarks in conjunction with Theorem 2.1 give

LEMMA 4.1. *There is a natural* 1:1 *correspondence between P-equivalence classes of representations of G on $L(X)$ by bounded positive operators belonging to a given flow and elements of the cohomology group $H^1(G, P)$ taken relative to the same flow. Under the correspondence, representations equivalent to the flow correspond to the identity in $H^1(G, P)$.*

The base space $C_0(X)$ could have been used rather than $L(X)$ in the above discussion. For G topological [resp. locally compact], an obvious variant of the discussion applies to strongly continuous representations on $L(X)$ [resp. $C_0(X)$] and continuous cocycles. Finally we note for G merely topological, that the cobounding continuous cocycles are necessarily bounded and hence are in 1:1 correspondence with the strongly continuous representations on $C_0(X)$ which are P-equivalent to the flow.

Example 4.1. Suppose that G is compact, and that a continuous action of G on X is given. Then any continuous cocycle $\theta(\cdot, \sigma)$ in $Z^1(G, P)$ is trivial (viz. a coboundary). In particular, therefore, any strongly continuous representation of a compact group on $L(X)$ by bounded positive operators is equivalent to a flow.

In fact, given the continuous cocycle $\theta(\cdot, \sigma)$, define

$$p(x) = \int_G \theta(x, \sigma)\, d\sigma,$$

where $d\sigma$ is an element of Haar measure of G. Trivially, p lies in $P(X)$. Further,

$$p(x\tau) = \int_G \theta(x\tau, \sigma)\, d\sigma = [\theta(x, \tau)]^{-1} \int_G \theta(x, \tau\sigma)\, d\sigma = [\theta(x, \tau)]^{-1} p(x).$$

This proves that θ is a coboundary, and the other assertions follow automatically.

5. On bounded representations.

By a *bounded positive representation* of G on $L(X)$ (or $C_0(X)$) we mean a representation by *uniformly* bounded positive operators. If $[U_\sigma]$ is such a representation, and if $\theta(\cdot, \sigma)$ is its cocycle, then the relation $||U_\sigma|| = ||\theta(\cdot, \sigma)||$ (from Lemma 2.1) shows that $\theta(\cdot, \sigma) < M$, for all σ and some constant M. This and the identity $\theta(\cdot, \sigma)^{-1} = \theta(\cdot\sigma, \sigma^{-1})$ show in turn that

$$M^{-1} < \theta(\cdot, \sigma) < M, \qquad \text{for all } \sigma. \tag{5.1}$$

We shall call a cocycle $\theta(\cdot, \sigma)$ bounded if it satisfies a relation (5.1). Our argument shows therefore that a positive representation of G on $L(X)$ is bounded if and only if its cocycle is bounded.

We shall deal in this section with the problem of determining when bounded positive representations are equivalent to pure flow representations. This comes in other words to determining conditions (on X, or on the flow) under which bounded cocycles are coboundaries. The following lemma (with M = the class of all positive functions on X bounded from 0 and infinity) shows that bounded cocycles do indeed cobound when X is discrete.

LEMMA 5.1. *Let $\theta(\cdot, \sigma)$ be a bounded cocycle in $Z^1(G, P)$. Let M be a class of positive functions on X, bounded away from 0 and infinity, which contains the functions $\theta(\cdot, \sigma)$ and contains, along with f, the function $\theta(\cdot, \sigma) f(\cdot \sigma)$. Then, if $h(\cdot) = GLB_\sigma\, \theta(\cdot, \sigma)$ exists relative to M, we have $\theta(\cdot, \sigma) = h(\cdot)/h(\cdot\sigma)$.*

Proof. We are assuming here that h lies in M, that $h(x) \leqslant \theta(x, \sigma)$, for all x and σ, and that any other f in M with this property must satisfy $f \leqslant h$.

Fix on τ in G. Then for all x and σ,

$$h(x\tau) \leqslant \theta(x\tau, \sigma) = \theta(x, \tau\sigma)/\theta(x, \tau),$$

or $\theta(x, \tau)\, h(x\,\tau) \leqslant \theta(x, \tau\sigma)$. Our assumptions about M and h imply now that $\theta(x, \tau)\, h(x\tau) \leqslant h(x)$. Substituting $x\tau$ for x and τ^{-1} for τ, and making use of the relation $\theta(x\tau, \tau^{-1}) = [\theta(x, \tau)]^{-1}$, gives the opposite inequality. Therefore, $\theta(x, \tau)\, h(x\tau) = h(x)$, as asserted.

When X is a Stone space—that is, a locally compact Hausdorff space for which $C(X)$ is a conditionally complete lattice—(see **(11)**), then we apply the Lemma with $M = P(X)$ to obtain

COROLLARY 5.1. *If X is a Stone space, then each strongly continuous bounded positive representation of a topological group G on $L(X)$ (or $C_0(X)$) is P-equivalent to a flow representation of G on $L(X)$ (resp. $C_0(X)$).*

We call a given action $(x, \sigma) \rightarrow x\sigma$ of a group G on X *ergodic* if each orbit $xG \equiv [x\sigma | \sigma \in G]$ is dense in X. As we now show under general conditions, this restriction on the flow suffices to eliminate non-trivial bounded cocycles.

THEOREM 5.1. *Let G be a group which acts ergodically on the Hausdorff space X. Then each bounded cocycle in $Z^1(G, P)$ is a coboundary.*

Proof. Fix on a bounded cocycle $\theta_1(\cdot, \sigma)$ in $Z^1(G, P)$. In order to simplify notation in the proof, we shall deal with $\theta(x, \sigma) \equiv \log \theta_1(x, \sigma)$ rather than with θ_1, so our conditions on θ are

$$(5.2) \qquad -M < \theta(x, \sigma) < M \text{ and } \theta(x, \sigma\tau) = \theta(x\sigma, \tau) + \theta(x, \sigma),$$

for all x, σ, τ. Our task is to exhibit a bounded continuous function $h(\cdot)$ on X satisfying

$$(5.3) \qquad \theta(x, \sigma) = h(x\sigma) - h(x),$$

for then $p(x) = \exp(-h(x))$ will give $p(x)/p(x\sigma) = \theta_1(x, \sigma)$ and $p \in P$.

To begin with we assume that X is a single orbit, $X = x_0G$, and establish the Theorem in this special case.

By Lemma 5.1, there exists a bounded (but not *a priori* continuous) function $h(\cdot)$ on X so that (5.3) holds. Since (5.3) will also hold when $h(\cdot)$ is replaced by $h(\cdot) + c$ (c constant), we can assume that $h(x_0) = 0$. Therefore,

$$(5.4) \qquad \text{if } x = x_0\sigma, \text{ then } h(x) = \theta(x_0, \sigma).$$

(Note from this that $x_0\sigma = x_0\tau$ will entail $\theta(x_0, \sigma) = \theta(x_0, \tau)$.) We shall prove that this function h is necessarily continuous.

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

Grant that we have proved continuity of h at x_0, namely,

$$\lim x_\alpha = x_0 \quad \text{entails} \quad \lim h(x_\alpha) = 0. \tag{5.5}$$

We show that h is then everywhere continuous. For suppose $\lim x_\alpha = y = x_0\sigma$. By (5.5), $\lim h(x_\alpha \sigma^{-1}) = 0$, so we have from (5.3) that

$$\begin{aligned}\lim h(x_\alpha) &= \lim h(x_\alpha \sigma^{-1}) - \lim \theta(x_\alpha, \sigma^{-1}) \\ &= -\theta(y, \sigma^{-1}) = -\theta(x_0\sigma, \sigma^{-1}) = \theta(x_0, \sigma) = h(y).\end{aligned}$$

We now prove (5.5). As the basis of an indirect proof, we can assume (replacing h, θ by $-h$, $-\theta$ if necessary) that

$$\limsup_{y \to x_0} h(y) > \epsilon > 0.$$

Each neighborhood of x_0 will then contain a point $y = x_0\sigma$ for which $\theta(x_0, \sigma) \geqslant \epsilon$. Choose any $\sigma = \sigma_1$ for which $\theta(x_0, \sigma_1) \geqslant \epsilon$. Assume elements $\sigma_1, \ldots, \sigma_n$ of G have been chosen so that

$$\theta(x_0, \sigma_n \ldots \sigma_1) \geqslant (n-1)\epsilon - \left[\frac{\epsilon}{2} + \ldots + \frac{\epsilon}{2^{n-1}}\right]. \tag{5.6}$$

Choose a neighborhood N of x_0 so that y in N gives

$$\theta(y, \sigma_n \ldots \sigma_1) \geqslant \theta(x_0, \sigma_n \ldots \sigma_1) - \frac{\epsilon}{2^n},$$

and then choose σ_{n+1} so that $x_0\sigma_{n+1}$ lies in N and $\theta(x_0, \sigma_{n+1}) \geqslant \epsilon$. We then have

$$\theta(x_0, \sigma_{n+1} \ldots \sigma_1) = \theta(x_0\sigma_{n+1}, \sigma_n \ldots \sigma_1) + \theta(x_0, \sigma_{n+1}) \geqslant n\epsilon - \left[\frac{\epsilon}{2} + \ldots + \frac{\epsilon}{2^n}\right],$$

so σ_n is defined for all n and (5.6) can be realized. This inequality shows that

$$\theta(x_0, \sigma_n \ldots \sigma_1) \geqslant (n-2)\epsilon, \qquad \text{for all } n. \tag{5.7}$$

But this contradicts the boundedness of θ. Therefore the Theorem is proved in the single orbit case.

We turn next to the general case. Choose any orbit $\Pi = x_0G$ and any point x in X. By what we have proved, there is a bounded function h, defined and continuous on Π, and satisfying

$$\theta(y, \sigma) = h(y\sigma) - h(y) \qquad (y \text{ in } \Pi, \sigma \text{ in } G). \tag{5.8}$$

Accordingly, for any subset S of X,

$$\operatorname{Var}_{S \cap \Pi} h(\cdot) \leqslant \operatorname{Var}_{S \cap \Pi} h(\cdot\sigma) + \operatorname{Var}_S \theta(\cdot, \sigma). \tag{5.9}$$

By the continuity of h on Π, given $\epsilon > 0$, there exists a neighborhood U of x_0 so that

$$\operatorname{Var}_{U \cap \Pi} h(\cdot) < \tfrac{1}{2}\epsilon.$$

Since xG is dense, by assumption, we can choose σ in G so that $x\sigma \in U$, and in turn, we can choose a neighborhood V of x so that $V\sigma \subset U$. Shrinking V

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

if necessary, we can assume that $\text{Var}_V\, \theta(\cdot, \sigma) < \frac{1}{2}\epsilon$, because $\theta(\cdot, \sigma)$ is continuous on X. With $S = V$, (5.9) then yields

$$\text{Var}_{V \cap \Pi}\, h(\cdot) < \epsilon. \tag{5.10}$$

Therefore, if x_α is a sequence in Π with $\lim x_\alpha = x$, then $\lim h(x_\alpha)$ will exist and depend only on x. This shows that $h(\cdot)$ extends to a function $\bar{h}(\cdot)$ on the closure X of Π which in particular satisfies

$$\text{Var}_V\, \bar{h}(\cdot) \leqslant \epsilon. \tag{5.11}$$

Since x is arbitrary, it follows that $\bar{h}$ is continuous on all X. In particular, continuity shows that (5.8) holds for this $\bar{h}$ and all y in X. This proves Theorem 5.1.

A fortiori, if G is a topological group acting continuously and ergodically on X, then any bounded cocycle is automatically continuous. We summarize the implications of this Theorem as they apply to bounded positive representations.

COROLLARY 5.2. *Let $\sigma \to U_\sigma$ be a strongly continuous bounded positive representation of the topological group G on $C_0(X)$. Assume that no non-trivial closed ideals of $C_0(X)$ are invariant under this representation. Then $[U_\sigma]$ is P-equivalent to a strongly continuous flow representation $\sigma \to T_\sigma$ of G on $C_0(X)$.*

Proof. Let $[T_\sigma]$ be the flow associated with $[U_\sigma]$ (as in Theorem 2.1). We know that $\sigma \to T_\sigma$ is strongly continuous. That the action of G on X implemented by T_σ is ergodic follows from the well-known characterization of closed ideals in $C_0(X)$ (viz. as the class of all functions in $C_0(X)$ vanishing on an arbitrary closed set). The bounded cocycle associated with $[U_\sigma]$ is therefore a coboundary, by the Theorem, and the conclusion follows from the remark following Lemma 4.1.

Example 5.1. We conclude this discussion by showing that bounded cocycles do not in general cobound.

For X take the two-point compactification $[-\infty, +\infty]$ of the reals, and for G the additive group of real numbers with the usual topology. Define the action (continuous) of G on X by setting $xt = x + t$ for x finite and $xt = x$ for x infinite. Next, define

$$p(x) = \begin{cases} 2 & \text{for } x = -\infty, \\ 2 + \sin |x|^{\frac{1}{2}} & \text{for } x \text{ finite}, \\ 2 & \text{for } x = +\infty, \end{cases}$$

and set $\theta(x, t) = p(x)/p(xt)$. A straightforward computation shows that $\theta(x, t)$ is a bounded (continuous) cocycle in $Z^1(G, P)$. However, $p(\cdot)$ does not belong to $P(X)$ since it is not continuous on the closed interval $[-\infty, \infty]$. Now, it is easy to see that any positive function q defined on $(-\infty, \infty)$ with the property $\theta(x, t) = q(x)/q(xt)$ must be a positive constant multiple of p; no such multiple can have a continuous extension on $[-\infty, \infty]$. It follows that $\theta(\cdot, t)$ cannot be a coboundary.

6. Automorphisms of groups of positive operators. Let F be a group of homeomorphisms of the locally compact Hausdorff space X. In abuse of notation, we shall also write F for the group of isometries $[T_\sigma]$ of $L(X)$ implemented by the σ in F. Denote by M the group of all multiplication operators $[L_p; p \in P]$ on $L(X)$. Finally, denote by $\mathfrak{G}$ the group of all regular positive operators with positive inverses on $L(X)$ whose canonical factorization L_pT_σ has $T_\sigma \in F$. We may describe $\mathfrak{G}$ as the group of all positive operators on $L(X)$ belonging to the given flow F. Group-theoretically, $\mathfrak{G}$ is the semi-direct product FM of the subgroup F and the (normal abelian) subgroup M. This section concerns a study of automorphisms of the group $\mathfrak{G}$, and our results here will serve to clarify the significance of some of the algebraic formalisms we have adopted.

LEMMA 6.1. *M is a normal maximal abelian subgroup of $\mathfrak{G}$, and any other normal abelian subgroup of $\mathfrak{G}$ already lies in M.*

Proof. If we write p^σ for the function $p^\sigma(x) \equiv p(x\sigma)$, then the relation

$$T_\sigma L_p T_\sigma^{-1} = L_{p^\sigma}$$

shows that M is normal. Suppose that $U = L_pT_\sigma$ is any element of $\mathfrak{G}$ commuting with all elements of M. We have

$$L_q(L_pT_\sigma) = L_pT_\sigma L_q = L_{pq^\sigma}T_\sigma,$$

or $qp = q^\sigma p$, or $q = q^\sigma$, for all q in P. This clearly entails $\sigma = e$ (the identity), and it follows that $U = L_p$ lies in M, proving that M is maximal abelian.

Suppose that $U = L_qT_\sigma$ is an element of some normal abelian subgroup N of $\mathfrak{G}$. For each p in P, N will then contain

$$L_p(L_qT_\sigma)L_{p^{-1}} = L_{(pq/p^\sigma)}T_\sigma,$$

and this element of N will in turn commute with U. The commutation relation

$$(L_{(pq/p^\sigma)}T_\sigma)(L_qT_\sigma) = (L_qT_\sigma)(L_{(pq/p^\sigma)}T_\sigma)$$

gives

$$pqq^\sigma/p^\sigma = qq^\sigma p^\sigma/p^{\sigma^2}, \text{ and hence } pp^{\sigma^2} = (p^\sigma)^2$$

for all p. Again, it is easy to conclude that $\sigma = e$, so that U lies in M. This proves the Lemma.

We call an automorphism φ of $\mathfrak{G}$ *bounded* if, for some constant K and all U in $\mathfrak{G}$,

$$K^{-1}||U|| \leqslant ||\varphi(U)|| \leqslant K||U||.$$

The inverse of a bounded automorphism is automatically bounded, and we may speak therefore of the group $\mathrm{Aut}_{bd}(\mathfrak{G})$ of bounded automorphisms of $\mathfrak{G}$.

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

LEMMA 6.2 *Assume X compact. Then each bounded automorphism φ of $\mathfrak{G}$ carries M onto itself, and on M has the form*

$$\varphi(L_p) = L_{p^\tau}, \tag{6.1}$$

where τ is a homeomorphism of X.

Proof. The characterization of Lemma 6.1 shows that any automorphism of $\mathfrak{G}$ will carry M onto itself. We shall deduce (6.1) from the corollary to Lemma 2.1 which asserts that any linear order isomorphism Ψ of $L(X)$ which conserves the identity (X compact) is implemented by a homeomorphism of X. For this, define

$$\Psi(f) = \log \varphi(\exp f), \qquad f \in L(X). \tag{6.2}$$

Then $\Psi^{-1}(f) = \log \varphi^{-1}(\exp f)$, and it is clear that Ψ is an additive isomorphism of $L(X)$ on itself. We next show that Ψ preserves order. For any x in X, q in $P(X)$, and $n \geqslant 1$, we have

$$[\varphi(q)(x)]^n = \varphi(q^n)(x) \leqslant K||q^n|| = K||q||^n,$$

and therefore

$$\varphi(q)(x) \leqslant ||q||. \tag{6.3}$$

In particular, if $f \geqslant 0$, then $q = \exp -f \leqslant 1$ and $\varphi(q) \leqslant 1$. Thus

$$\varphi(\exp f) = \varphi(q^{-1}) = [\varphi(q)]^{-1} \geqslant 1$$

and hence $\Psi(f) = \log \varphi(q^{-1}) \geqslant 0$. By the same token, Ψ^{-1} must preserve order, and it follows that Ψ is an order isomorphism and therefore linear. Substitution of a positive scalar c for q in (6.3) and the corresponding statement for c^{-1} shows that $\varphi(c) = c$. From this it follows that $\Psi(1) = 1$ so that Ψ conserves the identity. This yields (6.1).

LEMMA 6.3. *Let φ be any automorphism of $\mathfrak{G}$ with the property that its restriction to M has the form*

$$\varphi(L_p) = L_{p^\tau},$$

for some homeomorphism τ of X. Then τ lies in the normalizer $N(F)$ of the flow F, and for all L_pT_σ in $\mathfrak{G}$,

$$\varphi(L_pT_\sigma) = T_\tau(L_{\theta(\cdot,\,\sigma)}L_pT_\sigma)T_\tau^{-1}, \tag{6.4}$$

for some cocycle $\theta(\cdot, \sigma)$ in $Z^1(F, P)$.

Proof. Define $\varphi'(U) = T_\tau^{-1}\varphi(U)T_\tau$. φ' maps $\mathfrak{G}$ isomorphically on another group of positive operators on $L(X)$, and $\varphi'(L_p) = L_p$, for all p in P. Therefore

$$L_p\varphi'(T_\sigma) = \varphi'(L_pT_\sigma) = \varphi'(T_\sigma)(T_\sigma^{-1}L_pT_\sigma),$$

showing that the positive operator $\varphi'(T_\sigma)T_\sigma^{-1}$ commutes with all L_p. Lemma 6.1 applied to the group of all positive operators on $L(X)$ then shows that $\varphi'(T_\sigma)T_\sigma^{-1}$ lies in M, for each $\sigma \in F$. We write

$$\varphi'(T_\sigma) = L_{\theta(\cdot,\,\sigma)}T_\sigma.$$

A simple calculation shows that θ lies in $Z^1(F, P)$. Moreover,

$$\varphi(T_\sigma) = L_{\theta(\cdot,\,\sigma)} T_{\tau\sigma\tau^{-1}}.$$

This operator lies in $\mathfrak{G}$, and therefore $\tau F \tau^{-1} \subset F$. The same argument for θ^{-1}, τ^{-1} in place of φ, τ gives inclusion in the other direction, and we find that $F = \tau F \tau^{-1}$, or by definition, τ lies in $N(F)$. This proves the lemma.

We can now obtain some significant information about the structure of the automorphism group of $\mathfrak{G}$. One may note the formal similarity of the theorem to follow to a theorem of I. Singer on the automorphism group of a finite factor **(9**, Th. 3.3**)**.

THEOREM 6.1. *Suppose X compact. Then the group* $\mathrm{Aut}_{bd}(\mathfrak{G})$ *is isomorphic to a semi-direct product $N(F)\, Z_{bd}$ of groups isomorphic respectively to the normalizer of the flow F in the group of all homeomorphisms of X and the group of all bounded cocycles in $Z^1(F, P)$. Here $\tau \in N(F)$ implements the automorphism Θ_τ of Z_{bd} given by $\Theta_\tau(\theta(\cdot, \sigma)) = \theta(\cdot\tau, \tau^{-1}\sigma\tau)$.*

Proof. We associate with each τ in $N(F)$ the automorphism α_τ of $\mathfrak{G}$ defined by

$$\alpha_\tau(T_\sigma L_p) = T_\tau(T_\sigma L_p) T_{\tau^{-1}}.$$

It is trivial that $\alpha_\tau \neq e$ when $\tau \neq e$, and that

$$\alpha_{\tau_1\tau_2} = \alpha_{\tau_1}\alpha_{\tau_2}.$$

It follows that $\tau \to \alpha_\tau$ maps $N(F)$ isomorphically into $\mathrm{Aut}_{bd}(\mathfrak{G})$. Next, associate with the bounded cocycle θ in $Z^1(F, P)$ the bounded automorphism α_θ of $\mathfrak{G}$,

$$\alpha_\theta(T_\sigma L_p) = L_{\theta(\cdot,\,\sigma)} T_\sigma L_p.$$

Again, it follows readily that $\theta \to \alpha_\theta$ is an isomorphism of Z_{bd} into $\mathrm{Aut}_{bd}(\mathfrak{G})$. Moreover,

$$\alpha_\tau \alpha_\theta \alpha_{\tau^{-1}}(T_\sigma L_p) = L_{\theta(\cdot\tau,\,\tau^{-1}\sigma\tau)} T_\sigma L_p,$$

so that the image of Z_{bd} is normal in $\mathrm{Aut}_{bd}(\mathfrak{G})$. Lemmas 6.2 and 6.3 show that each bounded automorphism φ of $\mathfrak{G}$ has a factorization $\varphi = \alpha_\tau\alpha_\theta$, and it is readily seen that this factorization is unique. This proves the Theorem.

Following a similar pattern, we now give an interpretation of the cohomology group $H^1(F, P)$. For this, we call an automorphism φ of $\mathfrak{G}$ *flow related* if φ coincides on M with some automorphism α_σ (σ in F), in the sense that $\varphi(L_p) = L_{p^\sigma}$.

THEOREM 6.2. *There exists a natural isomorphism between $H^1(F, P)$ and the group* $\mathrm{Aut}_{fr}(\mathfrak{G})/\mathrm{Inaut}(\mathfrak{G})$ *of flow related automorphisms of $\mathfrak{G}$ modulo inner automorphisms.*

Proof. It follows from Lemma 6.3 that each flow related automorphism φ has a unique factorization $\varphi = \alpha_\tau\alpha_\theta$, ($\tau \in F, \theta \in Z^1 = Z^1(F, P)$), so in the notation of Theorem 6.1, we have

$$\mathrm{Aut}_{fr}(\mathfrak{G}) = F\, Z^1. \tag{6.5}$$

Suppose θ is an element of $B^1 = B^1(F, P)$, $\theta(\cdot, \sigma) = p(\cdot)/p(\cdot\sigma)$. Then

$$\alpha_\tau \alpha_\theta(T_\sigma L_q) = T_\tau L_p(T_\sigma L_q)L_p^{-1}T_\tau^{-1},$$

so that φ is inner. The argument here clearly reverses, and we see that

$$\text{Inaut } (\mathfrak{G}) = F B^1. \tag{6.6}$$

Suppose $\tau \in F$, $\theta \in Z^1$. Then

$$\alpha_{\tau^{-1}}\alpha_\theta\alpha_\tau\alpha_{\theta^{-1}} = \alpha_\beta, \text{ for some } \beta \in B^1. \tag{6.7}$$

In fact,

$$(\alpha_{\tau^{-1}}\alpha_\theta\alpha_\tau\alpha_{\theta^{-1}})(T_\sigma L_q) = L_{\theta(\cdot\tau^{-1},\ \tau\sigma\tau^{-1})/\theta(\cdot,\ \sigma)}T_\sigma L_q,$$

and

$$\begin{aligned}\beta(\cdot, \sigma) &\equiv \theta(\cdot\tau^{-1}, \tau\sigma\tau^{-1})/\theta(\cdot, \sigma) \qquad = \theta(\cdot, \sigma\tau^{-1})\, \theta(\cdot\tau^{-1}, \tau)/\theta(\cdot, \sigma)\\ &= \theta(\cdot, \sigma\tau^{-1})/\theta(\cdot, \tau^{-1})\, \theta(\cdot, \sigma) = \theta(\cdot\sigma, \tau^{-1})/\theta(\cdot, \tau^{-1}).\end{aligned}$$

If we set $p(\cdot) = [\theta(\cdot, \tau^{-1})]^{-1}$, then it is clear that $\beta(\cdot, \sigma) = p(\cdot)/p(\cdot\sigma) \in B^1$, proving (6.7).

As a characteristic subgroup, Inaut$(\mathfrak{G})$ is normal in $\text{Aut}_{f\tau}(\mathfrak{G})$. Observe now that the automorphisms

$$\alpha_{\tau_1}\alpha_{\theta_1},\ \alpha_{\tau_2}\alpha_{\theta_2}$$

lie in the same coset mod Inaut$(\mathfrak{G})$ if and only if θ_1 is cohomologous with θ_2. In fact, using (6.7), we have

$$\alpha_{\theta_2^{-1}}\alpha_{\tau_2^{-1}}\alpha_{\tau_1}\alpha_{\theta_1} = \alpha_{\tau_2^{-1}}\alpha_{\tau_1}\alpha_\beta\alpha_{\theta_2^{-1}}\alpha_{\theta_1} \qquad (\beta \in B^1),$$

and this automorphism is inner if and only if

$$\alpha_{\theta_2^{-1}}\alpha_{\theta_1} \in B^1,$$

or equivalently, if and only if θ_1 is in the same coset of Z^1 mod B^1 as θ_2. It follows that the mapping which carries the coset of the automorphism $\alpha_\tau\alpha_\theta$ on the coset of θ is an isomorphism onto.

We conclude our study of automorphisms of $\mathfrak{G}$ by a brief consideration of automorphisms implemented by bounded operators on $L(X)$. Since any such operator can be extended to be regular on $C_0(X)$ we may, without loss of generality, choose the latter as our base space.

LEMMA 6.4. *Let W be a bounded regular operator on $C_0(X)$ with the property that $U \to WUW^{-1}$ defines an automorphism of $\mathfrak{G}$. Then there exists a bounded positive operator V on $C_0(X)$ with a bounded positive inverse such that $WUW^{-1} = VUV^{-1}$ for all U in $\mathfrak{G}$.*

Proof. Let $\beta(X)$ be the Stone-Čech compactification of X and denote by $\mathfrak{G}'$ the unique extension of $\mathfrak{G}$ to a group of positive operators on $C(\beta(X))$. In the obvious way, $W\cdot W^{-1}$ defines a bounded automorphism of $\mathfrak{G}'$. By Lemma 6.2, therefore, there exists a homeomorphism τ of $\beta(X)$ such that $WL_pW^{-1} = L_{p^\tau}$, for all p in $P(\beta(X))$. For h in $L(X)$ and x in $\beta(X)$, we will have

$$W(ph)(x) = p(x\tau)\ W(h)(x).$$

Linearity shows that this must hold for all p in $C(\beta(X))$. We now show that $\tau(X) \subset X$. Suppose on the contrary that τ maps some x in X into $\tau(x) \in \beta(X) - X$. Choose h in $L(X)$ so that $(Wh)(x) \neq 0$, and then choose p in $C(\beta(X))$ so that $p(x) \equiv 1$ on the support of h (which lies in X since it is a compact subset of X) and so that $p(x\tau) = 0$. This gives $W(h)(x) = W(ph)(x) = 0$, which is impossible. This argument applies as well to W^{-1}, τ^{-1}, and we see therefore that $\tau(X) = X$. Hence if we set $W' = T_{\tau^{-1}}W$, then W' is a regular bounded operator on $C_0(X)$ which satisfies the relation

$$W'L_f = L_fW', \qquad \text{for all bounded } f \text{ in } C(X). \tag{6.8}$$

We shall complete the proof by showing that any bounded regular operator W' satisfying (6.8) has the form L_g, for some $g \in C(X)$ with $|g|$ in $P(X)$. The operator $V = T_\tau L_{|g|}$ will then be positive and will implement the same automorphism of $\mathfrak{G}$ as $W \cdot W^{-1}$. We now let $[0_\alpha]$ denote the collection of all open sets of X with compact closures, and for each α we choose a function h_α in $L(X)$ which is 1 on 0_α. Set $g_\alpha = W'h_\alpha$. If h in $L(X)$ vanishes off 0_α, then (6.8) gives

$$W'(h) = W'(h_\alpha h) = hW'(h_\alpha) = hg_\alpha. \tag{6.9}$$

Therefore, if $x \in 0_\alpha \cap 0_\beta$, $h(x) = 1$, and if h vanishes off $0_\alpha \cap 0_\beta$, then

$$g_\alpha(x) = h(x)\, g_\alpha(x) = (W'h)(x) = h(x)\, g_\beta(x) = g_\beta(x).$$

So $g_\alpha = g_\beta$ on $0_\alpha \cap 0_\beta$. Define a function g on X by setting $g(x) = g_\alpha(x)$ if $x \in 0_\alpha$. This function g is then well defined and continuous, and (6.9) shows that $W'h = gh$, for any h in $L(X)$. It follows from this that $||g|| = ||W'|| < \infty$ and that g does not vanish. If we apply this argument to W'^{-1}, to obtain a bounded k such that $W'^{-1}h = kh$ for all h in $L(X)$, then

$$h = W'(W'^{-1}(h)) = hkg,$$

so $kg = 1$, and $||g^{-1}||$ is finite. Therefore $|g|$ lies in P, and the Lemma is proved.

This lemma has application to the representation theory. Our work in §§4 and 5 was based on the notion of P-equivalence of representations. On the other hand, in the conventional sense, two representations $[U_\sigma]$ and $[V_\sigma]$ of a group G on $C_0(X)$ are equivalent if there exists a bounded regular operator W on $C_0(X)$ so that $WU_\sigma W^{-1} = V_\sigma$ for all $\sigma \in G$. If we require in addition that this operator W determine an automorphism of the group of all positive operators on $C_0(X)$, in the sense of the preceding lemma, then we can just as well assume to begin with that W is a positive operator. Knowing the form of positive operators with positive inverses, however, we infer from this

Corollary 6.1. *Let $\sigma \to U_\sigma$ be a bounded positive representation of the group G on $C_0(X)$, and suppose there exists a bounded regular operator W on $C_0(X)$ such that $\sigma \to WU_\sigma W^{-1}$ is a flow representation of G, and such that $U \to WUW^{-1}$ defines an automorphism of the group of all positive operators on $C_0(X)$. Then $[U_\sigma]$ is already P-equivalent to a flow representation.*

7. The adjoint representation. Suppose that $\sigma \to U_\sigma$ is a strongly continuous representation of a topological group G on $C_0(X)$. Denote by $C_0(X)^*$ the adjoint space of $C_0(X)$, that is, the space of bounded linear functionals λ on $C_0(X)$ with the norm $||\lambda|| = \sup_{||f||=1} |\lambda(f)|$. It is well known that elements of $C_0(X)^*$ can be represented as integrals on $C_0(X)$ relative to signed Borel measures on X of finite total variation (**4**, chap. X). Associated with the representation $[U_\sigma]$ is an anti-representation $\sigma \to U^*_\sigma$ of G on $C_0(X)^*$ defined by $(U^*_\sigma\lambda)(f) = \lambda(U_\sigma f)$. This anti-representation is in itself not a natural object to study, since it will in general fail to be strongly continuous. However, study of the forward diffusion equation in semi-group theory has suggested a natural refinement of these notions (see (**3** and **7**)).

Definition 7.1. By the *adjoint representation* $[U^0_\sigma, D(U^0_\sigma)]$ to a given strongly continuous representation $\sigma \to U_\sigma$ of G on $C_0(X)$, we mean the pair consisting of the representation $\sigma \to U^0_\sigma$ of G on $C_0(X)^*$ defined by

$$(U^0_\sigma\lambda)(f) = \lambda(U_{\sigma^{-1}}f),$$

together with a subspace $D(U^0_\sigma)$ of $C_0(X)^*$, called the *domain* of the adjoint representation, and consisting of all λ in $C_0(X)^*$ for which the mapping $\sigma \to U^0_\sigma\lambda$ is strongly continuous.

Two adjoint representations $[U^0_\sigma, D(U^0_\sigma)]$ and $[V^0_\sigma, D(V^0_\sigma)]$ of G will be called *equivalent* if $D(U^0_\sigma) = D(V^0_\sigma)$, and if there exists a bounded regular operator W on $C_0(X)^*$ such that $W^{-1}U^0_\sigma W = V^0_\sigma$ for all $\sigma \in G$. In the case $W[D(U^0_\sigma)] = D(U^0_\sigma)$.

To bring this notion of domain into clearer focus, we note that each operator U^0_σ maps $D(U^0_\sigma)$ into itself so that the restriction of U^0_σ to $D(U^0_\sigma)$ defines a strongly continuous representation of G. As to the extent of $D(U^0_\sigma)$, we now discuss the situation for G locally compact in the following

REMARK 7.1. Suppose that G is a locally compact group and that $\sigma \to U_\sigma$ is a strongly continuous bounded representation of G on $C_0(X)$. Given λ in $C_0(X)^*$ and h in $L(G)$, define

$$\lambda^h(f) = \int h(\sigma)\, \lambda(U_{\sigma^{-1}}f)\, d\sigma, \qquad f \text{ in } C_0(X), \tag{7.1}$$

where the integral is taken relative to left invariant haar measure on G. Then

(1) λ^h lies in $D(U^0_\sigma)$ so that $\sigma \to U^0_\sigma\lambda^h$ is strongly continuous,

(2) the set of all such λ^h (h and λ varying) is strongly dense in $D(U^0_\sigma)$, and

(3) $D(U^0_\sigma)$, the strong closure of the set in (2), is W^*-dense in $C_0(X)^*$.

Proof of (1):

$$|\lambda^h(U_{\tau^{-1}}f - f)| \leqslant \int |h(\tau^{-1}\sigma) - h(\sigma)| \cdot |\lambda(U_{\sigma^{-1}}f)|\, d\sigma \leqslant K||f|| \cdot ||h(\tau^{-1}\cdot) - h(\cdot)||_1,$$

for some constant K. Since $\tau \to h(\tau^{-1}\cdot)$ is continuous on G to $L_1(G)$, it follows that λ^h lies in $D(U^0_\sigma)$.

Proof of (2): Suppose $\lambda \in D(U^0_\sigma)$. Then given $\epsilon > 0$, there exists a neighborhood N of the identity e in G such that

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

$$|\lambda(U_{\sigma^{-1}}f - f)| < \epsilon||f||,$$

for all $f \in C_0(X)$, $\sigma \in N$. Choose a non-negative h in $L(G)$ vanishing off N and so that $\int h(\sigma)\, d\sigma = 1$. Then

(7.2) $\quad |\lambda(f) - \int h(\sigma)\, \lambda(U_{\sigma^{-1}}f)\, d\sigma| \leqslant \int h(\sigma) \cdot |\lambda(f - U_{\sigma^{-1}}f)|\, d\sigma < \epsilon||f||$

and therefore $||\lambda - \lambda^h|| \leqslant \epsilon$.

Proof of (3): Suppose next that λ is an arbitrary element of $C_0(X)^*$. Then given $\epsilon > 0$ there exists a neighborhood N of e in G, depending on f, such that $|\lambda(U_{\sigma^{-1}}f - f)| < \epsilon$ for all $\sigma \in N$. Choosing h as above, the first inequality in (7.2) shows that $|\lambda(f) - \lambda^h(f)| < \epsilon$ and therefore that $[\lambda^h;\ h \in L(G)]$ is W^*-dense in $C_0(X)^*$. Finally we note that $D(U^0{}_\sigma)$ is strongly closed since the $U^0{}_\sigma$ are uniformly bounded.

The above argument can readily be extended to the case where $[U_\sigma]$ is merely strongly continuous, if one makes greater use of the local compactness of G.

Consider now a strongly continuous bounded positive representation $[U_\sigma = L_{\theta(\cdot,\sigma)}T_\sigma]$ of the topological group G on $C_0(X)$. We recall (Theorem 2.1) that the flow representation $[T_\sigma]$ is also strongly continuous. We wish to determine conditions under which the adjoint representations $[U^0{}_\sigma, D(U^0{}_\sigma)]$ and $[T^0{}_\sigma, D(T^0{}_\sigma)]$ (the "adjoint flow representation") are equivalent. For this purpose, we shall say that the cocycle $\theta(\cdot, \sigma)$ of $[U_\sigma]$ has a *measurable factorization* if

(7.3) $$\theta(\cdot, \sigma) = p(\cdot)/p(\cdot\sigma),$$

for p a positive function on X, bounded away from 0 and infinity, and measurable, in the sense that its contraction to each Borel set is measurable. (Here, the Borel sets consist of the σ-ring generated by compact sets.) If p is such a function and if $\lambda \in C_0(X)^*$ is represented by the signed Borel measure $\lambda(E)$, then we see that the functional

$$\mu_p(f) \equiv \int p(x)\, f(x)\, \lambda(dx)$$

again lies in $C_0(X)^*$ and

$$\mu_p(E) = \int_E p(x)\, \lambda(dx)$$

for all Borel sets E. Heuristically one can write $\mu_p(f) = \lambda(pf)$. We can therefore define a bounded linear operator W on $C_0(X)^*$ by

(7.4) $$(W\lambda)(f) = \mu_p(f).$$

With this definition, we then have

(7.5) $$(W^{-1}T^0{}_\sigma W)\, \lambda = U^0{}_\sigma\lambda,$$

for all λ in $C_0(X)^*$. In order to prove this we note that

$$(T^0{}_\sigma\lambda)(f) = \lambda(T_{\sigma^{-1}}f) = \int f(x\sigma^{-1})\, \lambda(dx) = \int f(x)\, \lambda(dx\sigma),$$

so that $(T^0{}_\sigma\lambda)(E) = \lambda(E\sigma)$. Hence

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

$$
\begin{aligned}
[(W^{-1}T^0_\sigma W)(\lambda)](f) &= [(W^{-1}T^0_\sigma)(\textstyle\int .p(x)\,\lambda(dx))](f) = [W^{-1}(\textstyle\int ._\sigma p(x)\,\lambda(dx))](f)\\
&= \textstyle\int f(z)\{\int_{dz}[p(y)]^{-1}[\int_{dy\sigma}p(x)\,\lambda(dx)]\}\\
&= \textstyle\int f(z)\,[p(z)]^{-1}\,p(z\sigma)\,\lambda(dz\sigma)\\
&= \textstyle\int f(x\sigma^{-1})\,\theta(x,\sigma^{-1})\,\lambda(dx) = \lambda(U_{\sigma^{-1}}f) = (U^0_\sigma\lambda)(f),
\end{aligned}
$$

as asserted.

To establish the equivalence of $[U^0_\sigma, D(U^0_\sigma)]$ and $[T^0_\sigma, D(T^0_\sigma)]$ under the assumption (7.3), it remains to show that $D(U^0_\sigma) = D(T^0_\sigma)$. This will follow from

LEMMA 7.1. *Assume λ lies in $D(T^0_\sigma)$. Let p be a bounded non-negative function on X, measurable relative to each Borel set. Then the linear functional $\mu_p(f) = \lambda(pf)$ also lies in $D(T^0_\sigma)$.*

Proof. As is well known, λ can be expressed as the difference of two bounded positive functionals, $\lambda = \lambda_1 - \lambda_2$. The bounded positive functional $(\lambda_1 + \lambda_2)$ induces a regular Borel measure m on X with

$$m(X) \equiv LUB[m(C);\ C \text{ compact}] < \infty .$$

Take any $\delta > 0$. Choose a compact set K_1 so that $m(X) - m(K_1) < \delta$. By Lusin's theorem, we can find a compact $K \subset K_1$ so that $m(K_1) - m(K) < \delta$ and the restriction $p|K$ of p to K is continuous. Next, we can extend $p|K$ to a non-negative element $\bar{p}$ of $L(X)$ with preservation of the bound M of p. This gives

$$|\mu_p(f) - \mu_{\bar{p}}(f)| \leqslant (\lambda_1 + \lambda_2)(|f(p - \bar{p})|) \leqslant 2\delta M||f||.$$

It follows therefore that μ_p is a uniform limit of functionals $\mu_{\bar{p}}$, $\bar{p}$ in $L(X)$. Because $D(T^0_\sigma)$ is strongly closed, it will therefore suffice to prove the lemma under the initial assumption that $p \in L(X)$. In this case $pf \in L(X)$ and $\mu_p(f) = \lambda(pf)$ is strictly correct. For any τ in G and f in $C_0(X)$,

$$\text{(7.6)} \quad |\mu_p(T_{\tau^{-1}}f - f)| \leqslant |\lambda[T_{\tau^{-1}}(fp) - fp]| + (\lambda_1 + \lambda_2)[|(T_{\tau^{-1}}f)p - T_{\tau^{-1}}(fp)|].$$

Since $\lambda \in D(T^0_\sigma)$, there exists a neighborhood N_1 of e in G so that the first term is $< \frac{1}{2}\epsilon$, for all f of norm $\leqslant 1$. Choose a symmetric neighborhood N of e, $N \subset N_1$, so that $||p(\cdot\tau) - p(\cdot)|| < \delta$, $\tau \in N$. The second term in (7.6) becomes

$$\textstyle\int |f(x\tau^{-1})(p(x) - p(x\tau^{-1})|\, m(dx) \leqslant \delta \int |f(x\tau^{-1})|\, m(dx) \leqslant \delta ||f||\, (||\lambda_1|| + ||\lambda_2||),$$

for all τ in N. We can assume δ chosen to make this bound $< \frac{1}{2}\epsilon$, again for all f of norm $\leqslant 1$. Therefore, for τ in N and $||f|| \leqslant 1$, (7.6) has the bound ϵ, and the Lemma is proved.

It follows from this that the operator W of (7.4) will carry $D(T^0_\sigma)$ into itself. Since the same must be true of W^{-1}, we have

$$\text{(7.7)} \qquad W[D(T^0_\sigma)] = D(T^0_\sigma).$$

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

According to the relation (7.5), $U^0{}_\sigma\lambda$ and $T^0{}_\sigma W\lambda$ will be strongly continuous together so that $D(T^0{}_\sigma) = W[D(U^0{}_\sigma)]$. Consequently $D(U^0{}_\sigma) = D(T^0{}_\sigma)$, and we have established

THEOREM 7.1. *Let* $\sigma \to U_\sigma = L_{\theta(\cdot,\,\sigma)}T_\sigma$ *be a strongly continuous bounded positive representation of the topological group* G *on* $C_0(X)$, $[T_\sigma]$ *being the associated flow representation. If the cocycle* $\theta(\cdot, \sigma)$ *has a measurable factorization, then the corresponding adjoint representations* $[U^0{}_\sigma, D(U^0{}_\sigma)]$ *and* $[T^0{}_\sigma, D(T^0{}_\sigma)]$ *are equivalent.*

As an indication of the existence of measurable factorizations, we prove the following two lemmas. Here, as elsewhere, a real-valued function on the locally compact Hausdorff space X is called measurable if its contraction to each Borel set is measurable in the customary sense.

LEMMA 7.2. *If* G *is a separable topological group acting continuously on* X, *then each bounded cocycle in* $Z^1(G, P)$ *has a measurable factorization.*

Proof. Let $\{\sigma_n\}$ be a countable dense subset of G and set $h(x) = GLB_n$ $\theta(x, \sigma_n)$ (pointwise). Denoting by M the class of all measurable functions on X which are bounded away from 0 and infinity, we see that the function h lies in M. On the other hand, if for fixed x we apply Theorem 5.1 to the single orbit $\Pi = xG$, we perceive that $\theta(x, \sigma)$ is continuous in σ and hence $h(x) = GLB_\sigma\, \theta(x, \sigma)$. Employing Lemma 5.1, with M defined as above, we obtain $\theta(x, \sigma) = h(x)/h(x\sigma)$.

LEMMA 7.3. *If* G *is a* σ*-compact locally compact topological group acting continuously on* X, *then each continuous cocycle in* $Z^1(G, P)$ *has a measurable factorization.*

This result is an immediate consequence of Lemma 5.1 (with M defined as in the proof of Lemma 7.2) and the following

LEMMA 7.4. *Suppose that* G *and* X *are topological spaces,* G σ*-compact locally compact and* X *merely locally compact. Let* $f(x, \sigma)$ *be any real-valued continuous function on* $X \times G$ *with* $f(x, \sigma) \geqslant 0$. *Then the (pointwise)* GLB_σ $f(x, \sigma)$ *is a measurable function on* X.

Proof. Fix a compact subset C in G. We shall prove that $b(x) = GLB_{\sigma\epsilon C}$ $f(x, \sigma)$ is measurable. This will, in effect, establish the Lemma; for G is a union of an increasing sequence $\{C_n\}$ of compact sets, and if b_n denotes the b corresponding to C_n, then $GLB_n\, b_n(x)$ is measurable and equal to $GLB_\sigma f(x, \sigma)$.

To prove that b is measurable, let F be any compact subset in X. For each x in F, choose a neighborhood $N(x)$ of x so that $|f(x, \sigma) - f(y, \sigma)| < 1/2n$ for all y in $N(x)$ and σ in C. Further, given x, choose σ_x in C so that $f(x, \sigma_x) \leqslant f(x, \sigma)$ for all σ in C. Now a finite number $N(x_1), \ldots, N(x_r)$ of the $N(x)$'s cover F. Finally set

$$h_n(x) = \inf[f(x, \sigma_{x_i});\ i = 1, \ldots, r].$$

For all x in X, we clearly have $b(x) \leqslant h_n(x)$. Take a pair (x, σ) in $F \times C$, say x lies in $N(x_i)$. Then

$$h_n(x) \leqslant f(x, \sigma_{x_i}) \leqslant f(x_i, \sigma_{x_i}) + \frac{1}{2n} \leqslant f(x_i, \sigma) + \frac{1}{2n} \leqslant f(x, \sigma) + \frac{1}{n}.$$

Thus x in F entails $h_n(x) \leqslant b(x) + 1/n$. We now define $h_F(x) = GLB_n\, h_n(x)$. This function h_F is clearly measurable, satisfies $b(x) \leqslant h_F(x)$ for all x, and $b(x) = h_F(x)$ for all x in F. Finally we note that any Borel set can be covered by the union of an increasing sequence of compact sets, say $\{F_n\}$. But $GLB_n\, h_{F_n}(x)$ is measurable and equal to $b(x)$ on this union, that is, on the given Borel set. It follows that b is measurable in the generalized sense. This concludes the proof.

We see from the foregoing material that the equivalence of the adjoints of two strongly continuous positive representations is easier to establish than the equivalence of the original representations. On the other hand if the adjoints of two linear bounded operators, say U and V, are equivalent (in the sense that there exists a linear bounded regular operator W on $C_0(X)^*$ such that $V^* = WU^*W^{-1}$), then the spectra of U and V coincide (see, for instance, (**7**, Theorem 1.5)). In particular, if $[U_\sigma]$ is a strongly continuous bounded positive representation of a separable G or of a σ-compact locally compact G, then it follows from this fact together with Theorem 7.1 and Lemmas 7.2 and 7.3 that the spectrum of U_σ coincides with that of the associated flow operator T_σ for each $\sigma \in G$.

Actually, spectral problems are best dealt with in the setting of a complex linear space rather than a real linear space. For a complex linear $C_0(X)$, the notion of positivity remains the same as before and, in fact, everything we have established applies with obvious modifications.

8. Appendix. We close this paper with an application which is of interest in the theory of semi-groups of operators. We shall exhibit two one-parameter strongly continuous groups of operators on the complex linear space $C_0(X)$ having infinitesimal operators A_1 and A_2, respectively, with $D \equiv D(A_1) \cap D(A_2)$ dense in $C_0(X)$, such that no extension of $A_1 + A_2$ (defined on D) generates a strongly continuous semi-group of operators.

Set $X = (-\infty, \infty)$, let G be the additive group of real numbers with the usual topology, and define $xt = x + t$ and

$$\theta(x, t) = \exp\left[\int_x^{x+t} \beta(\tau)\, d\tau\right]. \tag{8.1}$$

If $\beta(x)$ is continuous in x and if

$$\sup\left[\left|\int_x^{x+t} \beta(\tau)\, d\tau\right| ; -\infty < x < \infty\right] < \infty$$

for each t, then it is easy to see that $\theta(x, t)$ is a continuous cocycle in $Z^1(G, P)$.

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

Moreover such a $\theta(x, t)$ will cobound if and only if it is bounded, that is, if and only if

$$\sup_x \left| \int_0^x \beta(\tau)\, d\tau \right| < \infty; \tag{8.2}$$

a suitable P-factor being

$$p(x) = \exp\left[- \int_0^x \beta(\tau)\, d\tau \right]. \tag{8.3}$$

We note that $p(x)$ is continuously differentiable and bounded away from 0 and infinity.

Let $[T_t]$ denote the flow representation: $T_t f(x) = f(x + t)$. A straightforward computation shows that the infinitesimal operator of $[T_t]$ is given by

$$A_0 f(x) = f'(x) \tag{8.4}$$

with

$$D(A_0) = [f; f(x) \text{ continuously differentiable}, f \text{ and } f' \in C_0(X)]. \tag{8.5}$$

Suppose next that $\beta(x)$ satisfies the condition (8.2). Then the corresponding representation: $U_t^{(1)} f(x) = \theta(x, t)\ f(x + t)$, is equivalent with the flow $[T_t]$; in fact,

$$U_t^{(1)} = L_p T_t L_p^{-1}, \qquad t \in G, \tag{8.6}$$

where $L_p f(x) = p(x) f(x)$. It follows from this that the infinitesimal operator of $[U_t^{(1)}]$ is given by

$$A_1 f(x) = [L_p A_0 L_p^{-1} f](x) = f'(x) + \beta(x) f(x) \tag{8.7}$$

and

$$D(A_1) = L_p[D(A_0)]. \tag{8.8}$$

We now choose

$$\beta(x) = \begin{cases} n\, j[n^3(x - n)] & \text{for } n < x < n + n^{-3}, \\ 0 & \text{for } x \leqslant 2 \text{ and } n + n^{-3} \leqslant x \leqslant n + 1, \end{cases} \qquad n = 2, 3, \ldots,$$

where $j(x) = \exp\{-[x(1 - x)]^{-1}\}$ for $0 < x < 1$. Then $\beta(x)$ is continuously differentiable (but not bounded) and

$$0 \leqslant - \log p(x) \leqslant \int_2^\infty \beta(\tau)\, d\tau = \left[\int_0^1 j(\tau)\, d\tau \right] \sum_{n=2}^{\infty} n^{-2} < \infty,$$

so that p lies in P and the above remarks are applicable.

Finally we choose $[U_t^{(2)}]$ to be the backward flow representation, that is $U_t^{(2)} = T_{-t}$. The infinitesimal operator for $[U_t^{(2)}]$, namely A_2, is now given by

(8.9) $$A_2 = -A_0 \text{ and } D(A_2) = D(A_0).$$

It is clear from (8.5) and (8.8) that $D \equiv D(A_1) \cap D(A_2)$ contains the class D_0 of all continuously differentiable functions with compact carrier. Thus D is dense in $C_0(X)$. For $f \in D$ we have

(8.10) $$[(A_1 + A_2)f](x) = \beta(x) f(x).$$

Suppose now that A_3 with domain $D(A_3)$ is an extension of $A_1 + A_2$ (with domain D). We wish to show that A_3 cannot generate a strongly continuous semi-group of linear bounded operators possessing even the mildest of regularity conditions at $t = 0$.[1] If the contrary were true, then there would be a constant ω such that the resolvent $R(\lambda; A_3)$ would exist and be bounded in norm for $\Re(\lambda) > \omega$. In this case the semi-group $[U_t^{(3)};\ t > 0]$ generated by A_3 could be computed from the inversion formula (cf. **5**, p. 239)

(8.11) $$U_t^{(3)} f = \lim_{\tau\to\infty} \frac{1}{2\pi i} \int_{\gamma - i\tau}^{\gamma + i\tau} e^{\lambda t} R(\lambda; A_3) f \, d\lambda, \qquad t > 0,$$

for $\gamma > \omega$ and each $f \in D[(A_3)^2]$; the integral can be taken either as an abstract Cauchy integral or the usual Cauchy integral for each x. For $f \in D_0$ we see by (8.10) that $A_3 f = (A_1 + A_2) f \in D_0$ so that such an f lies in $D[(A_3)^2]$. Let $C(f)$ denote the support for $f \in D_0$ and define $\gamma(f) = \sup[\beta(x);\ x \in C(f)]$. Then if $f \in D_0$ and $\Re(\lambda) \geqslant \gamma > \gamma(f)$, it is clear that

$$g(x) \equiv [\lambda - \beta(x)]^{-1} f(x) \in D_0$$

and hence that

$$R(\lambda; A_3) f = R(\lambda; A_3)(\lambda I - A_3)\, g = g.$$

Applying (8.11) we obtain

$$U_t^{(3)} f(x) = \lim_{\tau\to\infty} \frac{1}{2\pi i} \int_{\gamma - i\tau}^{\gamma + i\tau} e^{\lambda t} [\lambda - \beta(x)]^{-1} f(x) \, d\lambda = e^{t\beta(x)} f(x)$$

for all $t > 0$. Finally since D_0 is dense in $C_0(X)$ and $U_t^{(3)}$ is assumed to be a bounded operator, we must have $U_t^{(3)} f(x) = \exp[t\, \beta(x)] f(x)$ for all $f \in C_0(X)$ and each $t > 0$. However this is impossible since an obvious consequence of this relation would be $\log \|U_t^{(3)}\| = t \sup_x \beta(x) = \infty$ for each $t > 0$.

---

[1]More precisely, we shall prove that A_3 does not generate a semi-group of class (A) (**8**).

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press

References

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).
2. S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups* I, Ann. Math. (2) *48* (1947), 51–78.
3. W. Feller, *On positivity preserving semi-groups of transformations on* $C[r_1, r_2]$, Ann. Soc. Polonaise de Math. *25* (1952), 85–94.
4. P. R. Halmos, *Measure Theory* (New York, 1950).
5. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publ., XXXI (1948).
6. R. V. Kadison, *A generalized Schwartz inequality and algebraic invariants for operator algebras*, Ann. Math. *56* (1952), 494–503.
7. R. S. Phillips, *The adjoint semi-group*, Pacific J. Math. *5* (1955), 269–283.
8. ———, *Semi-groups of operators*, Bull. Amer. Math. Soc. *61* (1955), 16–33.
9. I. M. Singer, *Automorphisms of finite factors*, Amer. J. Math. *77* (1955), 117–133.
10. M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. *41* (1937), 375–481.
11. ———, *Boundedness properties in function-lattices*, Can. J. Math. *1* (1949), 176–186.

State University of Iowa
University of Southern California

https://doi.org/10.4153/CJM-1956-055-1 Published online by Cambridge University Press