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RESEARCH ARTICLE

# The Manin–Peyre conjecture for smooth spherical Fano varieties of semisimple rank one

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**Abstract**
The Manin–Peyre conjecture is established for a class of smooth spherical Fano varieties of semisimple rank one. This includes all smooth spherical Fano threefolds of type $T$ as well as some higher-dimensional smooth spherical Fano varieties.

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# 1. Introduction

## *1.1. Manin's conjecture*

Manin's conjecture [32] predicts an asymptotic formula for the number of rational points of bounded height on Fano varieties. Its most classical version is the following: Let $X$ be a smooth Fano variety over $\mathbb{Q}$ whose set of rational points is Zariski dense. Let $H\colon X(\mathbb{Q}) \to \mathbb{R}$ be an anticanonical height function. For an open subset $U$ of $X$, let $N_{X,U,H}(B)$ denote the number of $x \in U(\mathbb{Q})$ with $H(x) \leq B$. Then one expects that there is a dense open subset $U \subseteq X$ and a positive number $c$ such that

$$N_{X,U,H}(B) = (1 + o(1))cB(\log B)^{\operatorname{rk}\operatorname{Pic} X-1}. \tag{1.1}$$

Peyre [60] proposed a product formula for $c$, and in the sequel we refer to this predicted value of $c$ as Peyre's constant. It turned out that in its original form Manin's conjecture is not always correct (see [4]). The more recent thin set version (see [61], [51, Conjectures 1.2, 5.2]) is in line with all known results hitherto.

When the dimension is large compared to the degree of the variety, one may apply the circle method to estimate $N_{X,U,H}(B)$. In this way, Browning and Heath-Brown [19] confirmed Manin's conjecture whenever $X$ is geometrically integral and the inequality $\dim X \geq ((\deg X) - 1)2^{\deg X} - 1$ holds. The asymptotic formula (1.1) is also known for several classes of equivariant compactifications of algebraic groups or homogeneous spaces: for certain horospherical varieties (flag varieties [32], toric varieties [5] and toric bundles over flag varieties [66]), for wonderful compactifications of semisimple groups of adjoint type [68, 38], for certain other wonderful varieties [39] and for biequivariant compactifications of unipotent groups [67] (including equivariant $\mathbb{G}_\mathrm{a}^n$-compactifications [22]). Here, the proofs use harmonic analysis on adelic points.

In absence of additional structure, we only know four more low-dimensional cases: Manin's conjecture was verified for two smooth quintic del Pezzo surfaces [14, 16], for one smooth quartic del Pezzo surface [15] and (in the thin set version [51]) for a quadric bundle in $\mathbb{P}^3 \times \mathbb{P}^3$ [20]. Not surprisingly, there are many more results on versions of Manin's conjecture for *singular* varieties because usually analytic techniques are easier to implement in the presence of singularities.

In this paper, we take a different methodological approach and initiate a systematic study of Manin's conjecture for varieties for which we have access to the Cox ring, and where a universal torsor is given by a polynomial of the shape

$$\sum_{i=1}^{k} b_i \prod_{j=1}^{J_i} x_{ij}^{h_{ij}} = 0 \tag{1.2}$$

with integral coefficients $b_i$ and certain exponents $h_{ij} \in \mathbb{N}$. This includes a fairly large class of interesting cases, in particular numerous varieties with a torus action of complexity one or higher (see [42, 31, 41] and the references therein, for example), most weak del Pezzo surfaces whose universal torsor is given by one equation [27], (nontoric) spherical varieties of semisimple rank one, as well as several nonspherical smooth Fano threefolds [29] and many other varieties.

Our analytic approach towards Manin's conjecture, to be described later in more detail, is insensitive to the dimension of the variety (in contrast to the circle method) and independent of an additional group structure (in contrast to methods based on harmonic analysis on adelic points). A showcase for our approach is the proof the Manin–Peyre conjecture for all smooth spherical Fano threefolds of semisimple rank one and type $T$ in Theorem 1.1. We will give several more examples in Theorems 1.2 and 1.3 to shed light on the scope of the underlying method.

## *1.2. Spherical varieties*

Let $G$ be a connected reductive group. A normal $G$-variety $X$ is called spherical if a Borel subgroup of $G$ has a dense orbit in $X$. Spherical varieties have a rich theory. They include symmetric varieties, and

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the corresponding space $L^2(X)$ has been the subject of intense investigation from the point of view of (local) harmonic analysis and the (relative) Langlands program (e. g., [63, 64]). Spherical varieties also admit a combinatorial description. This is achieved by the recently completed Luna program [53, 13, 26, 52] and the Luna–Vust theory of spherical embeddings [54, 50]. We recall the relevant theory in Section 10 and refer to [12, 59, 71] as general references. In this paper, we are interested in the size of smooth spherical varieties in the context of Manin's conjecture.

If the acting group $G$ has semisimple rank zero, then $G$ is a torus and Manin's conjecture is known ([5]; see also [65]). The next interesting case is $G$ of semisimple rank one. Here, we may assume $G = \mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}^r$ by passing to a finite cover (see Section 10.2 for more details). Let $G/H = (\mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}^r)/H$ be the open orbit in $X$. Let $H' \times \mathbb{G}_\mathrm{m}^r = H \cdot \mathbb{G}_\mathrm{m}^r \subseteq \mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}^r$. Then the homogeneous space $\mathrm{SL}_2/H'$ is spherical, and hence either $H'$ is a maximal torus (*the case T*) or $H'$ is the normalizer of a maximal torus in $\mathrm{SL}_2$ (*the case N*) or the homogeneous space $\mathrm{SL}_2/H'$ is horospherical, in which case $X$ is isomorphic (as an abstract variety, possibly with a different group action) to a toric variety, so we may exclude this case from our discussion.

### 1.3. Spherical Fano threefolds

We start our discussion with dimension 3, the smallest dimension where nonhorospherical spherical varieties of semisimple rank one exist. A complete classification of nontoric smooth spherical Fano threefolds over $\overline{\mathbb{Q}}$ was established by Hofscheier [44], cf. Table 11.1. In this situation, the acting group always has semisimple rank one, so our present setup is in fact already the general picture, and the following discussion applies to all nontoric smooth spherical Fano threefolds.

There are precisely four nonhorospherical examples of type $T$ that are not equivariant $\mathbb{G}_\mathrm{a}^3$-compactifications. They have natural split forms $X_1, \dots, X_4$ over $\mathbb{Q}$, which we describe in Section 11 in detail; see Table 1.1 for an overview. In the classification of smooth Fano threefolds by Iskovskikh [48, 49] and Mori–Mukai [56], they have types III.24, III.20 (of Picard number 3), IV.8, IV.7 (of Picard number 4), respectively.

In Section 3.2, we will define natural anticanonical height functions $H_j\colon X_j(\mathbb{Q}) \to \mathbb{R}$ using the anticanonical monomials in their Cox rings. We establish the Manin–Peyre conjecture in all these cases. We write $N_j(B)$ for $N_{X_j,U_j,H_j}(B)$, where here and in all subsequent cases, the open subset $U_j$ will be the set of all points with nonvanishing Cox coordinates.

**Theorem 1.1.** *The Manin–Peyre conjecture holds for the smooth spherical Fano threefolds $X_1, \dots, X_4$ of semisimple rank one and type T. More precisely, there exist explicit constants $C_1, \dots, C_4$ such that*

$$N_j(B) = (1 + o(1))C_j B(\log B)^{\operatorname{rk}\operatorname{Pic} X_j - 1}$$

*for $1 \leq j \leq 4$. The values of $C_j$ are the ones predicted by Peyre.*

*Table 1.1. Our spherical varieties.*

| | dim | rk Pic | torsor equation | $N$ |
|---|---|---|---|---|
| $X_1$ | 3 | 3 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | 13 |
| $X_2$ | 3 | 3 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}^2$ | 13 |
| $X_3$ | 3 | 4 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | 14 |
| $X_4$ | 3 | 4 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | 17 |
| $X_5$ | 4 | 5 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}$ | 34 |
| $X_6$ | 5 | 3 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}^2$ | 24 |
| $X_7$ | 6 | 5 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}x_{34}x_{35}^2$ | 80 |
| $X_8$ | 7 | 6 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}^2x_{34}^2$ | 156 |
| $\widetilde{X}^\dagger$ | 3 | 4 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}^2$ | 13 |

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It is a fun exercise to compute $C_j$ explicitly (cf. Appendix A), for which the interesting and apparently previously unknown integral identities involving sin-integrals and Fresnel integrals in Lemma 1.1 play an important role. One obtains

$$C_1 = \frac{40-\pi^2}{12}\prod_p (1-p^{-2})^3, \quad C_3 = \frac{5(258-4\pi^2)}{1296}\prod_p \left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{4}{p^2}+\frac{1}{p^3}\right),$$
$$C_2 = \frac{170-\pi^2-96\log 2}{36}\prod_p (1-p^{-2})^3, \quad C_4 = \frac{94-2\pi^2}{72}\prod_p \left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{4}{p^2}+\frac{1}{p^3}\right).$$

Theorem 1.1 is an easy consequence of Theorem 10.1 that proves the Manin–Peyre conjecture for smooth split spherical Fano varieties of arbitrary dimension with semisimple rank one and type $T$, subject to a number of technical conditions that are straightforward to check in every given instance. Similar methods apply also to smooth spherical Fano varieties of type $N$, but these have some additional features to which we return in a subsequent paper.

Theorem 1.1 contains the first examples where Manin's conjecture is established for smooth Fano threefolds that do not follow from general results concerning equivariant compactifications of algebraic groups or homogeneous spaces. Theorem 1.1 in fact confirms the Manin–Peyre conjecture for *all* classes of *smooth spherical Fano threefolds of semisimple rank one and type T*. Previously, the knowledge of the number of rational points on these varieties has been much less precise. Manin [55] shows that smooth Fano threefolds have at least linear growth for rational points in Zariski dense open subsets of bounded anticanonical height over sufficiently large ground fields. A closer inspection of his arguments reveals in fact lower bounds of the correct order of magnitude: $N_j \gg B(\log B)^{\operatorname{rk}(\operatorname{Pic} X_j)-1}$ in the situation of Theorem 1.1 (cf. the proof of [55, Proposition 1.4] as the $X_j$ in Theorem 1.1 are blow-ups of toric varieties). Tanimoto [70, §7] proves the upper bounds $N_j \ll B^{5/2+\varepsilon}$ for $j = 1, 2, 4$ and $N_3 \ll B^{2+\varepsilon}$.

### *1.4. Higher-dimensional cases*

A classification of higher-dimensional spherical varieties is currently not available, but our methods work equally well in dimension exceeding three. For a given dimension, there are still only finitely many cases of smooth spherical Fano varieties of semisimple rank one, and we include some representative examples with interesting torsor equations and high Picard number. Many other examples are available by the same method. The four varieties $X_5, X_6, X_7, X_8$ that we investigate here are smooth spherical Fano varieties of semisimple rank one and type $T$ of dimension $4, 5, 6, 7$, respectively, with $\operatorname{rk}\operatorname{Pic} X_5 = 5$, $\operatorname{rk}\operatorname{Pic} X_6 = 3$, $\operatorname{rk}\operatorname{Pic} X_7 = 5$ and $\operatorname{rk}\operatorname{Pic} X_8 = 6$. We refer to Section 12 for their combinatorial description and Table 1.1 for a quick overview and remark that for neither of these varieties, Manin's conjecture follows from previous results (cf. Appendix B).

**Theorem 1.2.** *The Manin–Peyre conjecture holds for the smooth spherical Fano varieties $X_5, \dots, X_8$ of semisimple rank one and type T. More precisely, there exist explicit constants $C_5, \dots, C_8 > 0$ such that*

$$N_j(B) = (1+o(1))C_j B(\log B)^{\operatorname{rk}\operatorname{Pic} X_j - 1}$$

*for $j = 5, \dots, 8$. The values of $C_j$ are the ones predicted by Peyre.*

We remark that Theorems 1.1 and 1.2 are compatible with the thin set version of Manin's conjecture. Since our spherical varieties have a connected stabilizer for the open orbit, their sets of rational points are not thin [10, Corollary 2.5]. As in [51, Examples 5.12, 5.13], one can show that our results are compatible with [51, Conjecture 5.2].

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### 1.5. The methods

The starting point of the quantitative analysis of Fano varieties in this paper is a good understanding of their Cox ring. We use it to pass to a universal torsor and translate Manin's conjecture into an explicit counting problem whose structure we describe in a moment and that is amenable to analytic techniques. The descent to a universal torsor is a common technique in analytic approaches to Manin's conjecture, but in many cases it proceeds by ad hoc considerations. Here, we take a more systematic approach and derive the passage from the Cox ring to the explicit counting problem in considerable generality. This is summarized in Proposition 3.8. Next, we take the opportunity to express Peyre's constant in terms of Cox coordinates in Proposition 4.11 as a product of a surface integral, the volume of a polytope and an Euler product so that a verification of the complete Manin–Peyre conjecture is possible without additional ad hoc computations.

This first part of the paper is presented in greater generality than necessary for the direct applications to spherical varieties and should prove to be useful in other situations.

The second part of the paper is devoted to an explicit solution of counting problems having the structure required in Proposition 3.8. In many important cases, a universal torsor is given by a single equation of the shape (1.2). We may have additional variables $x_{01}, \dots, x_{0J_0}$ that do not appear in the torsor equation; for those, we put formally $h_{0j} = 0$. Equation (1.2) is then to be solved in nonzero integers $x_{ij}$. This seemingly simple diophantine problem has to be analyzed with certain coprimality constraints on the variables, and the variables are restricted to a highly cuspidal region. As specified in Proposition 3.8, the height condition translates into inequalities

$$\prod_{i=0}^{k} \prod_{j=1}^{J_i} |x_{ij}|^{\alpha_{ij}^{\nu}} \leq B \quad (1 \leq \nu \leq N) \tag{1.3}$$

for certain nonnegative exponents[1] $\alpha_{ij}^{\nu}$. In order to describe the coprimality conditions on the variables $x_{ij}$ in (1.2), let $S_\rho \subseteq \{(i, j) : i = 0, \dots, k, j = 1, \dots, J_i\}$ $(1 \leq \rho \leq r)$ be a collection of sets that define $r$ conditions

$$\gcd\{x_{ij} : (i, j) \in S_\rho\} = 1 \quad (1 \leq \rho \leq r). \tag{1.4}$$

Now, fix a set of coefficients $b_i$ in (1.2), and let $N_{\mathbf{b}}(B) = N(B)$ denote the number of $x_{ij} \in \mathbb{Z} \setminus \{0\}$ $(0 \leq i \leq k, 1 \leq i \leq J_i)$ satisfying (1.2), (1.3) and (1.4). We aim to establish an asymptotic formula of the shape

$$N(B) = (1 + o(1))c_1 B(\log B)^{c_2} \tag{1.5}$$

for some constants $c_1 > 0$, $c_2 \in \mathbb{N}_0$, and our method succeeds subject to quite general conditions. Of course, for a proper solution of the Manin–Peyre conjecture, we do not only have to establish (1.5) but to recover the geometric and arithmetic nature of $c_1$ and $c_2$ in terms of the Manin–Peyre predictions. This will require some natural consistency conditions involving the exponents $h_{ij}$ in the torsor equation (1.2) and $\alpha_{ij}^{\nu}$ in the height conditions (1.3), cf. in particular (7.4), (7.6) below.

We now describe in more detail the analytic machinery that yields asymptotic formulas of type (1.5) for the problem given by (1.2), (1.3), (1.4). Input of two types is required.

On the one hand, we need a preliminary upper bound of the expected order of magnitude for the count in question. The precise requirements are formulated in the form of Hypothesis 7.2 below. In many instances, the desired bounds can be verified by soft and elementary techniques. In particular, for smooth spherical Fano varieties of semisimple rank one and type $T$, this can be checked by computing dimensions and extreme points of certain polytopes; see Proposition 7.6.

[1]The superscript $\nu$ is not an exponent, but an index. This notation is chosen in accordance with the notation in Section 2.

On the other hand, we require an asymptotic formula for the number of integral solutions of (1.2) in potentially lopsided boxes, with variables restricted by $\frac{1}{2}X_{ij} \le |x_{ij}| \le X_{ij}$, say. As a notable feature of the method, the asymptotic information is required only when the $k$ products $\prod_j X_{ij}^{h_{ij}}$ $(1 \le i \le k)$ have roughly the same size. The circle method deals with this auxiliary counting problem in considerable generality, culminating in Proposition 5.2 that comes with a power saving in the shortest variable $\min_{ij} X_{ij}$.

The method described in Section 8 transfers the information obtained for counting in boxes to the strangely shaped region described by the conditions (1.3). In [7], we presented a combinatorial method to achieve this for certain regions of hyperbolic type. Here, we use complex analysis to do this work for us in a far more general context. A prototype of this idea, developed only in a special (and nonsmooth) case, can be found in [9]. The final result is Theorem 8.4 that we will state once the relevant notation has been developed. Again, we are working in greater generality than needed for the immediate applications in this paper, with future applications in mind.

In the case of smooth spherical Fano threefolds of semisimple rank one and type $T$ (and in many other examples that can be found in [29, 31, 42], for example), the torsor equation (1.2) is of the shape '2-by-2 determinant equals some monomial', that is (up to changing signs)

$$x_{11}x_{12} + x_{21}x_{22} + \prod_{j=1}^{J_3} x_{3j}^{h_{3j}} = 0. \tag{1.6}$$

While the general transition method is independent of the shape of the torsor equation, for the particular case (1.6), Theorem 8.4 together with Propositions 5.2 and 7.6 offers a 'black box' to obtain the Manin–Peyre conjecture in any given situation with a small amount of elementary computations. This is formalized in Theorem 10.1, which readily yields the proofs of Theorems 1.1 and 1.2 in Sections 11.4 and 12.4.

This leaves us with the task to establish an asymptotic formula for the number of solutions of the torsor equation (1.6), with suitable constraints on the variables. The equation (1.6) involves an isolated product $x_{11}x_{12}$, one way to proceed would be to view (1.6) as a congruence modulo $x_{11}$, thus eliminating $x_{12}$. This approach is very familiar to workers in the area of divisor sums; an exemplary and historic reference is Titchmarsh's work on the divisor problem that now bears his name. In contexts very closely related to the questions that concern us here, it has been successfully applied, too, for example in work of Le Boudec [11], in a collaboration of the first two authors of this paper with Salberger [9] and on many other occasions. However, there are a number of disadvantages stemming from the asymmetric use of the variables $x_{11}, x_{12}, x_{21}$ and $x_{22}$. In particular, our transition to counting solutions of (1.6) in spiky regions needs to be fed with information on the distribution of the solutions of (1.6) with *all* variables in dyadic ranges. We therefore eschew the elementary approach in favour of the circle method. The restriction to dyadic ranges is easy to implement in this environment, and the resulting leading terms in the asymptotic formulae lend themselves more easily to Peyre's predictions, too.

The following table summarizes the analytic data discussed in this subsection for the varieties $X_1, \dots, X_8$ featured in Theorems 1.1 and 1.2. Here, $N$ is the number of height conditions in (1.3); the total number of variables is $J = J_0 + \cdots + J_3 = \dim X_i + \operatorname{rk} \operatorname{Pic} X_i + 1$.

### *1.6. Another application*

Theorem 10.1 offers a promising line of attack to establish Manin's conjecture in many instances, not only those covered by Theorems 1.1 and 1.2. As proof of concept, we include a somewhat different application featuring a singular spherical Fano threefold. The last two authors [28] have studied some examples and have confirmed Manin's conjecture for two families of singular spherical Fano threefolds. One family was given by the equation $ad - bc - z^{n+1} = 0$ in weighted projective space $\mathbb{P}(1, n, 1, n, 1)$, the other was the family of hypersurfaces given by $ad - bc - y^n z^{n+1} = 0$ in a certain toric variety $(n \ge 2)$.

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For the counting problem on the torsor, elementary analytic techniques were enough. We believe that this is related to the fact that all the varieties have noncanonical (log terminal) singularities, with the exception of the first variety for $n = 2$, which is a slightly harder case with canonical singularities and a crepant resolution. However, for similar varieties, the elementary counting techniques in [28] do not seem to be of strength sufficient for a proof of Manin's conjecture.

In Section 13, we use the much stronger technology developed in this paper to discuss one such case. Let $X^\dagger$ be the anticanonical contraction of the blow-up of the hypersurface $\mathbb{V}(z_{11}z_{12} - z_{21}z_{22} - z_{31}z_{32})$ in $\mathbb{P}^2_\mathbb{Q} \times \mathbb{P}^2_\mathbb{Q}$ (with coordinates $(z_{11} : z_{21} : z_{31})$ and $(z_{12} : z_{22} : z_{32})$) in the two curves $\mathbb{V}(z_{31}) \times \{(0 : 0 : 1)\}$ and $\mathbb{V}(z_{31}, z_{32})$. This is a singular Fano threefold admitting a crepant resolution.

**Theorem 1.3.** *For the singular spherical Fano threefold $X^\dagger$, there exists a positive number $C^\dagger$ such that*

$$N^\dagger(B) = (1 + o(1))C^\dagger B(\log B)^3.$$

*The value of $C^\dagger$ is the one predicted by Peyre [61].*

Further applications are postponed to a separate paper.

*Notational remarks.* This work draws on results from various areas of mathematics. Due to the large number of topics covered it seemed impracticable to aim for an entirely consistent notation. Any attempt to do so would be in conflict with traditions in the respective fields. We opt for a pragmatic approach and use notation that, locally, seems natural to working mathematicians. For example, almost everywhere in the paper, the letter $B$ signals the threshold for the height of points in several counting problems, but in Section 10, a Borel subgroup of the group $G$ that occurs in the definition of a spherical variety is denoted by $B$. This is just one example of double booking for symbols that are often 'frozen' in less interdisciplinary writings. We therefore introduce notation at the appropriate stage of the argument.

## Part I Heights and Tamagawa measures in Cox coordinates

Universal torsors were introduced and studied by Colliot-Théléne and Sansuc; see [23]. Their first major application to Manin's conjecture can be found in the work of Salberger [65] on toric varieties.

Cox rings were defined by Hu and Keel [45], and they provide a global description of universal torsors; the Cox ring of a normal irreducible algebraic variety $X$ is roughly defined as $\mathscr{R}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathscr{O}_X(D))$, where specifying the multiplication law requires some care. Moreover, a quotient construction $\operatorname{Spec} \mathscr{R}(X) \supseteq \widetilde{X} \to X$ is obtained. This generalizes the homogeneous coordinate ring of $\mathbb{P}^n$ with quotient construction $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ as well as Cox's construction for toric varieties [24]. For details on toric varieties and Cox rings, we refer to the books [25, 2] and to [30].

Given a variety whose Cox ring with precisely one relation is known explicitly, we show (under mild conditions) how to write down an anticanonical height function (3.7), how to make the counting problem on a universal torsor explicit (Proposition 3.8) and how to express Peyre's constant (Proposition 4.11). This is achieved in terms of the Cox ring data, without constructing an anticanonical embedding in a projective space, widely generalizing results from [60, 62, 65, 8, 9].

## 2. Varieties and universal torsors in Cox coordinates

In this section, we recall how a variety $X$ with precisely one relation in its Cox ring can be described in *Cox coordinates* as a hypersurface in a toric variety (with affine charts as in Section 2.1 that will be used in in the following sections), and how this gives a description of their universal torsors as hypersurfaces in affine space (Section 2.2). This leads to an explicit description of the parameterization of the rational points on $X$ by integral points on a universal torsor (Proposition 2.4).

Let $X$ be a smooth split projective variety over $\mathbb{Q}$ with big and semiample anticanonical class $\omega_X^\vee$ whose Picard group is free of finite rank. (Here, *split* means that the natural map from the Picard group $\operatorname{Pic} X$ over the ground field to the geometric Picard group is an isomorphism.) Assume that it has a finitely generated Cox ring $\mathscr{R}(X)$ [45, Definition 2.6], [2, §1.4] with precisely one relation with integral coefficients.

In other words, $X$ has a Cox ring over $\mathbb{Q}$ [30] of the form

$$\mathscr{R}(X) \cong \mathbb{Q}[x_1, \dots, x_J]/(\Phi), \tag{2.1}$$

where $x_1, \dots, x_J$ is a system of pairwise nonassociated $\operatorname{Pic} X$-prime generators and the relation $\Phi \in \mathbb{Z}[x_1, \dots, x_J]$ is nonzero. According to [2, Construction 3.2.5.3], (2.1) defines a canonical embedding of $X$ into a (not necessarily complete) ambient toric variety $Y^\circ$.

**Lemma 2.1.** *The toric variety $Y^\circ$ can be completed to a projective toric variety $Y$ such that the natural map $\operatorname{Cl} Y \to \operatorname{Cl} X = \operatorname{Pic} X$ is an isomorphism and $-K_X$ is big and semiample on $Y$.*

*Proof.* By [2, Proposition 3.2.5.4(iii)], we have $\operatorname{Cl} Y^\circ = \operatorname{Cl} X$. We consider the Gelfand–Kapranov–Zelevinsky (GKZ) decomposition of $Y^\circ$ (see, for example, [2, §2.2.2]). According to [2, Construction 3.2.5.7], the chambers in the GKZ decomposition of $Y^\circ$ which contain ample divisors on $X$ give rise to completions $Y$ of $Y^\circ$ with $\operatorname{Cl} Y^\circ = \operatorname{Cl} Y$. Now, choose $Y$ corresponding to a chamber whose closure contains $-K_X$. Since $-K_X$ is semiample on $X$, this is possible by [2, Proposition 3.3.2.9]. Then $-K_X$ is semiample on $Y$ according to [2, Proposition 2.4.2.6].

By [2, Propositions 3.3.2.9 and 2.4.2.6], $-K_X$ is in the relative interior of the moving cone of $Y$, hence $-K_X$ is big on $Y$. □

We assume that $Y$ is chosen as in Lemma 2.1. Its Cox ring is $\mathscr{R}(Y) = \mathbb{Q}[x_1, \dots, x_J]$ [2, Construction 3.2.5.3]. Let $\Sigma$ be the fan of $Y$, and let $\Sigma_{\max}$ be the set of maximal cones. The generators $x_1, \dots, x_J$ have the same grading as in $\mathscr{R}(X)$ and are in bijection to the rays $\rho \in \Sigma(1)$; we also write $x_\rho$ for $x_i$ corresponding to $\rho$. We generally write

$$J = \#\Sigma(1), \quad N = \#\Sigma_{\max}, \tag{2.2}$$

and we assume:

$$\text{The projective toric variety } Y \text{ can be chosen to be regular.} \tag{2.3}$$

### 2.1. Affine charts in Cox coordinates

Since $\mathscr{R}(X) \cong \mathbb{Q}[x_\rho : \rho \in \Sigma(1)]/(\Phi)$ with $\operatorname{Pic} X$-homogeneous $\Phi$, our variety $X$ is a hypersurface defined by $\Phi$ (in Cox coordinates) in the toric variety $Y$ (with Cox ring $\mathscr{R}(Y) = \mathbb{Q}[x_\rho : \rho \in \Sigma(1)]$). On $Y$, we can regard $X$ as a prime divisor of class $\deg \Phi \in \operatorname{Cl} Y$.

We introduce further notation for the toric variety $Y$. In Part I, let $U$ be the open torus in $Y$. For each $\rho \in \Sigma(1)$, we have a $U$-invariant Weil divisor $D_\rho$ defined by $x_\rho$ of class $[D_\rho] = \deg(x_\rho) \in \operatorname{Cl} Y$ [25, §4.1]. Let

$$D_0 := \sum_{\rho \in \Sigma(1)} D_\rho, \tag{2.4}$$

which is an effective divisor of class $[D_0] = -K_Y$. For a $U$-invariant divisor $D = \sum_{\rho \in \Sigma(1)} \lambda_\rho D_\rho$, let

$$x^D := \prod_{\rho \in \Sigma(1)} x_\rho^{\lambda_\rho} \tag{2.5}$$

denote the corresponding monomial of degree $[D]$. For example,

$$x^{D_0} = \prod_{\rho \in \Sigma(1)} x_\rho. \tag{2.6}$$

**Lemma 2.2.** *Let $M$ and $N$ be the character and cocharacter lattices of the toric variety $Y$, respectively. Let $\rho_1, \dots, \rho_k \in \Sigma(1)$ be rays such that their primitive generators $u_{\rho_1}, \dots, u_{\rho_k} \in N$ form a basis of $N$. Then the set $\{[D_\rho] : \rho \neq \rho_1, \dots, \rho_k\}$ is a basis of $\operatorname{Cl} Y$.*

*Proof.* According to [2, Before Proposition 2.1.2.7], there are two exact sequences

$$0 \to L \to \mathbb{Z}^{\Sigma(1)} \to N \to 0,$$
$$0 \leftarrow \mathrm{Cl}(Y) \leftarrow \mathbb{Z}^{\Sigma(1)} \leftarrow M \leftarrow 0,$$

which are dual to each other. Here, $\mathbb{Z}^{\Sigma(1)}$ denotes the lattice with basis $\{e_\rho : \rho \in \Sigma(1)\}$, which is assumed to be dual to itself. The top right map sends $e_\rho$ to $u_\rho$ while the lower left map sends $e_\rho$ to $[D_\rho]$. Since the top right map sends $e_{\rho_1}, \dots e_{\rho_k}$ to a basis of $N$, the lower left map sends their complement to a basis of $\mathrm{Cl}(Y)$. □

It follows from Lemma 2.2 that, for each $\sigma \in \Sigma_{\max}$, the set $\{[D_\rho] : \rho \notin \sigma(1)\}$ is a basis of $\operatorname{Cl} Y$; in other words,

$$\{\deg(x_\rho) : \rho \notin \sigma(1)\} \tag{2.7}$$

is a basis of $\operatorname{Pic} X$.

**Lemma 2.3.** *For each $\sigma \in \Sigma_{\max}$, there is a unique effective Weil divisor $D(\sigma) = \sum_{\rho \notin \sigma(1)} \alpha_\rho^\sigma D_\rho$ of class $-K_X$ whose support is contained in $\bigcup_{\rho \notin \sigma(1)} D_\rho$.*

*Proof.* For the existence, choose an effective $U$-invariant $\mathbb{Q}$-Weil divisor $D$ on $Y$ with $[D] = -K_X$. Let $M$ be the character lattice of the torus $U$. We write $U_\sigma \subseteq Y$ for the open subset corresponding to the cone $\sigma$.

Choose $\chi_\sigma \in M_\mathbb{Q}$ such that $(\operatorname{div} \chi_\sigma)_{|U_\sigma} = D_{|U_\sigma}$. Define $D(\sigma) := D - \operatorname{div} \chi_\sigma$. Then $D(\sigma)$ is of class $-K_X$ and its support is contained in $\bigcup_{\rho \notin \sigma(1)} D_\rho$. Moreover, a multiple of $-K_X$ being globally generated means that we have $\chi_\sigma \le \chi_{\sigma'}$ on $\sigma'$ for every $\sigma' \in \Sigma_{\max}$ [25, Theorem 6.1.7]. Hence, $D(\sigma)$ is an effective $\mathbb{Q}$-divisor.

Because of (2.7), there is a unique $\mathbb{Z}$-linear combination of the $D_\rho$ with $\rho \notin \sigma(1)$ of class $-K_X$, which must be equal to $D(\sigma)$. □

For $\sigma \in \Sigma_{\max}$, notation (2.5) gives

$$x^{D(\sigma)} = \prod_{\rho \notin \sigma(1)} x_\rho^{\alpha_\rho^\sigma}, \tag{2.8}$$

where $\alpha_\rho^\sigma$ are the unique nonnegative integers satisfying $-K_X = \sum_{\rho \notin \sigma(1)} \alpha_\rho^\sigma \deg(x_\rho)$ in $\operatorname{Pic} X$ (as in Lemma 2.3).

Every $\sigma \in \Sigma_{\max}$ defines an affine chart on $Y$ as follows. For each $\rho' \in \Sigma(1)$, we can write

$$\deg(x_{\rho'}) = \sum_{\rho \notin \sigma(1)} \alpha_{\rho',\rho}^\sigma \deg(x_\rho) \tag{2.9}$$

with certain $\alpha_{\rho',\rho}^\sigma \in \mathbb{Z}$ by (2.7). Then

$$z_{\rho'}^\sigma := x_{\rho'} / \prod_{\rho \notin \sigma(1)} x_\rho^{\alpha_{\rho',\rho}^\sigma}$$

is a rational section of degree $0 \in \mathrm{Cl}\, Y$, with $z^\sigma_{\rho'} = 1$ for $\rho' \notin \sigma(1)$. By [25, Theorem 1.2.18], the sections $z^\sigma_{\rho'}$ for $\rho' \in \sigma(1)$ define an isomorphism

$$U^\sigma \to \mathbb{A}_{\mathbb{Q}}^{\sigma(1)}, \tag{2.10}$$

where $U^\sigma$ is the open subset of $Y$, where $x_\rho \neq 0$ for all $\rho \notin \sigma(1)$ (i. e., the complement of $\bigcup_{\rho \notin \sigma(1)} D_\rho$ in $Y$).

We also obtain affine charts on the open subset

$$X^\sigma := X \cap U^\sigma \tag{2.11}$$

of $X$. The image of $X^\sigma$ in $\mathbb{A}_{\mathbb{Q}}^{\sigma(1)}$ is defined by

$$\Phi^\sigma := \Phi(z^\sigma_\rho) = \Phi(x_\rho) / \prod_{\rho \notin \sigma(1)} x_\rho^{\beta^\sigma_\rho}, \tag{2.12}$$

where $\beta^\sigma_\rho \in \mathbb{Z}$ satisfy

$$\deg \Phi = \sum_{\rho \notin \sigma(1)} \beta^\sigma_\rho \deg(x_\rho) \tag{2.13}$$

since $x_\rho \neq 0$ on $U^\sigma$ for $\rho \notin \sigma(1)$. By the implicit function theorem, for every $P \in X^\sigma(\mathbb{Q}_v)$ with $\partial \Phi^\sigma / \partial z^\sigma_{\rho_0}(P) \neq 0$ for some $\rho_0 \in \sigma(1)$, there is an open $v$-adic neighborhood $U_0 \subseteq X^\sigma(\mathbb{Q}_v)$ such that the composition of $X^\sigma \to \mathbb{A}_{\mathbb{Q}}^{\sigma(1)}$ with the natural projection $\pi^\sigma_{\rho_0} \colon \mathbb{A}_{\mathbb{Q}}^{\sigma(1)} \to \mathbb{A}_{\mathbb{Q}}^{\sigma(1) \setminus \{\rho_0\}}$ that drops the $\rho_0$-coordinate induces a chart

$$U_0 \to \mathbb{Q}_v^{\sigma(1) \setminus \{\rho_0\}}. \tag{2.14}$$

Its inverse is obtained by computing the $\rho_0$-coordinate $z^\sigma_{\rho_0} = \phi((z^\sigma_\rho)_{\rho \in \sigma(1) \setminus \{\rho_0\}})$ using the implicit function $\phi$ obtained by solving $\Phi^\sigma$ for $z^\sigma_{\rho_0}$.

### *2.2. Universal torsors and models*

Let $T \cong \mathbb{G}_{\mathrm{m},\mathbb{Q}}^{\mathrm{rk}\,\mathrm{Pic}\, X}$ be the Néron–Severi torus of $X$ (i. e., the torus whose characters are $\mathrm{Pic}\, X = \mathrm{Cl}\, Y$). Cox's construction and the theory of Cox rings [65, §8] and [25, §5.1] give universal torsors $X_0 \subset Y_0$ (with inclusion morphism $\iota_0 \colon X_0 \to Y_0$) over $X \subset Y$ (with inclusion $\iota : X \to Y$). Here, $Y_0$ is the principal universal torsor over $Y$ under $T$. Both projections $X_0 \to X$ and $Y_0 \to Y$ are called $\pi$.

We have fans $\Sigma_1 \supset \Sigma_0 \to \Sigma$ (with the sets of rays $\Sigma_1(1) = \Sigma_0(1)$ in natural bijection to $\Sigma(1)$) corresponding to the toric varieties $\mathbb{A}^J_{\mathbb{Q}} = \mathbb{A}^{\Sigma(1)}_{\mathbb{Q}} = Y_1 \supset Y_0 \to Y$. We have $Y_0 = Y_1 \setminus Z_Y$, where $Z_Y$ is defined by the *irrelevant ideal* [25, §5.2] generated by the monomials

$$x^{\underline{\sigma}} := \prod_{\rho \notin \sigma(1)} x_\rho \tag{2.15}$$

for all maximal cones $\sigma \in \Sigma_{\max}$. By [25, Proposition 5.1.6], there are *primitive collections*

$$S_1, \dots, S_r \subseteq \Sigma(1) \tag{2.16}$$

(i. e., $S_j \not\subseteq \sigma(1)$ for all $\sigma \in \Sigma$, but for every proper subset $S'_j$ of $S_j$, there is a $\sigma \in \Sigma$ with $S'_j \subseteq \sigma(1)$) such that the $r$ irreducible components of $Z_Y$ are defined by the vanishing of $x_\rho$ for all $\rho \in S_j$.

The fans and their maps allow us to construct $\mathbb{Z}$-models $\widetilde{\pi} \colon \widetilde{Y}_1 \setminus \widetilde{Z}_Y = \widetilde{Y}_0 \to \widetilde{Y}$ with an action of $\widetilde{T} \cong \mathbb{G}_{\mathrm{m},\mathbb{Z}}^{\mathrm{rk}\,\mathrm{Cl}\, Y}$ on $\widetilde{Y}_0$ and $\widetilde{Y}_1$ (see [65, Remark 8.6b and later]).

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The characteristic space $X_0$ is defined in $Y_0$ by $\Phi$ (interpreted as an affine equation; see [2, §1.6.3]). Then $X_0 = X_1 \setminus Z_X$, where $X_1 = \operatorname{Spec} \mathscr{R}(X)$ is defined by $\Phi$ in $Y_1$, and $Z_X = Z_Y \cap X_1$.

We have $\widetilde{\pi}\colon \widetilde{X}_1 \setminus \widetilde{Z}_X = \widetilde{X}_0 \to \widetilde{X}$ for $\mathbb{Z}$-models of $X, X_0, X_1, Z_X$ defined in $\widetilde{Y}, \widetilde{Y}_0, \widetilde{Y}_1, \widetilde{Z}_Y$ by $\Phi$ (regarded as an affine equation for $\widetilde{X}_0, \widetilde{X}_1, \widetilde{Z}_X$ and as $\operatorname{Cl} Y$-homogeneous for $\widetilde{X}$).

**Proposition 2.4.** *We have*

$$\widetilde{X}_0(\mathbb{Z}) = \{\mathbf{x} = (x_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^{\Sigma(1)} : \Phi(\mathbf{x}) = 0,\ \gcd\{x_\rho : \rho \in S_j\} = 1 \text{for all} j = 1, \dots, r\},$$
$$\widetilde{X}_0(\mathbb{Z}_p) = \{\mathbf{x} = (x_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}_p^{\Sigma(1)} : \Phi(\mathbf{x}) = 0,\ p \nmid \gcd\{x_\rho : \rho \in S_j\} \text{for all} j = 1, \dots, r\}.$$

*The map $\widetilde{\pi}$ induces a $2^{\operatorname{rk} \operatorname{Pic} X} : 1$-map $\widetilde{X}_0(\mathbb{Z}) \to \widetilde{X}(\mathbb{Z}) = X(\mathbb{Q})$.*

*Proof.* Arguing as in [65, (11.5)], but using the description of $\widetilde{Z}_Y$ by the primitive collections shows

$$\widetilde{Y}_0(\mathbb{Z}) = \{\mathbf{y} \in \mathbb{Z}^{\Sigma(1)} : \gcd\{y_\rho : \rho \in S_j\} = 1 \text{ for all } j = 1, \dots, r\}.$$

Since $\widetilde{X}$ is defined by $\Phi$ in $\widetilde{Y}$, the first result follows. The description of $\widetilde{X}(\mathbb{Z}_p)$ is obtained similarly.

By [65, Lemma 11.4], $\widetilde{\pi}$ induces a $2^{\operatorname{rk} \operatorname{Cl} Y} : 1$-map $\widetilde{Y}_0(\mathbb{Z}) \to \widetilde{Y}(\mathbb{Z}) = Y(\mathbb{Q})$. Restricting to the points where $\Phi$ vanishes gives the result. □

## 3. Heights in Cox coordinates

In this section, we construct an explicit adelic metrization of the anticanonical bundle of our variety $X$ with one relation $\Phi$ in its Cox ring (Section 3.1), using the charts from Section 2.1 and Poincaré residues. This metrization is the basis for the construction of an anticanonical height function (Section 3.2) that we use to count points, and of the Tamagawa measure for Peyre's expected leading constant (Section 4). On the universal torsor, only the Archimedean factor of the height function remains (Section 3.6). This leads to the main result of this section: a completely explicit description of the counting problem (Proposition 3.8) in terms of the Cox ring of $X$. Section 3.5 contains some related linear algebra results that will be used later.

We keep the assumptions and notation from Section 2.

### 3.1. Adelic metrization of $\omega_X^{-1}$ via Poincaré residues

Here, we use the notation and results from Section 2.1. A special case of the following can be found in [8, §5]. There is a global nowhere vanishing section $s_Y$ of $\omega_Y(D_0)$ (2.4) whose restriction to every open subset $U^\sigma \subset Y$ as in (2.10) for $\sigma \in \Sigma_{\max}$ is $\pm \bigwedge_{\rho\in\sigma(1)} \frac{\mathrm{d}z_\rho^\sigma}{z_\rho^\sigma}$ (see [25, Proposition 8.2.3]). Recall the definition of $\Phi^\sigma$ (2.12).

**Lemma 3.1.** *For each $\sigma \in \Sigma_{\max}$, we define*

$$\varpi^\sigma := \frac{x^{D_0}}{x^{D(\sigma)}\Phi} s_Y \in \Gamma(Y, \omega_Y(D(\sigma) + X)); \tag{3.1}$$

*this is a nowhere vanishing global section of $\omega_Y(D(\sigma) + X)$. On $U^\sigma$, we have*

$$\varpi^\sigma = \frac{\pm 1}{\Phi^\sigma} \bigwedge_{\rho\in\sigma(1)} \mathrm{d}z_\rho^\sigma \in \Gamma(U^\sigma, \omega_Y(X)).$$

*Proof.* For the first statement, note that $x^{D_0}(x^{D(\sigma)}\Phi)^{-1}$ corresponds to the divisor $D_0 - D(\sigma) - X$.

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On $U^\sigma$, we have

$$\varpi^\sigma = \frac{\pm x^{D_0}}{x^{D(\sigma)}\Phi} \bigwedge_{\rho\in\sigma(1)} \frac{\mathrm{d}z_\rho^\sigma}{z_\rho^\sigma} \in \Gamma(U^\sigma, \omega_Y(X)) \tag{3.2}$$

where $\Gamma(U^\sigma, \omega_Y(X)) = \Gamma(U^\sigma, \omega_Y(D(\sigma)+X))$ since $D(\sigma)_{|U_\sigma} = 0$ by Lemma 2.3. With $\beta_\rho^\sigma$ as in (2.13), let

$$\lambda = \frac{x^{D_0}}{x^{D(\sigma)} \prod_{\rho\notin\sigma(1)} x_\rho^{\beta_\rho^\sigma}}.$$

In view of (2.12), we obtain

$$\varpi^\sigma = \frac{\pm\lambda}{\Phi^\sigma} \bigwedge_{\rho\in\sigma(1)} \frac{\mathrm{d}z_\rho^\sigma}{z_\rho^\sigma} \in \Gamma(U^\sigma, \omega_Y(X)).$$

On $U_\sigma$, we have

$$\operatorname{div}\lambda = (\operatorname{div} x^{D_0})_{|U_\sigma} - (\operatorname{div} x^{D(\sigma)})_{|U_\sigma} - \sum_{\rho\notin\sigma(1)} \beta_\rho^\sigma D_\rho = (\operatorname{div} x^{D_0})_{|U_\sigma} - 0 - 0 = (\operatorname{div} x^{D_0})_{|U_\sigma}.$$

We also have $\operatorname{div}\prod_{\rho\in\sigma(1)} z_\rho^\sigma = (\operatorname{div} x^{D_0})_{|U_\sigma}$. Therefore, $\lambda = \prod_{\rho\in\sigma(1)} z_\rho^\sigma$ on $U_\sigma$, and we obtain the second statement. □

The Poincaré residue map

$$\operatorname{Res}\colon \omega_Y(X) \longrightarrow \iota_*\omega_X \tag{3.3}$$

is a homomorphism of $\mathscr{O}_Y$-modules. On the smooth open subset $U^\sigma$ of $Y$, it sends $\varpi^\sigma \in \Gamma(U^\sigma, \omega_Y(X))$ to $\operatorname{Res}\varpi^\sigma \in \Gamma(U^\sigma, \iota_*\omega_X) = \Gamma(X^\sigma, \omega_X)$, which is given by

$$\operatorname{Res}\varpi^\sigma = \frac{\pm 1}{\partial\Phi^\sigma/\partial z_{\rho_0}^\sigma} \bigwedge_{\rho\in\sigma(1)\setminus\{\rho_0\}} \mathrm{d}z_\rho^\sigma \tag{3.4}$$

on the open subset of $X^\sigma$ (see (2.11)) where $\partial\Phi^\sigma/\partial z_{\rho_0}^\sigma \neq 0$, for any $\rho_0 \in \sigma(1)$.

**Lemma 3.2.** *The section* $\operatorname{Res}\varpi^\sigma$ *extends uniquely to a nowhere vanishing global section of* $\omega_X(D(\sigma)\cap X)$.

*Proof.* This is similar to [8, Lemma 13]. Since $s_Y$ generates the $\mathscr{O}_Y$-module $\omega_Y(D_0)$, each

$$\varpi^\sigma = \frac{x^{D_0}}{x^{D(\sigma)}\Phi} s_Y$$

generates the $\mathscr{O}_Y$-module $\omega_Y(X + D(\sigma))$. Since $\iota^*\mathscr{O}_Y(D(\sigma)) = \mathscr{O}_X(D(\sigma)\cap X)$ (using that $X \not\subseteq \operatorname{supp} D(\sigma)$), the isomorphism $\iota^*\omega_Y(X) \longrightarrow \omega_X$ adjoint to $\operatorname{Res}\colon \omega_Y(X) \longrightarrow \iota_*\omega_X$ induces an isomorphism $\iota^*\omega_Y(X+D(\sigma)) \longrightarrow \omega_X(D(\sigma)\cap X)$ that maps $\iota^*\varpi^\sigma$ to $\operatorname{Res}\varpi^\sigma$. Hence, $\operatorname{Res}\varpi^\sigma$ generates $\omega_X(D(\sigma)\cap X)$, that is, it is a nowhere vanishing global section. □

Therefore,

$$\tau^\sigma := (\operatorname{Res}\varpi^\sigma)^{-1} \tag{3.5}$$

is a nowhere vanishing global sections of $\omega_X^{-1}(-D(\sigma)\cap X)$, which we can also view as a global section of $\omega_X^{-1}$.

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**Lemma 3.3.** *The section $\tau^\sigma \in \Gamma(X, \omega_X^{-1})$ does not vanish anywhere on $X^\sigma$.*

*Proof.* The previous lemma shows that $\tau^\sigma$, as a global section of $\omega_X^{-1}$, has corresponding divisor $D(\sigma) \cap X$, whose support is contained in $X \cap \bigcup_{\rho \notin \sigma} D_\rho$, which is the complement of $X^\sigma$ (2.11). □

For any place $v$ of $\mathbb{Q}$, we define a $v$-adic norm (or metric) on $\omega_X^{-1}$ by

$$\|\tau(P)\|_v := \min_{\sigma \in \Sigma_{\max} : P \notin D(\sigma)} \left| \frac{\tau}{\tau^\sigma}(P) \right|_v \tag{3.6}$$

for any local section $\tau$ of $\omega_X^{-1}$ not vanishing in $P \in X(\mathbb{Q}_v)$. The next result shows that our family of local norms $\|\cdot\|_v$ for all places $v$ is an adelic anticanonical norm as in [61, Définition 2.3]; see also [9, Lemma 8.5].

**Lemma 3.4.** *Let $p$ be a prime such that $\widetilde{X}$ is smooth over $\mathbb{Z}_p$. On $\omega_X^{-1}$, the $p$-adic norm $\|\cdot\|_p$ defined by (3.6) coincides with the model norm $\|\cdot\|_p^*$ determined by $\widetilde{X}$ over $\mathbb{Z}_p$ as in [65, Definition 2.9].*

*Proof.* Let $P \in X(\mathbb{Q}_p)$, and let $\tau$ be a local section of $\omega_X^{-1}$ not vanishing in $P$. Choose $\xi \in \Sigma_{\max}$ such that $|(\tau^\xi/\tau)(P)|_p = \max_{\sigma \in \Sigma_{\max}} |(\tau^\sigma/\tau)(P)|_p$, which is positive by Lemma 3.3 and the fact that the sets $X^\sigma$ cover $X$ (2.11); in particular, $\tau^\xi$ does not vanish in $P$. Hence, we can compute

$$\|\tau^\xi(P)\|_p^{-1} = \max_{\sigma \in \Sigma_{\max}} \left| \frac{\tau^\sigma}{\tau^\xi}(P) \right|_p = \max_{\sigma \in \Sigma_{\max}} \frac{|(\tau^\sigma/\tau)(P)|_p}{|(\tau^\xi/\tau)(P)|_p} = 1.$$

On the other hand, for each $\sigma \in \Sigma_{\max}$, the section $\tau^\sigma$ extends to a global section $\widetilde{\tau}^\sigma$ of $\omega_{\widetilde{X}/\mathbb{Z}_p}^{-1}$, and $\omega_{\widetilde{X}/\mathbb{Z}_p}^{-1}$ is generated by the set of all these $\widetilde{\tau}^\sigma$ as an $\mathcal{O}_{\widetilde{X}}$-module. The computation above shows for every $\sigma \in \Sigma_{\max}$ that $\left|\frac{\tau^\sigma}{\tau^\xi}(P)\right|_p \le 1$, hence $\tau^\sigma(P) = a_\sigma \tau^\xi(P)$ for some $a_\sigma \in \mathbb{Z}_p$ in the $\mathbb{Q}_p$-module $\omega_X^{-1}(P)$, and hence also $\widetilde{\tau}^\sigma(P) = a_\sigma \widetilde{\tau}^\xi(P)$ in the $\mathbb{Z}_p$-module $\widetilde{P}^*(\omega_{\widetilde{X}/\mathbb{Z}_p}^{-1})$. Therefore, $\widetilde{P}^*(\omega_{\widetilde{X}/\mathbb{Z}_p}^{-1})$ is generated by $\tau^\xi(P)$ and consequently $\|\tau^\xi(P)\|_p^* = 1$ by definition of the model norm. Finally, we have

$$\|\tau(P)\|_p = |(\tau/\tau^\xi)(P)|_p \cdot \|\tau^\xi(P)\|_p = |(\tau/\tau^\xi)(P)|_p \cdot \|\tau^\xi(P)\|_p^* = \|\tau(P)\|_p^*.here$$

□

### *3.2. Height function*

As in [61, Définition 2.3], our adelic anticanonical norm $(\|\cdot\|_v)_v$ (3.6) allows us to define an anticanonical height $H : X(\mathbb{Q}) \to \mathbb{R}_{>0}$, namely

$$H(P) := \prod_v \|\tau(P)\|_v^{-1} \tag{3.7}$$

for any local section $\tau$ of $\omega_X^{-1}$ not vanishing in $P \in X(\mathbb{Q})$; here and elsewhere, the product is taken over all places $v$ of $\mathbb{Q}$. This anticanonical height on $X(\mathbb{Q})$ depends only on the choice of Cox coordinates on $X$ (2.1).

In the following lemma, $x^{D(\sigma)}$ and $F_0$ are homogeneous elements of $\mathbb{Q}[x_\rho : \rho \in \Sigma(1)]$ of the same degree in Pic $X$. Therefore, $x^{D(\sigma)}/F_0$ can be regarded as a rational function on $X$ that can be evaluated in $P \in X(\mathbb{Q})$ if $F_0$ does not vanish in $P$.

**Lemma 3.5.** *For any polynomial $F_0$ of degree $-K_X$ not vanishing in $P \in X(\mathbb{Q})$, one has*

$$H(P) = \prod_v \max_{\sigma \in \Sigma_{\max}} \left| \frac{x^{D(\sigma)}}{F_0}(P) \right|_v.$$

*Proof.* Since the sets $X^\sigma$ as in (2.11) for $\sigma \in \Sigma_{\max}$ cover $X$, our point $P$ is contained in $X^\xi(\mathbb{Q})$ for some $\xi \in \Sigma_{\max}$. By Lemma 3.3, we can compute $H(P)$ with $\tau := \tau^\xi$ as in (3.5). We have $\varpi^\sigma = x^{-D(\sigma)}x^{D(\xi)}\varpi^\xi$ by definition (3.1). Since Res is an $\mathscr{O}_Y$-module homomorphism (3.3), this implies $\tau^\sigma = x^{D(\sigma)}x^{-D(\xi)}\tau^\xi$. Therefore,

$$\|\tau^\xi(P)\|_v^{-1} = \max_{\sigma\in\Sigma_{\max}}\left|\frac{\tau^\sigma}{\tau^\xi}(P)\right|_v = \max_{\sigma\in\Sigma_{\max}}\left|\frac{x^{D(\sigma)}}{x^{D(\xi)}}(P)\right|_v, \tag{3.8}$$

hence our claim holds for $F_0 := x^{D(\xi)}$. By the product formula, it follows for arbitrary $F_0$ not vanishing in $P$. □

### 3.3. *Heights on torsors*

We lift the height function $H$ to the universal torsor $X_0$ as in Section 2.2 as follows. Let

$$H_0 : X_0(\mathbb{Q}) \to \mathbb{R}_{>0}$$

be the composition of $\pi : X_0(\mathbb{Q}) \to X(\mathbb{Q})$ and the height function $H$ defined in (3.7). The following is analogous to [65, Proposition 10.14].

**Lemma 3.6.** *For $P_0 \in X_0(\mathbb{Q})$, we have*

$$H_0(P_0) = \prod_v \max_{\sigma\in\Sigma_{\max}} |x^{D(\sigma)}(P_0)|_v.$$

*Proof.* Let $P = \pi(P_0) \in X(\mathbb{Q})$. For $F_0$ of degree $-K_X$ not vanishing in $P$ and $\sigma \in \Sigma_{\max}$, we can compute $(x^{D(\sigma)}/F_0)(P)$ as in Lemma 3.5, but we can also regard $x^{D(\sigma)}$ and $F_0$ as regular functions on $X_0$ that can be evaluated in $P_0$. Here, we have $x^{D(\sigma)}(P_0)/F_0(P_0) = (x^{D(\sigma)}/F_0)(P)$. Using Lemma 3.5, we obtain

$$H_0(P_0) = H(P) = \prod_v \max_{\sigma\in\Sigma_{\max}}\left|\frac{x^{D(\sigma)}}{F_0}(P)\right|_v = \prod_v \max_{\sigma\in\Sigma_{\max}}\left|\frac{x^{D(\sigma)}(P_0)}{F_0(P_0)}\right|,$$

and $\prod_v |F_0(P_0)|_v = 1$ by the product formula. □

The next result is analogous to [65, Proposition 11.3].

**Corollary 3.7.** *For any prime $p$ and $P_0 \in \widetilde{X}_0(\mathbb{Z}_p)$, we have*

$$\max_{\sigma\in\Sigma_{\max}} |x^{D(\sigma)}(P_0)|_p = 1.$$

*For $P_0 \in \widetilde{X}_0(\mathbb{Z})$, we have*

$$H_0(P_0) = \max_{\sigma\in\Sigma_{\max}} |x^{D(\sigma)}(P_0)|_\infty.$$

*Proof.* Let $p$ be a prime and $P_0 \in \widetilde{X}_0(\mathbb{Z}_p)$. Then $P_0$ mod $p$ is in $\widetilde{X}_0(\mathbb{F}_p)$. Since $\widetilde{X}_0$ is defined by the irrelevant ideal in $\widetilde{X}_1$ as in (2.15), there is a $\xi \in \Sigma_{\max}$ such that $x^{\underline{\xi}}(P_0 \bmod p) \neq 0 \in \mathbb{F}_p$. Since the support of $D(\xi)$ is as in Lemma 2.3, we have $x^{D(\xi)}(P_0 \bmod p) \neq 0 \in \mathbb{F}_p$, and hence $|x^{D(\xi)}(P_0)|_p = 1$. Using $x^{D(\sigma)}(P_0) \in \mathbb{Z}_p$ for all $\sigma \in \Sigma_{\max}$, we conclude $\max_{\sigma\in\Sigma_{\max}} |x^{D(\sigma)}(P_0)|_p = 1$.

Therefore, for $P_0 \in \widetilde{X}_0(\mathbb{Z})$, only the Archimedean factor in Lemma 3.6 remains. □

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### 3.4. Parameterization in Cox coordinates

The following proposition translates the analysis of $N_{X,U,H}(B)$ into a counting problem as described in the introduction that is amenable to methods of analytic number theory. It parameterizes the rational points on $X$ by integral points on the universal torsor $\widetilde{X}_0$ in terms of the torsor equation from the Cox ring (2.1), the height conditions from the anticanonical monomials (2.8) and the coprimality conditions from the primitive collections (2.16).

**Proposition 3.8.** *Let $X$ be a variety as in the first paragraph of Section 2 that satisfies the assumption (2.3). Let $U = X \setminus \bigcup_{\rho\in\Sigma(1)} D_\rho$ be the open subset of $X$ where all Cox coordinates $x_\rho$ are nonzero. Let $H$ be the anticanonical height function on $X(\mathbb{Q})$ defined in (3.7). Then*

$$N_{X,U,H}(B) = \frac{1}{2^{\operatorname{rk}\operatorname{Pic} X}}\#\left\{\mathbf{x}\in\mathbb{Z}_{\neq 0}^{\Sigma(1)} : \begin{array}{l}\Phi(\mathbf{x})=0,\ \max\limits_{\sigma\in\Sigma_{\max}} |\mathbf{x}^{D(\sigma)}|_\infty \le B,\\ \gcd\{x_\rho : \rho\in S_j\}=1 \textit{ for every } j=1,\dots,r\end{array}\right\},$$

*using the notation (2.1), (2.8), (2.16).*

*Proof.* We combine the $2^{\operatorname{rk}\operatorname{Pic} X}$ : 1-map and the description of $\widetilde{X}_0(\mathbb{Z})$ from Proposition 2.4 with the lifted height function in Corollary 3.7. The preimage of $U(\mathbb{Q})$ in $\widetilde{X}_0(\mathbb{Z})$ is the set where $x_\rho \neq 0$ for all $\rho\in\Sigma(1)$. □

### 3.5. Some linear algebra

The monomials $\mathbf{x}^{D(\sigma)}$ and the polynomial $\Phi$ that appear in Proposition 3.8 are not independent. In this subsection, we analyze this dependence and describe it in the form of a rank condition on a certain matrix. This will be useful later when we apply methods from complex analysis to obtain an asymptotic formula for $N_{X,U,H}(B)$.

We consider $\mathbb{Q}^J = \mathbb{Q}^{\Sigma(1)}$ (2.2) with standard basis $(e_\rho)_{\rho\in\Sigma(1)}$ indexed by the rays of $\Sigma$. Let

$$p\colon \mathbb{Q}^{\Sigma(1)} \to (\operatorname{Pic} X)_{\mathbb{Q}}$$

be the surjective linear map that sends $e_\rho$ to $[D_\rho] = \deg(x_\rho)$ as in (2.7). For $\mathbf{x} = (x_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Q}_v^{\Sigma(1)}$ for some place $v$ of $\mathbb{Q}$ and $\mathbf{v} = (v_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}_{\geq 0}^{\Sigma(1)}$, let $\mathbf{x}^{\mathbf{v}} := \prod_{\rho\in\Sigma(1)} x_\rho^{v_\rho}$.

**Lemma 3.9.** *The set $Q := p^{-1}(-K_X)\cap\mathbb{Q}_{\geq 0}^{\Sigma(1)}$ is a bounded polytope of dimension $J - \operatorname{rk}\operatorname{Pic} X$. Its set $\mathscr{V}$ of vertices of $Q$ lies in $\mathbb{Z}_{\geq 0}^{\Sigma(1)}$. Let $v$ be a place of $\mathbb{Q}$. For all nonzero $\mathbf{x}\in\mathbb{Q}_v^{\Sigma(1)}$, we have*

$$\max_{\sigma\in\Sigma_{\max}} |\mathbf{x}^{D(\sigma)}|_v = \max_{\mathbf{v}\in\mathscr{V}} |\mathbf{x}^{\mathbf{v}}|_v.$$

*Proof.* In the notation of the proof of Lemma 2.3, write $D = \sum_\rho a_\rho D_\rho$. Then the $-\chi_\sigma$ are the vertices, and possibly (if $-K_X$ is not ample) some other points, of the $\operatorname{rk} M$-dimensional polytope

$$P_D = \{\chi\in M_{\mathbb{Q}} : \langle n_\rho,\chi\rangle \ge -a_\rho \text{ for all } \rho\};$$

see [25, §4.3 and after Lemma 9.3.9].

Now, consider the injective affine map $\phi\colon M_{\mathbb{Q}} \to \mathbb{Q}^{\Sigma(1)}$, $\chi\mapsto \sum_\rho (a_\rho + \langle n_\rho,\chi\rangle)e_\rho$ as well as the linear surjective map $p\colon \mathbb{Q}^{\Sigma(1)} \to (\operatorname{Cl} Y)_{\mathbb{Q}}$. We have $\operatorname{rk} M = J - \operatorname{rk}\operatorname{Pic} X$ and $\operatorname{im}(p\circ\phi) = \{-K_X\}$. Moreover, the condition $\phi(\chi)\in\mathbb{Q}_{\geq 0}^{\Sigma(1)}$ is equivalent to $\langle n_\rho,\chi\rangle \ge -a_\rho$ for all $\rho$. It follows that $\phi$ restricts to a bijection $P_D \to Q = p^{-1}(-K_X)\cap\mathbb{Q}_{\geq 0}^{\Sigma(1)}$. Hence, $Q$ is bounded and of dimension $J - \operatorname{rk}\operatorname{Pic} X$.

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As we have $p(-\chi_\sigma) = D(\sigma)$, where $D(\sigma)$ is interpreted as an element of $\mathbb{Z}^{\Sigma(1)}$ in the obvious way, we obtain $\mathscr{V} \subseteq \phi(\{D(\sigma) : \sigma \in \Sigma_{\max}\}) \subseteq Q$. Hence, the equality

$$\max_{\sigma \in \Sigma_{\max}} |\mathbf{x}^{D(\sigma)}|_v = \max_{\mathbf{v} \in \mathscr{V}} |\mathbf{x}^{\mathbf{v}}|_v$$

holds, and, since $\phi(M) \subseteq \mathbb{Z}^{\Sigma(1)}$, we also obtain $\mathscr{V} \subset \mathbb{Z}^{\Sigma(1)}_{\geq 0}$. □

We recall (2.2) and the notation (2.8) for the exponents $\alpha_\rho^\sigma$ occurring in $\mathbf{x}^{D(\sigma)}$. We write the defining equation $\Phi$ from (2.1) in the form

$$\Phi = \sum_{i=1}^{k} b_i \prod_{\rho \in \Sigma(1)} x_\rho^{h_{i\rho}} \tag{3.9}$$

(i. e., $k$ is the number of monomials, and $\mathbf{h}_i = (h_{i\rho})_{\rho \in \Sigma(1)} \in \mathbb{Z}^{\Sigma(1)}_{\geq 0}$ is the exponent vector of the $i$-th term of $\Phi$). We now consider the block matrix

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_3 & \mathscr{A}_4 \end{pmatrix} \in \mathbb{R}^{(J+1)\times(N+k)}. \tag{3.10}$$

Here, $\mathscr{A}_1 = (\alpha_\rho^\sigma)_{(\rho,\sigma) \in \Sigma(1) \times \Sigma_{\max}} \in \mathbb{R}^{J \times N}$ is the height matrix for the height function from Proposition 3.8. We let $\mathscr{A}_2 \in \mathbb{R}^{J \times k}$ be the matrix whose $i$-th column is $\mathbf{h}_i - \mathbf{h}_k$ for $i = 1, \dots, k-1$ and whose $k$-th column is $\mathbf{h}_k - (1, \dots, 1)^\top$. Furthermore, let $\mathscr{A}_3 = (1, \dots, 1) \in \mathbb{R}^{1 \times N}$ and $\mathscr{A}_4 = (0, \dots, 0, -1) \in \mathbb{R}^{1 \times k}$.

The definition of $\mathscr{A}_2$ may appear to be somewhat artificial. Its purpose will become clear in (8.21) in Section 8.4.

**Lemma 3.10.** *We have* $\operatorname{rk} \mathscr{A} = \operatorname{rk} \mathscr{A}_1 = J - \operatorname{rk} \operatorname{Pic} X + 1$.

*Proof.* According to Lemma 3.9, the polytope $Q$ spans an affine subspace of dimension $J - \operatorname{rk}\operatorname{Pic} X$ in $\mathbb{R}^J$, which does not contain 0 since $-K_X \neq 0$. It follows that $Q$ spans a vector space of dimension $J - \operatorname{rk}\operatorname{Pic} X + 1$ in $\mathbb{R}^J$. This shows $\operatorname{rk} \mathscr{A}_1 = J - \operatorname{rk}\operatorname{Pic} X + 1$.

Since the columns of $\mathscr{A}_1$ lie in an affine subspace of $\mathbb{R}^J$ that does not contain 0, a linear combination of these columns can be 0 only if the sum of the coefficients is 0. It follows that we have $\operatorname{rk}\begin{pmatrix} \mathscr{A}_1 \\ \mathscr{A}_3 \end{pmatrix} = \operatorname{rk} \mathscr{A}_1$. Since $\Phi$ is $\operatorname{Pic} X$-homogeneous, the first $k-1$ columns of $\mathscr{A}_2$ lie in $p^{-1}(0)$. Moreover, note that the last column of $\mathscr{A}_2$ lies in $p^{-1}(K_X)$ since $\deg \Phi - \sum_{\rho \in \Sigma(1)} \deg(x_\rho) = K_X$ by [2, Proposition 3.3.3.2]. Together with the fact that the columns of $\mathscr{A}_1$ lie in $p^{-1}(-K_X)$, we obtain $\operatorname{rk} \mathscr{A} = \operatorname{rk}\begin{pmatrix} \mathscr{A}_1 \\ \mathscr{A}_3 \end{pmatrix}$. □

Let $\zeta = (\zeta_1, \dots, \zeta_k) \in \mathbb{R}^k$ be a vector satisfying

$$\zeta_i > 0 \text{ for all } 1 \leq i \leq k, \quad \sum_{i=1}^{k} h_{i\rho}\zeta_i < 1 \text{ for all } \rho \in \Sigma(1), \quad \sum_{i=1}^{k} \zeta_i = 1. \tag{3.11}$$

This condition will reappear in Part II as (5.10).

**Lemma 3.11.** *Let $\zeta$ be as in (3.11), $\tau_1 = (1 - \sum_{i=1}^k h_{i\rho}\zeta_i)_{\rho \in \Sigma(1)} = (1, \dots, 1) - \sum_{i=1}^k \zeta_i \mathbf{h}_i$, and let $\tau = (\tau_1, 1)^\top$. The system of $J+1$ linear equations*

$$\begin{pmatrix} \mathscr{A}_1 \\ \mathscr{A}_3 \end{pmatrix} \sigma = \tau$$

*has a solution $\sigma \in \mathbb{R}^N_{>0}$.*

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*Proof.* According to [2, Proposition 3.3.3.2], we have $\tau_1 \in p^{-1}(-K_X)$. It follows from $Q = p^{-1}(-K_X) \cap \mathbb{Q}_{\geq 0}^{\Sigma(1)}$ that the relative interior of $Q$ satisfies $Q^\circ \supseteq p^{-1}(-K_X) \cap \mathbb{Q}_{>0}^{\Sigma(1)}$. Since all coordinates of $\tau_1$ are positive, we obtain $\tau_1 \in Q^\circ$. Since the columns of $\mathscr{A}_1$ are the vertices of $Q$, the column $\tau_1^\top$ can be written as a linear combination of the columns of $\mathscr{A}_1$ with strictly positive coefficients whose sum is 1. The existence of $\sigma \in \mathbb{R}_{>0}^N$ as required follows. □

## 4. Tamagawa numbers in Cox coordinates

In this section, we use the adelic metrization (see Section 3.1) of the anticanonical bundle on our variety $X$ to make the local measures (Section 4.1) explicit that are used in the Tamagawa number (Section 4.2) in Peyre's constant. We lift the $p$-adic measures to the universal torsor (Section 4.3), which allows as to express the $p$-adic densities in the Tamagawa number in terms of the number of points on the universal torsor modulo $p^\ell$, which is the number of solutions modulo $p^\ell$ of the relation $\Phi$ in the Cox ring (Section 4.4). Furthermore, we rewrite the real density and Peyre's constant $\alpha$ (Section 4.5) in a way that will appear in our analytic method in Part II. In total, we obtain a description of Peyre's constant for $X$ in terms of the Cox ring of $X$ (Proposition 4.11).

We continue to work in the setting of Sections 2 and 3. Additionally, we assume that $X$ is an almost Fano variety (e. g., a smooth Fano variety) as in [61, Définition 3.1] (i. e., $X$ is smooth, projective and geometrically integral with $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$, free geometric Picard group of finite rank, and big $\omega_X^\vee$).

### 4.1. Local measures

By [60, (2.2.1)], [61, Notations 4.3] and [65, Theorem 1.10], the $v$-adic norm $\|\cdot\|_v$ on $\omega_X^{-1}$ defined in (3.6) induces a measure $\mu_v$ on $X(\mathbb{Q}_v)$. We express it using the Poincaré residues from Section 3.1 and the affine charts from Section 2.1; in particular, recall (2.8), (2.11), (3.1), (3.5). See [8, (5.8), (5.9)] for an example of the next result.

**Proposition 4.1.** *Let $\xi \in \Sigma_{\max}$. For a Borel subset $N_v$ of $X^\xi(\mathbb{Q}_v)$, we have*

$$\mu_v(N_v) = \int_{N_v} \frac{|\operatorname{Res}\varpi^\xi|_v}{\max_{\sigma\in\Sigma_{\max}} |\tau^\sigma \operatorname{Res}\varpi^\xi|_v} = \int_{N_v} \frac{|\operatorname{Res}\varpi^\xi|_v}{\max_{\sigma\in\Sigma_{\max}} |x^{D(\sigma)}/x^{D(\xi)}|_v}, \tag{4.1}$$

*where $|\operatorname{Res}\varpi^\xi|_v$ is the $v$-adic density on $X^\xi(\mathbb{Q}_v)$ of the volume form $\operatorname{Res}\varpi^\xi$ on $X^\xi$.*

*Let $\rho_0 \in \xi(1)$. If $N_v$ is contained in a sufficiently small open $v$-adic neighborhood of a point $P$ in $X^\xi(\mathbb{Q}_v)$ with $\partial\Phi^\xi/\partial z_{\rho_0}^\xi(P) \neq 0$, then*

$$\mu_v(N_v) = \int_{\pi_{\rho_0}^\xi(N_v)} \frac{\bigwedge_{\rho\in\xi(1)\setminus\{\rho_0\}} \mathrm{d}z_\rho^\xi}{|\partial\Phi^\xi/\partial z_{\rho_0}^\xi(\mathbf{z}^\xi)|_v \max_{\sigma\in\Sigma_{\max}} |x^{D(\sigma)}(\mathbf{z}^\xi)|_v} \tag{4.2}$$

*in the affine coordinates $\mathbf{z}^\xi = (z_\rho^\xi)_{\rho\in\xi(1)}$, where $\pi_{\rho_0}^\xi \colon U^\xi(\mathbb{Q}_v) = \mathbb{Q}_v^{\xi(1)} \to \mathbb{Q}_v^{\xi(1)\setminus\{\rho_0\}}$ is the natural projection and $z_{\rho_0}^\xi$ is expressed in terms of the other coordinates using the implicit function for $\Phi^\xi$.*

*Proof.* As in (2.14), the implicit function theorem gives a $v$-adic neighborhood $U_0 \subseteq X^\xi(\mathbb{Q}_v)$ of $P$ and an implicit function $\phi\colon V \to \mathbb{Q}_v$ for $V = \pi_{\rho_0}^\xi(U_0) \subseteq \mathbb{Q}_v^{\xi(1)\setminus\{\rho_0\}}$ such that $\Phi^\xi(\mathbf{z}^\xi) = 0$ for all $\mathbf{z}^\xi \in X^\xi(\mathbb{Q}_v)$ with $z_{\rho_0}^\xi$ the image of $(z_\rho^\xi)_{\rho\in\xi(1)\setminus\{\rho_0\}} \in V$ under $\phi$. We work with $\|\tau^\xi(P)\|_v$ as in (3.5) and use $x^{D(\xi)}(\mathbf{z}^\xi) = 1$ (see (2.8)) in our affine coordinates on $X^\xi(\mathbb{Q}_v)$. Then the formulas in [60, (2.2.1)] and [65, Theorem 1.10] give (4.2) for $N_v \subseteq U_0$. Indeed, our chart is

$$\pi := \pi_{\rho_0}^\xi \colon U_0 \to V \subseteq \mathbb{Q}_v^{\xi(1)\setminus\{\rho_0\}}.$$

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In this chart, by (3.4), the image of the local canonical section $\bigwedge_{\rho\in\xi(1)\setminus\{\rho_0\}} \mathrm{d}z_\rho^\xi$ under

$$\omega(\pi)\colon \pi^*\omega_{\mathbb{A}_\mathbb{Q}^{\xi(1)\setminus\{\rho_0\}}} \to \omega_X$$

is $\partial\Phi^\xi/\partial z_{\rho_0}^\xi \cdot \operatorname{Res}\varpi^\xi$. This implies that the image of the local anticanonical section $\bigwedge_{\rho\in\xi(1)\setminus\{\rho_0\}} \frac{\partial}{\partial z_\rho^\xi}$ under

$${}^t\omega(\pi)^{-1}\colon \pi^*\omega^{-1}_{\mathbb{A}_\mathbb{Q}^{\xi(1)\setminus\{\rho_0\}}} \to \omega_X^{-1}$$

is $(\partial\Phi^\xi/\partial z_{\rho_0}^\xi)^{-1}\cdot\tau^\xi$. Therefore, $\mu_v(N_v)$ for $N_v \subseteq U_0$ as defined in [Peyre95, (2.2.1)] is the integral over $\pi(N_v)$ of

$$\begin{aligned}\omega_v &= \|((\partial\Phi^\xi/\partial z_{\rho_0}^\xi)^{-1}\cdot\tau^\xi)(\pi^{-1}((z_\rho^\xi)_{\rho\in\xi(1)\setminus\{\rho_0\}}))\|_v \bigwedge_{\rho\in\xi(1)\setminus\{\rho_0\}} \mathrm{d}z_\rho^\xi\\ &= |\partial\Phi^\xi/\partial z_{\rho_0}^\xi(\mathbf{z}^\xi)|_v^{-1}\cdot\|\tau^\xi(\mathbf{z}^\xi)\|_v \bigwedge_{\rho\in\xi(1)\setminus\{\rho_0\}} \mathrm{d}z_\rho^\xi.\end{aligned}$$

Using (3.8) together with $x^{D(\xi)}(\mathbf{z}^\xi) = 1$, we obtain (4.2).

By (3.4), we see that the right-hand side of (4.1) coincides with (4.2) for $N_v \subseteq U_0$. Since $X$ is smooth, $X^\xi(\mathbb{Q}_v)$ can be covered with such $U_0$, hence $\mu_v(N_v)$ is equal to the right-hand side for all $N_v \subseteq X^\xi(\mathbb{Q}_v)$. Since $\varpi^\sigma/\varpi^\xi = x^{D(\xi)}/x^{D(\sigma)}$ by definition (3.1), we have $\tau^\sigma \operatorname{Res}\varpi^\xi = \tau^\sigma/\tau^\xi = x^{D(\sigma)}/x^{D(\xi)}$ by (3.5), and hence the integrals in (4.1) are equal. □

### *4.2. Tamagawa number*

Here, we use some standard notation as in [60, §2], [61, §4]. Let $S$ be a sufficiently large finite set of finite places of $\mathbb{Q}$ as in [61, Notations 4.5]. For any prime $p \in S$, let

$$L_p(s, \operatorname{Pic}\overline{X}) := \det(1 - p^{-s}\operatorname{Fr}_p \mid \operatorname{Pic}(X_{\overline{\mathbb{F}}_p})\otimes\mathbb{Q})^{-1}.$$

Since $X$ is split, $L_p(s,\operatorname{Pic}\overline{X}) = (1-p^{-s})^{-\operatorname{rk}\operatorname{Pic}X}$, hence

$$L_S(s,\operatorname{Pic}\overline{X}) := \prod_{p\notin S} L_p(s,\operatorname{Pic}\overline{X}) = \zeta(s)^{\operatorname{rk}\operatorname{Pic}X}\prod_{p\in S}(1-p^{-s})^{\operatorname{rk}\operatorname{Pic}X}.$$

Therefore, $\lim_{s\to1}(s-1)^{\operatorname{rk}\operatorname{Pic}X}L_S(s,\operatorname{Pic}\overline{X}) = \prod_{p\in S}(1-p^{-1})^{\operatorname{rk}\operatorname{Pic}X}$, and the convergence factors are

$$\lambda_p^{-1} := L_p(1,\operatorname{Pic}\overline{X})^{-1} = \left(1-p^{-1}\right)^{\operatorname{rk}\operatorname{Pic}X}$$

for $p \notin S$ and $\lambda_p^{-1} := 1$ for $p\in S$. Hence, Peyre's Tamagawa number [61, Définition 4.5] is

$$\tau_H(X) = \mu_\infty(X(\mathbb{R}))\prod_p (1-p^{-1})^{\operatorname{rk}\operatorname{Pic}X}\mu_p(X(\mathbb{Q}_p)). \tag{4.3}$$

The Euler product converges by [61, Remarque 4.6].

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### 4.3. Measures on the torsor

By [25, Proposition 8.2.3], we have a rational $\#\Sigma(1)$-form

$$s_{Y_0} = \bigwedge_{\rho\in\Sigma_0(1)} \frac{\mathrm{d}y_\rho}{y_\rho}$$

on the toric principal universal torsor $Y_0 \subset Y_1 = \mathbb{A}_{\mathbb{Q}}^{\Sigma_0(1)}$ as in Section 2.2, with coordinates $y_\rho$ for $\rho \in \Sigma_0(1)$, using our bijection $\Sigma_0(1) \to \Sigma(1)$. Now, we regard $\Phi$ and $y^D$ (defined as in (2.5) for $U$-invariant divisors $D$ on $Y$) as polynomials in $y_\rho$ and as functions on $Y_0$. As in [8, (5.12)] and using the notation (2.6), (2.8), we define

$$\varpi_{Y_0}^\sigma = \frac{y^{D_0}}{y^{D(\sigma)}\Phi} s_{Y_0}$$

for each $\sigma \in \Sigma_{\max}$, and

$$\varpi_{Y_0} = \frac{1}{\Phi} \bigwedge_{\rho\in\Sigma_0(1)} \mathrm{d}y_\rho.$$

We have

$$\varpi_{Y_0}^\sigma = \varpi_{Y_0}/y^{D(\sigma)} \tag{4.4}$$

on the open subset $Y_0^\sigma := \pi^{-1}(U^\sigma)$ of $Y_0$; see (2.10).

We have

$$\varpi_{Y_0}^\sigma \in \Gamma(Y_0^\sigma, \omega_{Y_0}(X_0))$$

with Poincaré residue $\operatorname{Res}\varpi_{Y_0}^\sigma \in \Gamma(X_0^\sigma, \omega_{X_0})$ on $X_0^\sigma = \pi^{-1}(X^\sigma) = X_0 \cap Y_0^\sigma$. As in Section 4.1, we obtain a $v$-adic measure $m_v$ on $X_0(\mathbb{Q}_v)$ defined by

$$m_v(M_v) = \int_{M_v} \frac{|\operatorname{Res}\varpi_{Y_0}^\xi|_v}{\max_{\sigma\in\Sigma_{\max}} |y^{D(\sigma)}/y^{D(\xi)}|_v}$$

for a Borel subset $M_v$ of $X_0^\xi(\mathbb{Q}_v)$. Alternatively, we can write

$$m_v(M_v) = \int_{M_v} \frac{|\operatorname{Res}\varpi_{Y_0}|_v}{\max_{\sigma\in\Sigma_{\max}} |y^{D(\sigma)}|_v}$$

because $\varpi_{Y_0} \in \Gamma(Y_0, \omega_{Y_0}(X_0))$ has a residue form $\operatorname{Res}\varpi_{Y_0} \in \Gamma(X_0, \omega_{X_0})$ that restricts to $y^{D(\xi)}\operatorname{Res}\varpi_{Y_0}^\xi$ on $X_0^\xi$ by (4.4). If $M_v$ is sufficiently small, this is explicitly

$$m_v(M_v) = \int_{\pi_{\rho_0}(M_v)} \frac{\bigwedge_{\rho\in\Sigma_0(1)\setminus\{\rho_0\}} \mathrm{d}y_\rho}{|\partial\Phi/\partial x_{\rho_0}(\mathbf{y})|_v \max_{\sigma\in\Sigma_{\max}} |\mathbf{y}^{D(\sigma)}|_v} \tag{4.5}$$

in the coordinates $\mathbf{y} = (y_\rho)_{\rho\in\Sigma_0(1)}$, where $\pi_{\rho_0}$ is the projection to all coordinates $y_\rho$ with $\rho \neq \rho_0$ and where $y_{\rho_0}$ is expressed in terms of these coordinates using the implicit function theorem.

**Lemma 4.2.** *Let $D_0^{Y_0} = \pi^* D_0$ be the sum of the prime divisors defined by $y_\rho = 0$ for $\rho \in \Sigma_0(1)$. Then there is a unique nowhere vanishing global section $s_{Y_0/Y} \in \Gamma(Y_0, \omega_{Y_0/Y})$ such that $s_{Y_0} = s_{Y_0/Y} \otimes \pi^* s_Y$ via the natural isomorphism $\omega_{Y_0}(D_0^{Y_0}) = \omega_{Y_0/Y} \otimes \pi^*\omega_Y(D_0)$.*

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*Let $s_{X_0/X}$ be the image of $\iota_0^* s_{Y_0/Y}$ under the isomorphism $\Gamma(X_0, \iota_0^*\omega_{Y_0/Y}) \to \Gamma(X_0, \omega_{X_0/X})$, and $s^\sigma_{X_0/X}$ be the restriction of $s_{X_0/X}$ to $X_0^\sigma$. Then $\operatorname{Res}\varpi^\sigma_{Y_0} = s^\sigma_{X_0/X} \otimes \pi^* \operatorname{Res}\varpi^\sigma$ under the canonical isomorphism $\omega_{X_0} = \omega_{X_0/X} \otimes \pi^*\omega_X$.*

*Proof.* See [8, Lemma 16]. □

**Lemma 4.3.** *For any prime p, we have $m_p(\widetilde{X}_0(\mathbb{Z}_p)) = (1-p^{-1})^{\operatorname{rk}\operatorname{Pic}X}\mu_p(X(\mathbb{Q}_p))$.*

*Proof.* Our proof follows [8, Lemma 18]. By [65, pp. 126–127], the map $\pi\colon X_0 \to X$ induces an $v$-adic analytic torsor $\pi_v\colon X_0(\mathbb{Q}_v) \to X(\mathbb{Q}_v)$ under $T(\mathbb{Q}_v)$. By [65, Theorem 1.22] and the previous lemma, the relative volume form $s_{X_0/X}$ defines $v$-adic measures on the fibers of $\pi_v$ over $X(\mathbb{Q}_v)$. Integrating along these fibers gives a linear functional $\Lambda_v\colon C_c(X_0(\mathbb{Q}_v)) \to C_c(X(\mathbb{Q}_v))$.

Let $\chi_p\colon X_0(\mathbb{Q}_p) \to \{0,1\}$ be the characteristic function of $\widetilde{X}_0(\mathbb{Z}_p) \subset \widetilde{X}_0(\mathbb{Q}_p) = X_0(\mathbb{Q}_p)$. Since $\chi_p \in C_c(X_0(\mathbb{Q}_p))$, we have $m_p(\widetilde{X}_0(\mathbb{Z}_p)) = \int_{X(\mathbb{Q}_p)} \Lambda_p(\chi_p)\mu_p$.

We claim that $(\Lambda_p(\chi_p))(P) = (1-p^{-1})^{\operatorname{rk}\operatorname{Pic}X}$ for every $P \in X(\mathbb{Q}_p) = \widetilde{X}(\mathbb{Z}_p)$. Indeed, we have $s_{\widetilde{Y}_0} = s_{\widetilde{Y}_0/\widetilde{Y}} \otimes \pi^* s_{\widetilde{Y}}$, where $s_{\widetilde{Y}_0/\widetilde{Y}}$ is the extension of $s_{Y_0/Y}$ to a $\widetilde{T}$-equivariant generator of $\omega_{\widetilde{Y}_0/\widetilde{Y}}$. Furthermore, $s_{X_0/X}$ extends to a $\widetilde{T}$-equivariant generator $s_{\widetilde{X}_0/\widetilde{X}}$ of $\omega_{\widetilde{X}_0/\widetilde{X}}$. For a point $P \in \widetilde{X}(\mathbb{Z}_p)$, the torsor $\widetilde{X}_0 \to \widetilde{X}$ can be pulled back to $(\widetilde{X}_0)_P \to P$, and hence $s_{\widetilde{X}_0/\widetilde{X}}$ pulls back to a $\widetilde{T}_{\mathbb{Z}_p}$-equivariant global section $s_{(\widetilde{X}_0)_P}$ on $\omega_{(\widetilde{X}_0)_P/\mathbb{Z}_p}$. But the torsor over $P$ is trivial, and $\widetilde{T} \cong \mathbb{G}_{\mathrm{m}}^r$ with $r = \operatorname{rk}\operatorname{Pic}X$, hence there are affine coordinates $(t_1,\dots,t_r)$ for the affine $\mathbb{Z}_p$-scheme $(\widetilde{X}_0)_P$ with $s_{(\widetilde{X}_0)_P} = \mathrm{d}t_1/t_1 \wedge \cdots \wedge \mathrm{d}t_r/t_r$. Therefore,

$$(\Lambda_p(\chi_p))(P) = \int_{(\widetilde{X}_0)_P(\mathbb{Z}_p)} |s_{(\widetilde{X}_0)_P}|_p = \left(\int_{\mathbb{Z}_p^\times} \frac{\mathrm{d}t}{t}\right)^r = (1-p^{-1})^r.$$ □

### 4.4. Comparison to the number of points modulo $p^\ell$

In this section, we describe $\mu_p(X(\mathbb{Q}_p))$ in terms of congruences. In the special case $Y = \mathbb{P}^n_{\mathbb{Q}}$, this was worked out in [62, Lemma 3.2].

Let $p$ be a prime. For $\ell \in \mathbb{Z}_{>0}$, using notation (2.16), we have

$$\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z}) = \{\mathbf{x} \in (\mathbb{Z}/p^\ell\mathbb{Z})^{\Sigma(1)} : \mathbf{\Phi}(\mathbf{x}) = 0 \in \mathbb{Z}/p^\ell\mathbb{Z},\ p \nmid \gcd\{x_\rho : \rho \in S_j\} \text{ for all } j = 1,\dots,r\}$$

as in Proposition 2.4 and define

$$c_p := \lim_{\ell\to\infty} \frac{\#\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})}{(p^\ell)^{\#\Sigma(1)-1}} \quad \text{and} \quad c_{\mathrm{fin}} := \prod_p c_p. \tag{4.6}$$

We will see in Proposition 4.5 that the sequence defining $c_p$ becomes stationary; in particular, the limit $\ell \to 4\infty$ exists. The convergence of $c_{\mathrm{fin}}$ will follow from Proposition 4.6; see (4.3). For $\mathbf{x} \in \widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})$, let

$$\widetilde{X}_0(\mathbb{Z}_p)_{\mathbf{x}} := \{\mathbf{y} \in \widetilde{X}_0(\mathbb{Z}_p) \mid \mathbf{y} \equiv \mathbf{x} \bmod p^\ell\}.$$

**Lemma 4.4.** *There is an $\ell_1 \in \mathbb{Z}_{>0}$ such that the following holds for all $\ell \geq \ell_1$: for any $\mathbf{x} \in \widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})$, there is a nonnegative integer $c_{\mathbf{x}} < \ell_1$ and an $\rho_{\mathbf{x}} \in \Sigma(1)$ such that for all $\mathbf{y} \in \widetilde{X}_0(\mathbb{Z}_p)_{\mathbf{x}}$ one has*

$$\inf_{\rho\in\Sigma(1)}\{v_p(\partial\Phi/\partial x_\rho(\mathbf{y}))\} = v_p(\partial\Phi/\partial x_{\rho_{\mathbf{x}}}(\mathbf{y})) = c_{\mathbf{x}}.$$

*Proof.* Since $X$ is smooth, $X_0$ is also smooth. Hence, for any $\mathbf{y} \in X_0(\mathbb{Q}_p)$, we have $\partial\Phi/\partial x_\rho(\mathbf{y}) \neq 0$ for some $\rho \in \Sigma(1)$. In particular, for any $\mathbf{y} \in \widetilde{X}_0(\mathbb{Z}_p)$, the valuation $v_p(\partial\Phi/\partial x_\rho(\mathbf{y}))$ is finite for some $\rho$. Hence, $I_p(\mathbf{y}) := \inf_{\rho\in\Sigma(1)}\{v_p(\partial\Phi/\partial x_\rho(\mathbf{y}))\}$ is finite.

There is an $\ell_1$ such that $I_p(\mathbf{y}) < \ell_1$ for all $\mathbf{y} \in \widetilde{X}_0(\mathbb{Z}_p)$. To see this, assume the contrary. Then there is a sequence $\mathbf{y}_1, \mathbf{y}_2, \ldots \in \widetilde{X}_0(\mathbb{Z}_p)$ with $I_p(\mathbf{y}_j) \geq j$ for all $j$. The description of $\widetilde{X}_0(\mathbb{Z}_p)$ in Proposition 2.4 shows that this sequence has an accumulation point $\mathbf{y}_0 \in \widetilde{X}_0(\mathbb{Z}_p)$: Infinitely many $\mathbf{y}_i$ have the same first $p$-adic digits, infinitely many of these have the same second $p$-adic digits and so on; we obtain $\mathbf{y}_0$ by using these $p$-adic digits; $\Phi(\mathbf{y}_0) = 0$ since $\Phi$ is continuous, and $\mathbf{y}_0$ satisfies the coprimality conditions since these depend only on the first $p$-adic digits. Passing to a subsequence, we may assume that $\mathbf{y}_0$ is the limit of the sequence $(\mathbf{y}_j)_j$. Then $\partial\Phi/\partial x_\rho(\mathbf{y}_0) = \lim_{j\to\infty} \partial\Phi/\partial x_\rho(\mathbf{y}_j) = 0$ for all $\rho \in \Sigma(1)$. This contradicts the smoothness of $X$ over $\mathbb{Q}_p$.

Let $\ell \geq \ell_1$ and $\mathbf{x} \in \widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})$. For any $\mathbf{y} \in \widetilde{X}_0(\mathbb{Z}_p)_\mathbf{x}$, the first $\ell$ digits of $\partial\Phi/\partial x_\rho(\mathbf{y})$ depend only on $\mathbf{x}$, and since $I_p(\mathbf{y}) < \ell_1 \leq \ell$, at least one of these digits is nonzero for some $\rho \in \Sigma(1)$. We choose $c_\mathbf{x}$ and $\rho_\mathbf{x}$ such that digit number $c_\mathbf{x}$ (i. e., the coefficient of $p^{c_\mathbf{x}}$ in the $p$-adic expansion) of $\partial\Phi/\partial x_{\rho_\mathbf{x}}(\mathbf{y})$ is nonzero, while all lower digits of $\partial\Phi/\partial x_\rho(\mathbf{y})$ for all $\rho \in \Sigma(1)$ are zero. □

**Proposition 4.5.** *For every prime $p$, there is an $\ell_0 \in \mathbb{Z}_{>0}$ such that for all $\ell \geq \ell_0$ we have*

$$m_p(\widetilde{X}_0(\mathbb{Z}_p)) = \frac{\#\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})}{(p^\ell)^{\dim X_0}}.$$

*Proof.* Let $\ell_1$ be as in Lemma 4.4. For $\mathbf{x} \in \widetilde{X}_0(\mathbb{Z}/p^{\ell_1}\mathbb{Z})$ and $\ell \geq \ell_1$, let

$$\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})_\mathbf{x} := \{\mathbf{y} \in (\mathbb{Z}/p^\ell\mathbb{Z})^{\Sigma(1)} \mid \Phi(\mathbf{y}) = 0 \in \mathbb{Z}/p^\ell\mathbb{Z},\ \mathbf{y} \equiv \mathbf{x} \bmod p^{\ell_1}\}.$$

We will see that

$$m_p(\widetilde{X}_0(\mathbb{Z}_p)_\mathbf{x}) = \frac{\#\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})_\mathbf{x}}{(p^\ell)^{\#\Sigma(1)-1}} \tag{4.7}$$

for all $\ell \geq \ell_1 + c_\mathbf{x}$ with $c_\mathbf{x} < \ell_1$ as in Lemma 4.4. Since $\widetilde{X}_0(\mathbb{Z}_p)$ is the disjoint union of the sets $\widetilde{X}_0(\mathbb{Z}_p)_\mathbf{x}$ and $\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})$ is the disjoint union of the sets $\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})_\mathbf{x}$ for $\mathbf{x} \in \widetilde{X}_0(\mathbb{Z}/p^{\ell_1}\mathbb{Z})$, our result follows for all $\ell \geq \ell_0 := 2\ell_1 - 1$.

For the proof of (4.7), we fix $\mathbf{x} \in \widetilde{X}_0(\mathbb{Z}/p^{\ell_1}\mathbb{Z})$ and let $c_\mathbf{x}, \rho_\mathbf{x}$ be as in Lemma 4.4. We claim that $\Phi(\mathbf{y}) \bmod p^{\ell_1+c_\mathbf{x}}$ is the same for all $\mathbf{y} \in \mathbb{Z}_p^{\Sigma(1)}$ with $\mathbf{y} \equiv \mathbf{x} \bmod p^{\ell_1}$; we write $\Phi^*(\mathbf{x})$ for this value in $\mathbb{Z}/p^{\ell_1+c_\mathbf{x}}\mathbb{Z}$. Indeed, for $\mathbf{y}, \mathbf{y}' \in \mathbb{Z}_p^{\Sigma(1)}$, we have

$$\Phi(\mathbf{y}') = \Phi(\mathbf{y}) + \sum_{\rho\in\Sigma(1)} (y'_\rho - y_\rho) \cdot \partial\Phi/\partial x_\rho(\mathbf{y}) + \sum_{\rho',\rho''\in\Sigma(1)} \Psi_{\rho',\rho''}(\mathbf{y}, \mathbf{y}')(y'_{\rho'} - y_{\rho'})(y'_{\rho''} - y_{\rho''})$$

for certain polynomials $\Psi_{\rho',\rho''} \in \mathbb{Z}_p[X_\rho, X'_\rho : \rho \in \Sigma(1)]$ by Taylor expansion. If $\mathbf{y}' \equiv \mathbf{y} \bmod p^{\ell_1}$, we conclude $\Phi(\mathbf{y}') \equiv \Phi(\mathbf{y}) \bmod p^{\ell_1+c_\mathbf{x}}$.

If $\Phi^*(\mathbf{x}) \neq 0 \in \mathbb{Z}/p^{\ell_1+c_\mathbf{x}}\mathbb{Z}$, then there is no $\mathbf{y} \in \mathbb{Z}_p^{\Sigma(1)}$ with $\mathbf{y} \equiv \mathbf{x} \bmod p^{\ell_1}$ and $\Phi(\mathbf{y}) = 0$, hence the set $\widetilde{X}_0(\mathbb{Z}_p)_\mathbf{x}$ is empty, and the same holds for $\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})_\mathbf{x}$ for all $\ell \geq \ell_1 + c_\mathbf{x}$ for similar reasons.

Now, assume $\Phi^*(\mathbf{x}) = 0 \in \mathbb{Z}/p^{\ell_1+c_\mathbf{x}}\mathbb{Z}$. By Hensel's lemma, the map $\pi_{\rho_\mathbf{x}}$ that drops the $\rho_\mathbf{x}$-coordinate defines an isomorphism from the integration domain $\widetilde{X}_0(\mathbb{Z}_p)_\mathbf{x}$ to the set

$$\begin{aligned} &\{(y_\rho)_{\rho\in\Sigma(1)\setminus\{\rho_\mathbf{x}\}} \in \mathbb{Z}_p^{\Sigma(1)\setminus\{\rho_\mathbf{x}\}} \mid y_\rho \equiv x_\rho \bmod p^{\ell_1} \text{ for all } \rho \in \Sigma(1) \setminus \{\rho_\mathbf{x}\}\} \\ &= \{(x_\rho + z_\rho)_{\rho\in\Sigma(1)\setminus\{\rho_\mathbf{x}\}} \mid z_\rho \in p^{\ell_1}\mathbb{Z}_p\} \cong (p^{\ell_1}\mathbb{Z}_p)^{\Sigma(1)\setminus\{\rho_\mathbf{x}\}}. \end{aligned}$$

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Therefore, by (4.5) and the first statement in Corollary 3.7,

$$m_p(\widetilde{X}_0(\mathbb{Z}_p)_{\mathbf{x}}) = \int_{\pi_{\rho_{\mathbf{x}}}(\widetilde{X}_0(\mathbb{Z}_p)_{\mathbf{x}})} \frac{\bigwedge_{\rho\in\Sigma(1)\setminus\{\rho_{\mathbf{x}}\}} \mathrm{d}y_\rho}{|\partial\Phi/\partial x_{\rho_{\mathbf{x}}}(\mathbf{y})|_p},$$

where $y_{\rho_{\mathbf{x}}}$ is expressed in terms of the other coordinates using $\pi_{\rho_{\mathbf{x}}}^{-1}$. We have $|\partial\Phi/\partial x_{\rho_{\mathbf{x}}}(\mathbf{y})|_p = p^{-c_{\mathbf{x}}}$ on the integration domain (Lemma 4.4). Thus,

$$m_p(\widetilde{X}_0(\mathbb{Z}_p)_{\mathbf{x}}) = \int_{(p^{\ell_1}\mathbb{Z}_p)^{\Sigma(1)\setminus\{\rho_{\mathbf{x}}\}}} \frac{\bigwedge_{\rho\in\Sigma(1)\setminus\{\rho_{\mathbf{x}}\}} \mathrm{d}z_\rho}{p^{-c_{\mathbf{x}}}} = p^{c_{\mathbf{x}}-\ell_1(\#\Sigma(1)-1)}.$$

On the other hand, by the discussion above, $\Phi^*(\mathbf{x}) = 0 \in \mathbb{Z}/p^{\ell_1+c_{\mathbf{x}}}\mathbb{Z}$ means $\Phi(\mathbf{y}) = 0 \in \mathbb{Z}/p^{\ell_1+c_{\mathbf{x}}}\mathbb{Z}$ for all $\mathbf{y} \equiv \mathbf{x} \bmod p^{\ell_1}$. Therefore,

$$\frac{\#\widetilde{X}_0(\mathbb{Z}/p^{\ell_1+c_{\mathbf{x}}}\mathbb{Z})_{\mathbf{x}}}{(p^{\ell_1+c_{\mathbf{x}}})^{\#\Sigma(1)-1}} = \frac{p^{c_{\mathbf{x}}\#\Sigma(1)}}{(p^{\ell_1+c_{\mathbf{x}}})^{\#\Sigma(1)-1}} = p^{c_{\mathbf{x}}-\ell_1(\#\Sigma(1)-1)}.$$

Using Hensel's lemma as before, we see that $\#\widetilde{X}_0(\mathbb{Z}/p^\ell\mathbb{Z})_{\mathbf{x}}/(p^\ell)^{\#\Sigma(1)-1}$ has the same value for all $\ell \geq \ell_1 + c_{\mathbf{x}}$. This completes the proof of (4.7). □

**Proposition 4.6.** *We have*

$$(1-p^{-1})^{\mathrm{rk}\,\mathrm{Pic}\,X}\mu_p(X(\mathbb{Q}_p)) = c_p.$$

*Proof.* We combine Lemma 4.3 and Proposition 4.5 with (4.6). □

### *4.5. The real density*

In this section, we compute the real density and Peyre's $\alpha$-constant in terms of quantities that come up naturally in the analytic method in Sections 8 and 9. For the case $Y = \mathbb{P}^n_{\mathbb{Q}}$, see [60, §5.4].

For any $\sigma \in \Sigma_{\max}$, we can write

$$-K_X = \sum_{\rho\notin\sigma(1)} \alpha_\rho^\sigma \deg(x_\rho)$$

with $\alpha_\rho^\sigma \in \mathbb{Z}$ by Lemma 2.3. In this section, we assume for convenience:

$$\begin{aligned}&\text{Every variable } x_\rho \text{ appears in at most one monomial of } \Phi.\\ &\text{There are } \sigma\in\Sigma_{\max}, \rho_0\in\sigma(1) \text{ and } \rho_1\in\Sigma(1)\setminus\sigma(1) \text{ such that } \alpha^\sigma_{\rho_1}\neq 0,\\ &\text{the variable } x_{\rho_0} \text{ appears with exponent 1 in } \Phi \text{ and}\\ &\text{no } x_\rho \text{ with } \rho\in\sigma(1)\cup\{\rho_1\}\setminus\{\rho_0\} \text{ appears in the same monomial of } \Phi \text{ as } x_{\rho_0}.\end{aligned} \tag{4.8}$$

This assumption will be satisfied and easy to check in all our applications. It implies assumption (9.2) below and hence will allow us to compare Peyre's real density with $c_\infty$ as in Section 9.

We fix $\sigma, \rho_0, \rho_1$ as in (4.8). Let $\sigma(1)' := \sigma(1) \cup \{\rho_1\}$. When we write $\rho \notin \sigma(1)'$, we mean $\rho \in \Sigma(1) \setminus \sigma(1)'$. Because of $\alpha^\sigma_{\rho_1} \neq 0$ and (2.7), $\{\deg(x_\rho) : \rho \notin \sigma(1)'\} \cup \{K_X\}$ is an $\mathbb{R}$-basis of $(\mathrm{Pic}\,X)_{\mathbb{R}}$. Hence, we can define the real numbers $b_{\rho,\rho'}$ and $b_{\rho'}$ to satisfy

$$\deg(x_{\rho'}) = -b_{\rho'}K_X - \sum_{\rho\notin\sigma(1)'} b_{\rho,\rho'}\deg(x_\rho)$$

for $\rho' \in \sigma(1)'$.

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We consider the height matrix $\mathscr{A}_1 = (\alpha_\rho^\sigma)_{(\rho,\sigma)\in\Sigma(1)\times\Sigma_{\max}} \in \mathbb{R}^{\Sigma(1)\times\Sigma_{\max}} = \mathbb{R}^{J\times N}$ as in (3.10). Let $Z_\rho$ for $\rho \in \Sigma(1)$ be the rows of this matrix. The following shows that our definition of $b_{\rho,\rho'}$ and $b_{\rho'}$ is consistent with definitions (8.23) and (8.24) that will be needed in Section 8.

**Lemma 4.7.** *We have*

$$Z_\rho = \sum_{\rho'\in\sigma(1)'} b_{\rho,\rho'} Z_{\rho'} \quad \textit{and} \quad (1,\dots,1) = \sum_{\rho'\in\sigma(1)'} b_{\rho'} Z_{\rho'}$$

*for all $\rho \notin \sigma(1)'$. In particular, with*

$$R = 2 + \dim X = J - \operatorname{rk}\operatorname{Pic} X + 1, \tag{4.9}$$

*the $R$ rows $\{Z_{\rho'} : \rho' \in \sigma(1)'\}$ form a maximal linearly independent subset.*

*Proof.* As in (3.10), let $\mathscr{A}_3 = (1,\dots,1) \in \mathbb{R}^{1\times\Sigma_{\max}} = \mathbb{R}^{1\times N}$. Let $\{e_\rho : \rho \in \Sigma(1)\} \cup \{e_0\}$ be the standard basis of $\mathbb{R}^{\Sigma(1)}\times\mathbb{R}$. We define $\deg(e_\rho) = \deg(x_\rho)$ for $\rho \in \Sigma(1)$ and $\deg(e_0) = K_X$. Consider the sequence of linear maps

$$\mathbb{R}^{\Sigma_{\max}} \xrightarrow{\begin{pmatrix}\mathscr{A}_1\\ \mathscr{A}_3\end{pmatrix}} \mathbb{R}^{\Sigma(1)} \times \mathbb{R} \xrightarrow{\deg} (\operatorname{Pic} X)_{\mathbb{R}} \longrightarrow 0.$$

The second map is surjective, and the image of the first is contained in the kernel of the second. Since we have $\operatorname{rk}\mathscr{A}_1 = \#\Sigma(1) + 1 - \operatorname{rk}\operatorname{Pic} X$ by Lemma 3.10, this sequence is exact. It follows that the dual sequence

$$\mathbb{R}^{\Sigma_{\max}} \xleftarrow{\begin{pmatrix}\mathscr{A}_1^\top & \mathscr{A}_3^\top\end{pmatrix}} \mathbb{R}^{\Sigma(1)} \times \mathbb{R} \xleftarrow{\deg^\vee} (\operatorname{Pic} X)_{\mathbb{R}}^\vee \longleftarrow 0$$

is exact as well. Let $\{d_\rho^\vee : \rho \notin \sigma(1)'\} \cup \{K_X^\vee\}$ be the $\mathbb{R}$-basis of $(\operatorname{Pic} X)_{\mathbb{R}}^\vee$ dual to the $\mathbb{R}$-basis of $(\operatorname{Pic} X)_{\mathbb{R}}$ given above. We have

$$\deg^\vee(d_\rho^\vee) = e_\rho - \sum_{\rho'\in\sigma(1)'} b_{\rho,\rho'} e_{\rho'} \quad \text{and} \quad \deg^\vee(K_X^\vee) = e_0 - \sum_{\rho'\in\sigma(1)'} b_{\rho'} e_{\rho'}$$

for all $\rho \notin \sigma(1)'$. Since these elements lie in the kernel of the leftmost map in the dual exact sequence, this gives the required relations between the rows of the matrix $\mathscr{A}_1$ and the row $\mathscr{A}_3$. □

We compare the factor $\alpha(X)$ of Peyre's constant as in [60, Définition 2.4] to

$$c^* := \operatorname{vol}\Big\{\mathbf{r} \in [0,\infty]^{\Sigma(1)\setminus\sigma(1)'} : b_{\rho'} - \sum_{\rho\notin\sigma(1)'} r_\rho b_{\rho,\rho'} \ge 0 \text{ for all } \rho' \in \sigma(1)'\Big\}, \tag{4.10}$$

which will appear in (8.34).

**Lemma 4.8.** *We have*

$$\alpha(X) = \frac{1}{|\alpha_{\rho_1}^\sigma|} c^*.$$

*Proof.* Let $\operatorname{vol}_{\mathbb{Z}}$ be the volume on $(\operatorname{Pic} X)_{\mathbb{R}}$ defined by the lattice $\operatorname{Pic} X$, and let $\operatorname{vol}_{\mathbb{R}}$ be the volume on $(\operatorname{Pic} X)_{\mathbb{R}}$ defined by the basis $\{K_X\} \cup \{\deg(x_\rho) : \rho \notin \sigma(1)'\}$. Since the determinant of the transformation matrix is $-\alpha_{\rho_1}^\sigma$, we have $\operatorname{vol}_{\mathbb{Z}} = |\alpha_{\rho_1}^\sigma| \operatorname{vol}_{\mathbb{R}}$. For the corresponding dual volumes on $(\operatorname{Pic} X)_{\mathbb{R}}^\vee$, we have $\operatorname{vol}_{\mathbb{Z}}^\vee = |\alpha_{\rho_1}^\sigma|^{-1} \operatorname{vol}_{\mathbb{R}}^\vee$.

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Peyre considers the unique $(\operatorname{rk}\operatorname{Pic} X - 1)$-form $\operatorname{vol}_{\mathrm{P}}$ on $(\operatorname{Pic} X)_{\mathbb{R}}^\vee$ such that $\operatorname{vol}_{\mathrm{P}} \wedge K_X = \operatorname{vol}_{\mathbb{Z}}^\vee$. We also consider the form $\operatorname{vol}_V = \bigwedge_{\rho \notin \sigma(1)'} \deg(x_\rho)$. Note that we have $\operatorname{vol}_V \wedge K_X = \operatorname{vol}_{\mathbb{R}}^\vee$. It follows that we have $\operatorname{vol}_{\mathrm{P}} = |\alpha_{\rho_1}^\sigma|^{-1} \operatorname{vol}_V$. These forms can be restricted to volumes on any affine subspace parallel to the subspace $V = \{\phi \in (\operatorname{Pic} X)_{\mathbb{R}}^\vee : \langle \phi, K_X \rangle = 0\}$. Hence,

$$\begin{aligned}
\alpha(X) &= \operatorname{vol}_P \{r \in (\operatorname{Eff} X)^\vee : \langle r, K_X \rangle = -1\} \\
&= |\alpha_{\rho_1}^\sigma|^{-1} \operatorname{vol}_V \{r \in (\operatorname{Pic} X)_{\mathbb{R}}^\vee : \langle r, K_X \rangle = -1, \langle r, \deg x_\rho \rangle \geq 0 \text{ for all } \rho \in \Sigma(1)\} \\
&= |\alpha_{\rho_1}^\sigma|^{-1} \operatorname{vol}_V \left\{ r_0 K_X^\vee + \sum_{\rho \notin \sigma(1)'} r_\rho d_\rho^\vee : \begin{array}{l} r_0 = -1, r_\rho \geq 0 \text{ for all } \rho \notin \sigma(1)', \\ b_{\rho'} - \sum_{\rho \notin \sigma(1)'} r_\rho b_{\rho,\rho'} \geq 0 \text{ for all } \rho' \in \sigma(1)' \end{array} \right\},
\end{aligned}$$

and the claim follows. □

Next, we analyze Peyre's real density $\mu_\infty(X(\mathbb{R}))$ as given in Proposition 4.1. By our assumption (4.8), the equation $\Phi = 0$ can be solved for $x_{\rho_0}$ when all $x_\rho$ with $\rho \notin \sigma(1)'$ are nonzero; here, the implicit function $\phi$ is a rational function in $\{x_\rho : \rho \in \Sigma(1) \setminus \{\rho_0\}\}$ whose total Pic $X$-degree is $\deg(x_{\rho_0})$. Whenever $S \subseteq \sigma(1)' \setminus \{\rho_0\}$ and $\mathbf{u} = (u_\rho) \in \mathbb{R}^S$, we write $\phi(\mathbf{u}, \mathbf{1})$ for $\phi((x_\rho)_{\rho \in \Sigma(1) \setminus \{\rho_0\}})$ with $x_\rho = u_\rho$ for $\rho \in S$ and $x_\rho = 1$ otherwise; this is a polynomial expression in $\mathbf{u}$. Using notation (2.8), we write

$$H_\infty(\mathbf{x}) := \max_{\sigma' \in \Sigma_{\max}} |\mathbf{x}^{D(\sigma')}|$$

for any $\mathbf{x} \in \mathbb{R}^{\Sigma(1)}$.

For the computation of $\mu_\infty(X(\mathbb{R}))$, we work with (4.2) and the chart (2.14) from the subset of $X^\sigma(\mathbb{R})$ to $\mathbb{R}^{\sigma(1) \setminus \{\rho_0\}}$ that drops the $\rho_0$-coordinate. Its inverse is induced by the map

$$f : \mathbb{R}^{\sigma(1) \setminus \{\rho_0\}} \to \mathbb{R}^{\Sigma(1)}, \quad \mathbf{z} = (z_\rho) \mapsto (x_\rho) \text{ with } x_\rho := \begin{cases} \phi(\mathbf{z}, \mathbf{1}), & \rho = \rho_0, \\ z_\rho, & \rho \in \sigma(1) \setminus \{\rho_0\}, \\ 1, & \rho \notin \sigma(1) \end{cases}$$

if we interpret the right-hand side in Cox coordinates. Since $f(\mathbb{R}^{\sigma(1) \setminus \{\rho_0\}})$ and $X(\mathbb{R})$ differ by a set of measure zero, Peyre's real density can be expressed as

$$\omega_\infty := \mu_\infty(X(\mathbb{R})) = \int_{\mathbf{z} \in \mathbb{R}^{\sigma(1) \setminus \{\rho_0\}}} \frac{\mathrm{d}\mathbf{z}}{|\partial\Phi/\partial x_{\rho_0}(f(\mathbf{z}))| \cdot H_\infty(f(\mathbf{z}))}. \tag{4.11}$$

Using the map

$$g : \mathbb{R}^{\sigma(1)' \setminus \{\rho_0\}} \to \mathbb{R}^{\Sigma(1)}, \quad \mathbf{t} = (t_\rho) \mapsto (x_\rho) \text{ with } x_\rho := \begin{cases} \phi(\mathbf{t}, \mathbf{1}), & \rho = \rho_0, \\ t_\rho, & \rho \in \sigma(1)' \setminus \{\rho_0\}, \\ 1, & \rho \notin \sigma(1)', \end{cases}$$

we define

$$c_\infty := 2^{\#\Sigma(1) - \#\sigma(1) - 1} \int_{\mathbf{t} \in \mathbb{R}^{\sigma(1)' \setminus \{\rho_0\}},\ H_\infty(g(\mathbf{t})) \leq 1} \frac{\mathrm{d}\mathbf{t}}{|\partial\Phi/\partial x_{\rho_0}(g(\mathbf{t}))|}, \tag{4.12}$$

which will reappear in (9.3) and (9.7).

To compare $\omega_\infty$ and $c_\infty$, we use the following substitution.

**Lemma 4.9.** *Let $\Psi$ be a* Pic $X$*-homogeneous rational function in $\{x_\rho : \rho \in \Sigma(1)\}$ of degree*

$$\sum_{\rho \notin \sigma(1)} \alpha^\sigma_{\Psi,\rho} \deg(x_\rho).$$

*Let $\alpha^\sigma_{\rho',\rho} \in \mathbb{Z}$ for $\rho' \in \Sigma(1)$ and $\rho \notin \sigma(1)$ be as in (2.9). Then the substitution $z_{\rho'} = t_{\rho_1}^{-\alpha^\sigma_{\rho',\rho_1}} t_{\rho'}$ for $\rho' \in \sigma(1) \setminus \{\rho_0\}$ gives $\Psi(f(\mathbf{z})) = t_{\rho_1}^{-\alpha^\sigma_{\Psi,\rho_1}} \Psi(g(\mathbf{t}))$. In particular, $\phi(\mathbf{z}, \mathbf{1}) = t_{\rho_1}^{-\alpha^\sigma_{\rho_0,\rho_1}} \phi(\mathbf{t}, \mathbf{1})$.*

*If $t_{\rho_1}$ appears in $\phi(\mathbf{t}, \mathbf{1})$ with odd exponent, then there is another $t_\rho$ with odd exponent in the same monomial or there is a $t_\rho$ with odd exponent in each of the other monomials of $\phi(\mathbf{t}, \mathbf{1})$.*

*Proof.* Consider the case $\Psi = x_\rho$ first. For $\rho \in \sigma(1) \setminus \{\rho_0\}$, the claim holds by definition of the substitution. For $\rho = \rho_1$, we have $\Psi(f(\mathbf{z})) = 1 = t_{\rho_1}^{-1} \cdot t_{\rho_1} = t_{\rho_1}^{-\alpha^\sigma_{\Psi,\rho_1}} \Psi(g(\mathbf{t}))$. For $\rho \notin \sigma(1)'$, we have $\Psi(f(\mathbf{z})) = 1 \cdot 1 = t_{\rho_1}^{-\alpha^\sigma_{\Psi,\rho_1}} \Psi(g(\mathbf{t}))$. Therefore, the claim holds for all monomials and hence also for all homogeneous polynomials and all homogeneous rational functions in $\{x_\rho : \rho \in \Sigma(1) \setminus \{\rho_0\}\}$. In particular, in the case $\Psi = x_{\rho_0}$, since $\phi$ is such a rational function of degree $\deg(x_{\rho_0})$, the substitution gives $\Psi(f(\mathbf{z})) = \phi(\mathbf{z}, \mathbf{1}) = t_{\rho_1}^{-\alpha^\sigma_{\rho_0,\rho_1}} \phi(\mathbf{t}, \mathbf{1}) = t_{\rho_1}^{-\alpha^\sigma_{\Psi,\rho_1}} \Psi(g(\mathbf{t}))$. Now, the claim follows for all monomials, homogeneous polynomials and finally all homogeneous rational functions in $\{x_\rho : \rho \in \Sigma(1)\}$.

Let $\psi$ be the numerator of $\phi$. Because of (4.8), $t_{\rho_1}$ appears in at most one monomial of $\psi(\mathbf{t}, \mathbf{1})$; we assume that it appears in the first monomial with odd exponent. Therefore, either the exponent of $t_{\rho_1}$ in the first monomial of $t_{\rho_1}^{-\alpha^\sigma_{\psi,\rho_1}} \psi(\mathbf{t}, \mathbf{1})$ is odd, or the exponents of $t_{\rho_1}$ in all other monomials of this expression are odd. But since our substitution gives $\psi(\mathbf{z}, \mathbf{1}) = t_{\rho_1}^{-\alpha^\sigma_{\psi,\rho_1}} \psi(\mathbf{t}, \mathbf{1})$, the exponent of $t_{\rho_1}$ in a certain monomial of $t_{\rho_1}^{-\alpha^\sigma_{\psi,\rho_1}} \psi(\mathbf{t}, \mathbf{1})$ can only be odd if there is a $z_\rho$ with odd exponent in the corresponding monomial of $\psi(\mathbf{z}, \mathbf{1})$, and then the exponent of $t_\rho$ in this monomial of $\psi(\mathbf{t}, \mathbf{1})$ is also odd. □

**Proposition 4.10.** *We have*

$$\mu_\infty(X(\mathbb{R})) = \frac{|\alpha^\sigma_{\rho_1}|}{2^{\operatorname{rk} \operatorname{Pic} X}} c_\infty.$$

*Proof.* Our starting point is (4.11). We use the identity (for positive real $s$)

$$\frac{1}{s} = \int_{z_{\rho_1} > 0,\ s z_{\rho_1} \le 1} \mathrm{d}z_{\rho_1}$$

to deduce

$$\omega_\infty = \int_{(\mathbf{z}, z_{\rho_1}) \in \mathbb{R}^{\sigma(1) \setminus \{\rho_0\}} \times \mathbb{R}_{>0},\ H_\infty(f(\mathbf{z})) \cdot z_{\rho_1} \le 1} \frac{\mathrm{d}\mathbf{z}\, \mathrm{d}z_{\rho_1}}{|\partial \Phi / \partial x_{\rho_0}(f(\mathbf{z}))|}.$$

We use the transformation $z_{\rho_1} = t_{\rho_1}^{\alpha^\sigma_{\rho_1}}$ (with positive $t_{\rho_1}$) and the transformations from Lemma 4.9. The latter give $H_\infty(f(\mathbf{z})) = t_{\rho_1}^{-\alpha^\sigma_{\rho_1}} H_\infty(g(\mathbf{t}))$ since all monomials appearing in the definition of the anticanonical height function $H_\infty$ have degree $-K_X$; therefore, $H_\infty(f(\mathbf{z})) \cdot z_{\rho_1} = H_\infty(g(\mathbf{t}))$. Furthermore, $|\partial \Phi / \partial x_{\rho_0}(f(\mathbf{z}))| = |t_{\rho_1}^{-\alpha^\sigma_{\partial \Phi / \partial x_{\rho_0},\rho_1}} \partial \Phi / \partial x_{\rho_0}(g(\mathbf{t}))|$ (even without using the observation that these are the same constants by (4.8)). We obtain $\mathrm{d}z_{\rho_1} = |\alpha^\sigma_{\rho_1} t_{\rho_1}^{\alpha^\sigma_{\rho_1}-1}|\,\mathrm{d}t_{\rho_1}$ and

$$\mathrm{d}\mathbf{z} = |t_{\rho_1}^{-\sum_{\rho'\in\sigma(1)\setminus\{\rho_0\}}\alpha^\sigma_{\rho',\rho_1}}| \bigwedge_{\rho'\in\sigma(1)\setminus\{\rho_0\}} \mathrm{d}t_{\rho'}.$$

The integration domain is unchanged.

We have $-K_X = \sum_{\rho'\in\Sigma(1)} \deg(x_{\rho'}) - \deg(\Phi)$ by [2, Proposition 3.3.3.2], and $\deg(\partial\Phi/\partial x_{\rho_0}) = \deg(\Phi) - \deg(x_{\rho_0})$. Therefore, $\alpha^\sigma_{\rho_1} = \sum_{\rho'\in\Sigma(1)} \alpha^\sigma_{\rho',\rho_1} - \alpha^\sigma_{\Phi,\rho_1}$ and $\alpha^\sigma_{\partial\Phi/\partial x_{\rho_0},\rho_1} = \alpha^\sigma_{\Phi,\rho_1} - \alpha^\sigma_{\rho_0,\rho_1}$. Since $\alpha^\sigma_{\rho',\rho} = \delta_{\rho'=\rho}$ for all $\rho',\rho \notin \sigma(1)$, we conclude that

$$\alpha^\sigma_{\rho_1} = \sum_{\rho'\in\sigma(1)\setminus\{\rho_0\}} \alpha^\sigma_{\rho',\rho_1} + 1 - \alpha^\sigma_{\partial\Phi/\partial x_{\rho_0},\rho_1}.$$

This shows that the powers of $t_{\rho_1}$ cancel out so that $\mathrm{d}\mathbf{z}\,\mathrm{d}z_{\rho_1}/|\partial\Phi/\partial x_{\rho_0}(f(\mathbf{z}))| = \mathrm{d}\mathbf{t}/|\partial\Phi/\partial x_{\rho_0}(g(\mathbf{t}))|$. Therefore,

$$\omega_\infty = |\alpha^\sigma_{\rho_1}| \int_{\mathbf{t}\in\mathbb{R}^{\sigma(1)\setminus\{\rho_0\}}\times\mathbb{R}_{>0},\ H_\infty(g(\mathbf{t}))\le 1} \frac{\mathrm{d}\mathbf{t}}{|\partial\Phi/\partial x_{\rho_0}(g(\mathbf{t}))|}.$$

We claim that

$$\omega^-_\infty := |\alpha^\sigma_{\rho_1}| \int_{\mathbf{t}\in\mathbb{R}^{\sigma(1)\setminus\{\rho_0\}}\times\mathbb{R}_{<0},\ H_\infty(g(\mathbf{t}))\le 1} \frac{\mathrm{d}\mathbf{t}}{|\partial\Phi/\partial x_{\rho_0}(g(\mathbf{t}))|}$$

has the same value as $\omega_\infty$. Indeed, $\phi(\mathbf{t},\mathbf{1})$ (the $\rho_0$-component of $g(\mathbf{t})$) is the only place where the sign of $t_{\rho_1}$ might matter. Our claim is clearly true if $t_{\rho_1}$ does not appear in $\phi(\mathbf{t},\mathbf{1})$ or if $t_{\rho_1}$ has an even exponent in $\phi(\mathbf{t},\mathbf{1})$. If $t_{\rho_1}$ appears in $\phi(\mathbf{t},\mathbf{1})$ with odd exponent, then the change of variables $t'_{\rho_1} := -t_{\rho_1}$ and $t'_\rho := -t_\rho$ for all $t_\rho$ appearing in the final statement of Lemma 4.9 in $\omega^-_\infty$ shows that $\omega^-_\infty = \omega_\infty$. Therefore,

$$\mu_\infty(X(\mathbb{R})) = \omega_\infty = \frac{1}{2}(\omega_\infty + \omega^-_\infty) = \frac{|\alpha^\sigma_{\rho_1}|}{2} \int_{\mathbf{t}\in\mathbb{R}^{\sigma(1)\setminus\{\rho_0\}}\times\mathbb{R}_{\neq 0},\ H_\infty(g(\mathbf{t}))\le 1} \frac{\mathrm{d}\mathbf{t}}{|\partial\Phi/\partial x_{\rho_0}(g(\mathbf{t}))|}.$$

Since $\operatorname{rk}\operatorname{Pic} X = \#\Sigma(1) - \#\sigma(1)$ and replacing $\mathbb{R}^{\sigma(1)\setminus\{\rho_0\}}\times\mathbb{R}_{\neq 0}$ by $\mathbb{R}^{\sigma(1)'\setminus\{\rho_0\}}$ does not change the integral, this completes the proof. □

### *4.6. Peyre's constant in Cox coordinates*

**Proposition 4.11.** *Let $X$ be a split almost Fano variety over $\mathbb{Q}$ with semiample $\omega_X^\vee$ that has a finitely generated Cox ring $\mathscr{R}(X)$ with precisely one relation $\Phi$ with integral coefficients and satisfies the assumptions (2.3) and (4.8). Then Peyre's constant for $X$ with respect to the anticanonical height $H$ as in (3.7) is*

$$c = \frac{1}{2^{\operatorname{rk}\operatorname{Pic} X}} c^* c_\infty c_{\mathrm{fin}},$$

*using the notation (4.6), (4.10), (4.12).*

*Proof.* According to [61, 5.1], Peyre's constant for $X$ is $c = \alpha(X)\beta(X)\tau_H(X)$. Here, the cohomological constant is

$$\beta(X) = \#H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(X\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})) = 1$$

since $X$ is split. Recall (4.3) for $\tau_H(X)$. By Lemma 4.8 and Proposition 4.10, $\alpha(X)\mu_\infty(X(\mathbb{R})) = c^* c_\infty$. Furthermore, we use Proposition 4.6 for the $p$-adic densities. □

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## Part II The asymptotic formula

This part, culminating in Theorem 8.4, is devoted to a proof of the asymptotic formula (1.5) for the counting problem described by (1.2), (1.3) and (1.4), subject to certain conditions to be specified in due course. The nature of our results will be similar to Proposition 3.8, except that we specialize the general polynomial $\Phi$ to a polynomial of the shape (1.2). In other words, every variable appears in at most one monomial, and for better readability in comparison with (3.9), we relabel the variables and their exponents as in (1.2). In the notation of (1.2), we have

$$J = J_0 + J_1 + \cdots + J_k$$

variables, where $J_0$ is the number of variables that do not occur in any of the monomials. As mentioned in the introduction, the particular shape (1.2) is not an atypical situation; it appears sufficiently often in practice that it deserves special attention. In Section 9, we will also show that if the conditions (1.2)–(1.4) come from an algebraic variety satisfying the hypotheses of Proposition 4.11, then the leading constant in (1.5) agrees with Peyre's prediction, as computed in Proposition 4.11.

Before we begin, we fix some notation for use in the remainder of the paper. Vector operations are to be understood componentwise. In particular, just like the common addition of vectors, for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{C}^n$, we write $\mathbf{x} \cdot \mathbf{y} = (x_1 y_1, \dots, x_n y_n) \in \mathbb{C}^n$. If $\mathbf{x} \in \mathbb{R}^n_{>0}$, $\mathbf{y} \in \mathbb{C}^n$, we write $\mathbf{x}^{\mathbf{y}} = x_1^{y_1} \cdots x_n^{y_n}$. We also use this notation when $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{N}^n$. We put $\langle \mathbf{x} \rangle = x_1 x_2 \cdots x_n$. We write $|\cdot|_1$ for the usual 1-norm, and $|\cdot|$ denotes the maximum norm. For $q \in \mathbb{N}$, we write $\mu(q)$ for the Möbius function of $q$, the Euler totient is denoted $\phi(q)$ and we write $\sum^*_{a \bmod q}$ for a sum over reduced residue classes modulo $q$. The greatest common divisor of nonzero integers $a$, $b$ is denoted by $(a, b)$; confusion with elements of $\mathbb{Z}^2$ should not arise. The lowest common multiple is $[a, b]$. As usual, $e(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. Finally, we apply the following convention concerning the letter $\varepsilon$: Whenever $\varepsilon$ occurs in a statement, it is asserted that the statement is true for any positive real number $\varepsilon$. Note that this allows implicit constants in Landau or Vinogradov symbols to depend on $\varepsilon$, and that one may conclude from $A_1 \ll B^\varepsilon$ and $A_2 \ll B^\varepsilon$ that one has $A_1 A_2 \ll B^\varepsilon$, for example.

## 5. Diophantine analysis of the torsor

In this section and the next, we study the torsor equation (1.2) with its variables *restricted to boxes*. For the number of its integral solutions, we seek an asymptotic expansion whose leading term features a product of local densities. All estimates are required uniformly relative to the coefficients $b_1, \dots, b_k \in \mathbb{Z} \setminus \{0\}$ that occur in (1.2). We assume $k \geq 3$ throughout.

The building blocks of the local densities are Gauß sums and their continuous analogues, and we begin by defining the former. Let $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{N}^n$ be a 'chain of exponents'. In the following, all implied constants may depend on $\mathbf{h}$. Then, for $a \in \mathbb{Z}$, $q \in \mathbb{N}$ let

$$E(q, a; \mathbf{h}) = q^{-n} \sum_{\substack{1 \leq x_j \leq q \\ 1 \leq j \leq n}} e\Big(\frac{a x_1^{h_1} x_2^{h_2} \cdots x_n^{h_n}}{q}\Big) = q^{-n} \sum_{\substack{1 \leq x_j \leq q \\ 1 \leq j \leq n}} e\Big(\frac{a \mathbf{x}^{\mathbf{h}}}{q}\Big). \tag{5.1}$$

For a continuous counterpart, let $\mathbf{Y} \in [\frac{1}{2}, \infty)^n$, put $\mathscr{Y} = \{\mathbf{y} \in \mathbb{R}^n : \frac{1}{2} Y_j < |y_j| \leq Y_j \ \ (1 \leq j \leq n)\}$ and define

$$I(\beta, \mathbf{Y}; \mathbf{h}) = \int_{\mathscr{Y}} e(\beta y_1^{h_1} y_2^{h_2} \cdots y_n^{h_n})\, \mathrm{d}\mathbf{y}. \tag{5.2}$$

This exponential integral satisfies the simple bound

$$I(\beta, \mathbf{Y}; \mathbf{h}) \ll \langle \mathbf{Y} \rangle (1 + \mathbf{Y}^{\mathbf{h}} |\beta|)^{-1}. \tag{5.3}$$

Indeed, if $n = 1$, then integration by parts yields the bound $O(Y^{1-h}|\beta|^{-1})$, which together with the trivial bound $O(Y)$ confirms (5.3). If $n > 1$, then one uses the obvious relation

$$I(\beta, \mathbf{Y}; \mathbf{h}) = \int_{\frac{1}{2}Y_1 \le |y| \le Y_1} I(\beta y^{h_1}, (Y_2, \ldots, Y_n); (h_2, \ldots, h_n))\, \mathrm{d}y$$

together with induction. With (5.3) in hand for $n - 1$ in place of $n$, one infers (5.3) for $n$ from

$$I(\beta, \mathbf{Y}; \mathbf{h}) \ll Y_2 Y_3 \cdots Y_n \int_{\frac{1}{2}Y_1 \le |y| \le Y_1} (1 + Y_2^{h_2} \cdots Y_n^{h_n} |y^{h_1}\beta|)^{-1}\, \mathrm{d}y.$$

We now describe the counting problem at the core of this section. For $\mathbf{b} \in (\mathbb{Z} \setminus \{0\})^k$ and $\mathbf{X} = (X_{ij}) \in [1, \infty)^J$, let $\mathcal{N}_{\mathbf{b}}(\mathbf{X})$ denote the number of solutions $\mathbf{x} \in \mathbb{Z}^J$ to (1.2) satisfying $\frac{1}{2}X_{ij} \le |x_{ij}| \le X_{ij}$. Associated with each summand in (1.2) are a chain of exponents $\mathbf{h}_i = (h_{i1}, \ldots, h_{iJ_i})$ and boxing vectors $\mathbf{X}_i = (X_{i1}, \ldots, X_{iJ_i})$. In the interest of brevity, we now put

$$E_i(q, a) = E(q, a; \mathbf{h}_i), \quad I_i(\beta, \mathbf{X}) = I(\beta, \mathbf{X}_i; \mathbf{h}_i) \quad (1 \le i \le k). \tag{5.4}$$

The *singular integral* for this counting problem is then defined by

$$\mathcal{J}_{\mathbf{b}}(\mathbf{X}) = \langle \mathbf{X}_0 \rangle \int_{-\infty}^{\infty} I_1(b_1\beta, \mathbf{X}) I_2(b_2\beta, \mathbf{X}) \cdots I_k(b_k\beta, \mathbf{X})\, \mathrm{d}\beta, \tag{5.5}$$

and the *singular series* is

$$\mathcal{E}_{\mathbf{b}} = \sum_{q=1}^{\infty} \sum_{a \bmod q}^{*} E_1(q, ab_1) E_2(q, ab_2) \cdots E_k(q, ab_k). \tag{5.6}$$

By (5.3), the singular integral converges absolutely provided only that $k \ge 2$. Unfortunately, it is not as easy to determine whether the singular series converges; this depends on the chains of exponents in a subtle manner. However, we note that an argument paralleling that in the proof of [72, Lemma 2.11] shows that the sum

$$\sum_{a \bmod q}^{*} E_1(q, ab_1) E_2(q, ab_2) \cdots E_k(q, ab_k) \tag{5.7}$$

is a multiplicative function of $q$. Hence, based on the hypothesis that the singular series is absolutely convergent, one has the alternative representation

$$\mathcal{E}_{\mathbf{b}} = \prod_p \sum_{l=0}^{\infty} \sum_{a \bmod p^l} {}^{*} E_1(p^l, ab_1) E_2(p^l, ab_2) \cdots E_k(p^l, ab_k).$$

By orthogonality of additive characters, the partial sums $0 \le l \le L$ count congruences modulo $p^L$, and (still under the assumption of absolute convergence) we can therefore express the singular series as a product of ‘local densities’:

$$\mathcal{E}_{\mathbf{b}} = \prod_p \lim_{L \to \infty} \frac{1}{p^{L(J_1 + \cdots + J_k - 1)}} \#\Big\{(\mathbf{x}_1, \ldots, \mathbf{x}_k) \bmod p^L : b_1 \mathbf{x}_1^{\mathbf{h}_1} + \cdots + b_k \mathbf{x}_k^{\mathbf{h}_k} \equiv 0 \bmod p^L\Big\}. \tag{5.8}$$

The transition method to be detailed in Section 8 works with the proviso that the product $\mathcal{E}_{\mathrm{b}}\mathcal{J}_{\mathbf{b}}(\mathbf{X})$ is a good approximation to $\mathcal{N}_{\mathbf{b}}(\mathbf{X})$. We detail these requirements as follows; note that (5.10) is (3.11) specialized to the equation (1.2).

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**Hypothesis 5.1.** *The singular series $\mathscr{E}_{\mathbf{b}}$ converges absolutely. There are real numbers $\beta_1, \dots, \beta_k \le 1$ with*

$$\mathscr{E}_{\mathbf{b}} \ll |b_1|^{\beta_1}|b_2|^{\beta_2}\cdots|b_k|^{\beta_k}. \tag{5.9}$$

*Further, there exists $\zeta \in \mathbb{R}^k$ with*

$$\zeta_i > 0 \text{ for all } 1 \le i \le k, \quad h_{ij}\zeta_i < 1 \text{ for all } i, j, \quad \sum_{i=1}^{k} \zeta_i = 1, \tag{5.10}$$

*and there exist real numbers $0 < \lambda \le 1$, $\delta_1 > 0$ and $C \ge 0$ with the property that whenever $\mathbf{X} \in [1, \infty)^J$ obeys the condition that*

$$\min_{1\le i\le k} \mathbf{X}_i^{\mathbf{h}_i} \ge \big(\max_{1\le i\le k} \mathbf{X}_i^{\mathbf{h}_i}\big)^{1-\lambda}, \tag{5.11}$$

*then uniformly in $\mathbf{b} \in (\mathbb{Z}\setminus\{0\})^k$, one has*

$$\mathscr{N}_{\mathbf{b}}(\mathbf{X}) - \mathscr{E}_{\mathbf{b}}\mathscr{I}_{\mathbf{b}}(\mathbf{X}) \ll |b_1\cdots b_k|^C (\min_{ij} X_{ij})^{-\delta_1} \prod_{i=0}^{k}\prod_{j=1}^{J_i} X_{ij}^{1-h_{ij}\zeta_i+\varepsilon}, \tag{5.12}$$

*wherein we wrote $\zeta_0 = h_{0j} = 0$ $(1 \le j \le J_0)$.*

In the situation of (1.6), Hypothesis 5.1 is in fact a theorem.

**Proposition 5.2.** *Suppose that $k = 3$, $J_1 \ge J_2 \ge 2$ and $h_{ij} = 1$ for $i = 1, 2$, $1 \le j \le J_i$. Then Hypothesis 5.1 is true.*

We prove this in the next section. As the proof will show, much more is true. We are free to choose $\zeta$ according to (5.10), and one can specify the parameters $\beta$, $\lambda$ and $C$. In terms of the number $\omega$ defined in (6.5) below, one may take

$$\lambda = 2^{-4-|\mathbf{h}_3|_1}\omega, \quad C = 300/\omega$$

and

$$\beta = \Big(\frac{1}{2}(1-\mu)+\varepsilon, \frac{1}{2}(1-\mu)+\varepsilon, \mu\Big), \tag{5.13}$$

for any $\varepsilon > 0$, and any $\mu$ with $\varepsilon < \mu < |\mathbf{h}_3|^{-1}$.

In the rest of this section, we prepare the proof of Proposition 5.2 with some bounds for the local factors, and we begin with an upper bound for the singular integral. At the same time, we compare the singular integral with a truncated version of it. To define the latter, let $Z_0$ be the maximum of the numbers $\mathbf{X}_i^{\mathbf{h}_i}$ $(1 \le i \le k)$, and let $Q \ge 1$. Then put

$$\mathscr{I}_{\mathbf{b}}(\mathbf{X}, Q) = \langle \mathbf{X}_0\rangle \int_{-QZ_0^{-1}}^{QZ_0^{-1}} I_1(b_1\beta, \mathbf{X}) I_2(b_2\beta, \mathbf{X}) \cdots I_k(b_k\beta, \mathbf{X})\, \mathrm{d}\beta.$$

**Lemma 5.3.** *Let $k \ge 3$, let $\zeta_0 = 0$, and let $\zeta_i$ $(1 \le i \le k)$ be positive real numbers with $\zeta_1 + \zeta_2 + \cdots + \zeta_k = 1$. Then*

$$\mathscr{I}_{\mathbf{b}}(\mathbf{X}) \ll |b_1|^{-\zeta_1}\cdots|b_k|^{-\zeta_k} \prod_{i=0}^{k}\prod_{j=1}^{J_i} X_{ij}^{1-h_{ij}\zeta_i}.$$

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*Further, there is a number* $\delta > 0$ *such that whenever* $Q \geq 1$ *one has*

$$\mathscr{I}_{\mathbf{b}}(\mathbf{X}) - \mathscr{I}_{\mathbf{b}}(\mathbf{X}, Q) \ll Q^{-\delta} \prod_{i=0}^{k} \prod_{j=1}^{J_i} X_{ij}^{1-h_{ij}\zeta_i}.$$

*Proof.* By Hölder's inequality,

$$\int_{-\infty}^{\infty} \prod_{i=1}^{k} (1 + \mathbf{X}_i^{\mathbf{h}_i} |b_i\beta|)^{-1} \, \mathrm{d}\beta \leq \prod_{i=1}^{k} \Big( \int_{-\infty}^{\infty} (1 + \mathbf{X}_i^{\mathbf{h}_i} |b_i\beta|)^{-1/\zeta_i} \, \mathrm{d}\beta \Big)^{\zeta_i},$$

and by (5.5) and (5.3) the first statement in the lemma is immediate. For the second, one picks $\iota$ with $Z_0 = \mathbf{X}_\iota^{\mathbf{h}_\iota}$ and observes that

$$\int_{QZ_0^{-1}}^{\infty} (1 + \mathbf{X}_\iota^{\mathbf{h}_\iota} |b_\iota\beta|)^{-1/\zeta_\iota} \, \mathrm{d}\beta \ll Q^{1-(1/\zeta_\iota)} \mathbf{X}_\iota^{-\mathbf{h}_\iota}.$$

If this bound is used within the preceding application of Hölder's inequality, one arrives at the second statement in the lemma. □

We continue with some general remarks on Gauß sums.

**Lemma 5.4.** *Let* $\mathbf{h} \in \mathbb{N}^n$. *Let* $b \in \mathbb{Z}$, $q \in \mathbb{N}$ *and* $q' = q/(q,b)$, $b' = b/(q,b)$. *Then* $E(q,b;\mathbf{h}) = E(q',b';\mathbf{h})$. *If* $n \geq 2$, $h_1 = 1$ *and* $(b,q) = 1$, *then*

$$E(q,b,\mathbf{h}) = q^{1-n} \#\{x_2, \ldots, x_n : 1 \leq x_j \leq q,\ x_2^{h_2} x_3^{h_3} \cdots x_n^{h_n} \equiv 0 \bmod q\}.$$

*Further,*

$$E(q,b,(1,\ldots,1)) = q^{1-n} \sum_{\substack{d_j | q \\ q | d_2 d_3 \cdots d_n}} \varphi\Big(\frac{q}{d_2}\Big) \cdots \varphi\Big(\frac{q}{d_n}\Big).$$

*In particular,* $E(q,b,(1,\ldots,1)) \ll q^{\varepsilon-1}$ *and* $E(q,b,(1,1)) = q^{-1}$.

*Proof.* We have $b/q = b'/q'$ whence $e(bx_1^{h_1} \cdots x_n^{h_n}/q)$ has period $q'$ in all $x_j$. Summing over all $x_j$ modulo $q$ gives the first statement at once. The second statement follows from (5.1) and orthogonality, after carrying out the sum over $x_1$. If we specialize the second statement to $h_j = 1$ for all $j$ and sort the $x_j$ according to the values of $d_j = (x_j, q)$, then we arrive at the formula for $E(q,b,(1,\ldots,1))$, from which the remaining claims are immediate. □

**Lemma 5.5.** *Let* $\mathbf{h} \in \mathbb{N}^n$ *with* $h_1 \leq h_2 \leq \cdots \leq h_n$. *Then, for each* $b \in \mathbb{Z}$, *the sum*

$$D(q,b,\mathbf{h}) = \sum_{a \bmod q}^{*} E(q,ab,\mathbf{h})$$

*is multiplicative as a function of* $q$, *and one has* $D(q,b,\mathbf{h}) \ll (q,b)^{1/h_n} q^{1+\varepsilon-1/h_n}$.

*Proof.* Within this proof the numbers $h_j$ are fixed. Therefore, we remove $\mathbf{h}$ from the notation temporarily. Thus, $D(q,b)$ abbreviates $D(q,b,\mathbf{h})$, for example.

By (5.7), the function $D(q,b)$ is multiplicative in $q$, and we proceed to evaluate it for $q = p^l$ with $p$ prime and $l \in \mathbb{N}$. Let $M_b(q)$ denote the number of $\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^n$ with $bx_1^{h_1} \cdots x_n^{h_n} \equiv 0 \bmod q$. Now, first applying Lemma 5.4, and then (5.1) and orthogonality, one confirms the identities

$$D(p^l,b) = \sum_{a \bmod p^l} E(p^l,ab,\mathbf{h}) - \sum_{a \bmod p^{l-1}} E(p^{l-1},ab,\mathbf{h}) = p^{l(1-n)} M_b(p^l) - p^{(l-1)(1-n)} M_b(p^{l-1}).$$

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Let $\beta$ be the number with $p^\beta \mid b$ and $p^{\beta+1} \nmid b$. Obviously, if $l \le \beta$, then $M_b(p^l) = p^{ln}$, and the preceding formula gives $D(p^l, b) = \phi(p^l)$. If $l > \beta$, then $M_b(p^l)$ is the number of solutions of $x_1^{h_1} \cdots x_n^{h_n} \equiv 0 \bmod p^{l-\beta}$ with $1 \le x_j \le p^l$ $(1 \le j \le n)$. Thus, $M_b(p^l) = p^{\beta n} M_1(p^{l-\beta})$. We now estimate $M_1(p^\sigma)$. Consider $x_1, \dots, x_n$ with $p^{\nu_j} \mid x_j$. The congruence $x_1^{h_1} \cdots x_n^{h_n} \bmod p^\sigma$ is equivalent with

$$h_1\nu_1 + \cdots + h_n\nu_n \ge \sigma. \tag{5.14}$$

Thus, for a fixed tuple $\nu_1, \dots \nu_n$, there are at most $p^{n\sigma-\nu_1-\cdots-\nu_n}$ solutions counted by $M_1(p^\sigma)$. Further, if (5.14) holds, then

$$\nu_1 + \cdots + \nu_n \ge \frac{1}{h_n}(h_1\nu_1 + \cdots + h_n\nu_n) \ge \frac{\sigma}{h_n}.$$

Since the number of tuples $\nu_1, \dots, \nu_n$ that arise here certainly does not exceed $\sigma^n$, we deduce that $M_1(p^\sigma) \le \sigma^n p^{n\sigma - \lceil \sigma/h_n \rceil}$. This implies $M_b(p^l) \le l^n p^{ln - \lceil (l-\beta)/h_n \rceil}$. On inserting this bound in the identity for $D(p^l, b)$, one first confirms the desired estimate for $D(q, b)$ for prime powers $q$ and then for general $q$ by multiplicativity. □

We now use these results to discuss the singular series that arises in Proposition 5.2. Then we have $k = 3$, $J_1 \ge J_2 \ge 2$, and we may use the last clause of Lemma 5.4 with $\mathbf{h}_1$ and $\mathbf{h}_2$. Further, we put $h = \max h_{3j}$ and use Lemma 5.5 to confirm that

$$\sum_{a \bmod q}^{*} E_1(q, ab_1)E_2(q, ab_2)E_3(q, ab_3) \ll q^{\varepsilon-1-1/h}(q, b_1)(q, b_2)(q, b_3)^{1/h}. \tag{5.15}$$

It is now immediate that the singular series converges absolutely. Further, on using crude bounds of the type $(x, y) \le x^\sigma y^{1-\sigma}$ with $0 \le \sigma \le 1$, it follows from (5.15) that whenever $0 < \varepsilon < \mu < 1/h$ one has from (5.15) that

$$\sum_{q=1}^{\infty} \Big| \sum_{a \bmod q}^{*} E_1(q, ab_1)E_2(q, ab_2)E_3(q, ab_3) \Big| \ll \sum_{q=1}^{\infty} q^{\varepsilon-1-\mu}(q, b_1)(q, b_2)b_3^\mu$$
$$\ll b_3^\mu \sum_{c_1|b_1} \sum_{c_2|b_2} (c_1c_2)^{\varepsilon-\mu}(c_1, c_2)^{1+\mu-\varepsilon} \le b_3^\mu \sum_{c_1|b_1} \sum_{c_2|b_2} (c_1c_2)^{\frac{1}{2}(1-\mu+\varepsilon)} \ll b_3^\mu (b_1b_2)^{\frac{1}{2}(1-\mu)+\varepsilon}. \tag{5.16}$$

This establishes all the statements in Proposition 5.2 that concern the singular series, and it also confirms the comment following Proposition 5.2 about an admissible choice of $\beta$.

## 6. The circle method

### *6.1. Weyl sums*

In this section, we apply the circle method to establish Proposition 5.2. We prepare the ground with a discussion of the generalized Weyl sums

$$W(\alpha, \mathbf{Y}; \mathbf{h}) = \sum_{\mathbf{y} \in \mathbb{Z}^n \cap \mathscr{Y}} e(\alpha \mathbf{y}^{\mathbf{h}}).$$

Here and in the sequel, we continue to use the notation from the previous section, and in particular, $\mathbf{h}$, $\mathbf{Y}$ and $\mathscr{Y}$ are as in (5.2). The upper bound for the mean square

$$\int_0^1 |W(\alpha, \mathbf{Y}; \mathbf{h})|^2 \, \mathrm{d}\alpha \ll \langle \mathbf{Y} \rangle^{1+\varepsilon} \tag{6.1}$$

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is pivotal and is readily checked: By orthogonality, the integral in question equals the number of solutions of the diophantine equation $\mathbf{x}^{\mathbf{h}} = \mathbf{y}^{\mathbf{h}}$ with $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n \cap \mathscr{Y}$. There are $\langle \mathbf{Y} \rangle$ choices for $\mathbf{x}$, and $y_1 \cdots y_n$ is a divisor of $\mathbf{x}^{\mathbf{h}}$, leaving $\langle \mathbf{Y} \rangle^{\varepsilon}$ choices for $\mathbf{y}$, once $\mathbf{x}$ is chosen.

The next result is a version of Weyl's inequality.

**Lemma 6.1.** *Let $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $|q\alpha - a| \le q^{-1}$. Suppose that $Y_1 \ge Y_2 \ge \cdots \ge Y_n$. Then*

$$|W(\alpha, \mathbf{Y}; \mathbf{h})|^{2^{|\mathbf{h}|_1 - n}} \ll \langle \mathbf{Y} \rangle^{2^{|\mathbf{h}|_1 - n + \varepsilon}} \Big( \frac{1}{q} + \frac{1}{Y_n} + \frac{q}{\mathbf{Y}^{\mathbf{h}}} \Big).$$

*Proof.* If $n = 1$ this is the familiar form of Weyl's inequality. If $n \ge 2$, then we apply repeated Weyl differencing. Let $h \in \mathbb{N}$. On combining [72, Lemma 2.4] with [72, Exercise 2.8.1], one has

$$\Big| \sum_{X < x \le 2X} e(\beta x^h) \Big|^{2^{h-1}} \le (2X)^{2^{h-1} - h} \sum_{\substack{|u_j| \le X \\ 1 \le j < h}} \sum_{x \in I(\mathbf{u})} e\big(h! \beta u_1 u_2 \ldots u_{h-1} (x + \tfrac{1}{2} |\mathbf{u}|_1)\big),$$

where the $I(\mathbf{u})$ are certain subintervals of $[X, 2X]$. Note here that the sum on the right is real and nonnegative. One trivially has

$$\Big| \sum_{-2X \le x < -X} e(\beta x^h) \Big| = \Big| \sum_{X < x \le 2X} e(\beta x^h) \Big|,$$

and hence it follows that

$$\Big| \sum_{X < |x| \le 2X} e(\beta x^h) \Big|^{2^{h-1}} \ll X^{2^{h-1} - h} \sum_{\substack{|u_j| \le X \\ 1 \le j < h}} \sum_{x \in I(\mathbf{u})} e\big(h! \beta u_1 u_2 \ldots u_{h-1} (x + \tfrac{1}{2} |\mathbf{u}|_1)\big). \tag{6.2}$$

By Hölder's inequality,

$$|W(\alpha, \mathbf{Y}; \mathbf{h})|^{2^{h_1 - 1}} \le (Y_2 \cdots Y_n)^{2^{h_1 - 1} - 1} \sum_{\substack{\frac{1}{2} Y_\nu < |y_\nu| \le Y_\nu \\ 2 \le \nu \le n}} \Big| \sum_{\frac{1}{2} Y_1 < |y_1| \le Y_1} e(\alpha y_1^{h_1} y_2^{h_2} \cdots y_n^{h_n}) \Big|^{2^{h_1 - 1}}.$$

We apply (6.2) with $\beta = \alpha y_2^{h_2} \cdots y_n^{h_n}$ to the sum over $y_1$. We write $\mathbf{h}' = (h_2, h_3, \ldots, h_n)$, $\mathbf{Y}' = (Y_2, Y_3, \ldots, Y_n)$ and then find that

$$|W(\alpha, \mathbf{Y}; \mathbf{h})|^{2^{h_1 - 1}} \ll Y_1^{2^{h_1 - 1} - h_1} \langle \mathbf{Y}' \rangle^{2^{h_1 - 1} - 1} \sum_{\substack{|u_j| \le Y_1 \\ 1 \le j < h_1}} \sum_{y \in I_1(\mathbf{u})} W(h_1! \alpha u_1 u_2 \cdots u_{h_1 - 1} (y + \tfrac{1}{2} |\mathbf{u}|_1), \mathbf{Y}'; \mathbf{h}'),$$

where $I_1(\mathbf{u})$ are certain subintervals of $[\frac{1}{2} Y_1, Y_1]$. Now, we apply Hölder's inequality again to bring in $|W(\beta, \mathbf{Y}'; \mathbf{h}')|^{2^{h_2 - 1}}$. We may then estimate the sum over $y_2$ by (6.2). Repeated use of this process produces the inequality

$$|W(\alpha, \mathbf{Y}; \mathbf{h})|^{2^{h_1 - 1} \ldots 2^{h_n - 1}} \ll \langle \mathbf{Y} \rangle^{2^{h_1 + \cdots + h_n - n}} \mathbf{Y}^{-\mathbf{h}} \sum_{\mathbf{u}_1, \ldots, \mathbf{u}_n} \sum_{\substack{y_\nu \in I_\nu(\mathbf{u}_\nu) \\ 1 \le \nu < n}} \Big| \sum_{y_n \in I_n(\mathbf{u}_n)} e(\alpha v y_n) \Big| \tag{6.3}$$

in which $\mathbf{u}_\nu \in \mathbb{Z}^{h_\nu - 1}$ runs over integer vectors with $|\mathbf{u}_\nu| \le Y_\nu$ for $1 \le \nu \le n$, the $I_\nu(\mathbf{u}_\nu)$ are certain subintervals of $[\frac{1}{2} Y_\nu, Y_\nu]$ and

$$v = h_1! h_2! \cdots h_n! \langle \mathbf{u}_1 \rangle \cdots \langle \mathbf{u}_n \rangle y_1 y_2 \cdots y_{n-1}.$$

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Note that $\nu = 0$ will occur in (6.3) only when one of the $\mathbf{u}_\nu$ has a zero entry so that the total contribution to (6.3) from summands with $\nu = 0$ does not exceed $\langle \mathbf{Y}\rangle^{2^{h_1+\cdots+h_n-n}} Y_n^{-1}$, which is acceptable. For nonzero $\nu$, the innermost sum in (6.3) does not exceed $\min(Y_n, \|\alpha\nu\|^{-1})$. Further, we have $\nu \ll \mathbf{Y}^{\mathbf{h}} Y_n^{-1}$, and a divisor function estimate shows that there are no more than $O(|\nu|^\varepsilon)$ choices for $\mathbf{u}_\nu$, $y_\nu$ that correspond to the same $\nu$. This shows that

$$|W(\alpha, \mathbf{Y}; \mathbf{h})|^{2^{h_1-1}\cdots 2^{h_n-1}} \ll \langle \mathbf{Y}\rangle^{2^{|\mathbf{h}|_1-n}} Y_n^{-1} + \langle \mathbf{Y}\rangle^{2^{|\mathbf{h}|_1-n}+\varepsilon} \mathbf{Y}^{-\mathbf{h}} \sum_{1\le \nu \ll \mathbf{Y}^{\mathbf{h}} Y_n^{-1}} \min(Y_n, \|\alpha\nu\|^{-1}).$$

Reference to [72, Lemma 2.1] completes the proof. □

We complement this result with an approximate evaluation of $W$.

**Lemma 6.2.** *Let $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\alpha = (a/q) + \beta$. Suppose that $Y_1 \ge Y_2 \ge \cdots \ge Y_n$. Then*

$$W(\alpha, \mathbf{Y}; \mathbf{h}) = E(q, a; \mathbf{h}) I(\beta, \mathbf{Y}; \mathbf{h}) + O\big(Y_1 Y_2 \cdots Y_{n-1} q (1 + \mathbf{Y}^{\mathbf{h}}|\beta|)\big).$$

*Proof.* The case $n = 1$ is a rough and elementary version of [72, Theorem 4.1]. We now induct on $n$ and suppose that the lemma is already available with $n - 1$ in place of $n$. As before, we write $\mathbf{Y}' = (Y_2, Y_3, \ldots, Y_n)$ and so on, isolate the sum over $y_1$ and invoke the induction hypothesis with $\alpha y_1^{h_1}$ for $\alpha$. This yields

$$\begin{aligned} W(\alpha, \mathbf{Y}; \mathbf{h}) &= \sum_{\frac12 Y_1 < |y_1| \le Y_1} \Big( E(q, a y_1^{h_1}; \mathbf{h}') I(\beta y_1^{h_1}, \mathbf{Y}'; \mathbf{h}') + O\big(Y_2 \cdots Y_{n-1} q (1 + \mathbf{Y}'^{\mathbf{h}'} |y_1|^{h_1} |\beta|)\big)\Big) \\ &= \sum_{\frac12 Y_1 < |y_1| \le Y_1} E(q, a y_1^{h_1}; \mathbf{h}') I(\beta y_1^{h_1}, \mathbf{Y}'; \mathbf{h}') + O\big(Y_1 Y_2 \cdots Y_{n-1} q (1 + \mathbf{Y}^{\mathbf{h}}|\beta|)\big). \end{aligned}$$

In view of (5.1) and (5.2), we may rewrite the sum over $y_1$ on the right-hand side as

$$q^{1-n} \sum_{\substack{1 \le x_\nu \le q \\ 2 \le \nu \le n}} \int_{\mathscr{Y}'} \sum_{\frac12 Y_1 < |y_1| \le Y_1} e\Big( y_1^{h_1} \Big( \beta \mathbf{y}'^{\mathbf{h}'} + \frac{a \mathbf{x}'^{\mathbf{h}'}}{q} \Big)\Big) \, \mathrm{d}\mathbf{y}',$$

where $\mathscr{Y}'$ is the analogue of $\mathscr{Y}$ in the coordinates $\mathbf{y}'$. We may now apply the case $n = 1$ with $\beta \mathbf{y}'^{\mathbf{h}'}$ for $\beta$ and $a\mathbf{x}'^{\mathbf{h}'}$ for $a$ to conclude that

$$\begin{aligned} &\sum_{\frac12 Y_1 < |y_1| \le Y_1} e\Big( y_1^{h_1} \Big( \beta \mathbf{y}'^{\mathbf{h}'} + \frac{a \mathbf{x}'^{\mathbf{h}'}}{q} \Big)\Big) \\ &\qquad = q^{-1} \sum_{x_1=1}^{q} e\Big( \frac{a x_1^{h_1} x'^{\mathbf{h}'}}{q} \Big) \int_{\frac12 Y_1 < |y_1| \le Y_1} e(\beta y_1^{h_1} \mathbf{y}'^{\mathbf{h}'}) \, \mathrm{d}y_1 + O\big(q + q Y_1^{h_1} |y'^{\mathbf{h}'} \beta|\big). \end{aligned}$$

The induction is now completed by inserting this last formula into the two preceding displays. □

### *6.2. Towards the circle method*

We are ready to embark on the proof of Proposition 5.2. We work in the broader framework of Hypothesis 5.1 in large parts of the argument but will restrict to the situation described in Proposition 5.2 whenever the bounds for Gauss sums are entering the argument. We hope that the wider scope of our presentation will be helpful in related investigations.

We begin with a general remark concerning the 'dummy variables' $x_{0j}$ that do not occur explicitly in the torsor equation. Suppose that Hypothesis 5.1 has been established for a given torsor equation,

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without any dummy variables, that is, with $J_0 = 0$. Now, consider the same torsor equation with $J_0 \geq 1$ dummy variables. For this new problem, the count $\mathcal{N}_{\mathbf{b}}(\mathbf{X})$ factorizes as $\mathcal{N}_{\mathbf{b}}(\mathbf{X}) = W_0(\mathbf{X}_0)\mathcal{N}^*$, say, where $\mathcal{N}^*$ is the number of solutions counted by $\mathcal{N}_{\mathbf{b}}(\mathbf{X})$ but with the variables $\mathbf{x}_0$ ignored, and $W_0(\mathbf{X}_0)$ is the number of $\mathbf{x}_0 \in \mathbb{Z}^{J_0}$ with $\frac{1}{2}X_{0j} < |x_{0j}| \leq X_{0j}$ for $1 \leq j \leq J_0$. A trivial lattice point count yields

$$W_0(\mathbf{X}_0) = \langle \mathbf{X}_0 \rangle + O(\langle \mathbf{X}_0 \rangle (\min X_{0j})^{-1}),$$

and if one multiplies this with the asymptotic formula for $\mathcal{N}^*$ that we have assumed to be available to us, then one derives the claims in Hypothesis 5.1 with dummy variables. This shows that it suffices to address the problem of verifying Hypothesis 5.1 only in the case where $J_0 = 0$, and we will assume this for the rest of this section.

To launch the circle method argument, recall the definition of $\mathcal{N}_{\mathbf{b}}(\mathbf{X})$ in the paragraph encapsulating displays (5.4)–(5.6). In the notation of that section, we define

$$W_i(\alpha, \mathbf{X}) = W(\alpha, \mathbf{X}_i; \mathbf{h}_i) \quad (1 \leq i \leq k).$$

By orthogonality,

$$\mathcal{N}_{\mathbf{b}}(\mathbf{X}) = \int_0^1 W_1(b_1\alpha, \mathbf{X}) \cdots W_k(b_k\alpha, \mathbf{X})\, \mathrm{d}\alpha.$$

Our main parameters are

$$Z = \min_{1 \leq i \leq k} \mathbf{X}_i^{\mathbf{h}_i}, \quad Z_0 = \max_{1 \leq i \leq k} \mathbf{X}_i^{\mathbf{h}_i}, \quad M = \min_{ij} X_{ij},$$

and we find it convenient to renumber variables to ensure that

$$X_{i1} \leq X_{i2} \leq \cdots \leq X_{iJ_i} \quad (1 \leq i \leq k). \tag{6.4}$$

Once and for all, fix positive numbers $\zeta_i$ as in (5.10), and the number $\omega$ defined by

$$\omega^{-1} = 40k \max_{1 \leq i \leq k} J_i |\mathbf{h}_i|. \tag{6.5}$$

In particular, we have $0 < \omega \leq 1/120$. Hence, the intervals $\mathfrak{M}(q, a)$, defined as the set of $\alpha \in \mathbb{R}$ with $|\alpha - (a/q)| \leq Z^{\omega-1}$, are disjoint as $a, q$ range over $1 \leq a \leq q \leq Z^{\omega}$, $(a, q) = 1$. The union of these intervals we denote by $\mathfrak{M}$. Let $\mathfrak{m} = [Z^{\omega-1}, 1 + Z^{\omega-1}] \setminus \mathfrak{M}$. On writing

$$\mathcal{N}_{\mathfrak{A}} = \int_{\mathfrak{A}} W_1(b_1\alpha, \mathbf{X}) \cdots W_k(b_k\alpha, \mathbf{X})\, \mathrm{d}\alpha$$

one has

$$\mathcal{N}_{\mathbf{b}}(\mathbf{X}) = \mathcal{N}_{\mathfrak{M}} + \mathcal{N}_{\mathfrak{m}}. \tag{6.6}$$

The circle method treatment depends on the relative size of $M$ and $Z$. We first give a proof of Proposition 5.2 in the case where $M \geq Z^{10k\omega}$ (the *tame* case).

### *6.3. The tame case: major arcs*

For $\alpha \in \mathfrak{M}$, there is a unique pair $a, q$ with $1 \leq a \leq q \leq Z^{\omega}$, $(a, q) = 1$ and a number $\beta \in \mathbb{R}$ with $|\beta| \leq Z^{\omega-1}$ and $\alpha = (a/q) + \beta$. By Lemma 6.2,

$$W_i(b_i\alpha, \mathbf{X}) = E_i(q, ab_i) I_i(\beta b_i, \mathbf{X}_i) + O(\langle \mathbf{X}_i^{\dagger} \rangle q (1 + \mathbf{X}_i^{\mathbf{h}_i} |b_i\beta|)), \tag{6.7}$$

where, temporarily, $\mathbf{X}_i^\dagger = (X_{i2}, \ldots, X_{iJ_i})$ is the vector that is $\mathbf{X}_i$ with its smallest entry deleted. Since we are in the tame case, this implies that $\langle \mathbf{X}_i^\dagger \rangle \leq \langle \mathbf{X}_i \rangle Z^{-10k\omega}$. Further, by hypothesis and (5.11), we have $\mathbf{X}_i^{\mathbf{h}_i} \leq Z_0 \leq Z^{1/(1-\lambda)}$. Now, since $\lambda \leq \omega/2$, it follows that $(1-\lambda)^{-1} \leq 1+\omega$, and therefore

$$\mathbf{X}_i^{\mathbf{h}_i} \leq Z_0 \leq Z^{1+\omega} \quad (1 \leq i \leq k). \tag{6.8}$$

We shall use these bounds frequently. Here, we apply (6.8) to obtain the estimate

$$W_i(b_i\alpha, \mathbf{X}) = E_i(q, ab_i) I_i(\beta b_i, \mathbf{X}_i) + O(\langle \mathbf{X}_i \rangle Z^{-9k\omega} |b_i|).$$

Noting the trivial bounds

$$W_i(b_i\alpha, \mathbf{X}) \ll \langle \mathbf{X}_i \rangle, \qquad E_i(q, ab_i) I_i(\beta b_i, \mathbf{X}_i) \ll \langle \mathbf{X}_i \rangle$$

and the identity

$$W_1 W_2 \cdots W_k - T_1 T_2 \cdots T_k = \sum_{i=1}^k (W_i - T_i) W_1 \cdots W_{i-1} T_{i+1} \cdots T_k,$$

we conclude that

$$\prod_{i=1}^k W_i(b_i\alpha, \mathbf{X}) = \prod_{i=1}^k E_i(q, ab_i) I_i(\beta b_i, \mathbf{X}_i) + O(\langle \mathbf{X}_1 \rangle \cdots \langle \mathbf{X}_k \rangle |\mathbf{b}|_1 Z^{-9k\omega}).$$

We integrate this over $\mathfrak{M}$. Since the measure of $\mathfrak{M}$ is $O(Z^{3\omega-1})$, the error will contribute an amount not exceeding

$$\langle \mathbf{X}_1 \rangle \cdots \langle \mathbf{X}_k \rangle |\mathbf{b}|_1 Z^{-8k\omega-1} \leq \langle \mathbf{X}_1 \rangle \cdots \langle \mathbf{X}_k \rangle |\mathbf{b}|_1 M^{-1/5} Z^{-6k\omega-1} \leq \langle \mathbf{X}_1 \rangle \cdots \langle \mathbf{X}_k \rangle |\mathbf{b}|_1 M^{-1/5} Z_0^{-1}.$$

It follows that

$$\mathcal{N}_{\mathfrak{M}} = \mathcal{E}_{\mathbf{b}}(Z^\omega) \mathcal{I}_{\mathbf{b}}(\mathbf{X}, Z^\omega) + O(\langle \mathbf{X}_1 \rangle \cdots \langle \mathbf{X}_k \rangle |\mathbf{b}|_1 M^{-1/5} Z_0^{-1}), \tag{6.9}$$

where

$$\mathcal{E}_{\mathbf{b}}(Q) = \sum_{q \leq Q} \sum_{a \bmod q}^{*} E_1(q, ab_1) E_2(q, ab_2) \cdots E_k(q, ab_k).$$

Note here that the error estimate in (6.9) is good enough to be absorbed in the error term in (5.12).

We are now required to complete the singular series. At this stage, we have to be content with the setup in Proposition 5.2, but then have recourse to (5.15), which provides us with the bound

$$\mathcal{E}_{\mathbf{b}}(Z^\omega) = \mathcal{E}_{\mathbf{b}} + O(Z^{-\omega/(2h)} |b_1 b_2 b_3|).$$

In combination with Lemma 5.3, we then infer that there is a number $\delta > 0$ with

$$\mathcal{E}_{\mathbf{b}}(Z^\omega) \mathcal{I}_{\mathbf{b}}(\mathbf{X}, Z^\omega) = \mathcal{E}_{\mathbf{b}} \mathcal{I}_{\mathbf{b}}(\mathbf{X}) + O(|b_1 b_2 b_3| Z^{-\omega\delta} \langle \mathbf{X}_1 \rangle \langle \mathbf{X}_2 \rangle \langle \mathbf{X}_3 \rangle \mathbf{X}_1^{-\zeta_1 \mathbf{h}_1} \mathbf{X}_2^{-\zeta_2 \mathbf{h}_2} \mathbf{X}_3^{-\zeta_3 \mathbf{h}_3}).$$

It follows that in the tame case, there is indeed a number $\delta_1 > 0$ such that

$$\mathcal{N}_{\mathfrak{M}} = \mathcal{E}_{\mathbf{b}} \mathcal{I}_{\mathbf{b}}(\mathbf{X}) + O(|b_1 b_2 b_3| M^{-\delta_1} \langle \mathbf{X}_1 \rangle \langle \mathbf{X}_2 \rangle \langle \mathbf{X}_3 \rangle \mathbf{X}_1^{-\zeta_1 \mathbf{h}_1} \mathbf{X}_2^{-\zeta_2 \mathbf{h}_2} \mathbf{X}_3^{-\zeta_3 \mathbf{h}_3}). \tag{6.10}$$

### *6.4. The tame case: minor arcs*

In our treatment of the minor arcs, we again work subject to the conditions in Proposition 5.2. There are two cases.

First, suppose that $|b_3| \le Z^{\omega/2}$. We apply Weyl's inequality to $W_3(b_3\alpha, \mathbf{X})$. Let

$$H = 2^{h_{31}+\cdots+h_{3J_3}-J_3}.$$

We claim that uniformly for $\alpha \in \mathfrak{m}$, one has

$$W_3(b_3\alpha, \mathbf{X}) \ll \langle \mathbf{X}_3\rangle Z^{-\omega/(3H)}. \tag{6.11}$$

Indeed, if $Z$ is large and $\alpha \in \mathbb{R}$ is such that $|W_3(b_3\alpha, \mathbf{X})| \ge \langle \mathbf{X}_3\rangle Z^{-\omega/(3H)}$, then a familiar coupling of Lemma 6.1 with Dirichlet's theorem on diophantine approximation shows that there are coprime numbers $a, q$ with $|qb_3\alpha - a| \le Z^{\omega/2}\mathbf{X}_3^{-\mathbf{h}_3} \le Z^{(\omega/2)-1}$ and $1 \le q \le Z^{\omega/2}$. But then $1 \le |b_3|q \le Z^{\omega}$, and hence $\alpha$ cannot be in $\mathfrak{m}$.

By (6.1) and an obvious substitution,

$$\int_0^1 |W_i(b_i\alpha, \mathbf{X})|^2 \,\mathrm{d}\alpha \ll \langle \mathbf{X}_i\rangle^{1+\varepsilon}.$$

Hence, by Schwarz's inequality and (6.11),

$$\mathscr{N}_{\mathfrak{m}} \ll \big(\langle \mathbf{X}_1\rangle\langle \mathbf{X}_2\rangle\big)^{1/2+\varepsilon} \sup_{\alpha\in\mathfrak{m}} |W_3(b_3\alpha, \mathbf{X})| \ll \langle \mathbf{X}_1\rangle\langle \mathbf{X}_2\rangle\langle \mathbf{X}_3\rangle Z^{\varepsilon-1-\omega/(3H)}.$$

We have $\lambda \le \omega/(12H)$, and so

$$(1-\lambda)(1+\omega/(3H)) \ge 1+\omega/(6H). \tag{6.12}$$

Hence, $Z^{-1-(\omega/3H)} \ll Z_0^{-1-\omega/(6H)}$, which shows that $\mathscr{N}_{\mathfrak{m}}$ is an acceptable error in Proposition 5.2. This combines with (6.6) to complete the proof of Proposition 5.2 in the case under consideration.

Next, consider the case where $|b_3| > Z^{\omega/2}$. Here the claim in Proposition 5.2 reduces to a trivial upper bound, as we now explain. The triangle inequality give $|W_i(\alpha)| \le \langle \mathbf{X}_i\rangle$, and therefore, the integral representation of $\mathscr{N}_{\mathbf{b}}(\mathbf{X})$ gives $\mathscr{N}_{\mathbf{b}}(\mathbf{X}) \le \langle \mathbf{X}_1\rangle\langle \mathbf{X}_2\rangle\langle \mathbf{X}_3\rangle$. Similarly, on combing (5.16) with Lemma 5.3, we have the crude bound

$$\mathscr{E}_{\mathbf{b}}\mathscr{I}_{\mathbf{b}}(\mathbf{X}) \ll |b_1b_2b_3|^{1/2}\langle \mathbf{X}_1\rangle\langle \mathbf{X}_2\rangle\langle \mathbf{X}_3\rangle.$$

We take $C = 300/\omega$ in (5.12). Then $|b_3|^C \ge Z^{150}$, and so

$$|b_1b_2b_3|^{1/2}\langle \mathbf{X}_1\rangle\langle \mathbf{X}_2\rangle\langle \mathbf{X}_3\rangle \le |b_1b_2b_3|^C Z_0^{-2}$$

which is more than is required to confirm (5.12) in this final case. It should be noted that the discussion of the case $|b_3| > Z^{\omega/2}$ did not use that we are in the tame case, but applies in general. Also, we have now completed the proof of Proposition 5.2 in the tame case.

### *6.5. Major arcs again*

It remains to deal with the case where $M < Z^{10k\omega}$. We assume this inequality from now on. Again, we work in the broader framework of Sections 6.2 and 6.3 and refine the circle method approach to cover the current situation as well. We say that a variable $x_{ij}$ is small if $X_{ij} < Z^{10k\omega}$. By hypothesis, there is

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at least one small variable. Also, by (6.4), there is a number $J_i'$ such that the $x_{ij}$ with $j \le J_i'$ are small, and those with $j > J_i'$ are not. We proceed to show that

$$\prod_{j \le J_i'} X_{ij} \le \langle \mathbf{X}_i \rangle^{1/4}. \tag{6.13}$$

To see this, note that the definition of $J_i'$ gives

$$\prod_{j \le J_i'} X_{ij} \le Z^{10k\omega J_i'} \le Z^{10k\omega J_i}. \tag{6.14}$$

But $Z \le \mathbf{X}_i^{\mathbf{h}_i} \le \langle \mathbf{X}_i \rangle^{|\mathbf{h}_i|}$. We insert this in the previous display and apply the inequality

$$10k\omega J_i |\mathbf{h}_i| \le \frac{1}{4}$$

(which is immediate from (6.5)) to derive (6.13).

The significance of (6.13) is that it implies that for each $i$, there are variables $x_{ij}$ that are not small. This is important throughout this section. We put

$$\mathbf{X}_i' = (X_{i1}, \dots, X_{iJ_i'}), \quad \mathbf{X}_i'' = (X_{i,J_i'+1}, \dots, X_{iJ_i}), \quad \mathbf{X}_i = (\mathbf{X}_i', \mathbf{X}_i''),$$

where $\mathbf{X}_i'$ is void if $x_{i1}$ is not small. In the same way, we dissect the variable $\mathbf{x}_i = (\mathbf{x}_i', \mathbf{x}_i'')$ and the chain of exponents $\mathbf{h}_i = (\mathbf{h}_i', \mathbf{h}_i'')$. By orthogonality, we then have

$$\mathscr{N}_{\mathbf{b}}(\mathbf{X}) = \sum_{(\mathbf{x}_1', \dots, \mathbf{x}_k') \in \mathscr{Y}' \cap \mathbb{Z}^{J'}} \int_0^1 W(b_1 \alpha \mathbf{x}_1'^{\mathbf{h}_1'}, \mathbf{X}_1''; \mathbf{h}_1'') \dots W(b_k \alpha \mathbf{x}_k'^{\mathbf{h}_k'}, \mathbf{X}_k''; \mathbf{h}_k'')\, \mathrm{d}\alpha, \tag{6.15}$$

where $J' = J_1' + \dots + J_k'$ and

$$\mathscr{Y}' := \{\mathbf{x}' \in \mathbb{R}^{J'} : \tfrac{1}{2} X_{ij} < |x_{ij}| \le X_{ij} \text{ for } 1 \le i \le k, 1 \le j \le J_i'\}. \tag{6.16}$$

We apply the circle method to the integral in (6.15). By Lemma 6.2, when $\alpha = (a/q) + \beta$, one finds that subject to (6.16), one has

$$W\big(b_i \alpha \mathbf{x}_i'^{\mathbf{h}_i'}, \mathbf{X}_i''; \mathbf{h}_i''\big) = E\big(q, ab_i \mathbf{x}_i'^{\mathbf{h}_i'}; \mathbf{h}_i''\big) I\big(\beta b_i \mathbf{x}_i'^{\mathbf{h}_i'}, \mathbf{X}_i''; \mathbf{h}_i''\big) + O\big(\langle \mathbf{X}_i'' \rangle Z^{-10k\omega} q (1 + |b_i \beta| \mathbf{X}_i^{\mathbf{h}_i})\big).$$

Here, it is worth recalling that $\mathbf{X}_i''$ is not void and has all its components at least as large as $Z^{10k\omega}$. We now apply (6.8) to confirm that for $\alpha \in \mathfrak{M}$, the error in the preceding display does not exceed

$$\langle \mathbf{X}_i'' \rangle Z^{\omega - 10k\omega} + \langle \mathbf{X}_i'' \rangle Z^{-10k\omega} |b_i| Z^{2\omega - 1} \mathbf{X}_i^{\mathbf{h}_i} \le \langle \mathbf{X}_i'' \rangle |b_i| Z^{3\omega - 10k\omega} \le \langle \mathbf{X}_i'' \rangle |b_i| Z^{-9k\omega}.$$

Let $\mathsf{S}$ denote the integrand in (6.15), and let $\mathsf{M}$ denote the product of the expressions

$$E(q, ab_i \mathbf{x}_i'^{\mathbf{h}_i'}, \mathbf{h}_i'') I(\beta b_i \mathbf{x}_i'^{\mathbf{h}_i'}, \mathbf{X}_i''; \mathbf{h}_i''),$$

with $1 \le i \le k$. Then, following the discussion in the initial part of Section 6.3, we obtain

$$\mathsf{S} - \mathsf{M} \ll \langle \mathbf{X}_1'' \rangle \cdots \langle \mathbf{X}_k'' \rangle |\mathbf{b}|_1 Z^{-9k\omega}. \tag{6.17}$$

We integrate over $\mathfrak{M}$ and sum over the integral points in $\mathscr{Y}'$. Then, again as in Section 6.3, this gives

$$\mathscr{N}_{\mathbf{b}}(\mathbf{X}) = \sum_{(\mathbf{x}'_1,\dots,\mathbf{x}'_k)\in\mathscr{Y}'\cap\mathbb{Z}^{J'}} \mathscr{E}'\mathscr{I}' + \mathscr{N}^{\dagger} + O(\langle\mathbf{X}_1\rangle\cdots\langle\mathbf{X}_k\rangle|\mathbf{b}|_1 Z^{-8k\omega-1}), \tag{6.18}$$

where

$$\mathscr{E}' = \sum_{q\le Z^{\omega}}\ \sideset{}{^*}\sum_{a \bmod q} E(q, ab_1\mathbf{x}_1'^{\mathbf{h}_1'}, \mathbf{h}_1'')\cdots E(q, ab_k\mathbf{x}_k'^{\mathbf{h}_k'}, \mathbf{h}_k''), \tag{6.19}$$

$$\mathscr{I}' = \int_{-Z^{\omega-1}}^{Z^{\omega-1}} I(\beta b_1\mathbf{x}_1'^{\mathbf{h}_1'}, \mathbf{X}_1''; \mathbf{h}_1'')\cdots I(\beta b_k\mathbf{x}_k'^{\mathbf{h}_k'}, \mathbf{X}_k''; \mathbf{h}_k'')\,\mathrm{d}\beta,$$

and where $\mathscr{N}^{\dagger}$ is the same expression as in (6.15) but with integration over the minor arcs $\mathfrak{m}$. Exchanging the sum with the integral in (6.15), we see that $\mathscr{N}^{\dagger} = \mathscr{N}_{\mathfrak{m}}$. Note that the error in (6.18) also occurred in Section 6.3 and, in the display preceding (6.9), was shown to be of acceptable size.

The difficulty now is that the moduli $q$ in (6.19) are too large for the small variables to be arranged in residue classes modulo $q$. We therefore prune the sum over $q$. In preparation for this manoeuvre, we bound $\mathscr{I}'$ uniformly in $\mathbf{x}_i'$. Whenever $\mathbf{x}_i' \in \mathscr{Y}'$, one finds from (5.3) that

$$I(\beta b_i\mathbf{x}_i'^{\mathbf{h}_i'}, \mathbf{X}_i''; \mathbf{h}_i'') \ll \langle\mathbf{X}_i''\rangle(1 + \mathbf{X}_i''^{\mathbf{h}_i''}|\mathbf{x}_i'^{\mathbf{h}_i'} b_i\beta|)^{-1} \ll \langle\mathbf{X}_i''\rangle(1 + \mathbf{X}_i^{\mathbf{h}_i}|b_i\beta|)^{-1}.$$

Hence, by Hölder's inequality,

$$\mathscr{I}' \ll \prod_{i=1}^{k}\langle\mathbf{X}_i''\rangle\Big(\int_{-\infty}^{\infty}(1 + \mathbf{X}_i^{\mathbf{h}_i}|b_i\beta|)^{-1/\zeta_i}\,\mathrm{d}\beta\Big)^{\zeta_i} \ll \prod_{i=1}^{k}\langle\mathbf{X}_i''\rangle\mathbf{X}_i^{-\zeta_i\mathbf{h}_i}. \tag{6.20}$$

Now, let $\mathscr{E}^{\dagger}$ be the portion of the sum defining $\mathscr{E}$ where $q \le M^{1/8}$, and let $\mathscr{E}^{\ddagger}$ be the portion with $M^{1/8} < q \le Z^{\omega}$. Then $\mathscr{E}' = \mathscr{E}^{\dagger} + \mathscr{E}^{\ddagger}$, and (6.19) and (6.20) yield

$$\sum_{(\mathbf{x}'_1,\dots,\mathbf{x}'_k)\in\mathscr{Y}'} \mathscr{E}^{\ddagger}\mathscr{I}' \ll \Big(\prod_{i=1}^{k}\langle\mathbf{X}_i''\rangle\mathbf{X}_i^{-\zeta_i\mathbf{h}_i}\Big)\sum_{M^{1/8}<q<Z^{\omega}}\ \sum_{(\mathbf{x}'_1,\dots,\mathbf{x}'_k)\in\mathscr{Y}'}\Big|\sideset{}{^*}\sum_{a \bmod q}\prod_{i=1}^{k}E(q, ab_i\mathbf{x}_i'^{\mathbf{h}_i'}; \mathbf{h}_i'')\Big|. \tag{6.21}$$

At this point, we require a workable upper bound for the innermost sum. In the situation of Proposition 5.2, we have $k = 3$, and such a bound is provided by (5.15). With $h = \max h_{3j}$, this yields

$$\sideset{}{^*}\sum_{a \bmod q}\prod_{i=1}^{3}E(q, ab_i\mathbf{x}_i'^{\mathbf{h}_i'}; \mathbf{h}_i'') \ll \frac{(q, b_1\langle\mathbf{x}_1'\rangle)(q, b_2\langle\mathbf{x}_2'\rangle)(q, b_3\mathbf{x}_3'^{\mathbf{h}_3'})^{1/h}}{q^{1+1/h}}. \tag{6.22}$$

Now, $(q, b_1\langle\mathbf{x}_1'\rangle) \le |b_1|(q, x_{11})\cdots(q, x_{1J_1'})$ and likewise for $(q, b_2\langle\mathbf{x}_2'\rangle)$. Similarly,

$$(q, b_3\mathbf{x}_1'^{\mathbf{h}_3'})^{1/h} \le |b_3|(q, x_{31}^{h_{31}})^{1/h}\cdots(q, x_{3J_3'}^{h_{3J_3'}})^{1/h} \le |b_3|(q, x_{31})\cdots(q, x_{3J_3'}).$$

We may sum (6.22) over $\mathbf{x}_i' \in \mathscr{Y}'$, using the simple bound

$$\sum_{x\le X}(q, x) \ll q^{\varepsilon}X.$$

It then follows that the right-hand side of (6.21) does not exceed

$$\ll \Big(\prod_{i=1}^{3}|b_i|\langle\mathbf{X}_i'\rangle\langle\mathbf{X}_i''\rangle\mathbf{X}_i^{-\zeta_i\mathbf{h}_i}\Big)\sum_{M^{1/8}<q<Z^{\omega}} q^{\varepsilon-1-1/h} \ll M^{-1/(9h)}|b_1b_2b_3|\prod_{i=1}^{3}\langle\mathbf{X}_i\rangle\mathbf{X}_i^{-\zeta_i\mathbf{h}_i}. \tag{6.23}$$

In the specific situation of Proposition 5.2, this is an acceptable error term.

We now turn to the product $\mathscr{E}^{\dagger}\mathscr{I}'$. Here, we prune the range of integration. Let

$$\mathscr{I}^{\dagger} = \int_{-M^{1/8}Z_0^{-1}}^{M^{1/8}Z_0^{-1}} I(\beta b_1\mathbf{x}_1'^{\mathbf{h}_1'},\mathbf{X}_1'';\mathbf{h}_1'')\cdots I(\beta b_k\mathbf{x}_k'^{\mathbf{h}_k'},\mathbf{X}_k'';\mathbf{h}_k'')\,\mathrm{d}\beta,$$

and let $\mathscr{I}^{\ddagger}$ be the complementary integral over $M^{1/8}Z_0^{-1} < |\beta| \le Z^{\omega-1}$ so that $\mathscr{I}' = \mathscr{I}^{\dagger} + \mathscr{I}^{\ddagger}$. To obtain an upper bound for $\mathscr{I}^{\ddagger}$, choose an index $\iota$ with $Z_0 = \mathbf{X}_\iota^{\mathbf{h}_\iota}$. Then

$$\int_{M^{1/8}Z_0^{-1}}^{\infty}(1+\mathbf{X}_\iota^{\mathbf{h}_\iota}|b_\iota\beta|)^{-1/\zeta_\iota}\,\mathrm{d}\beta \ll \mathbf{X}_\iota^{-\mathbf{h}_\iota}M^{(\zeta_\iota-1)/8},$$

and since $\zeta_\iota < 1$, we observe that the exponent of $M$ is negative. With this adjustment, the argument in (6.20) shows that uniformly for $\mathbf{x}_i' \in \mathscr{Y}'$ one has

$$\mathscr{I}^{\ddagger} \ll M^{(\zeta_\iota-1)\zeta_\iota/8}\prod_{i=1}^{k}\langle\mathbf{X}_i''\rangle\mathbf{X}_i^{-\zeta_i\mathbf{h}_i}. \tag{6.24}$$

We can now imitate the argument from (6.21)–(6.23), this time applying (6.24) and summing over $q \le M^{1/8}$. In the cases covered by Proposition 5.2, this yields

$$\sum_{(\mathbf{x}_1',\ldots,\mathbf{x}_3')\in\mathscr{Y}'}\mathscr{E}^{\dagger}\mathscr{I}^{\ddagger} \ll M^{(\zeta_\iota-1)\zeta_\iota/9}|b_1b_2b_3|\prod_{i=1}^{3}\langle\mathbf{X}_i\rangle\mathbf{X}_i^{-\zeta_i\mathbf{h}_i},$$

which can be absorbed in the error term when $\delta_1 < \frac{1}{9}\min(1-\zeta_i)\zeta_i$. On collecting together, we deduce from (6.18) and the discussion above that

$$\mathscr{N}_{\mathbf{b}}(\mathbf{X}) = \sum_{(\mathbf{x}_1',\ldots,\mathbf{x}_k')\in\mathscr{Y}'}\mathscr{E}^{\dagger}\mathscr{I}^{\dagger} + \mathscr{N}_{\mathfrak{m}} + O(F), \tag{6.25}$$

where $F$ is an acceptable error provided that $C > 1$ and $\delta_1$ is small enough.

It would now be possible to exchange the sums over $\mathbf{x}_i'$ with the summations present in the definition of $\mathscr{E}^{\dagger}$ and to evaluate these sums by arranging the $x_{ij}$ in arithmetic progressions, as suggested earlier. However, we prefer an indirect argument that is technically simpler. Let $\mathfrak{N}$ denote the union of the pairwise disjoint intervals $|\alpha-(a/q)| \le M^{1/8}Z_0^{-1}$ with $1 \le a \le q \le M^{1/8}$ and $(a,q)=1$. Observe that $\mathfrak{N} \subset \mathfrak{M}$. Hence, integrating (6.17) over $\mathfrak{N}$ we find that

$$\sum_{(\mathbf{x}_1',\ldots,\mathbf{x}_k')\in\mathscr{Y}'}\int_{\mathfrak{N}} W(b_1\alpha\mathbf{x}_1'^{\mathbf{h}_1'},\mathbf{X}_1'';\mathbf{h}_1'')\cdots W(b_k\alpha\mathbf{x}_k'^{\mathbf{h}_k'},\mathbf{X}_k'';\mathbf{h}_k'')\,\mathrm{d}\alpha = \sum_{(\mathbf{x}_1',\ldots,\mathbf{x}_k')\in\mathscr{Y}'}\mathscr{E}^{\dagger}\mathscr{I}^{\dagger} + O(F'), \tag{6.26}$$

where $F'$ is an error that certainly does not exceed the error present in (6.18) because the measure of $\mathfrak{N}$ is smaller than that of $\mathfrak{M}$. Exchanging sum and integral, it transpires that the left-hand side of (6.26) is

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simply the major arc contribution $\mathscr{N}_{\mathfrak{N}}$. To evaluate the latter, we can run an argument from Section 6.3 with $\mathfrak{N}$ in place of $\mathfrak{M}$. The bound (6.7) becomes

$$W_i(b_i\alpha, \mathbf{X}) = E_i(q, ab_i)I_i(\beta b_i, \mathbf{X}_i) + O(\langle \mathbf{X}_i\rangle M^{-3/4}|b_i\beta|),$$

and then the result in (6.9) changes to

$$\mathscr{N}_{\mathfrak{N}} = \mathscr{E}_{\mathbf{b}}(M^{1/8})\mathscr{I}_{\mathbf{b}}(\mathbf{X}, M^{1/8}) + O(\langle \mathbf{X}_1\rangle \cdots \langle \mathbf{X}_k\rangle |\mathbf{b}|_1 M^{-3/8}Z_0^{-1}).$$

We can now complete the singular series and the singular integral as in Section 6.3. The argument that produced (6.10) now delivers exactly the same asymptotics for $\mathscr{N}_{\mathfrak{N}}$. Via (6.25) and (6.26), it follows that $\mathscr{N}_{\mathbf{b}}(\mathbf{X}) = \mathscr{E}_{\mathbf{b}}\mathscr{I}_{\mathbf{b}}(\mathbf{X}) + \mathscr{N}_{\mathfrak{m}} + O(F'')$, where $F''$ is an error acceptable to Hypothesis 5.1. Consequently, it remains to estimate the contribution from the minor arcs.

### 6.6. *Minor arcs again*

The argument of Section 6.4 yields an acceptable bound for $\mathscr{N}_{\mathfrak{m}}$ provided that the estimate (6.11) remains valid in cases that are not tame. Hence, we now complete the proof of Proposition 5.2 by showing that indeed (6.11) holds in the wider context, uniformly for $\alpha \in \mathfrak{m}$ and $1 \le |b_3| \le Z^{\omega/2}$. In doing so, we may suppose that $x_{31}$ is small, for otherwise our previous argument leading to (6.11) still applies. We write

$$T(\alpha, \mathbf{x}_3') = W(b_3\alpha \mathbf{x}_3'^{\mathbf{h}_3'}, \mathbf{X}_3''; \mathbf{h}_3'').$$

Then

$$W_3(b_3\alpha, \mathbf{X}) = \sum_{\mathbf{x}_3'} T(\alpha, \mathbf{x}_3'),$$

with the sum extending over $\frac{1}{2}X_{3j} \le |x_{3j}| \le X_{3j}$ $(1 \le j \le J_3')$.

We apply Weyl's inequality to $T(\alpha, \mathbf{x}_3')$. Let $K = 2^{|\mathbf{h}_3''|_1 - J_3 + J_3'}$, and note that all entries in $\mathbf{X}_3''$ are at least as large as $Z^\omega$. Hence, whenever the real number $\gamma$ and $c \in \mathbb{Z}$ and $t \in \mathbb{N}$ are such that $|t\gamma - c| \le t^{-1}$, then by Lemma 6.1, one has

$$|W(\gamma, \mathbf{X}_3''; \mathbf{h}_3'')|^K \ll \langle \mathbf{X}_3''\rangle^{K+\varepsilon}\Big(\frac{1}{t} + \frac{1}{Z^\omega} + \frac{t}{\mathbf{X}_3''^{\mathbf{h}_3''}}\Big). \tag{6.27}$$

By Dirichlet's theorem on diophantine approximation, there are $c$ and $t$ with $t \le Z^{-\omega}\mathbf{X}_3''^{\mathbf{h}_3''}$ and $|t\gamma - c| \le Z^\omega \mathbf{X}_3''^{-\mathbf{h}_3''}$. Then, on applying a familiar transference principle (see [72, Exercise 2.8.2]) to (6.27), we find that

$$|W(\gamma, \mathbf{X}_3''; \mathbf{h}_3'')|^K \ll \langle \mathbf{X}_3''\rangle^{K+\varepsilon}\Big(\frac{1}{Z^\omega} + \frac{1}{t + \mathbf{X}_3''^{\mathbf{h}_3''}|t\gamma - c|}\Big).$$

Since there is a variable that is not small, we have $K < H$, and hence that $K \le H/2$. Consequently, for a given $\mathbf{x}_3'$, we either have $T(\alpha, \mathbf{x}_3') \ll \langle \mathbf{X}_3''\rangle Z^{-\omega/(3H)}$ or there are $t = t(\mathbf{x}_3')$ and $c = c(\mathbf{x}_3')$ with $t \le Z^{\omega/3}$ and

$$\Big|b_3\alpha \mathbf{x}_3'^{\mathbf{h}_3'} - \frac{c}{t}\Big| \le \frac{Z^{\omega/3}}{t\mathbf{X}_3''^{\mathbf{h}_3''}}. \tag{6.28}$$

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Let $\mathscr{X}$ be the set of all $\mathbf{x}_3'$ where the latter case occurs. Then

$$W_3(b_3\alpha, \mathbf{X}) \ll \langle \mathbf{X}_3 \rangle Z^{-\omega/(3H)} + \langle \mathbf{X}_3'' \rangle \sum_{\mathbf{x}_3' \in \mathscr{X}} \left(t + \mathbf{X}_3''^{\mathbf{h}_3''} |tb_3\alpha \mathbf{x}_3'^{\mathbf{h}_3'} - c|\right)^{-1/H}. \tag{6.29}$$

We write $Q = \mathbf{X}_3'^{\mathbf{h}_3'} Z^{\omega}$ and apply Dirichlet's theorem again to find coprime numbers $a$, $q$ with $1 \leq q \leq Q$ and $|qb_3\alpha - a| \leq Q^{-1}$. On comparing this approximation to $b_3\alpha$ with that given by (6.28), we find that whenever $\mathbf{x}_3' \in \mathscr{X}$, then

$$|at\mathbf{x}_3'^{\mathbf{h}_3'} - cq| \leq QZ^{\omega/3}\mathbf{X}_3''^{-\mathbf{h}_3''} + Q^{-1}t\mathbf{X}_3'^{\mathbf{h}_3'}. \tag{6.30}$$

But $t \leq Z^{\omega/3}$, and therefore, the second summand on the right does not exceed $Z^{-\omega/2}$. For the first summand, we note that

$$QZ^{\omega/3}\mathbf{X}_3''^{-\mathbf{h}_3''} = Z^{4\omega/3}\mathbf{X}_3'^{2\mathbf{h}_3'}\mathbf{X}_3^{-\mathbf{h}_3} \leq Z^{4\omega/3-1}\mathbf{X}_3'^{2\mathbf{h}_3'}. \tag{6.31}$$

Further, by (6.14), we have $\langle \mathbf{X}_3' \rangle \leq Z^{10k\omega J_3}$, and hence that $\mathbf{X}_3'^{2\mathbf{h}_3'} \leq \langle \mathbf{X}_3' \rangle^{2|\mathbf{h}|} \leq Z^{20k\omega J_3|\mathbf{h}_3|}$. However, it is immediate from (6.5) that

$$\frac{4}{3}\omega + 20k\omega J_3|\mathbf{h}_3| < 1,$$

so that the expression in (6.31) tends to zero as $Z \to \infty$. By (6.30), we see that for large $Z$ we must have $at\mathbf{x}_3'^{\mathbf{h}_3'} = cq$. Hence, $t = q/(q, \mathbf{x}_3'^{\mathbf{h}_3'})$, and (6.29) simplifies to

$$W_3(b_3\alpha, \mathbf{X}) \ll \langle \mathbf{X}_3 \rangle Z^{-\omega/(3H)} + \langle \mathbf{X}_3'' \rangle \sum_{\mathbf{x}_3' \in \mathscr{X}} (q, \mathbf{x}_3'^{\mathbf{h}_3'})^{1/H} \left(q + \mathbf{X}_3^{-\mathbf{h}_3} |b_3||qb_3\alpha - a|\right)^{-1/H}.$$

Here, we can sum over *all* $\mathbf{x}_3'$ and apply an argument paralleling that leading from (6.22) to (6.23). This produces

$$W_3(b_3\alpha, \mathbf{X}) \ll \langle \mathbf{X}_3 \rangle Z^{-\omega/(3H)} + \langle \mathbf{X}_3 \rangle q^{\varepsilon} \left(q + \mathbf{X}_3^{-\mathbf{h}_3} |qb_3\alpha - a|\right)^{-1/H}.$$

The bound (6.11) is now evident, and the proof of Proposition 5.2 is complete.

## 7. Upper bound estimates

### *7.1. The upper bound hypothesis*

As we mentioned in the introduction, not only asymptotic information of the type encoded in Hypothesis 5.1 is required as an input for the transition method in Section 8, but also certain upper bound estimates that are needed, for example, to handle the contribution to the count that comes from solutions of (1.2) where the summands are very unbalanced. Again, we formulate the requirements as a hypothesis that can then be checked in the particular cases at hand. We recall the definition of the block matrix

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_1 & \mathscr{A}_2 \\ \mathscr{A}_3 & \mathscr{A}_4 \end{pmatrix} \in \mathbb{R}^{(J+1)\times(N+k)} \tag{7.1}$$

in (3.10). In the slightly simpler setup of the torsor equation (1.2) and the height conditions (1.3) we have

$$\mathscr{A}_1 = (\alpha_{ij}^{\nu}) \in \mathbb{R}_{\geq 0}^{J\times N} \tag{7.2}$$

with $0 \le i \le k$, $1 \le j \le J_i$, $1 \le \nu \le N$ and

$$\mathscr{A}_2 = (e^{\mu}_{ij}) \in \mathbb{R}^{J\times k} \text{ with} e^{\mu}_{ij} = \begin{cases} \delta_{\mu=i} h_{ij} & i < k, \mu < k, \\ -h_{kj} & i = k, \mu < k, \\ -1 & i < k, \mu = k, \\ h_{kj} - 1 & i = k, \mu = k. \end{cases} \tag{7.3}$$

This notation is more convenient for the analytic manipulations in the following sections.

Throughout, we assume that

$$\mathrm{rk}(\mathscr{A}_1) = \mathrm{rk}(\mathscr{A}) = R \quad \text{(say)}. \tag{7.4}$$

In our applications, this will be satisfied by Lemma 3.10, and $R$ plays by Lemma 4.7 the same role as in (4.9). We define

$$c_2 = J - R \tag{7.5}$$

so that by (4.9) this choice of $c_2$ is the expected exponent in (1.5). For any vector $\zeta$ satisfying the properties specified in (5.10), where we allow more generally also $\zeta_i \ge 0$, and for arbitrary $\zeta_0 > 0$, we also assume that the system of $J + 1$ linear equations

$$\begin{pmatrix} \mathscr{A}_1 \\ \mathscr{A}_3 \end{pmatrix} \sigma = \Big(1 - h_{01}\zeta_0, \dots, 1 - h_{kJ_k}\zeta_k, 1\Big)^{\top} \tag{7.6}$$

in $N$ variables has a solution $\sigma \in \mathbb{R}^N_{>0}$. In our applications, this is ensured by Lemma 3.11 (whose proof also works for $\zeta_i \ge 0$).

**Remark 7.1.** The condition $\mathrm{rk}\,\mathscr{A} = \mathrm{rk}\,\mathscr{A}_1$ puts some restrictions on the height matrix $\mathscr{A}_1$. For instance, no row of $\mathscr{A}_1$ can vanish completely (since every column of $\mathscr{A}_2$ is linearly dependent on the columns of $\mathscr{A}_1$). For future reference, we remark that this implies that the set of conditions (1.3) for $x_{ij} \in \mathbb{Z} \setminus \{0\}$ implies $|x_{ij}| \le B$ for all $(i, j)$.

Now, let $H \ge 1, 0 < \lambda \le 1$ and $\mathbf{b}, \mathbf{y} \in \mathbb{N}^J$. Let $N_{\mathbf{b},\mathbf{y}}(B, H, \lambda)$ be the number of solutions $\mathbf{x} \in (\mathbb{Z}\setminus\{0\})^J$ satisfying the conditions

$$\sum_{i=1}^{k} \prod_{j=1}^{J_i} (b_{ij}x_{ij})^{h_{ij}} = 0, \quad \prod_{i=0}^{k} \prod_{j=1}^{J_i} |y_{ij}x_{ij}|^{\alpha^{\nu}_{ij}} \le B \quad (1 \le \nu \le N), \tag{7.7}$$

and at least one of the inequalities

$$\min_{ij} |x_{ij}| \le H, \qquad \min_{1\le i\le k} \prod_{j=1}^{J_i} |x_{ij}|^{h_{ij}} < \Big( \max_{1\le i\le k} \prod_{j=1}^{J_i} |2x_{ij}|^{h_{ij}} \Big)^{1-\lambda}. \tag{7.8}$$

Note that for $\mathbf{x} \in (\mathbb{Z} \setminus \{0\})^J$ satisfying (7.7), the first condition in (7.8) is always satisfied for $H = B$ and the second condition in (7.8) is never satisfied for $\lambda = 1$. Let $\mathscr{S}_{\mathbf{y}}(B, H, \lambda)$ denote the set of all $\mathbf{x} \in [1,\infty)^J$ that satisfy (7.8) and the $N$ inequalities in the second part of (7.7). As in (1.4), we denote by $S_\rho$, $1 \le \rho \le r$, subsets of the set of pairs $(i, j)$ with $0 \le i \le k$, $1 \le j \le J_i$ corresponding to the coprimality conditions.

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**Hypothesis 7.2.** *Let $c_2$ be the number introduced in (7.5), and let $\lambda$ be as in Hypothesis 5.1. Suppose that there exist $\eta = (\eta_{ij}) \in \mathbb{R}^J_{>0}$ and $\delta_2, \delta_2^* > 0$ with the following properties:*

$$C_1(\eta): \quad \sum_{(i,j)\in S_\rho} \eta_{ij} \geq 1 + \delta_2 \quad \textit{for all} \quad 1 \leq \rho \leq r, \tag{7.9}$$

$$N_{\mathbf{b},\mathbf{b}\cdot\mathbf{y}}(B, H, \lambda) \ll B(\log B)^{c_2-1+\varepsilon}(1 + \log H)\mathbf{b}^{-\eta}\langle \mathbf{y}\rangle^{-\delta_2^*} \tag{7.10}$$

*and*

$$\int_{\mathcal{S}_y(B,H,\lambda)} \prod_{ij} x_{ij}^{-h_{ij}\zeta_i}\,\mathrm{d}\mathbf{x} \ll B(\log B)^{c_2-1+\varepsilon}(1 + \log H)\langle \mathbf{y}\rangle^{-\delta_2^*} \tag{7.11}$$

*for any $\varepsilon > 0$ and some $\zeta$ satisfying (5.10).*

The bound (7.10) is the desired upper bound $B(\log B)^{c_2+\varepsilon}$ with some saving in the coefficients $\mathbf{b}$, $\mathbf{y}$ and with some extra logarithmic saving in the situation of condition (7.8), that is, if one variable is short (that is, $\log H = o((\log B)^{1+\varepsilon})$) or the blocks $\prod_j |x_{ij}|^{h_{ij}}$ for $1 \leq i \leq k$ are unbalanced in size (so that the second assumption in (7.8) holds and we may choose $H$ very small even if all $x_{ij}$ are large).

### 7.2. Reduction to linear algebra

Our main applications involve the torsor equation (1.6). In this case, the verification of Hypothesis 7.2 can be checked simply by a linear program. This will be established in Proposition 7.6 below. We start with two elementary lemmas. Here, $(.,.,.)$ denotes the greatest common divisor, $[.,.,.]$ denotes the least common multiple and $\tau$ is the divisor function.

**Lemma 7.3.** *Let $\mathbf{v} \in \mathbb{Z}^3$ be primitive, and let $H_1, H_2, H_3 > 0$. Then the number of primitive $\mathbf{u} \in \mathbb{Z}^3$ that satisfy $u_1v_1 + u_2v_2 + u_3v_3 = 0$ and that lie in the box $|u_i| \leq H_i$ $(1 \leq i \leq 3)$ is $O(1 + H_1H_2|v_3|^{-1})$.*

This is [43, Lemma 3].

**Lemma 7.4.** *Let $\alpha, \beta, \gamma \in \mathbb{N}$, $A, B, X_1, \ldots, X_r \geq 1$, $h_1, \ldots, h_r \in \mathbb{N}$ with $h_1 \leq \cdots \leq h_r$. Then*

$$\sum_{a\leq A}\sum_{b\leq B}\sum_{\substack{x_j\leq X_j\\1\leq j\leq r}} (\alpha a, \beta b, \gamma \mathbf{x}^{\boldsymbol{h}}) \ll (\alpha,\beta,\gamma)^{1/h_r}(\alpha,\beta)^{1-1/h_r}\tau(\alpha)\tau(\beta)\tau(\gamma)\tau_r(\alpha\beta\gamma)AB\langle \boldsymbol{X}\rangle.$$

*Proof.* The left-hand side of the formula is at most

$$\sum_f f \sum_{\substack{a\leq A\\ f\mid\alpha a}}\sum_{\substack{b\leq B\\ f\mid\beta b}}\sum_{\substack{x_j\leq X_j\ (1\leq j\leq r)\\ f\mid\gamma\mathbf{x}^{\mathbf{h}}}} 1 \leq AB\sum_f \frac{(f,\alpha)(f,\beta)}{f}\sum_{f_1\cdots f_r = f/(f,\gamma)}\sum_{\substack{x_j\leq X_j\ (1\leq j\leq r)\\ f_j\mid x_j^{h_j}}} 1$$

$$\leq AB\langle \mathbf{X}\rangle \sum_f \frac{(f,\alpha)(f,\beta)(f,\gamma)^{1/h_r}\tau_r(f)}{f^{1+1/h_r}} \leq \zeta(1+1/h_r)^r AB\langle \mathbf{X}\rangle \sum_{a\mid\alpha}\sum_{b\mid\beta}\sum_{c\mid\gamma}\frac{abc^{1/h_r}\tau_r([a,b,c])}{[a,b,c]^{1+1/h_r}}.$$

Since $abc^\delta[a,b,c]^{-1-\delta} \leq (a,b)^{1-\delta}(a,b,c)^\delta$ for $0 \leq \delta \leq 1$, the lemma follows. □

We apply the previous two lemmas to analyze the number of solutions $\mathbf{x} \in (\mathbb{Z}\setminus\{0\})^J$ to the first equation in (7.7) in the special case where $k = 3$, $J_1 = J_2 = 2$ and $h_{11} = h_{12} = h_{21} = h_{22} = 1$, cf. (1.6). In this case, the equation reads

$$b_{11}b_{12}x_{11}x_{12} + b_{21}b_{22}x_{21}x_{22} + \prod_{j=1}^{J_3}(b_{3j}x_{3j})^{h_{3j}} = 0. \tag{7.12}$$

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Without loss of generality, assume

$$h_{31} \leq \cdots \leq h_{3J_3}, \text{ and let } \nu \text{be the largest index with } h_{3\nu} = 1. \tag{7.13}$$

If no such index exists, we put $\nu = 0$. For notational simplicity, we write

$$\mu = 1 - h_{3J_3}^{-1} \in [0, 1). \tag{7.14}$$

Suppose first that $\nu \geq 1$. Let us temporarily restrict to $\mathbf{x}$ satisfying

$$(x_{11}x_{12}, x_{21}x_{22}, x_{31} \cdots x_{3\nu}) = 1. \tag{7.15}$$

For $X_{ij} \leq |x_{ij}| \leq 2X_{ij}$ in dyadic boxes, by Lemma 7.3 with $x_{12}, x_{22}, x_{31}$ in the roles of $u_1, u_2, u_3$ and

$$v_3 = \frac{x_{31}^{-1} \prod_j (b_{3j}x_{3j})^{h_{3j}}}{(b_{11}b_{12}x_{11}, b_{21}b_{22}x_{21}, x_{31}^{-1} \prod_j (b_{3j}x_{3j})^{h_{3j}})}$$

(since $\mathbf{v}$ must be primitive) and Lemma 7.4, the number of such solutions to (7.12) is

$$\begin{aligned}
&\ll \langle \mathbf{X}_0 \rangle \sum_{\substack{X_{11} \leq x_{11} \leq 2X_{11} \\ X_{21} \leq x_{21} \leq 2X_{21}}} \sum \sum_{\substack{X_{3j} \leq x_{3j} \leq 2X_{3j} \\ 2 \leq j \leq J_3}} \left(1 + \frac{X_{12}X_{22}}{x_{31}^{-1} \prod_j (b_{3j}x_{3j})^{h_{3j}}} \Big(b_{11}b_{12}x_{11}, b_{21}b_{22}x_{21}, x_{31}^{-1} \prod_j (b_{3j}x_{3j})^{h_{3j}}\Big)\right) \\
&\ll \langle \mathbf{X}_0 \rangle \left(X_{11}X_{21} \frac{\langle \mathbf{X}_3 \rangle}{X_{31}} + |\mathbf{b}|^{\varepsilon} \left(\frac{(b_{11}b_{12}, b_{21}b_{22})}{\mathbf{b}_3^{\mathbf{h}_3}}\right)^{\mu} X_{11}X_{12}X_{21}X_{22} \prod_j X_{3j}^{1-h_{3j}}\right)
\end{aligned}$$

for every $\varepsilon > 0$ and $\mu$ as in (7.14). By symmetry, this improves itself to

$$\langle \mathbf{X}_0 \rangle \left(\frac{\min(X_{11}, X_{12}) \min(X_{21}, X_{22}) \langle \mathbf{X}_3 \rangle}{\max(X_{31}, \ldots, X_{3\nu})} + |\mathbf{b}|^{\varepsilon} \left(\frac{(b_{11}b_{12}, b_{21}b_{22})}{\mathbf{b}_3^{\mathbf{h}_3}}\right)^{\mu} X_{11}X_{12}X_{21}X_{22} \prod_j X_{3j}^{1-h_{3j}}\right). \tag{7.16}$$

Permuting the roles of $u_1, u_2, u_3$ in Lemma 7.3, we obtain similarly the bound

$$\begin{aligned}
&\ll \langle \mathbf{X}_0 \rangle \sum_{\substack{X_{11} \leq x_{11} \leq 2X_{11} \\ X_{21} \leq x_{21} \leq 2X_{21}}} \sum \sum_{\substack{X_{3j} \leq x_{3j} \leq 2X_{3j} \\ 2 \leq j \leq J_3}} \left(1 + \frac{X_{12}X_{31}}{b_{21}b_{22}x_{21}} \Big(b_{11}b_{12}x_{11}, b_{21}b_{22}x_{21}, \prod_j (b_{3j}x_{3j})^{h_{3j}}\Big)\right) \\
&\ll \langle \mathbf{X}_0 \rangle \Big(X_{11}X_{21}X_{32} \cdots X_{3J_3} + |\mathbf{b}|^{\varepsilon} X_{11}X_{12} \langle \mathbf{X}_3 \rangle\Big).
\end{aligned}$$

Again by symmetry, this improves itself to

$$\langle \mathbf{X}_0 \rangle \left(\frac{\min(X_{11}, X_{12}) \min(X_{21}, X_{22}) \langle \mathbf{X}_3 \rangle}{\max(X_{31}, \ldots, X_{3\nu})} + |\mathbf{b}|^{\varepsilon} \min(X_{11}X_{12}, X_{21}X_{22}) \langle \mathbf{X}_3 \rangle\right).$$

Together with (7.16), we now see that the number of $\mathbf{x} \in (\mathbb{Z} \setminus \{0\})^J$ satisfying (7.12), (7.15) and $X_{ij} \le |x_{ij}| \le 2X_{ij}$ does not exceed

$$|\mathbf{b}|^{\varepsilon} \langle \mathbf{X}_0 \rangle \Big( \frac{\min(X_{11}, X_{12}) \min(X_{21}, X_{22}) \langle \mathbf{X}_3 \rangle}{\max(X_{31}, \dots, X_{3\nu})} + \frac{X_{11}X_{12}X_{21}X_{22}\langle \mathbf{X}_3 \rangle}{\max(X_{11}X_{12}, X_{21}X_{22}, (\mathbf{b}_3^{\mathbf{h}_3}(b_{11}b_{12}, b_{21}b_{22})^{-1})^{\mu} \mathbf{X}_3^{\mathbf{h}_3})} \Big). \tag{7.17}$$

We now replace the minima and maxima in (7.17) by suitable geometric means. With future applications in mind, we keep the result as general as is possible.

For $\ell = 1, 2$ and $\tau^{(\ell)} = (\tau_{ij}^{(\ell)}) \in \mathbb{R}^J_{>0}$ with

$$\tau_{0j}^{(\ell)} = 1, \quad \tau_{11}^{(\ell)} + \tau_{12}^{(\ell)} \ge 1, \quad \tau_{21}^{(\ell)} + \tau_{22}^{(\ell)} \ge 1, \quad \sum_{j=1}^{\nu} \tau_{3j}^{(\ell)} \ge \nu - 1, \quad \tau_{3j}^{(\ell)} = 1 \; (j > \nu),$$
$$\min(\tau_{11}^{(\ell)}, \tau_{12}^{(\ell)}) + \min(\tau_{21}^{(\ell)}, \tau_{22}^{(\ell)}) + \min(\tau_{31}^{(\ell)}, \dots, \tau_{3\nu}^{(\ell)}) > 1 \tag{7.18}$$

(where $\nu$ is as in (7.13)), we have

$$\frac{\langle \mathbf{X}_0 \rangle \min(X_{11}, X_{12}) \min(X_{21}, X_{22}) \langle \mathbf{X}_3 \rangle}{\max(X_{31}, \dots, X_{3\nu})} \le \mathbf{X}^{\tau^{(\ell)}}.$$

(The second line in (7.18) is not needed here but will be required later when we remove condition (7.15).) Let $\zeta, \zeta'$ satisfy (5.10), and let $\zeta_0, \zeta_0' \in \mathbb{R}$ be arbitrary. Then

$$\frac{\langle \mathbf{X}_0 \rangle X_{11}X_{12}X_{21}X_{22}\langle \mathbf{X}_3 \rangle}{\max(X_{11}X_{12}, X_{21}X_{22}, (\mathbf{b}_3^{\mathbf{h}_3}(b_{11}b_{12}, b_{21}b_{22})^{-1})^{\mu} \mathbf{X}_3^{\mathbf{h}_3})} \le \Big( \frac{(b_{11}b_{12}b_{21}b_{22})^{1/2}}{\mathbf{b}_3^{\mathbf{h}_3}} \Big)^{\mu \zeta_3'} \prod_{ij} X_{ij}^{1 - h_{ij}\zeta_i'}.$$

Thus, we can bound (7.17) by

$$|\mathbf{b}|^{\varepsilon} \Big( \mathbf{X}^{\tau^{(1)}} + \Big( \frac{(b_{11}b_{12}b_{21}b_{22})^{1/2}}{\mathbf{b}_3^{\mathbf{h}_3}} \Big)^{\mu \zeta_3'} \prod_{ij} X_{ij}^{1 - h_{ij}\zeta_i'} \Big)$$

and also by

$$|\mathbf{b}|^{\varepsilon + 1} \Big( \mathbf{X}^{\tau^{(2)}} + \prod_{ij} X_{ij}^{1 - h_{ij}\zeta_i} \Big)$$

and so, for any $0 < \alpha \le 1$, by

$$|\mathbf{b}|^{\varepsilon + \alpha} \Big( \mathbf{X}^{\tau^{(1)}} + \Big( \frac{(b_{11}b_{12}b_{21}b_{22})^{1/2}}{\mathbf{b}_3^{\mathbf{h}_3}} \Big)^{\mu \zeta_3'} \prod_{ij} X_{ij}^{1 - h_{ij}\zeta_i'} \Big)^{1-\alpha} \Big( \mathbf{X}^{\tau^{(2)}} + \prod_{ij} X_{ij}^{1 - h_{ij}\zeta_i} \Big)^{\alpha}. \tag{7.19}$$

We will apply this with $\alpha$ very small (but fixed). The idea of this maneuver is to separate the **b**- and **y**-decay in (7.10) from the bound in $B$ and $H$. Before we proceed with the estimation, we remove the condition (7.15). Let us therefore assume that $(x_{11}x_{12}, x_{21}x_{22}, x_{31} \cdots x_{3\nu}) = d$. Then we can apply the previous analysis with $X_{ij}/d_{ij}$ in place of $X_{ij}$ for numbers $d_{ij}$ satisfying $d_{11}d_{12} = d_{21}d_{22} = d_{31} \cdots d_{3\nu} = d$ for $i = 1, 2, 3$. The second line in (7.18) and (5.10) (recall that $h_{11} = h_{12} = h_{21} = h_{22} = h_{31} = \cdots = h_{3\nu} = 1$) ensure that summing (7.19) over all $d$ (and all such combinations of $d_{ij}$) yields a convergent sum. Thus the bound (7.19) remains true for the number of all $\mathbf{x} \in (\mathbb{Z} \setminus \{0\})^J$ satisfying (7.12) and $X_{ij} \le |x_{ij}| \le 2X_{ij}$.

We are currently working under the assumption $\nu \geq 1$, but this is only for notational convenience. Indeed, if $\nu = 0$, we apply Lemma 7.3 with one of $u_1, u_2, u_3$ equal to 1, and in (7.17) we agree on the convention that the maximum of the empty set is 1. Condition (7.15) is automatically satisfied in this case (the empty product being defined as 1), and hence the second line in (7.18) is not needed so that we may define as usual the minimum of the empty set as $\infty$. With these conventions, (7.19) remains true also if $\nu = 0$.

We now invoke the $N$ inequalities in (7.7). We choose

$$\zeta' = (\zeta_1', \zeta_2', \zeta_3') = \Big(\frac{1}{2} - \frac{1}{5h_{3J_3}}, \frac{1}{2} - \frac{1}{5h_{3J_3}}, \frac{2}{5h_{3J_3}}\Big)$$

and

$$\tau^{(1)} = \big(1 - h_{01}\zeta_0'', \dots, 1 - h_{kJ_k}\zeta_k''\big) \tag{7.20}$$

where $\zeta'' = (\zeta_1'', \zeta_2'', \zeta_3'')$ satisfies

$$\zeta'' = (\zeta_1'', \zeta_2'', \zeta_3'') = \begin{cases} (1/3, 1/3, 1/3), & h_{3J_3} = 1, \\ (1/2, 1/2, 0), & h_{3J_3} > 1. \end{cases}$$

Then $\tau^{(1)}$ satisfies (7.18). By (7.6), there exists $\sigma^{(1)} \in \mathbb{R}^N_{>0}$ with

$$|\sigma^{(1)}|_1 \leq 1, \quad \mathscr{A}_1\sigma^{(1)} = \tau^{(1)}. \tag{7.21}$$

Such a vector also exists if $\tau^{(1)}$ is replaced by $\tau = (1 - h_{00}\zeta_0', \dots, 1 - h_{3J_3}\zeta_3')$.

Now, taking suitable combinations of the $N$ inequalities of the second condition in (7.7), we see that every $\mathbf{x}$ satisfying these also satisfies

$$\prod_{ij} |x_{ij}|^{\tau_{ij}^{(1)}} \leq B\mathbf{y}^{-\tau^{(1)}}, \quad \prod_{ij} |x_{ij}|^{1-h_{ij}\zeta_i'} \leq B\prod_{ij} y_{ij}^{h_{ij}\zeta_i'-1}.$$

Define

$$\zeta^* = \Big(\zeta_1' - \frac{1}{2}\mu\zeta_3', \zeta_2' - \frac{1}{2}\mu\zeta_3', \ \zeta_3'(1+\mu)\Big) = \Big(\frac{1}{2} - \frac{1}{5(1+\mu)h_{3J_3}}, \frac{1}{2} - \frac{1}{5(1+\mu)h_{3J_3}}, \frac{2}{5(1+\mu)h_{3J_3}}\Big)$$

with $\mu$ as in (7.14) and $\tilde{\tau} = (1 - h_{ij}\zeta_i^*)_{ij}$. We summarize our findings in the following lemma.

**Lemma 7.5.** *In the situation of equation (7.12), suppose that* $\mathbf{b}, \mathbf{y} \in \mathbb{N}^J$, $1 \leq H \leq B$, $0 < \alpha, \lambda \leq 1$, $\tau_* := \min_{ij}(\tau_{ij}^{(1)}, 1 - h_{ij}\zeta_i') > 0$. *Let* $\zeta$ *satisfy (5.10) and* $\tau^{(2)} \in \mathbb{R}^J_{>0}$ *as in (7.18). Then*

$$N_{\boldsymbol{b},\boldsymbol{b}\cdot\boldsymbol{y}}(B, H, \lambda) \ll |\boldsymbol{b}|^{\varepsilon+\alpha}\Big(\langle\boldsymbol{y}\rangle^{-\tau_*}\big(\boldsymbol{b}^{-\tau^{(1)}} + \boldsymbol{b}^{-\tilde{\tau}}\big)B\Big)^{1-\alpha}\sideset{}{^*}\sum_{\boldsymbol{X}}\Big(\boldsymbol{X}^{\tau^{(2)}\alpha} + \prod_{ij} X_{ij}^{(1-h_{ij}\zeta_i)\alpha}\Big), \tag{7.22}$$

*where* $\boldsymbol{X} = (X_{ij})$ *and the asterisk indicates that each* $X_{ij} = 2^{\xi_{ij}}$ *runs over powers of 2 and is subject to* $\prod_{ij} X_{ij}^{\alpha_{ij}^{\nu}} \leq B$ *for* $1 \leq \nu \leq N$ *and at least one of the inequalities*

$$\min_{ij} X_{ij} \leq H, \qquad \min_{1\leq i\leq k}\prod_{j=1}^{J_i} X_{ij}^{h_{ij}} < \Big(\max_{1\leq i\leq k}\prod_{j=1}^{J_i}(2X_{ij})^{h_{ij}}\Big)^{1-\lambda}.$$

Similarly, but in a much simpler way, we derive the continuous analogue

$$\int_{\mathscr{S}_{\mathbf{y}}(B,H,\lambda)} \prod_{ij} x_{ij}^{-h_{ij}\zeta_i}\,\mathrm{d}\mathbf{x} \ll \big(\langle \mathbf{y}\rangle^{-\tau^\dagger} B\big)^{1-\alpha} \sum_{\mathbf{X}}{}^{*} \prod_{ij} X_{ij}^{(1-h_{ij}\zeta_i)\alpha} \tag{7.23}$$

with $\tau^\dagger = \min_{ij}(1-h_{ij}\zeta_i) > 0$ and the sum is subject to the same conditions.

As mentioned above, we will choose $\alpha$ in (7.22) very small. The key property of $\tau^{(\mathbf{1})}$ and $\tilde{\tau}$ is that all their entries are $\geq 1/2$ where equality is only possible for $\tau^{(\mathbf{1})}$ at indices $(ij)$ with $i \in \{1,2\}$ if $h_{3J_3} \geq 2$. Since $|S_\rho| \geq 2$ for all $1 \leq \rho \leq r$, we conclude that the conditions

$$C_1\big((1-\alpha)\tau^{(\mathbf{1})}\big), \quad C_1\big((1-\alpha)\tilde{\tau}\big)$$

in (7.9) hold for sufficiently small $\alpha > 0$ provided that

$$\max_{ij} h_{ij} = 1 \text{ or there exists no } \rho \text{ with } S_\rho = \{(i_1,j_1),(i_2,j_2)\}, i_1,i_2 \in \{1,2\}. \tag{7.24}$$

We now transform the $X$-sums in (7.22) and (7.23). For an arbitrary vector $\tau \in \mathbb{R}^J_{\geq 0}$, we rewrite a sum $\sum_{\mathbf{X}}^* \mathbf{X}^{\tau\alpha}$ of the type appearing in (7.22) and (7.23) as

$$\sum_{\xi \in \mathbb{N}_0^J}{}^{*} B^{\alpha\tilde{\xi}^\top \tau}, \qquad \tilde{\xi} = \frac{\log 2}{\log B}\xi, \tag{7.25}$$

and now $\sum^*$ indicates that the sum is subject to

$$\mathscr{A}_1^\top \tilde{\xi} \leq (1,\dots,1)^\top \in \mathbb{R}^N \tag{7.26}$$

(the inequality being understood componentwise) and at least one of the inequalities

$$\tilde{\xi}_{ij} \leq \frac{\log H}{\log B} \quad \text{for some } i,j, \tag{7.27}$$

$$\min_{1\leq i\leq k} \sum_{j=1}^{J_i} \tilde{\xi}_{ij} h_{ij} < \max_{1\leq i\leq k} \sum_{j=1}^{J_i} \Big(\tilde{\xi}_{ij} + \frac{\log 2}{\log B}\Big) h_{ij}(1-\lambda). \tag{7.28}$$

For future reference, we note that

$$\max_{1\leq i\leq k} \sum_{j=1}^{J_i} \Big(\tilde{\xi}_{ij} + \frac{\log 2}{\log B}\Big) h_{ij}(1-\lambda) = \max_{1\leq i\leq k} \sum_{j=1}^{J_i} \tilde{\xi}_{ij} h_{ij}(1-\lambda) + O\Big(\frac{1}{\log B}\Big). \tag{7.29}$$

For $0 \leq i \leq k$, $1 \leq j \leq J_i$, $0 < \lambda \leq 1$ and a permutation $\pi \in S_k$, we consider the closed, convex polytopes

$$\begin{aligned}
\mathscr{P} &= \{\psi \in \mathbb{R}^J : \psi \geq 0,\ \mathscr{A}_1^\top \psi \leq (1,\dots,1)^\top\},\\
\mathscr{P}_{ij} &= \{\psi \in \mathscr{P} : \psi_{ij} = 0\},\\
\mathscr{P}(\lambda,\pi) &= \Big\{\psi \in \mathscr{P} : \sum_{j=1}^{J_{\pi(1)}} \psi_{\pi(1),j} h_{\pi(1),j} \leq \cdots \leq \sum_{j=1}^{J_{\pi(k)}} \psi_{\pi(k),j} h_{\pi(k),j},\\
&\qquad \sum_{j=1}^{J_{\pi(1)}} \psi_{\pi(1),j} h_{\pi(1),j} \leq (1-\lambda) \sum_{j=1}^{J_{\pi(k)}} \psi_{\pi(k),j} h_{\pi(k),j}\Big\}.
\end{aligned} \tag{7.30}$$

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We assume that

$$C_2(\tau): \quad \max\{\psi^\top\tau : \psi \in \mathscr{P}\} = 1. \tag{7.31}$$

The intersection of the hyperplane $\mathscr{H} : \psi^\top\tau = 1$ with any of the above polytopes is again a closed convex polytope, and we assume that the dimensions satisfy

$$C_3(\tau): \quad \begin{array}{l} \dim(\mathscr{H} \cap \mathscr{P}) \leq c_2, \\ \dim(\mathscr{H} \cap \mathscr{P}_{ij}) \leq c_2 - 1, \quad 0 \leq i \leq k, 1 \leq j \leq J_i, \\ \dim(\mathscr{H} \cap \mathscr{P}(\lambda, \pi)) \leq c_2 - 1, \quad \pi \in S_k. \end{array} \tag{7.32}$$

With this notation and the assumptions (7.31) and (7.32), we return to (7.25). Clearly, the sum has $O((\log B)^J)$ terms, so the contribution of $\xi$ with

$$\tilde{\xi}^\top\tau \leq 1 - \frac{J\log\log B}{\alpha\log B}$$

to (7.25) is $O(B^\alpha)$. By (7.31), we may now restrict to

$$1 - \frac{J\log\log B}{\alpha\log B} \leq \tilde{\xi}^\top\tau \leq 1 \tag{7.33}$$

in the sense that

$$\sum_{\xi\in\mathbb{N}_0^J}{}^{*} B^{\alpha\,\tilde{\xi}^\top\tau} \ll B^\alpha\big(1 + \#\mathscr{X}_1 + \#\mathscr{X}_2\big), \tag{7.34}$$

where

$$\mathscr{X}_1 = \{\xi \in \mathbb{N}_0^J : (7.26), (7.27), (7.33)\}, \quad \mathscr{X}_2 = \{\xi \in \mathbb{N}_0^J : (7.26), (7.28), (7.33)\}.$$

We define

$$\mathscr{Y}_1 = \{\xi \in \mathbb{R}_{\geq 0}^J : (7.26), (7.27), (7.33)\}, \quad \mathscr{Y}_2 = \{\xi \in \mathbb{R}_{\geq 0}^J : (7.26), (7.28), (7.33)\}$$

and bound $\#\mathscr{X}_1$ resp. $\#\mathscr{X}_2$ by the Lipschitz principle, that is, by the volume and the volume of the boundary of $\mathscr{Y}_1$ resp. $\mathscr{Y}_2$ (or a superset thereof). By the third condition in (7.32) as well as (7.29) and (7.33) we see that $\mathscr{Y}_2$ is contained in an $O_\alpha(\log\log B)$ neighborhood of a union of polytopes of dimension at most $c_2 - 1$ and side lengths $O(\log B)$ so that

$$\#\mathscr{X}_2 \ll_{\alpha,\lambda} (\log B)^{c_2-1}(\log\log B)^{J-(c_2-1)} \ll (\log B)^{c_2-1+\varepsilon}.$$

Similarly, by the first two conditions in (7.32) and (7.33) we see that $\mathscr{Y}_2$ is contained in an $O_\alpha(\log\log B)$ neighborhood of a union of parallelepipeds of dimension at most $c_2$, where at most $c_2 - 1$ of the side lengths of each parallelepiped are of size $O(\log B)$ and the remaining ones (if any) are of size $O(\log H)$. We conclude

$$\#\mathscr{X}_1 \ll_\alpha (\log B)^{c_2-1}(\log H + \log\log B)(\log\log B)^{J-c_2} \ll (\log B)^{c_2-1+\varepsilon}(1 + \log H).$$

We substitute the bounds for $\#\mathscr{X}_1, \#\mathscr{X}_2$ into (7.34) and use this in (7.22) and (7.23). From Lemma 7.5, we conclude the following result.

**Proposition 7.6.** *In the situation of equation (7.12), let $\lambda$ be as in Hypothesis 5.1 and $\zeta$ as in (5.10). Define the matrix $\mathscr{A}_1$ as in (7.2) and the polytopes $\mathscr{P}, \mathscr{P}_{ij}, \mathscr{P}(\lambda, \pi)$ as in (7.30). Choose $\tau^{(2)}$ satisfying (7.18).*

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*Suppose that (7.24) holds as well as the conditions*

$$C_2(\tau^{(2)}),\quad C_3(\tau^{(2)}),\quad C_2((1-h_{ij}\zeta_i)_{ij}),\quad C_3((1-h_{ij}\zeta_i)_{ij}) \tag{7.35}$$

*hold as in (7.31) and (7.32). Then Hypothesis 7.2 is true.*

Condition (7.35) requires a linear program. In principle, this can be done by hand (we show this in a special case in Appendix A), but a straightforward computer-assisted verification is more time efficient. We can replace (7.24) by the following condition: There exist vectors $\tau^{(1)} \in \mathbb{R}^J$, $\sigma \in \mathbb{R}^N$ satisfying (7.20) and (7.21) such that $C_1(\tau^{(1)})$ holds.

## 8. The transition method

In this section, we describe a method that derives an asymptotic formula for $N(B)$ as in (1.5) from the input provided by Hypotheses 5.1 and 7.2. In fact, we will only need these hypotheses for certain choices of parameters to be discussed in a moment. Our main result will be formulated at the end of the section. In the interest of brevity, we now choose $b_1 = \cdots = b_k = 1$ in (1.2). No extra difficulties arise should one wish to handle the more general case, but a more elaborate notation would be needed. All equations that occur in the examples treated in this paper may be interpreted to have coefficients 1 only.

We begin with some more notation. We continue to use the vector operations introduced in Section 5. In addition, if $\mathscr{R} \subseteq \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} \cdot \mathscr{R} = \{\mathbf{x} \cdot \mathbf{y} : \mathbf{y} \in \mathscr{R}\} \subseteq \mathbb{R}^n$. For $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we write

$$\widetilde{\mathbf{v}} = (2^{v_1}, \ldots, 2^{v_n}) \in \mathbb{R}^n. \tag{8.1}$$

For $\mathbf{g} \in \mathbb{N}^r$, we write $\mu(\mathbf{g}) = \prod_{\rho=1}^r \mu(g_\rho)$ where $\mu$ denotes the Möbius function. We write $\mathbf{1} = (1, \ldots, 1)$, the dimension of the vector being understood from the context.

For $0 < \Delta < 1$, let $f_\Delta : [0, \infty) \to [0, 1]$ be a smooth function with

$$\operatorname{supp}(f_\Delta) \subseteq [0, 1 + \Delta), \quad f_\Delta = 1 \text{ on } [0, 1], \quad \frac{d^j}{dx^j} f_\Delta(x) \ll_j \Delta^{-j} \tag{8.2}$$

whose Mellin transform $\widehat{f_\Delta}$ obeys, once $\delta_3 > 0$ and $A \geq 0$ are fixed, the inequality

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j} \widehat{f_\Delta}(s) \ll_{j,A,\delta_3} \frac{(1 + \Delta|s|)^{-A}}{|s|} \tag{8.3}$$

for all $j \in \mathbb{N}_0$, uniformly in $\delta_3 \leq \Re s < 2$. A construction of $f_\Delta$ is given in [8, (2.3)]. From (8.3), we infer the useful estimate

$$\mathscr{D}\Big(\mathbf{s}^{\mathbf{a}} \prod_{\nu=1}^N \widehat{f_\Delta}(s_\nu)\Big) \ll \Delta^{-\|\mathbf{a}\|_1 - c} |\mathbf{s}|^{-c} \langle \mathbf{s} \rangle^{-1} \tag{8.4}$$

for $\mathbf{s} = (s_1, \ldots, s_N) \in \mathbb{C}^N$ with $2 > \Re s_\nu \geq \delta_3 > 0$, $\mathbf{a} \in \mathbb{N}_0^N$, $c \geq 1$ and any linear differential operator $\mathscr{D}$ with constant coefficients in $s_1, \ldots, s_N$, the implied constant being dependent on $\mathbf{a}, N, c, \mathscr{D}$.

We write $\int^{(n)}$ for an iterated $n$-fold Mellin–Barnes integral. The lines of integration will be clear from the context or otherwise specified in the text. If all $n$ integrations are over the same line $(c)$, then we write this as $\int_{(c)}^{(n)}$.

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We continue to work subject to the conditions (7.4), (7.6). Also, we suppose that Hypotheses 5.1 and 7.2 are available to us. With $\beta_i$ as in Hypothesis 5.1 and $S_\rho$ as in (1.4), we suppose that there is some $\delta_4 > 0$ with

$$\sum_{(i,j)\in S_\rho} (1-\beta_i h_{ij}) \geq 1+\delta_4 \ \ (1 \leq \rho \leq r) \quad \text{and} \quad \beta_i h_{ij} \leq 1 \ \ (1 \leq i \leq k, 1 \leq j \leq J_i). \tag{8.5}$$

In order to efficiently work with the asymptotic formula in Hypothesis 5.1, it is necessary to rewrite the singular integral as a Mellin transform. With $\zeta$ as in Hypothesis 5.1 (in particular satisfying (5.10)), we assume that

$$J_i \geq 2 \quad \text{whenever} \quad \zeta_i \geq 1/2. \tag{8.6}$$

We also define

$$J^* = J_1 + \cdots + J_k$$

for the number of variables appearing in the torsor equation.

**Lemma 8.1.** *Let* $\mathbf{b} \in (\mathbb{Z}\setminus\{0\})^k$ *and* $\mathbf{X} \in [1/2,\infty)^J$. *For* $1 \leq i \leq k$, *put*

$$\mathcal{K}_i(z) = \begin{cases} \Gamma(z)\cos(\pi z/2), & h_{ij} \text{ odd for some } 1 \leq j \leq J_i, \\ \Gamma(z)\exp(\mathrm{i}\pi z/2), & h_{ij} \text{ even for all } 1 \leq j \leq J_i. \end{cases} \tag{8.7}$$

*Then, on writing* $z_k = 1 - z_1 - \cdots - z_{k-1}$, *one has*

$$\mathcal{I}_{\mathbf{b}}(\mathbf{X}) = \frac{2^{J^*}}{\pi}\langle X_0\rangle \int_{(\zeta_1)} \cdots \int_{(\zeta_{k-1})} \prod_{i=1}^{k} \frac{\mathcal{K}_i(z_i)}{b_i^{z_i}} \prod_{j=1}^{J_i}\left(X_{ij}^{1-h_{ij}z_i}\frac{1-2^{h_{ij}z_i-1}}{1-h_{ij}z_i}\right)\frac{\mathrm{d}z_1\cdots \mathrm{d}z_{k-1}}{(2\pi\mathrm{i})^{k-1}}.$$

Note that (5.10) implies that $\Re z_k = \zeta_k$.

*Proof.* We start with the absolutely convergent Mellin identity

$$e(w) = \int_{\mathcal{C}} \Gamma(s)\exp\left(\frac{1}{2}\mathrm{sgn}(w)\mathrm{i}\pi s\right)|2\pi w|^{-s}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$$

for $w \in \mathbb{R}\setminus\{0\}$ and $\mathcal{C}$ the contour

$$(-1-\mathrm{i}\infty, -1-\mathrm{i}] \cup [-1-\mathrm{i}, \tfrac{1}{k}-\mathrm{i}] \cup [\tfrac{1}{k}-\mathrm{i}, \tfrac{1}{k}+\mathrm{i}] \cup [\tfrac{1}{k}+\mathrm{i}, -1+\mathrm{i}] \cup [-1+\mathrm{i}] \cup [-1+\mathrm{i}\infty),$$

which can simply be checked by moving the contour to the left and comparing power series. Integrating this over $\mathcal{Y}$ as in (5.2) based on

$$\int_{\frac{1}{2}Y\leq y\leq Y} y^{-hs}\,\mathrm{d}y = \frac{1-2^{hs}}{1-hs}Y^{1-hs}$$

and using the definition (5.4), we obtain

$$I_i(b_i\beta, \mathbf{X}_i) = 2^{J_i}\int_{\mathcal{C}} \frac{\mathcal{K}_i(z_i)}{(2\pi|b_i\beta|)^{z_i}} \prod_{j=1}^{J_i}\left(X_{ij}^{1-h_{ij}z_i}\frac{1-2^{h_{ij}z_i-1}}{1-h_{ij}z_i}\right)\frac{\mathrm{d}z_i}{2\pi\mathrm{i}} \tag{8.8}$$

for every $i$. Note that $\mathrm{sgn}(\mathbf{y}_i^{\mathbf{h}_i})$ is always 1 if and only if $h_{ij}$ is even for all $1 \leq j \leq J_i$. At this point, we can straighten the contour and replace it with $\Re z_i = \zeta_i$. The expression is still absolutely convergent, provided that (8.6) holds. We insert this formula into (5.5) for $i = 1, \dots, k-1$ getting

$$\mathscr{I}_{\mathbf{b}}(\mathbf{X}) = \langle \mathbf{X}_0 \rangle \int_{-\infty}^{\infty} 2^{J_1+\cdots+J_{k-1}} \int_{\Re z_i = \zeta_i}^{(k-1)} \prod_{i=1}^{k-1} \frac{\mathscr{K}_i(z_i)}{(2\pi|b_i|)^{z_i}} \prod_{j=1}^{J_i} \left( X_{ij}^{1-h_{ij}z_i} \frac{1-2^{h_{ij}z_i-1}}{1-h_{ij}z_i} \right) \frac{d\mathbf{z}}{(2\pi \mathrm{i})^{k-1}} \times I_k(b_k\beta, \mathbf{X}_k)|\beta|^{-z_1-\cdots-z_{k-1}} d\beta.$$

The integral in $\beta$ is still absolutely convergent, by (5.3) and (5.10). It is the two-sided Mellin transform of $I_k(b_k\beta, \mathbf{X}_k)$ in $\beta$ at $z_k = 1 - z_1 - \cdots - z_{k-1}$. An evaluation can be read off from (8.8) by Mellin inversion, and the lemma follows. □

We are now prepared to describe our method in detail.

### *8.1. Step 1: initial manipulations*

Let $\chi\colon (\mathbb{Z} \setminus \{0\})^J \to [0,1]$ be the characteristic function on the set of solutions to the torsor equation (1.2) subject to $b_1 = \cdots = b_k = 1$, and let $\psi\colon (\mathbb{Z} \setminus \{0\})^J \to [0,1]$ be the characteristic function on $J$-tuples of nonzero integers satisfying the coprimality conditions (1.4). For $1 \le \nu \le N$, let

$$P_\nu(\mathbf{x}) = \prod_{ij} |x_{ij}|^{\alpha_{ij}^\nu} \tag{8.9}$$

denote the monomials appearing in the height conditions (1.3). We start with some smoothing. Let $0 < \Delta < 1/10$ and define

$$F_{\Delta,B}(\mathbf{x}) = \prod_{\nu=1}^{N} f_\Delta\left(\frac{P_\nu(\mathbf{x})}{B}\right).$$

Then the counting function

$$N_\Delta(B) = \sum_{\mathbf{x} \in (\mathbb{Z}\setminus\{0\})^J} \psi(\mathbf{x})\chi(\mathbf{x})F_{\Delta,B}(\mathbf{x})$$

satisfies

$$N_\Delta(B(1-\Delta)) \le N(B) \le N_\Delta(B). \tag{8.10}$$

We remove the coprimality conditions encoded in $\psi$ by Möbius inversion. As in [9, Lemma 2.1], we have

$$N_\Delta(B) = \sum_{\mathbf{g} \in \mathbb{N}^r} \mu(\mathbf{g}) \sum_{\mathbf{x} \in (\mathbb{Z}\setminus\{0\})^J} \chi(\gamma \cdot \mathbf{x})F_{\Delta,B}(\gamma \cdot \mathbf{x}),$$

where for given $\mathbf{g} \in \mathbb{N}^r$, we wrote

$$\gamma = (\gamma_{ij}) \in \mathbb{N}^J, \quad \gamma_{ij} = \mathrm{lcm}\{g_\rho \mid (i,j) \in S_\rho\} \tag{8.11}$$

for $0 \le i \le k$, $1 \le j \le J_i$. In the following, we will need (7.10) of Hypothesis 7.2 only for $\mathbf{b} = \gamma$. For later purposes, we state the following elementary lemma.

**Lemma 8.2.** *For $\gamma \in \mathbb{N}^J$ as in (8.11), $\delta > 0$, $1 \le \rho \le r$, and $\eta = (\eta_{ij}) \in \mathbb{R}^J_{\ge 0}$, the series*

$$\sum_{\mathbf{g} \in \mathbb{N}^r} \gamma^{-\eta} g_\rho^\delta$$

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*is convergent provided that*

$$\sum_{(i,j)\in S_\rho} \eta_{ij} > 1+\delta$$

*holds for all* $1 \le \rho \le r$.

*Proof.* Suppose that $\sum_{(i,j)\in S_\rho} \eta_{ij} \ge 1+\delta+\delta_0$ for all $\rho$ and some $\delta_0 > 0$. The sum in question can be written as an Euler product, and a typical Euler factor has the form

$$\sum_{\alpha\in\mathbb{N}_0^r} p^{f(\alpha)}, \quad f(\alpha) = \delta\alpha_\rho - \sum_{i,j} \eta_{ij} \max_{(i,j)\in S_t} \alpha_t.$$

This is

$$1 + O\Big(\sum_{\alpha=1}^{\infty} \frac{(1+\alpha)^r}{p^{\alpha(1+\delta_0)}}\Big).$$

The statement is now clear. □

For $1 \le T \le B$, we define

$$N_{\Delta,T}(B) = \sum_{|\mathbf{g}|\le T} \mu(\mathbf{g}) \sum_{\mathbf{x}\in(\mathbb{Z}\setminus\{0\})^J} \chi(\gamma\cdot\mathbf{x}) F_{\Delta,B}(\gamma\cdot\mathbf{x}).$$

By (7.10), (7.9) (recall $\Delta \le 1/10$) and Lemma 8.2, and by an estimate that is often called Rankin's trick,

$$\begin{aligned} |N_{\Delta,T}(B) - N_\Delta(B)| &\le \sum_{|\mathbf{g}|>T} N_{\gamma,\gamma}(2B,2B,1) \ll B(\log B)^{c_2+\varepsilon} \sum_{|\mathbf{g}|>T} \gamma^{-\eta} \\ &\le B(\log B)^{c_2+\varepsilon} \sum_{\mathbf{g}} \gamma^{-\eta}\Big(\frac{|\mathbf{g}|}{T}\Big)^{\delta_2-\varepsilon} \ll B(\log B)^{c_2+\varepsilon} T^{-\delta_2}. \end{aligned} \tag{8.12}$$

Next, we write each factor $f_\Delta$ in the definition of $F_{\Delta,B}$ as its own Mellin inverse so that

$$N_{\Delta,T}(B) = \sum_{|\mathbf{g}|\le T} \mu(\mathbf{g}) \int_{(1)}^{(N)} \sum_{\mathbf{x}\in(\mathbb{Z}\setminus\{0\})^J} \frac{\chi(\gamma\cdot\mathbf{x})}{\gamma^{\mathbf{v}}} \prod_{ij} |x_{ij}|^{-v_{ij}} \prod_{\nu=1}^{N} \Big(\widehat{f_\Delta}(s_\nu)B^{s_\nu}\Big) \frac{\mathrm{d}\mathbf{s}}{(2\pi \mathrm{i})^N},$$

where

$$\mathbf{v} = (v_{ij}) = \mathscr{A}_1\mathbf{s} \in \mathbb{C}^J \tag{8.13}$$

and $\mathscr{A}_1 = (\alpha_{ij}^\nu) \in \mathbb{R}^{J\times N}$ is as before. By partial summation, we obtain

$$\begin{aligned} \sum_{\mathbf{x}\in(\mathbb{Z}\setminus\{0\})^J} \frac{\chi(\gamma\cdot\mathbf{x})}{\gamma^{\mathbf{v}}} \prod_{ij} |x_{ij}|^{-v_{ij}} &= \frac{1}{\gamma^{\mathbf{v}}}\Big(\prod_{i,j} v_{ij}\Big) \int_{[1,\infty)^J} \sum_{0<|x_{ij}|\le X_{ij}} \chi(\gamma\cdot\mathbf{x}) \mathbf{X}^{-\mathbf{v}-\mathbf{1}}\,\mathrm{d}\mathbf{X} \\ &= \frac{1}{\gamma^{\mathbf{v}}}\Big(\prod_{i,j} \frac{v_{ij}}{1-2^{-v_{ij}}}\Big) \int_{[1,\infty)^J} \sum_{\frac12 X_{ij}<|x_{ij}|\le X_{ij}} \chi(\gamma\cdot\mathbf{x}) \mathbf{X}^{-\mathbf{v}-\mathbf{1}}\,\mathrm{d}\mathbf{X}, \end{aligned}$$

so that

$$N_{\Delta,T}(B) = \sum_{|\mathbf{g}|\le T} \mu(\mathbf{g}) \int_{(1)}^{(N)} \frac{1}{\gamma^{\mathbf{v}}}\Big(\prod_{i,j} \frac{v_{ij}}{1-2^{-v_{ij}}}\Big) \int_{[1,\infty)^J} \frac{\mathscr{N}_{\gamma^*}(\mathbf{X})}{\mathbf{X}^{\mathbf{v}+\mathbf{1}}}\,\mathrm{d}\mathbf{X} \prod_{\nu=1}^{N} \Big(\widehat{f_\Delta}(s_\nu)B^{s_\nu}\Big) \frac{\mathrm{d}\mathbf{s}}{(2\pi \mathrm{i})^N}$$

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in the notation of Hypothesis 5.1, where

$$\gamma^* = \Big(\prod_{j=1}^{J_i} \gamma_{ij}^{h_{ij}}\Big)_{1\le i\le k} \in \mathbb{N}^k. \tag{8.14}$$

We emphasize that we need (5.9) of Hypothesis 5.1 only for $\mathbf{b} = \gamma^*$.

### 8.2. Step 2: removing the cusps

We would like to insert the asymptotic formula from Hypothesis 5.1. This gives a meaningful error term only if $\min X_{ij}$ is not too small, and the formula is only applicable if (5.11) holds. Thus, for $0 < \delta < 1, 0 < \lambda \le 1$ we define the set

$$\mathscr{R}_{\delta,\lambda} = \Big\{\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_k) \in [1,\infty)^J : \min_{i,j} X_{ij} \ge \max X_{ij}^{\delta},\ \min_{1\le i\le k} \mathbf{X}_i^{\mathbf{h}_i} \ge \big(\max_{1\le i\le k} \mathbf{X}_i^{\mathbf{h}_i}\big)^{1-\lambda}\Big\}.$$

Correspondingly, we put

$$N_{\Delta,T,\delta,\lambda} = \sum_{|\mathbf{g}|\le T} \mu(\mathbf{g}) \int_{(1)}^{(N)} \frac{1}{\gamma^{\mathbf{v}}}\Big(\prod_{i,j} \frac{v_{ij}}{1-2^{-v_{ij}}}\Big) \int_{\mathscr{R}_{\delta,\lambda}} \frac{\mathscr{N}_{\gamma^*}(\mathbf{X})}{\mathbf{X}^{\mathbf{v}+\mathbf{1}}}\, \mathrm{d}\mathbf{X} \prod_{\nu=1}^{N} \Big(\widehat{f}_{\Delta}(s_\nu) B^{s_\nu}\Big) \frac{\mathrm{d}\mathbf{s}}{(2\pi \mathrm{i})^N}. \tag{8.15}$$

While $\lambda$ is fixed, $\delta$ is allowed to depend on $B$ and will later be chosen as a negative power of $\log B$. In particular, all subsequent estimates will be uniform in $\delta$.

**Lemma 8.3.** *We have*

$$N_{\Delta,T}(B) - N_{\Delta,T,\delta,\lambda} \ll T^r B(\log B)^{c_2+\varepsilon}(\delta + (\log B)^{-1}).$$

*Proof.* This is essentially [9, Lemma 5.1]. The idea is to revert all steps from Section 8.1 and apply the bound (7.10). By a change of variables, we have

$$\begin{aligned} N_{\Delta,T,\delta,\lambda} = \sum_{|\mathbf{g}|\le T} \mu(\mathbf{g}) \int_{(1)}^{(N)} \frac{1}{\gamma^{\mathbf{v}}}\Big(\prod_{i,j} \frac{v_{ij}}{1-2^{-v_{ij}}}\Big) \sum_{\sigma\in\{0,1\}^J} (-1)^{|\sigma|_1} \\ \times \int_{\widetilde{\sigma}\cdot\mathscr{R}_{\delta,\lambda}} \sum_{0<|x_{ij}|\le X_{ij}} \chi(\gamma\cdot\mathbf{x})(\widetilde{\sigma}\cdot\mathbf{X})^{-\mathbf{v}} \frac{\mathrm{d}\mathbf{X}}{\langle\mathbf{X}\rangle} \prod_{\nu=1}^{N} \Big(\widehat{f}_{\Delta}(s_\nu) B^{s_\nu}\Big) \frac{\mathrm{d}\mathbf{s}}{(2\pi \mathrm{i})^N}, \end{aligned}$$

where we recall the notation (8.1). By partial summation, this equals

$$\begin{aligned} \sum_{|\mathbf{g}|\le T} \mu(\mathbf{g}) \int_{(1)}^{(N)} \Big(\prod_{i,j} \frac{1}{1-2^{-v_{ij}}}\Big) \sum_{\sigma\in\{0,1\}^J} (-1)^{|\sigma|_1} 2^{-\sum_{ij}\sigma_{ij}v_{ij}} \\ \times \sum_{\mathbf{x}\in\widetilde{\sigma}\cdot\mathscr{R}_{\delta,\lambda}} \frac{\chi(\gamma\cdot\mathbf{x})}{\gamma^{\mathbf{v}}\mathbf{x}^{\mathbf{v}}} \prod_{\nu=1}^{N} \Big(\widehat{f}_{\Delta}(s_\nu) B^{s_\nu}\Big) \frac{\mathrm{d}\mathbf{s}}{(2\pi \mathrm{i})^N}. \end{aligned}$$

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We conclude that

$$|N_{\Delta,T}(B) - N_{\Delta,T,\delta,\lambda}| \leq \sum_{|\mathbf{g}|\leq T} \sum_{\sigma\in\{0,1\}^J} \Big| \int_{(1)}^{(N)} \Big( \prod_{i,j} \frac{1}{1-2^{-v_{ij}}} \Big) \times 2^{-\sum_{ij}\sigma_{ij}v_{ij}} \sum_{\mathbf{x}\in(\mathbb{Z}\setminus\{0\})^J\setminus\widetilde{-\sigma}\cdot\mathscr{R}_{\delta,\lambda}} \frac{\chi(\gamma\cdot\mathbf{x})}{\gamma^{\mathbf{v}}\mathbf{x}^{\mathbf{v}}} \prod_{\nu=1}^{N} \Big(\widehat{f_\Delta}(s_\nu)B^{s_\nu}\Big) \frac{\mathrm{d}\mathbf{s}}{(2\pi\mathrm{i})^N} \Big|.$$

Finally, we write each factor $(1-2^{-v_{ij}})$ as a geometric series and apply Mellin inversion to recast the right-hand side as

$$\sum_{|\mathbf{g}|\leq T} \sum_{\sigma\in\{0,1\}^J} \sum_{\mathbf{k}\in\mathbb{N}_0^J} \sum_{\mathbf{x}\in(\mathbb{Z}\setminus\{0\})^J\setminus\widetilde{-\sigma}\cdot\mathscr{R}_{\delta,\lambda}} \chi(\gamma\cdot\mathbf{x}) F_{\Delta,B}(\gamma\cdot(\widetilde{\mathbf{k}+\sigma})\cdot\mathbf{x}).$$

Note that any $\mathbf{x}\notin\widetilde{-\sigma}\cdot\mathscr{R}_{\delta,\lambda}$ in the support of $F_{\Delta,B}(\gamma\cdot(\widetilde{\mathbf{k}+\sigma})\cdot\mathbf{x})$ satisfies

$$\min_{ij}|x_{ij}| \leq ((1+\Delta)B)^\delta \quad \text{or} \quad \min_{1\leq i\leq k} \prod_{j=1}^{J_i} |x_{ij}|^{h_{ij}} \leq \Big(\max_{1\leq i\leq k} \prod_{j=1}^{J_i} |2x_{ij}|^{h_{ij}}\Big)^{1-\lambda}$$

so that

$$|N_{\Delta,T}(B) - N_{\Delta,T,\delta,\lambda}| \leq 2^J \sum_{|\mathbf{g}|\leq T} \sum_{\mathbf{k}\in\mathbb{N}_0^J} N_{\gamma,\gamma\cdot\widetilde{\mathbf{k}}}((1+\Delta)B, ((1+\Delta)B)^\delta, \lambda)$$

by (7.8). The lemma follows from (7.10). Note that $\delta_2^* > 0$ in (7.10) ensures that the $\mathbf{k}$-sum converges. □

### 8.3. Step 3: the error term in the asymptotic formula

We insert Hypothesis 5.1 into (8.15). For convenience, we now write $\Psi_{\mathbf{b}}(\mathbf{X}) = N_{\mathbf{b}}(\mathbf{X}) - \mathscr{E}_{\mathbf{b}}\mathscr{I}_{\mathbf{b}}(\mathbf{X})$. In this section, we estimate the contribution of the error $\Psi_{\mathbf{b}}(\mathbf{X})$, which amounts to bounding

$$E_{\Delta,T,\delta,\lambda} = \sum_{|\mathbf{g}|\leq T} \Big| \int_{(1)}^{(N)} \frac{1}{\gamma^{\mathbf{v}}} \Big( \prod_{i,j} \frac{v_{ij}}{1-2^{-v_{ij}}} \Big) \int_{\mathscr{R}_{\delta,\lambda}} \frac{\Psi_{\gamma^*}(\mathbf{X})}{\mathbf{X}^{\mathbf{v}+\mathbf{1}}} \,\mathrm{d}\mathbf{X} \prod_{\nu=1}^{N} \Big(\widehat{f_\Delta}(s_\nu)B^{s_\nu}\Big) \frac{\mathrm{d}\mathbf{s}}{(2\pi\mathrm{i})^N} \Big|.$$

For $\mathbf{X}\in\mathscr{R}_{\delta,\lambda}$, we use (5.12) and $\min X_{ij}^{-\delta\delta_1} \leq \prod_{ij} X_{ij}^{-\delta\delta_1/J}$ to conclude that

$$\Psi_{\gamma^*}(\mathbf{X}) \ll \gamma^{C\mathbf{h}} \Big( \prod_{i=0}^{k} \prod_{j=1}^{J_i} X_{ij}^{1-h_{ij}\zeta_i+\varepsilon-\delta\delta_1/J} \Big).$$

Thus, the $\mathbf{X}$-integral is absolutely convergent provided that

$$\Re v_{ij} > 1 - h_{ij}\zeta_i - \delta\delta_1/J \tag{8.16}$$

holds for each $i, j$. We now choose appropriate contours for the $\mathbf{s}$-integral. By (8.13), the choice $\Re\mathbf{s} = \sigma = (\sigma_\nu) \in \mathbb{R}_{>0}^N$ as in (7.6) is admissible to ensure (8.16). These contours stay also to the right of the poles of $\widehat{f_\Delta}$ at $s = 0$ (and in fact inside the validity of (8.3) and (8.4) if $\delta_3$ is sufficiently small) and to the right of the poles of $(1-2^{-v_{ij}})^{-1}$ at $\Re v_{ij} = 0$ by (5.10) if $\delta$ is sufficiently small. By (7.6), this $\sigma$

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satisfies $\sum \sigma_\nu = 1$. We now shift each $s_\nu$-contour to $\Re s_\nu = \sigma_\nu - \delta\delta_1/(2JA)$, where

$$A = \max_{ij} \sum_\nu \alpha_{ij}^\nu.$$

Then $\Re v_{ij} \geq 1 - h_{ij}\zeta_i - \delta\delta_1/(2J)$ in accordance with (8.16), and poles of any $(1-2^{-v_{ij}})^{-1}$ or $\widehat{f_\Delta}(s_\nu)$ remain on the left of the lines of integration provided that $\delta$ is less than a sufficiently small constant (it will later tend to zero as $B \to \infty$). Having shifted the $\mathbf{s}$-contour in this way, we estimate trivially. The $\mathscr{R}_{\delta,\lambda}$-integral is $\ll \delta^{-J}$ so that

$$\begin{aligned} E_{\Delta,T,\delta,\lambda} &\ll \delta^{-J} B^{1-\frac{\delta\delta_1 N}{2JA}} \sum_{|\mathbf{g}|\leq T} \gamma^{C\mathbf{h}} \int^{(N)} \left| \langle \mathbf{v} \rangle \prod_\nu \widehat{f_\Delta}(s_\nu) \right| |\mathrm{d}\mathbf{s}| \\ &\ll T^{CS+r} \delta^{-J} B^{1-\frac{\delta\delta_1 N}{2JA}} \Delta^{-J+\varepsilon} \end{aligned} \tag{8.17}$$

by (8.4) (which is still applicable if $\delta_3$ is sufficiently small) with $\mathscr{D} = \mathrm{id}$, $c = \varepsilon$, $\|\mathbf{a}\|_1 = J$, where

$$S = \sum_{\rho=1}^{r} \sum_{(i,j)\in S_\rho} h_{ij}. \tag{8.18}$$

### 8.4. Step 4: inserting the asymptotic formula

We now insert the main term in Hypothesis 5.1 into (8.15). In order to compute this properly, we reinsert the cuspidal contribution and replace the range $\mathscr{R}_{\delta,\lambda}$ of integration with $[1,\infty)^J$. In this section, we estimate the error

$$E^*_{\Delta,T,\delta,\lambda} = \sum_{|\mathbf{g}|\leq T} \left| \int_{(1)}^{(N)} \frac{1}{\gamma^{\mathbf{v}}} \left( \prod_{i,j} \frac{v_{ij}}{1-2^{-v_{ij}}} \right) \int_{[1,\infty)^J \setminus \mathscr{R}_{\delta,\lambda}} \frac{\mathscr{E}_{\gamma^*} \mathscr{I}_{\gamma^*}(\mathbf{X})}{\mathbf{X}^{\mathbf{v}+1}} \mathrm{d}\mathbf{X} \prod_{\nu=1}^{N} \left( \widehat{f_\Delta}(s_\nu) B^{s_\nu} \right) \frac{\mathrm{d}\mathbf{s}}{(2\pi \mathrm{i})^N} \right|.$$

We interchange the $\mathbf{s}$- and $\mathbf{X}$-integral and compute the $\mathbf{s}$-integral first. Writing as before each $(1-2^{-v_{ij}})^{-1}$ as a geometric series, we obtain

$$\begin{aligned} &\int_{(1)}^{(N)} \frac{1}{\gamma^{\mathbf{v}} \mathbf{X}^{\mathbf{v}}} \left( \prod_{i,j} \frac{v_{ij}}{1-2^{-v_{ij}}} \right) \prod_{\nu=1}^{N} \left( \widehat{f_\Delta}(s_\nu) B^{s_\nu} \right) \frac{\mathrm{d}\mathbf{s}}{(2\pi \mathrm{i})^N} \\ &= \sum_{\mathbf{k}\in\mathbb{N}_0^J} \int_{(1)}^{(N)} (\widetilde{\mathbf{k}} \cdot \gamma \cdot \mathbf{X})^{-\mathbf{v}} \langle \mathbf{v} \rangle \prod_{\nu=1}^{N} \left( \widehat{f_\Delta}(s_\nu) B^{s_\nu} \right) \frac{\mathrm{d}\mathbf{s}}{(2\pi \mathrm{i})^N}, \end{aligned}$$

and $\langle \mathbf{v} \rangle \prod_\nu (\widehat{f_\Delta}(s_\nu) B^{s_\nu})$ is a linear combination of terms of the form $\prod_{\nu=1}^N s_\nu^{a_\nu} \widehat{f_\Delta}(s_\nu) B^{s_\nu}$ for vectors $\mathbf{a} = (a_\nu) \in \mathbb{N}_0^N$ with $\|\mathbf{a}\|_1 = J$. The inverse Mellin transform of $s^a \widehat{f_\Delta}(s)$ is $\mathtt{D}^a f_\Delta$, where $\mathtt{D}$ is the differential operator $f(x) \mapsto -xf'(x)$. Hence, defining

$$F^{(\mathbf{a})}_{\Delta,B}(\mathbf{x}) = \prod_{\nu=1}^{N} \mathtt{D}^{a_\nu} f_\Delta \left( \frac{|P_\nu(\mathbf{x})|}{B} \right)$$

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with $P_\nu$ as in (8.9), we see that $E^*_{\Delta,T,\delta,\lambda}$ is bounded by a linear combination of terms of the form

$$\begin{aligned}\sum_{|\mathbf{g}|\le T}\int_{[1,\infty)^J\setminus\mathscr{R}_{\delta,\lambda}}\frac{|\mathscr{E}_{\gamma^*}\mathscr{I}_{\gamma^*}(\mathbf{X})|}{\langle\mathbf{X}\rangle}\sum_{\mathbf{k}\in\mathbb{N}_0^J}|F^{(\mathbf{a})}_{\Delta,B}(\widetilde{\mathbf{k}}\cdot\gamma\cdot\mathbf{X})|\,\mathrm{d}\mathbf{X}\\ \ll\Delta^{-J}\sum_{|\mathbf{g}|\le T}\gamma^{\mathbf{h}}\sum_{\mathbf{k}\in\mathbb{N}_0^J}\int_{[1,\infty)^J\setminus\mathscr{R}_{\delta,\lambda}}\Big(\prod_{ij}X_{ij}^{-h_{ij}\zeta_i}\Big)F_{0,B(1+\Delta)}(\widetilde{\mathbf{k}}\cdot\gamma\cdot\mathbf{X})\,\mathrm{d}\mathbf{X}\end{aligned}$$

by Lemma 5.3, (5.9) and (8.2). By (7.11) with $\mathbf{b}=(1,\dots,1)$, $\mathbf{y}=\widetilde{\mathbf{k}}\cdot\gamma$ and $H=((1+\Delta)B)^\delta$, we obtain

$$E^*_{\Delta,T,\delta,\lambda}\ll T^{S+r}\Delta^{-J}B(\log B)^{c_2+\varepsilon}(\delta+(\log B)^{-1})\tag{8.19}$$

with $S$ as in (8.18). Again, $\delta_2^*>0$ in (7.11) ensures that the $\mathbf{k}$-sum converges. Combining Lemma 8.3, (8.17) and (8.19) and choosing $\delta=(\log B)^{-1+\varepsilon}$, we have shown

$$N_{\Delta,T}(B)=N^{(1)}_{\Delta,T}(B)+O(T^{S+r}\Delta^{-J}B(\log B)^{c_2-1+\varepsilon}),\tag{8.20}$$

where

$$N^{(1)}_{\Delta,T}(B)=\sum_{|\mathbf{g}|\le T}\mu(\mathbf{g})\int_{(1)}^{(N)}\frac{1}{\gamma^{\mathbf{v}}}\Big(\prod_{i,j}\frac{v_{ij}}{1-2^{-v_{ij}}}\Big)\int_{[1,\infty)^J}\frac{\mathscr{E}_{\gamma^*}\mathscr{I}_{\gamma^*}(\mathbf{X})}{\mathbf{X}^{\mathbf{v}+\mathbf{1}}}\,\mathrm{d}\mathbf{X}\prod_{\nu=1}^N\Big(\widehat{f_\Delta}(s_\nu)B^{s_\nu}\Big)\frac{\mathrm{d}\mathbf{s}}{(2\pi\mathrm{i})^N}.$$

We insert Lemma 8.1 and integrate over $\mathbf{X}$. This gives

$$\begin{aligned}N^{(1)}_{\Delta,T}(B)=\frac{2^{J^*}}{\pi}\sum_{|\mathbf{g}|\le T}\mu(\mathbf{g})\int_{(1)}^{(N)}\int_{\Re z_i=\zeta_i}^{(k-1)}\frac{\mathscr{E}_{\gamma^*}}{\gamma^{\mathbf{v}}(\gamma^*)^{\mathbf{z}}}\Big(\prod_{i=1}^k\mathscr{K}_i(z_i)\prod_{j=1}^{J_i}\frac{1-2^{h_{ij}z_i-1}}{1-h_{ij}z_i}\Big)\\ \times\Big(\prod_{i=0}^k\prod_{j=1}^{J_i}\frac{v_{ij}}{(1-2^{-v_{ij}})w_{ij}}\Big)\prod_{\nu=1}^N\Big(\widehat{f_\Delta}(s_\nu)B^{s_\nu}\Big)\frac{\mathrm{d}\mathbf{z}}{(2\pi\mathrm{i})^{k-1}}\frac{\mathrm{d}\mathbf{s}}{(2\pi\mathrm{i})^N},\end{aligned}$$

where $w_{ij}=v_{ij}+h_{ij}z_i-1$ and we recall our convention $z_k=1-z_1-\cdots-z_{k-1}$. If we write $\mathbf{w}=(w_{ij})\in\mathbb{C}^J$, then by (8.13) and (7.3), we have

$$\mathbf{w}=\mathscr{A}_1\mathbf{s}+\mathscr{A}_2\mathbf{z}^*,\quad \mathbf{z}^*=(z_1,\dots,z_{k-1},1).\tag{8.21}$$

This explains the seemingly artificial definition of $\mathscr{A}_2$. We can simplify this first by recalling the definition (8.14) of $\gamma^*$, which implies $\gamma^{\mathbf{v}}(\gamma^*)^{\mathbf{z}}=\gamma^{\mathbf{w}+\mathbf{1}}$. Next, we use our convention $h_{0j}=0$ and insert a redundant factor $2^{J_0}\prod_{j=1}^{J_0}(1-2^{h_{0j}z_0-1})$. We also write $\kappa=k-1$. In this way, we can recast $N^{(1)}_{\Delta,T}(B)$ as

$$\frac{2^J}{\pi}\sum_{|\mathbf{g}|\le T}\mu(\mathbf{g})\int_{(1)}^{(N)}\int_{\Re z_i=\zeta_i}^{(\kappa)}\frac{\mathscr{E}_{\gamma^*}}{\gamma^{\mathbf{w}+\mathbf{1}}}\Big(\prod_{i=1}^k\mathscr{K}_i(z_i)\Big)\frac{1}{\langle\mathbf{w}\rangle}\frac{\phi(\mathbf{v})}{\phi(\mathbf{v}-\mathbf{w})}\prod_{\nu=1}^N\Big(\widehat{f_\Delta}(s_\nu)B^{s_\nu}\Big)\frac{\mathrm{d}\mathbf{z}}{(2\pi\mathrm{i})^\kappa}\frac{\mathrm{d}\mathbf{s}}{(2\pi\mathrm{i})^N}$$

where

$$\phi(\mathbf{v})=\prod_{i=0}^k\prod_{j=1}^{J_i}\frac{v_{ij}}{1-2^{-v_{ij}}}.\tag{8.22}$$

### 8.5. Step 5: contour shifts

In this section, we evaluate asymptotically $N^{(1)}_{\Delta,T}(B)$ by contour shifts. Let $\sigma = (\sigma_\nu) \in \mathbb{R}^N_{>0}$ be as in (7.6). For some small $\varepsilon > 0$, we shift the $\mathbf{s}$-contour to $\Re s_\nu = \sigma_\nu + \varepsilon$ without crossing any poles. Shifting a little further to the left will pick up the poles at $\mathbf{w} = 0$, whose residues produce the main term for $N(B)$. To make this transparent, we make a change of variables as follows.

By (7.4), we have $\mathrm{rk}(\mathscr{A}) = \mathrm{rk}(\mathscr{A}_1\, \mathscr{A}_2) = R$, so we can choose $R$ linearly independent members of the linear forms $w_{ij}$ in $\mathbf{s}$ and $\mathbf{z}^* = (z_1, \dots, z_{k-1}, 1)$, say $w^{(1)}, \dots, w^{(R)}$, and then the remaining $w_{ij}$ are linearly dependent. Since also $\mathrm{rk}(\mathscr{A}_1) = R$, we may, for fixed $\mathbf{z}$, change variables in the $\mathbf{s}$-integral by completing the $R$ functions $w^{(1)}, \dots, w^{(R)}$ to a basis in any way such that the determinant of the Jacobian is $\pm 1$. We call the new variables $\mathbf{y} = (y_1, \dots, y_N)$.

We can describe this also in terms of matrices. We pick a maximal linearly independent set of $R$ rows $Z_1, \dots, Z_R$ of the matrix $(\mathscr{A}_1\, \mathscr{A}_2)$. Let $Z_{R+1}, \dots, Z_J$ denote the remaining rows of $(\mathscr{A}_1\, \mathscr{A}_2)$, and let $\mathscr{B} = (b_{kl}) \in \mathbb{R}^{(J-R)\times R}$ be the unique matrix satisfying

$$\mathscr{B}\begin{pmatrix} Z_1 \\ \vdots \\ Z_R \end{pmatrix} = \begin{pmatrix} Z_{R+1} \\ \vdots \\ Z_J \end{pmatrix}. \tag{8.23}$$

That is, $\mathscr{B}$ expresses the remaining $w_{ij}$ in terms of the selected linearly independent set. Again by (7.4), we can also write the last row $(\mathscr{A}_3\, \mathscr{A}_4)$ of $\mathscr{A}$ as a linear combination of $Z_1, \dots, Z_R$, say

$$\sum_{\ell=1}^{R} b_\ell Z_\ell = (\mathscr{A}_3\, \mathscr{A}_4). \tag{8.24}$$

The coefficients $b_{kl}$ and $b_\ell$ play the same role as in Lemma 4.7. Choose a matrix

$$\mathscr{C} = (\mathscr{C}_1\, \mathscr{C}_2) = \begin{pmatrix} & Z_1 & \\ & \vdots & \\ & Z_R & \\ \boxed{\ast} & & \boxed{0} \end{pmatrix} \in \mathbb{R}^{N\times(N+k)}, \quad (\mathscr{C}_1 \in \mathbb{R}^{N\times N}, \mathscr{C}_2 \in \mathscr{R}^{N\times k}), \tag{8.25}$$

with $\boxed{\ast} \in \mathbb{R}^{(N-R)\times N}$ chosen such that $\mathscr{C}_1 \in \mathbb{R}^{N\times N}$ satisfies $\det \mathscr{C}_1 = 1$. This is possible since $\mathrm{rk}(\mathscr{A}_1) = R$ by (7.4). Given $\mathbf{s} \in \mathbb{C}^N$, $\mathbf{z} \in \mathbb{C}^{k-1}$, we define the vector

$$(y_1, \dots, y_N)^\top = \mathbf{y} = \mathbf{y}(\mathbf{s}, \mathbf{z}^*) = \mathscr{C}(\mathbf{s}, \mathbf{z}^*)^\top = \mathscr{C}_1 \mathbf{s}^\top + \mathscr{C}_2 \mathbf{z}^{*\top}. \tag{8.26}$$

We write

$$\eta = \mathbf{y}(\sigma, (\zeta_1, \dots, \zeta_{k-1}, 1)) \in \mathbb{R}^N, \quad \eta^* = \mathbf{y}(\sigma + \varepsilon \cdot \mathbf{1}, (\zeta_1, \dots, \zeta_{k-1}, 1)) \in \mathbb{R}^N$$

with $\sigma$ as in (7.6) and some fixed $\varepsilon > 0$. In the new variables $\mathbf{y}$, the path of integration $\Re s_\nu = \sigma_\nu + \varepsilon$ becomes $\Re y_\nu = \eta^*_\nu$. Moreover, by (8.23) and (8.24), we have

$$\langle \mathbf{w} \rangle = y_1 \cdots y_R \prod_{\iota=1}^{J-R} \mathscr{L}_\iota(\mathbf{y}), \quad \mathscr{L}_\iota(\mathbf{y}) = \sum_{\ell=1}^{R} b_{\iota\ell} y_\ell \tag{8.27}$$

and

$$-1 + \sum_{\nu=1}^{N} s_\nu = \mathscr{L}(\mathbf{y}), \quad \mathscr{L}(\mathbf{y}) = \sum_{\ell=1}^{R} b_\ell y_\ell. \tag{8.28}$$

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Thus, we can recast $N^{(1)}_{\Delta,T}(B)$ as

$$\frac{2^J}{\pi}\sum_{|\mathbf{g}|\le T}\mu(\mathbf{g})\int^{(\kappa)}_{\Re z_i=\zeta_i}\int^{(N)}_{\Re y_\nu=\eta^*_\nu}\frac{\mathscr{E}_{\gamma^*}}{\gamma^{\mathbf{w+1}}}\frac{\phi(\mathbf{v})}{\phi(\mathbf{v}-\mathbf{w})}\Big(\prod_{\nu=1}^N\widehat{f_\Delta}(s_\nu)\Big)\Big(\prod_{i=1}^k\mathscr{K}_i(z_i)\Big)$$
$$\times\frac{B^{1+\mathscr{L}(\mathbf{y})}}{y_1\cdots y_R\prod_{\iota=1}^{J-R}\mathscr{L}_\iota(\mathbf{y})}\frac{\mathrm{d}\mathbf{y}}{(2\pi\mathrm{i})^N}\frac{\mathrm{d}\mathbf{z}}{(2\pi\mathrm{i})^\kappa},\tag{8.29}$$

where now $\mathbf{s},\mathbf{v},\mathbf{w}$ are linear forms in $\mathbf{y},\mathbf{z}^*$ given by (8.13), (8.21), (8.23) and (8.26). We now shift the $y_1,\dots,y_R$-contours appropriately within a sufficiently small $\varepsilon$-neighborhood of $\eta$ (in which in particular $\phi(\mathbf{v})/\phi(\mathbf{v}-\mathbf{w})\prod_\nu\widehat{f_\Delta}(s_\nu)$ is holomorphic), always keeping $\Re z_i=\zeta_i$. Recalling definitions (8.22) and (8.7) as well as $\mathbf{v}-\mathbf{w}=(1-h_{ij}z_{ij})_{ij}$, we record the bound

$$\mathscr{D}\bigg(\frac{\mathscr{E}_{\gamma^*}}{\gamma^{\mathbf{w+1}}}\Big(\phi(\mathbf{v})\prod_{\nu=1}^N\widehat{f_\Delta}(s_\nu)\Big)\Big(\frac{1}{\phi(\mathbf{v}-\mathbf{w})}\prod_{i=1}^k\mathscr{K}_i(z_i)\Big)\bigg)\ll T^S\Delta^{-J-c}|\mathbf{s}|_\infty^{-c}\Big(\prod_{i=1}^k|z_i|^{\zeta_i-\frac12-J_i+\varepsilon}\Big)$$
$$=T^S\Delta^{-J-c}\Big(\prod_{i=1}^k|z_i|^{\zeta_i-\frac12-J_i+\varepsilon}\Big)\big|\mathscr{C}_1^{-1}\mathbf{y}-\mathscr{C}_1^{-1}(\mathscr{C}_2\mathbf{z}^*)\big|_\infty^{-c}\tag{8.30}$$

that holds for any fixed linear differential operator $\mathscr{D}$ with constant coefficients in $s_1,\dots,s_N,z_1,\dots,z_{k-1}$ and any fixed $c>0$. This follows from Stirling's formula, (8.4), (5.9) and (8.18). In particular, choosing $c>N$ and recalling (8.6), this expression is absolutely integrable over $\mathbf{z}$ and $\mathbf{y}$. We return to (8.29) and evaluate the $(y_1,\dots,y_R)$-integral asymptotically by appropriate contour shifts. The integrals that arise are of the form

$$B(\log B)^{\alpha_0}\int^{(R)}\frac{B^{\ell(\tilde{\mathbf{y}})}H(\tilde{\mathbf{y}})}{\ell_1(\tilde{\mathbf{y}})\cdots\ell_{J_0}(\tilde{\mathbf{y}})}\frac{\mathrm{d}\tilde{\mathbf{y}}}{(2\pi i)^{R_0}},$$

where $\alpha_0\in\mathbb{N}_0$, $\ell_1,\dots,\ell_{J_0}$ are linear forms in $R_0$ variables spanning a vector space of dimension $R_0$, $\ell$ is a linear form, the contours of integration are in an $\varepsilon$-neighborhood of $\Re y_\nu=0$ and $H$ is a holomorphic function in this region satisfying the bound (8.30); initially, we have $R_0=R$, $J_0=J$, $\alpha_0=0$. As long as $\Re\ell(\tilde{\mathbf{y}})>0$, we can shift one of the variables to the left (if appearing with positive coefficient) or to the right (if appearing with negative coefficient), getting a small power saving in $B$ in the remaining integral and picking up the residues on the way. Inductively, we see that in each step $J_0-R_0+\alpha_0$ is nonincreasing. Recalling the definition of $c_2$ in (7.5), we obtain eventually

$$N^{(1)}_{\Delta,T}(B)=c^*c_{\mathrm{fin}}(T)c_\infty(\Delta)B(\log B)^{c_2}+O(T^{S+r+\varepsilon}\Delta^{-J-N-\varepsilon}B(\log B)^{c_2-1})\tag{8.31}$$

for some constant $c^*\in\mathbb{Q}$ (to be computed in a moment) and

$$c_{\mathrm{fin}}(T)=\sum_{|\mathbf{g}|\le T}\mu(\mathbf{g})\frac{\mathscr{E}_{\gamma^*}}{\langle\gamma\rangle},$$
$$c_\infty(\Delta)=\frac{2^J}{\pi}\int^{(\kappa)}_{\Re z_i=\zeta_i}\int^{(N-R)}_{\Re y_\nu=\eta^*_\nu}\Big(\prod_{\nu=1}^N\widehat{f_\Delta}(s_\nu)|_{y_1=\dots=y_R=0}\Big)\Big(\prod_{i=1}^k\mathscr{K}_i(z_i)\Big)\frac{\mathrm{d}y_{R+1}\cdots\mathrm{d}y_N}{(2\pi\mathrm{i})^{N-R}}\frac{\mathrm{d}\mathbf{z}}{(2\pi\mathrm{i})^\kappa}.\tag{8.32}$$

That the multiple integral in the formula for $c_\infty(\Delta)$ is absolutely convergent follows again from (8.30). Combining (8.31) with (8.12) and (8.20), we have shown

$$N_\Delta(B)=c^*c_{\mathrm{fin}}(T)c_\infty(\Delta)B(\log B)^{c_2}+O\big(B(\log B)^{c_2-1+\varepsilon}(T^{S+r}\Delta^{-J-N-\varepsilon}+T^{-\delta_2}\log B)\big)\tag{8.33}$$

for any $1<T<B$.

### 8.6. Step 6: computing the leading constant

We proceed to compute explicitly the leading constant in (8.33). In this subsection, we consider $c^*$ and $c_{\rm fin}(T)$, and we start with the former. To this end, we observe that in the course of the contour shifts, only the polar behavior at $\mathbf{w} = 0$ is relevant so that

$$c^* = \lim_{B\to\infty} \frac{1}{(\log B)^{c_2}} \int^{(R)} B^{\mathscr{L}(y)} \prod_{\ell=1}^{R} F(y_\ell) \prod_{\iota=1}^{J-R} \mathscr{L}_\iota(\mathbf{y})^{-1} \frac{\mathrm{d}\mathbf{y}}{(2\pi \mathrm{i})^R}$$

for any function $F$ that is holomorphic except for a simple pole at 0 with residue 1, provided the integral is absolutely convergent. We choose $F = \widehat{f_{\Delta_0}}$ for some $\Delta_0 > 0$ as in (8.2)–(8.3), recall the notation (8.27)–(8.28) and insert the formula $s^{-1} = \int_0^1 t^{s-1}\,\mathrm{d}t$ for $\Re s > 0$. In this way, we get the absolutely convergent expression

$$\begin{aligned} c^* &= \lim_{B\to\infty} \frac{1}{(\log B)^{c_2}} \int^{(R)} B^{\mathscr{L}(y)} \prod_{\ell=1}^{R} \widehat{f_{\Delta_0}}(y_\ell) \int_{[0,1]^{J-R}} \prod_{\iota=1}^{J-R} t_\iota^{\mathscr{L}_\iota(\mathbf{y})-1}\,\mathrm{d}\mathbf{t} \frac{\mathrm{d}\mathbf{y}}{(2\pi \mathrm{i})^R} \\ &= \lim_{B\to\infty} \int^{(R)} B^{\mathscr{L}(y)} \prod_{\ell=1}^{R} \widehat{f_{\Delta_0}}(y_\ell) \int_{[0,\infty]^{J-R}} \prod_{\iota=1}^{J-R} B^{-r_\iota \mathscr{L}_\iota(\mathbf{y})}\,\mathrm{d}\mathbf{r} \frac{\mathrm{d}\mathbf{y}}{(2\pi \mathrm{i})^R} \\ &= \lim_{B\to\infty} \int_{[0,\infty]^{J-R}} \int^{(R)} \Big(\prod_{\ell=1}^{R} \widehat{f_{\Delta_0}}(y_\ell)\Big) B^{\sum_\ell (b_\ell - \sum_\iota r_\iota b_{\iota\ell}) y_\ell} \frac{\mathrm{d}\mathbf{y}}{(2\pi \mathrm{i})^R}\,\mathrm{d}\mathbf{r} \\ &= \lim_{B\to\infty} \int_{[0,\infty]^{J-R}} \prod_{\ell=1}^{R} f_{\Delta_0}\big(B^{-b_\ell + \sum_\iota r_\iota b_{\iota\ell}}\big)\,\mathrm{d}\mathbf{r}. \end{aligned}$$

Here, we used a change of variables along with $c_2 = J - R$ in the first step, cf. (7.5), and Mellin inversion in the last step. This formula holds for every $\Delta_0 > 0$, so we can take the limit $\Delta_0 \to 0$ getting

$$c^* = \mathrm{vol}\Big\{\mathbf{r} \in [0,\infty]^{J-R} : b_\ell - \sum_{\iota=1}^{J-R} r_\iota b_{\iota\ell} \geq 0 \text{ for all } 1 \leq \ell \leq R\Big\}. \tag{8.34}$$

Next, we investigate $c_{\rm fin}(T)$. We can complete the $\mathbf{g}$-sum at the cost of an error

$$\sum_{|\mathbf{g}|>T} \Big|\frac{\mathscr{E}_{\gamma^*}}{\langle\gamma\rangle}\Big| \ll \sum_{\mathbf{g}} \Big(\prod_{ij} \gamma_{ij}^{-1+h_{ij}\beta_i}\Big)\Big(\frac{|\mathbf{g}|}{T}\Big)^{\delta_4-\varepsilon} \ll T^{-\delta_4+\varepsilon}$$

by (5.9), (8.11), (8.14), (8.5) and Lemma 8.2 so that

$$c_{\rm fin}(T) = c_{\rm fin} + O(T^{-\delta_4+\varepsilon}), \quad c_{\rm fin} = \sum_{\mathbf{g}} \mu(\mathbf{g}) \frac{\mathscr{E}_{\gamma^*}}{\langle\gamma\rangle}. \tag{8.35}$$

Using (5.8), we can rewrite $c_{\rm fin}$ in terms of local densities (note that the sum is absolutely convergent). Recall that $\mathbf{g} = (g_1,\ldots,g_r)$ is indexed by the coprimality conditions $S_1,\ldots,S_r$ in (1.4). For a given choice of $\alpha_1,\ldots,\alpha_r \in \{0,1\}$, let

$$S(\alpha) = \bigcup_{\alpha_\rho=1} S_\rho, \quad \delta(ij,\alpha) = \begin{cases} 1, & (i,j) \in S(\alpha), \\ 0, & (i,j) \notin S(\alpha). \end{cases}$$

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Then

$$c_{\text{fin}} = \prod_p \sum_{\alpha \in \{0,1\}^r} \frac{(-1)^{|\alpha|_1}}{p^{\#S(\alpha)}} \cdot \lim_{L\to\infty} \frac{1}{p^{L(J-1)}} \#\Big\{\mathbf{x} \bmod p^L : \sum_{i=1}^k \prod_{j=1}^{J_i} (p^{\delta(ij,\alpha)} x_{ij})^{h_{ij}} \equiv 0 \bmod p^L\Big\}.$$

By inclusion-exclusion, this equals

$$c_{\text{fin}} = \prod_p \lim_{L\to\infty} \frac{1}{p^{L(J-1)}} \#\left\{\mathbf{x} \bmod p^L : \begin{array}{l} \sum_{i=1}^k \prod_{j=1}^{J_i} x_{ij}^{h_{ij}} \equiv 0 \bmod p^L, \\ (\{x_{ij} : (i,j) \in S_\rho\}, p) = 1 \text{ for } 1 \le \rho \le r \end{array}\right\}. \tag{8.36}$$

Combining (8.33) and (8.35), we conclude

$$N_\Delta(B) = c^* c_{\text{fin}} c_\infty(\Delta) B(\log B)^{c_2} + O\Big(B(\log B)^{c_2 - 1 - \delta_0} \Delta^{-J-N-\varepsilon}\Big)$$

for $\delta_0 = \min(\delta_2, \min(\delta_4, 1)(S + r + 1)^{-1}) > 0$, upon choosing $T = (\log B)^{1/(S+r+1)}$. Since $N_\Delta(B)$ is obviously nonincreasing in $\Delta$, we conclude from (8.10) and the previous display that $N(B) = (1 + o(1))c^* c_{\text{fin}} c_\infty B(\log B)^{c_2}$ as $B \to \infty$ with

$$c_\infty = \lim_{\Delta\to 0} c_\infty(\Delta), \tag{8.37}$$

and this limit must exist. We have proved

**Theorem 8.4.** *Suppose that we are given a diophantine equation (1.2) with $b_1 = \cdots = b_k = 1$ and height conditions (1.3) whose variables are restricted by coprimality conditions (1.4). Suppose that Hypotheses 5.1 and 7.2 and (7.4), (7.6), (8.5), (8.6) hold. Then we have the asymptotic formula*

$$N(B) = (1 + o(1))c^* c_{\text{fin}} c_\infty B(\log B)^{c_2}, \quad B \to \infty. \tag{8.38}$$

*Here, $c^*$ is given in (8.34) (using the notation (8.27)–(8.28)), $c_{\text{fin}}$ in (8.36), $c_\infty$ in (8.37) and (8.32) and $c_2$ in (7.5).*

More precisely, we need (5.9) of Hypothesis 5.1 only for $\mathbf{b} = \gamma^*$ and (7.10) of Hypothesis 7.2 only for $\mathbf{b} = \gamma$.

## 9. The Manin–Peyre conjecture

In Sections 5–8, we established an asymptotic formula for a certain counting problem, subject to several hypotheses. By design, we presented this in an axiomatic style without recourse to the underlying geometry. In the section, we relate the asymptotic formula in Theorem 8.4 to the Manin–Peyre conjecture. In particular, we compute $c_\infty$ explicitly, and we will show (under conditions that are easy to check) that the leading constant $c^* c_{\text{fin}} c_\infty$ agrees with Peyre's constant for almost Fano varieties as in Part I. This applies in particular to the spherical Fano varieties in Part III of the paper.

### 9.1. Geometric interpretation of $c_\infty$

In this subsection, we establish the following alternative formulation of the constant $c_\infty$. Recall – cf. (8.25) – that the first $R$ rows of $\mathscr{C} = (\mathscr{C}_1 \mathscr{C}_2)$ are $R$ linearly independent rows of $(\mathscr{A}_1 \mathscr{A}_2)$, let's say indexed by a set $I$ of pairs $(i,j)$ with $0 \le i \le k$, $1 \le j \le J_i$ with $|I| = R$. Let

$$\Phi^*(\mathbf{t}) = \sum_{i=1}^{k} \prod_{(i,j)\in I} t_{ij}^{h_{ij}}, \tag{9.1}$$

and let $\mathscr{F}$ be the affine $(R-1)$-dimensional hypersurface $\Phi^*(\mathbf{t}) = 0$ over $\mathbb{R}$. Let $\chi_I$ be the characteristic function on the set

$$\prod_{(i,j)\in I} |t_{ij}|^{\alpha_{ij}^{\mu}} \le 1, \quad 1 \le \mu \le N.$$

In order to avoid technical difficulties that are irrelevant for the applications we have in mind, we make the simplifying assumption that

$$\text{one of the } k \text{ monomials in } \Phi^* \text{ consists of only one variable, which has exponent } 1. \tag{9.2}$$

Without loss of generality, we can assume that this is the first monomial. (Assumption (9.2) can be removed if necessary and follows from assumption (4.8).)

**Lemma 9.1.** *Suppose that* $\{(1,j) \in I\} = \{(1,1)\}$ *and* $h_{11} = 1$. *Then* $c_\infty$ *is given by the surface integral*

$$c_\infty = 2^{J-R} \int_{\mathscr{F}} \frac{\chi_I(\boldsymbol{t})}{\|\nabla\Phi^*(\boldsymbol{t})\|}\, \mathrm{d}\mathscr{F}\boldsymbol{t}. \tag{9.3}$$

*Proof.* We return to the definition (8.32) of $c_\infty(\Delta)$ and compute the $\mathbf{y}$-integral for fixed $\mathbf{z}$. Let us write $\widehat{F}(\mathbf{y}) = \prod_{\nu=1}^{N} \widehat{f}_\Delta(s_\nu)$. We recall from (8.26) that $\mathbf{y} = \mathscr{C}_1\mathbf{s} + \mathscr{C}_2\mathbf{z}^*$ with $\det\mathscr{C}_1 = 1$, and we view $\mathbf{s}$ as a function of $\mathbf{y}$ (for fixed $\mathbf{z}$). By Mellin inversion one confirms the formula

$$\int_{\Re y_\nu = \eta_\nu^*}^{(N-R)} \widehat{F}(0,\dots,0,y_{R+1},\dots y_N)\frac{\mathrm{d}y_{R+1}\cdots\mathrm{d}y_N}{(2\pi\mathrm{i})^{N-R}} = \int_{\mathbb{R}_{>0}^R}\int_{\Re y_\nu=\eta_\nu^*}^{(N)} \widehat{F}(\mathbf{y})t_1^{y_1}\cdots t_R^{y_R}\frac{\mathrm{d}\mathbf{y}}{(2\pi\mathrm{i})^N}\frac{\mathrm{d}\mathbf{t}}{\langle\mathbf{t}\rangle}.$$

Note that by Mellin inversion, the $\mathbf{t}$-integral on the right-hand side is absolutely convergent, even though the combined $\mathbf{y},\mathbf{t}$-integral is not. (This formula is a distributional version of the 'identity' $\int_0^\infty t^{y-1}\,\mathrm{d}t = \delta_{y=0}$.) Let us write $\mathscr{C} = (\mathscr{C}_1\,\mathscr{C}_2) = (c_{\nu\mu}) \in \mathbb{R}^{N\times(N+k)}$ and $\mathscr{C}_2\mathbf{z}^* = \tilde{\mathbf{z}} \in \mathbb{C}^N$. We change back to $\mathbf{s}$-variables and compute the $\mathbf{s}$-integral in the preceding display by Mellin inversion, getting

$$\int_{\mathbb{R}_{>0}^R}\prod_{\mu=1}^{N} f_\Delta\Big(\prod_{\ell=1}^{R} t_\ell^{-c_{\ell,\mu}}\Big)t_1^{\tilde{z}_1}\cdots t_R^{\tilde{z}_R}\frac{\mathrm{d}\mathbf{t}}{\langle\mathbf{t}\rangle}.$$

By construction this integral is absolutely convergent for every fixed $\mathbf{z}$ with $\Re z_i = \zeta_i$. Plugging back into the definition, we obtain

$$c_\infty(\Delta) = \frac{2^J}{\pi}\int_{\Re z_i=\zeta_i}^{(\kappa)}\prod_{i=1}^{k}\mathscr{K}_i(z_i)\int_{\mathbb{R}_{>0}^R}\prod_{\mu=1}^{N} f_\Delta\Big(\prod_{\ell=1}^{R} t_\ell^{-c_{\ell,\mu}}\Big)t_1^{\tilde{z}_1}\cdots t_R^{\tilde{z}_R}\frac{\mathrm{d}\mathbf{t}}{\langle\mathbf{t}\rangle}\frac{\mathrm{d}\mathbf{z}}{(2\pi\mathrm{i})^\kappa}.$$

Here, the $\mathbf{z}$-integral is absolutely convergent since the multiple integral in (8.32) was absolutely convergent. The combined $\mathbf{t},\mathbf{z}$-integral, however, is not absolutely convergent. Recall that $\kappa = k-1$, $z_k = 1 - z_1 - \cdots - z_\kappa$ and $\mathscr{K}_i(z)$ was defined in (8.7) with inverse Mellin transform $x \mapsto K_i(x)$, say,

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where $K_i(x) = \cos(x)$ or $\exp(\mathrm{i}x)$. In order to avoid convergence problems, we define, for $\varepsilon > 0$, the function

$$K_i^{(\varepsilon)}(x) = K_i(x)e^{-(\varepsilon x)^2} = \begin{cases} \cos(x)e^{-(\varepsilon x)^2}, & h_{ij} \text{ odd for some } 1 \le j \le J_i, \\ e^{ix}e^{-(\varepsilon x)^2}, & h_{ij} \text{ even for all } 1 \le j \le J_i, \end{cases} \tag{9.4}$$

and its Mellin transform $\mathscr{K}_i^{(\varepsilon)}(z) = \int_0^\infty K_i^{(\varepsilon)}(x)x^{z-1}\,\mathrm{d}x$. This can be expressed explicitly in terms of confluent hypergeometric functions by [40, 3.462.1], but we do not need this. It suffices to know that $\mathscr{K}_i^{(\varepsilon)}(z)$ is holomorphic in $\Re z > 0$, rapidly decaying on vertical lines, and we have the pointwise limit $\lim_{\varepsilon\to 0}\mathscr{K}_i^{(\varepsilon)}(z) = \mathscr{K}_i(z)$ for $0 < \Re z < 1$. The latter follows elementarily with one integration by parts by writing

$$\int_0^\infty (K_i(x) - K_i^{(\varepsilon)}(x))x^{z-1}\,\mathrm{d}x = \int_0^{\varepsilon^{-1/2}} + \int_{\varepsilon^{-1/2}}^\infty \ll \varepsilon^{1/2} + \varepsilon^{1/2} \to 0$$

for $\varepsilon \to 0$. Correspondingly, we write

$$c_\infty^{(\varepsilon)}(\Delta) = \frac{2^J}{\pi}\int_{\Re z_i = \zeta_i}^{(\kappa)} \prod_{i=1}^{k} \mathscr{K}_i^{(\varepsilon)}(z_i) \int_{\mathbb{R}_{>0}^R} \prod_{\mu=1}^{N} f_\Delta\Big(\prod_{\ell=1}^{R} t_\ell^{-c_{\ell,\mu}}\Big) t_1^{\tilde{z}_1}\cdots t_R^{\tilde{z}_R}\frac{\mathrm{d}\mathbf{t}}{\langle \mathbf{t}\rangle}\frac{\mathrm{d}\mathbf{z}}{(2\pi \mathrm{i})^\kappa}.$$

This multiple integral is now absolutely convergent, and by dominated convergence we have

$$c_\infty(\Delta) = \lim_{\varepsilon\to 0} c_\infty^{(\varepsilon)}(\Delta). \tag{9.5}$$

We interchange the $\mathbf{t}$- and $\mathbf{z}$-integral, fix $\mathbf{t}$ and compute the $\mathbf{z}$-integral. Mellin inversion yields

$$\mathscr{K}_k^{(\varepsilon)}(1 - z_1 - \cdots - z_\kappa) = \int_0^\infty \int_{(\frac{1}{2}\zeta_k)} \mathscr{K}_k^{(\varepsilon)}(z_k)x^{-z_1-\cdots-z_k}\frac{\mathrm{d}z_k}{2\pi \mathrm{i}}\,\mathrm{d}x$$

for $\Re z_i = \zeta_i$, $1 \le i \le \kappa$. Note that on the right-hand side $\Re(z_1 + \cdots + z_k) < 1$ (which is why we chose $\Re z_k = \frac{1}{2}\zeta_k$). Again, the double integral is not absolutely convergent, but the $x$-integral is absolutely convergent. In particular, after substituting this into the definition of $c_\infty^{(\varepsilon)}(\Delta)$, we may interchange the $x$-integral and the $z_1,\dots,z_\kappa$-integral to conclude

$$c_\infty^{(\varepsilon)}(\Delta) = \frac{2^J}{\pi}\int_{\mathbb{R}_{>0}^R}\int_0^\infty \int^{(k)} \prod_{i=1}^{k}\mathscr{K}_i^{(\varepsilon)}(z_i)\prod_{\mu=1}^{N} f_\Delta\Big(\prod_{\ell=1}^{R} t_\ell^{-c_{\ell,\mu}}\Big)t_1^{\tilde{z}_1}\cdots t_R^{\tilde{z}_R}x^{-z_1-\cdots-z_k}\frac{\mathrm{d}\mathbf{z}}{(2\pi\mathrm{i})^k}\,\mathrm{d}x\frac{\mathrm{d}\mathbf{t}}{\langle\mathbf{t}\rangle},$$

where $\Re z_i = \zeta_i$, $1 \le i \le \kappa$, $\Re z_k = \frac{1}{2}\zeta_k$. By Mellin inversion, we can now compute each of the $z_1,\dots,z_\kappa$-integrals. We recall our notation $\tilde{\mathbf{z}} = \mathscr{C}_2\mathbf{z}^*$, so

$$\tilde{z}_j = \sum_{i=1}^{\kappa} c_{j,N+i}z_i + c_{j,N+k}.$$

This gives

$$c_\infty^{(\varepsilon)}(\Delta) = \frac{2^J}{\pi}\int_{\mathbb{R}_{>0}^R}\int_0^\infty \Big[\prod_{\mu=1}^{N} f_\Delta\Big(\prod_{\ell=1}^{R} t_\ell^{-c_{\ell,\mu}}\Big)\Big]\Big[K_k^{(\varepsilon)}(x)\prod_{i=1}^{\kappa}K_i^{(\varepsilon)}\Big(x\prod_{\nu=1}^{R}t_\nu^{-c_{\nu,N+i}}\Big)\Big]\prod_{\nu=1}^{R}t_\nu^{c_{\nu,N+k}}\,\mathrm{d}x\frac{\mathrm{d}\mathbf{t}}{\langle\mathbf{t}\rangle}.$$

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Changing variables $t_\nu \mapsto t_\nu^{-1}$ and then $x \mapsto 2\pi x \prod_{\nu=1}^R t_\nu^{1+c_{\nu,N+k}}$, this becomes

$$2^J \int_{\mathbb{R}^R_{>0}} \int_{-\infty}^{\infty} \Big[ \prod_{\mu=1}^N f_\Delta\Big( \prod_{\ell=1}^R t_\ell^{c_{\ell,\mu}} \Big) \Big] \Big[ K_k^{(\varepsilon)}(2\pi x \prod_{\nu=1}^R t_\nu^{1+c_{\nu,N+k}}) \prod_{i=1}^{\kappa} K_i^{(\varepsilon)}\Big( 2\pi x \prod_{\nu=1}^R t_\nu^{c_{\nu,N+i}+1+c_{\nu,N+k}} \Big) \Big] \, \mathrm{d}x \, \mathrm{d}\mathbf{t}.$$

We reindex the variables $t_\nu$ as $t_{ij}$ with $(i,j) \in I$, as described prior to the statement of the lemma. By the definition of $(\mathscr{A}_1\mathscr{A}_2)$ in (3.10), we then have

$$\prod_{\nu=1}^R t_\nu^{c_{\nu,N+i}+1+c_{\nu,N+k}} = \prod_{(i,j)\in I} t_{ij}^{h_{ij}} \quad (1 \le i \le \kappa), \qquad \prod_{\nu=1}^R t_\nu^{1+c_{\nu,N+k}} = \prod_{(k,j)\in I} t_{kj}^{h_{kj}}$$

so that

$$c_\infty^{(\varepsilon)}(\Delta) = 2^J \int_{-\infty}^{\infty} \int_{\mathbb{R}^R_{>0}} \Big[ \prod_{\mu=1}^N f_\Delta\Big( \prod_{(i,j)\in I} t_{ij}^{\alpha_{ij}^\mu} \Big) \Big] \Big[ \prod_{i=1}^k K_i^{(\varepsilon)}\Big( 2\pi x \prod_{(i,j)\in I} t_{ij}^{h_{ij}} \Big) \Big] \, \mathrm{d}x \, \mathrm{d}\mathbf{t}.$$

By symmetry, we may extend $\mathbf{t}$-integral to all of $\mathbb{R}^R$, recall (9.4) and write

$$c_\infty^{(\varepsilon)}(\Delta) = 2^{J-R} \int_{-\infty}^{\infty} \int_{\mathbb{R}^R} \Psi_\Delta(\mathbf{t}) e\big(x\Phi^*(\mathbf{t})\big) \exp\big( -(\pi\varepsilon x)^2 \tilde{\Phi}(\mathbf{t})\big) \, \mathrm{d}x \, \mathrm{d}\mathbf{t}$$

with $\Phi^*$ as in (9.1) and

$$\Psi_\Delta(\mathbf{t}) = \prod_{\mu=1}^N f_\Delta\Big( \prod_{(i,j)\in I} |t_{ij}|^{\alpha_{ij}^\mu} \Big), \qquad \tilde{\Phi}(\mathbf{t}) = 4 \sum_{i=1}^k \prod_{(i,j)\in I} t_{ij}^{2h_{ij}}.$$

We compute the $x$-integral, getting

$$c_\infty^{(\varepsilon)}(\Delta) = \frac{2^{J-R}}{\sqrt{\pi}\varepsilon} \int_{\mathbb{R}^R} \Psi_\Delta(\mathbf{t}) \exp\Big( -\frac{(\Phi^*)^2(\mathbf{t})}{\varepsilon^2 \tilde{\Phi}(\mathbf{t})} \Big) \frac{\mathrm{d}\mathbf{t}}{\sqrt{\tilde{\Phi}(\mathbf{t})}}.$$

By construction, this is absolutely convergent for every fixed $\varepsilon > 0$, and the limit as $\varepsilon \to 0$ exists by (9.5). Let $\mathscr{U} := \{\mathbf{t} \in \mathbb{R}^R : |(\Phi^*)^2(\mathbf{t})/\tilde{\Phi}(\mathbf{t})| \le 1/25\}$. Writing

$$\exp\Big( -\frac{(\Phi^*)^2(\mathbf{t})}{\varepsilon^2 \tilde{\Phi}(\mathbf{t})} \Big) = \exp\Big( -\frac{(\Phi^*)^2(\mathbf{t})}{\tilde{\Phi}(\mathbf{t})} \Big) \exp\Big( (1-\varepsilon^{-2}) \frac{(\Phi^*)^2(\mathbf{t})}{\tilde{\Phi}(\mathbf{t})} \Big),$$

we obtain

$$c_\infty^{(\varepsilon)}(\Delta) = \frac{2^{J-R}}{\sqrt{\pi}\varepsilon} \int_{\mathscr{U}} \Psi_\Delta(\mathbf{t}) \exp\Big( -\frac{(\Phi^*)^2(\mathbf{t})}{\varepsilon^2 \tilde{\Phi}(\mathbf{t})} \Big) \frac{\mathrm{d}\mathbf{t}}{\sqrt{\tilde{\Phi}(\mathbf{t})}} + O\Big( \frac{1}{\varepsilon} e^{(1-\varepsilon^{-2})/25} \Big).$$

We consider now the equation

$$\Phi^*(\mathbf{t})/\sqrt{\tilde{\Phi}(\mathbf{t})} - u = 0 \tag{9.6}$$

for $|u| \le 1/5$. It is only at this point that we use (9.2). We write $\mathbf{t} = (t_{11}, \mathbf{t}')$ and

$$\Phi^*(\mathbf{t}) = t_{11} + (\Phi^*)'(\mathbf{t}'), \quad \tilde{\Phi}(\mathbf{t}) = 4t_{11}^2 + \tilde{\Phi}'(\mathbf{t}').$$

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Then for $u = 0$, the equation (9.6) has the unique solution $t_{11} = -(\Phi^*)'(\mathbf{t}')$, while for $0 < |u| \leq 1/5$, both $u$ and $-u$ lead to two solutions

$$t_{11} = \frac{-(\Phi^*)'(\mathbf{t}') \pm |u|\sqrt{4(\Phi^*)'(\mathbf{t}')^2 + \tilde{\Phi}'(\mathbf{t}')(1-4u^2)}}{1-4u^2} =: \phi_u^{\pm}(\mathbf{t}').$$

For $u = 0$, we have $\phi_0^+ = \phi_0^-$, and for notational simplicity we write $\phi_0^{\pm} = \phi = -(\Phi^*)'$. Changing variables, we obtain

$$\frac{2^{J-R}}{\sqrt{\pi}\varepsilon}\int_{\mathcal{U}} \Psi_\Delta(\mathbf{t}) \exp\Big(-\frac{(\Phi^*)^2(\mathbf{t})}{\varepsilon^2\tilde{\Phi}(\mathbf{t})}\Big)\frac{\mathrm{d}\mathbf{t}}{\sqrt{\tilde{\Phi}(\mathbf{t})}} = \frac{2^{J-R}}{\sqrt{\pi}\varepsilon}\int_{-1/5}^{1/5} \exp\Big(-\frac{u^2}{\varepsilon^2}\Big)\Theta(u)\,\mathrm{d}u,$$

where

$$\Theta(u) = \int_{\mathbb{R}^{R-1}} \Xi(\phi_u^+(\mathbf{t}'),\mathbf{t}')\,\mathrm{d}\mathbf{t}', \quad \Xi = \frac{2\tilde{\Phi}\Psi_\Delta}{|2\tilde{\Phi}\Phi^*_{t_{11}} - \Phi^*\tilde{\Phi}_{t_{11}}|}.$$

By a Taylor expansion, we have $\Theta(u) = \Theta(0) + O(|u|)$ for $|u| \leq 1/5$ so that

$$\begin{aligned} c_\infty(\Delta) = \lim_{\varepsilon\to 0}\frac{2^{J-R}}{\sqrt{\pi}\varepsilon}\int_{-\eta}^{\eta}\exp\Big(-\frac{u^2}{\varepsilon^2}\Big)\Theta(u)\,\mathrm{d}u = 2^{J-R}\Theta(0) &= 2^{J-R}\int_{\mathbb{R}^{R-1}}\Xi(\phi(\mathbf{t}'),\mathbf{t}')\,\mathrm{d}\mathbf{t}' \\ &= 2^{J-R}\int_{\mathbb{R}^{R-1}}\frac{\Psi_\Delta(\phi(\mathbf{t}'),\mathbf{t}')}{|\Phi^*_{t_{11}}(\phi(\mathbf{t}'),\mathbf{t}')|}\,\mathrm{d}\mathbf{t}'. \end{aligned}$$

Here, we can let $\Delta \longrightarrow 0$, obtaining

$$c_\infty = 2^{J-R}\int_{\mathbb{R}^{R-1}}\frac{\chi_I(\phi(\mathbf{t}'),\mathbf{t}')}{|\Phi^*_{t_{11}}(\phi(\mathbf{t}'),\mathbf{t}')|}\,\mathrm{d}\mathbf{t}'. \tag{9.7}$$

(Note that the denominator is 1 by (9.2), but that this formula should also hold without this assumption.) We write this more symmetrically as follows. If $t_{ij}$ is any component of $\mathbf{t}'$, then by implicit differentiation, we have

$$\phi_{t_{ij}}(\mathbf{t}) = -\frac{\Phi^*_{t_{ij}}(\phi(\mathbf{t}'),\mathbf{t}')}{\Phi^*_{t_{11}}(\phi(\mathbf{t}')\mathbf{t}')},$$

so that we can write $c_\infty$ as a surface integral

$$2^{J-R}\int_{\mathbb{R}^{R-1}}\frac{\chi_I(\phi(\mathbf{t}'),\mathbf{t}')}{|\Phi^*_{t_{11}}(\phi(\mathbf{t}'),\mathbf{t}')|}d\mathbf{t}' = 2^{J-R}\int_{\mathscr{F}}\frac{\chi_I(\mathbf{t})}{\|\nabla\Phi^*(\mathbf{t})\|}d\mathscr{F}(\mathbf{t})$$

as claimed. □

### *9.2. Comparison with the Manin–Peyre conjecture*

**Theorem 9.2.** *Let $X, H$ be as in Proposition 4.11. Suppose that the corresponding counting problem for $U \subset X$ given by Proposition 3.8 satisfies all assumptions of Theorem 8.4. Then the Manin–Peyre conjecture holds for $X$ with respect to $H$, that is,*

$$N_{X,U,H}(B) = (1+o(1))cB(\log B)^{\operatorname{rk}\operatorname{Pic}X-1}$$

*with Peyre's constant $c$.*

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*Proof.* By Proposition 3.8,

$$N_{X,U,H}(B) = 2^{-\operatorname{rk}\operatorname{Pic} X} N(B)$$

for $N(B)$ as in (1.5). Formula (8.38) in Theorem 8.4 states that

$$N(B) = (1 + o(1))c^* c_{\mathrm{fin}} c_\infty B(\log B)^{c_2}.$$

Comparing definition (4.6) with expression (8.36) for $c_{\mathrm{fin}}$, the definitions (4.10) and (8.34) of $c^*$, and definition (4.12) with expression (9.7) for $c_\infty$ (which are both valid since assumption (4.8) implies (9.2)), then Proposition 4.11 shows that the leading constant for $N_{X,U,H}(B)$ is Peyre's constant, and $c_2 = J - R = \operatorname{rk}\operatorname{Pic} X - 1$ by (4.9), (7.5) and Lemma 3.10. Therefore, Proposition 3.8 combined with (8.38) agrees with the Manin–Peyre conjecture. □

The following part provides numerous applications and shows how to apply this in practice.

## Part III Application to spherical varieties

Having established the relevant theory in Part I and Part II of the paper, we are now prepared to prove Manin's conjecture for concrete families of varieties. In particular, as a consequence of Theorem 10.1, we obtain Manin's conjecture for all smooth spherical Fano threefolds of semisimple rank one and type $T$.

## 10. Spherical varieties

### *10.1. Luna–Vust invariants*

Let $G$ be a connected reductive group over $\overline{\mathbb{Q}}$. Let $\overline{\mathbb{Q}}(X)$ be the function field of a spherical $G$-variety $X$ over $\overline{\mathbb{Q}}$. Only in this section and in Section 11.1, let $B$ denote a Borel subgroup of $G$ with character group $\mathfrak{X}(B)$. The *weight lattice* is defined as

$$\mathscr{M} = \left\{ \chi \in \mathfrak{X}(B) : \begin{array}{l} \text{there exists } f_\chi \in \overline{\mathbb{Q}}(X)^\times \text{ such that} \\ b \cdot f_\chi = \chi(b) \cdot f_\chi \text{ for every } b \in B \end{array} \right\}.$$

Note that for every $\chi \in \mathscr{M}$, the function $f_\chi$ is uniquely determined up to a constant factor because of the dense $B$-orbit in $X$. The *set of colors* $\mathscr{D}$ is the set of $B$-invariant prime divisors on $X$ that are not $G$-invariant. Moreover, we have the *valuation cone* $\mathscr{V} \subseteq \mathscr{N}_\mathbb{Q} = \operatorname{Hom}(\mathscr{M}, \mathbb{Q})$, which can be identified with the $\mathbb{Q}$-valued $G$-invariant discrete valuations on $\overline{\mathbb{Q}}(X)^\times$. By Losev's uniqueness theorem [52, Theorem 1], the combinatorial invariants $(\mathscr{M}, \mathscr{V}, \mathscr{D})$ uniquely determine the birational class of (i. e., the open $G$-orbit in) the spherical $G$-variety $X$ over $\overline{\mathbb{Q}}$.

Now, let $\Delta$ be the set of all $B$-invariant prime divisors on $X$. There is a map $\mathfrak{c}\colon \Delta \to \mathscr{N}_\mathbb{Q}$ defined by $\langle \mathfrak{c}(D), \chi \rangle = \nu_D(f_\chi)$, where $\nu_D$ is the valuation on $\overline{\mathbb{Q}}(X)^\times$ induced by the prime divisor $D$. For every $G$-orbit $Z \subseteq X$, we define $\mathscr{W}_Z = \{D \in \Delta : Z \subseteq D\}$. Then the collection

$$\operatorname{CF} X = \{(\operatorname{cone}(\mathfrak{c}(\mathscr{W}_Z)), \mathscr{W}_Z \cap \mathscr{D}) : Z \subseteq X \text{ is a } G-\text{orbit}\}$$

is called the *colored fan* of $X$. According to the Luna–Vust theory of spherical embeddings [54, 50], the colored fan $\operatorname{CF} X$ uniquely determines the spherical $G$-variety $X$ over $\overline{\mathbb{Q}}$ among those in the same birational class.

The divisor class group $\operatorname{Cl} X$ can be computed from $\operatorname{CF} X$: By [18, Proposition 4.1.1], the maps $\mathscr{M} \to \mathbb{Z}^\Delta$, $\chi \mapsto \operatorname{div} f_\chi$ and $\mathbb{Z}^\Delta \to \operatorname{Cl} X$, $D \mapsto [D]$ fit into the exact sequence $\mathscr{M} \to \mathbb{Z}^\Delta \to \operatorname{Cl} X \to 0$.

Spherical varieties with $\mathscr{V} = \mathscr{N}_\mathbb{Q}$ are called *horospherical*. These include flag varieties and toric varieties. In the latter case, $G = B = T$ is a torus, and we have $\mathscr{V} = \mathscr{N}_\mathbb{Q}$ and $\mathscr{D} = \emptyset$.

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### *10.2. Semisimple rank one*

Let $X$ be a spherical $G$-variety over $\overline{\mathbb{Q}}$. If the connected reductive group $G$ has semisimple rank one, we may assume $G = \mathrm{SL}_2 \times \mathbb{G}_{\mathrm{m}}^r$ by passing to a finite cover. As a further simplification, we replace the action by a *smart action* as introduced in [1, Definition 4.3]. As before, let $G/H = (\mathrm{SL}_2 \times \mathbb{G}_{\mathrm{m}}^r)/H$ be the open orbit in $X$. Let $H' \times \mathbb{G}_{\mathrm{m}}^r = H \cdot \mathbb{G}_{\mathrm{m}}^r \subseteq \mathrm{SL}_2 \times \mathbb{G}_{\mathrm{m}}^r$. Then the homogeneous space $\mathrm{SL}_2/H'$ is spherical, and hence either $H'$ is a maximal torus in $\mathrm{SL}_2$ (*the case* $T$) or $H'$ is the normalizer of a maximal torus in $\mathrm{SL}_2$ (*the case* $N$) or the homogeneous space $\mathrm{SL}_2/H'$ is horospherical. Since the action is smart, in the horospherical case $H'$ is either a Borel subgroup in $\mathrm{SL}_2$ (*the case* $B$) or the whole group $\mathrm{SL}_2$ (*the case* $G$).

Now, let $T \subset G = \mathrm{SL}_2 \times \mathbb{G}_{\mathrm{m}}^r$ be a maximal torus, and let $\alpha \in \mathfrak{X}(T) \cong \mathfrak{X}(B)$ be the simple root with respect to a Borel subgroup $B \subset G$. It follows from the general theory of spherical varieties that in the cases $T$ and $N$, we always have $\mathscr{V} = \{v \in \mathscr{N}_{\mathbb{Q}} : \langle v, \alpha \rangle \le 0\}$. The colored cones of the form $(\mathbb{Q}_{\ge 0} \cdot u, \emptyset) \in \mathrm{CF}\, X$, where $u \in \mathscr{M} \cap \mathscr{V}$ is a primitive element, correspond to the $G$-invariant prime divisors in $X$. Let $(\mathbb{Q}_{\ge 0} \cdot u_{0j}, \emptyset) \in \mathrm{CF}\, X$ for $j = 1, \ldots, J_0$ be those with $u \in \mathscr{V} \cap (-\mathscr{V})$, and let $(\mathbb{Q}_{\ge 0} \cdot u_{3j}, \emptyset) \in \mathrm{CF}\, X$ for $j = 1, \ldots, J_3$ be those with $u \notin \mathscr{V} \cap (-\mathscr{V})$. We denote by $D_{ij}$ the $G$-invariant prime divisor in $X$ corresponding to $(\mathbb{Q}_{\ge 0} \cdot u_{ij}, \emptyset) \in \mathrm{CF}\, X$. Then we have $\mathfrak{c}(D_{ij}) = u_{ij}$.

We define $h_{3j} = -\langle u_{3j}, \alpha \rangle$. The following descriptions of the Cox rings in the different cases can be explicitly obtained from [18, Theorem 4.3.2] or [33, Theorem 3.6].

Case $T$: There are two colors $D_{11}, D_{12} \in \mathscr{D}$, and we have $\mathfrak{c}(D_{11}) + \mathfrak{c}(D_{12}) = \alpha^\vee|_{\mathscr{M}}$. The Cox ring is given by

$$\mathscr{R}(X) = \overline{\mathbb{Q}}[x_{01}, \ldots, x_{0J_0}, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, \ldots, x_{3J_3}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}^{h_{31}} \cdots x_{3J_3}^{h_{3J_3}}), \quad (10.1)$$

cf. (1.6), with

$$\deg(x_{11}) = \deg(x_{21}) = [D_{11}] \in \mathrm{Cl}\, X, \quad \deg(x_{12}) = \deg(x_{22}) = [D_{12}] \in \mathrm{Cl}\, X, \text{ and}$$
$$\deg(x_{ij}) = [D_{ij}] \in \mathrm{Cl}\, X \text{ for } i \in \{0, 3\}.$$

Case $N$: There is one color $D_{11} \in \mathscr{D}$, and we have $\mathfrak{c}(D_{11}) = \frac{1}{2}\alpha^\vee|_{\mathscr{M}}$. The Cox ring is given by

$$\mathscr{R}(X) = \overline{\mathbb{Q}}[x_{01}, \ldots, x_{0J_0}, x_{11}, x_{12}, x_{21}, x_{31}, \ldots, x_{3J_3}]/(x_{11}x_{12} - x_{21}^2 - x_{31}^{h_{31}} \cdots x_{3J_3}^{h_{3J_3}})$$

with

$$\deg(x_{11}) = \deg(x_{12}) = \deg(x_{21}) = [D_{11}] \in \mathrm{Cl}\, X, \quad \deg(x_{ij}) = [D_{ij}] \in \mathrm{Cl}\, X \text{ for } i \in \{0, 3\}.$$

Case $B$: We mention this case only for completeness since $X$ is isomorphic to a toric variety here (as an abstract variety with a different group action). There is one color $D_{11} \in \mathscr{D}$, and we have $\mathfrak{c}(D_{11}) = \alpha^\vee|_{\mathscr{M}}$. The Cox ring is given by $\mathscr{R}(X) = \overline{\mathbb{Q}}[x_{01}, \ldots, x_{0J_0}, x_{11}, x_{12}]$ with

$$\deg(x_{11}) = \deg(x_{12}) = [D_{11}] \in \mathrm{Cl}\, X, \quad \deg(x_{0j}) = [D_{0j}] \in \mathrm{Cl}\, X.$$

Case $G$: We mention this case only for completeness since $X$ is a toric $\mathbb{G}_m^r$-variety here. We have $\mathscr{D} = \emptyset$. The Cox ring is given by $\mathscr{R}(X) = \overline{\mathbb{Q}}[x_{01}, \ldots, x_{0J_0}]$ with $\deg(x_{0j}) = [D_{0j}] \in \mathrm{Cl}\, X$.

### *10.3. Ambient toric varieties*

Every quasiprojective variety $X$ with finitely generated Cox ring may be embedded into a toric variety $Y^\circ$ with nice properties, as described in [2, 3.2.5].

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For a spherical variety $X$, this is explicitly described in [35]. According to [18, Theorem 4.3.2], the Cox ring of $X$ is generated by the union of sets $x_{D1}, \dots, x_{Dr_D} \in \mathscr{R}(X)$ for every $D \in \Delta$. We have $r_D = 1$ if $D \notin \mathscr{D}$ and $r_D \geq 2$ if $D \in \mathscr{D}$. Each $x_{Di}$ corresponds to a ray $\rho_{Di}$ in the fan $\Sigma^\circ$ of the ambient toric variety $Y^\circ$.

Even if $X$ is projective, the quasiprojective toric variety $Y^\circ$ might not be projective. This is the case if and only if the colored cones in CF $X$ do not cover $\mathscr{N}_{\mathbb{Q}}$.

Any $\mathscr{W} \subseteq \Delta$ defines a pair $(\mathrm{cone}(\mathfrak{c}(\mathscr{W})), \mathscr{W} \cap \mathscr{D})$. If $\mathrm{cone}(\mathfrak{c}(\mathscr{W}))$ is strictly convex, we call the pair a *supported colored cone* if $\mathrm{cone}(\mathfrak{c}(\mathscr{W}))^\circ \cap \mathscr{V} \neq \emptyset$ and an *unsupported colored cone* if $\mathrm{cone}(\mathfrak{c}(\mathscr{W}))^\circ \cap \mathscr{V} = \emptyset$. If we can extend CF $X$ by some of these unsupported colored cones to a collection $(\mathrm{CF}\,X)_{\mathrm{ext}}$ such that every face (in the sense of [71, Definition 15.3]) of a colored cone is again in $(\mathrm{CF}\,X)_{\mathrm{ext}}$ such that different colored cones intersect in faces and such that the colored cones cover the whole space $\mathscr{N}_{\mathbb{Q}}$, then $(\mathrm{CF}\,X)_{\mathrm{ext}}$ yields a toric variety $Y$ that completes $Y^\circ$.

We recall here how to obtain the fan $\Sigma$ of the toric variety $Y$ from the (possibly extended) colored fan $(\mathrm{CF}\,X)_{\mathrm{ext}}$. Let $\Psi_D = \{\rho_{D1}, \dots, \rho_{Dr_D}\}$, and define $\Psi_D^j = \Psi_D \setminus \{\rho_{Dj}\}$ for every $1 \leq j \leq r_D$. For every subset $\mathscr{W} \subseteq \Delta$, consider the sets of cones

$$\Phi(\mathscr{W}) = \left\{ \mathrm{cone}\left( \bigcup_{D \in \mathscr{W}} \Psi_D \cup \bigcup_{D \in \Delta \setminus \mathscr{W}} \Psi_D^{j(D)} \right) : j \in \mathbb{N}^{\Delta \setminus \mathscr{W}},\ 1 \leq j(D) \leq r_D \right\}.$$

Then we have

$$\Sigma = \bigcup_{(\mathrm{cone}(\mathfrak{c}(\mathscr{W})), \mathscr{W} \cap \mathscr{D}) \in (\mathrm{CF}\,X)_{\mathrm{ext}}} \Phi(\mathscr{W}) \quad \text{and} \quad \Sigma_{\max} = \bigcup_{(\mathrm{cone}(\mathfrak{c}(\mathscr{W})), \mathscr{W} \cap \mathscr{D}) \in (\mathrm{CF}\,X)_{\mathrm{ext},\max}} \Phi(\mathscr{W}). \tag{10.2}$$

### *10.4. Manin's conjecture*

We present now the main result of this paper, which implies all theorems stated in the introduction.

**Theorem 10.1.** *Let $X$ be a smooth split spherical almost Fano variety of semisimple rank one and type $T$ over $\mathbb{Q}$ with semiample $\omega_X^\vee$ satisfying (2.3) whose colored fan* CF $X$ *contains a maximal cone without colors.*

*The corresponding counting problem as in Proposition 3.8 features a torsor equation (1.6) with exponents $h_{ij}$, a height matrix $\mathscr{A}$ as in (7.1) and coprimality conditions $S_1, \dots S_r$ as in (1.4). Choose $\zeta$ satisfying (5.10) and (8.6), let $\lambda$ be as in (5.13) and choose $\tau^{(2)}$ as in (7.18).*

*With these data, assume that (7.24) and (7.35) hold. Then the Manin–Peyre conjecture holds for $X$ with respect to the anticanonical height function (3.7).*

*Proof.* It is enough to check all assumptions of Theorem 9.2.

We observe that $X$ is as in Proposition 4.11 by our assumptions. In particular by (10.1), its Cox ring is as required. By (10.2), a maximal cone without colors in CF $X$ gives four maximal cones $\sigma \in \Sigma_{\max}$ such that the variables corresponding to the rays of $\sigma$ include precisely one of $x_{11}, x_{21}$ and precisely one of $x_{12}, x_{22}$ in (10.1); it is not hard to see that one of these four cones satisfies (4.8).

Next, we check that Theorem 8.4 applies. The counting problem is of the required form by Proposition 3.8 and (10.1). Hypothesis 5.1 holds by Proposition 5.2, whose assumptions are satisfied by (10.1) and which allows us to choose

$$\beta = \left( \frac{1}{2} - \frac{1}{5 \max_{ij} h_{ij}}, \frac{1}{2} - \frac{1}{5 \max_{ij} h_{ij}}, \frac{2}{5 \max_{ij} h_{ij}} \right),$$

so that (8.5) holds. Condition (8.6) means $\zeta_3 < 1/2$ which is consistent with (5.10). Hypothesis 7.2 holds by Proposition 7.6. The conditions (7.4), (7.6) hold by Lemmas 3.10 and 3.11. □

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The assumption (2.3) can be read off of the colored fan CF $X$, using the method described in Section 10.3. The existence of a maximal cone without colors in CF $X$ is straightforward to check and clearly holds in all our examples below; alternatively, (4.8) can be checked directly. As mentioned after Proposition 7.6, if (7.24) fails, we can apply an alternative, but slightly more complicated criterion. Assumption (7.35) requires elementary linear algebra (and can be checked quickly by computer if desired).

**Remark 10.2.** If the torsor equation is $x_{11}x_{12}+x_{21}x_{22}+x_{31}x_{33} = 0$, we can use [9, Proposition 1.2] instead of Proposition 5.2 to verify Hypothesis 5.1, which conveniently yields again $\beta = (1/3+\varepsilon, 1/3+\varepsilon, 1/3+\varepsilon)$ and more importantly

$$\lambda = 1.$$

The advantage is that the third line of (7.32) is trivially satisfied (the polytope is empty) so that checking (7.35) requires a little less computational effort.

## 11. Spherical Fano threefolds

### *11.1. Geometry*

According to [44, §6.3], all horospherical smooth Fano threefolds are either toric or flag varieties. Furthermore, there are nine smooth Fano threefolds over $\overline{\mathbb{Q}}$ that are spherical but not horospherical; they are equipped with an action of $G = \mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}$. The notation $T$ and $N$ in [44, Table 6.5] and in our Table 11.1 refers to the cases in Section 10.2.

We proceed to describe the four $T$ cases $X_1, \dots, X_4$ in Table 11.1 that are not equivariant $\mathbb{G}_\mathrm{a}^3$-compactifications [46] in more detail. In each case, we first construct a split form over $\mathbb{Q}$ following the elementary description from the Mori–Mukai classification, and then we give the description using the Luna–Vust theory of spherical embeddings from Hofscheier's list. Finally, we describe in each case an ambient toric variety $Y_i$ satisfying (2.3) that can be used with Sections 2–4.

Let $\varepsilon_1 \in \mathfrak{X}(B)$ be a primitive character of $\mathbb{G}_\mathrm{m}$ composed with the natural inclusion $\mathfrak{X}(\mathbb{G}_\mathrm{m}) \to \mathfrak{X}(B)$.

#### 11.1.1. $X_1$ of type III.24 and $X_4$ of type IV.7

Consider $\mathbb{P}^2_\mathbb{Q} \times \mathbb{P}^2_\mathbb{Q}$ with coordinates $(z_{11} : z_{21} : z_{31})$ and $(z_{12} : z_{22} : z_{32})$, and the hypersurface $W_4 = \mathbb{V}(z_{11}z_{12} - z_{21}z_{22} - z_{31}z_{32}) \subset \mathbb{P}^2_\mathbb{Q} \times \mathbb{P}^2_\mathbb{Q}$ of bidegree $(1, 1)$. This is a smooth Fano threefold of type II.32. It contains the curves

$$C_{01} = \mathbb{V}(z_{11}, z_{21}, z_{32}) = \{(0:0:1)\} \times \mathbb{V}(z_{32}),$$
$$C_{02} = \mathbb{V}(z_{12}, z_{22}, z_{31}) = \mathbb{V}(z_{31}) \times \{(0:0:1)\}$$

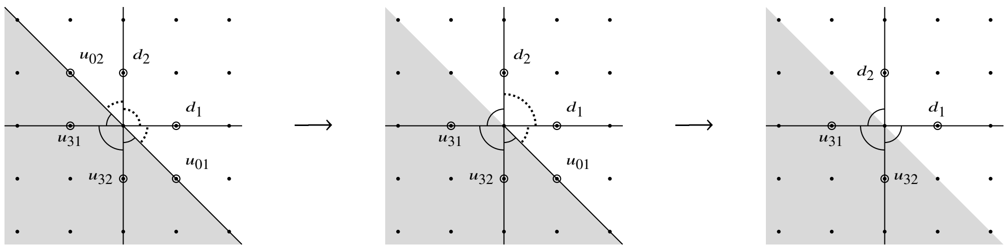
***Table 11.1.*** *Smooth Fano threefolds that are spherical but not horospherical.*

| rk Pic | Hofscheier | Mori–Mukai | torsor equation | remark |
|---|---|---|---|---|
| 2 | $T_1 12$ | II.31 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}^2$ | eq. $\mathbb{G}_\mathrm{a}^3$-cpct. |
| 2 | $N_1 6$, $N_1 7$ | II.30 | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}$ | eq. $\mathbb{G}_\mathrm{a}^3$-cpct. |
| 2 | $N_1 8$ | II.29 | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}^2x_{33}$ | |
| 3 | $T_1 18$ | III.24 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | variety $X_1$ |
| 3 | $T_1 21$ | III.20 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}^2$ | variety $X_2$ |
| 3 | $N_0 3$ | III.22 | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}$ | |
| 3 | $N_1 9$ | III.19 | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}$ | |
| 4 | $T_0 3$ | IV.8 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | variety $X_3$ |
| 4 | $T_1 22$ | IV.7 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | variety $X_4$ |

of bidegrees $(0,1)$ and $(1,0)$, respectively. Let $X_1$ be the blow-up of $W_4$ in the curve $C_{01}$. This is a smooth Fano threefold of type III.24. Moreover, let $X_4$ be the further blow-up in the curve $C_{02}$ (which is disjoint from the curve $C_{01}$ in $W_4$). This is a smooth Fano threefold of type IV.7. We may define an action of $G = \mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}$ on $W_4$ by

$$(A,t)\cdot\left(\begin{pmatrix} z_{11} & z_{22} \\ z_{21} & z_{12}\end{pmatrix}, z_{31}, z_{32}\right) = \left(A\cdot\begin{pmatrix} z_{11} & z_{22} \\ z_{21} & z_{12}\end{pmatrix}\cdot\begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}, z_{31}, z_{32}\right),$$

which turns $W_4$ into a spherical variety. The following description using the Luna–Vust theory of spherical embeddings can be easily verified. The lattice $\mathscr{M}$ has basis $(\frac{1}{2}\alpha+\varepsilon_1, \frac{1}{2}\alpha-\varepsilon_1)$. We denote the corresponding dual basis of the lattice $\mathscr{N}$ by $(d_1, d_2)$. Then there are two colors with valuations $d_1$ and $d_2$, and the valuation cone is given by $\mathscr{V} = \{v \in \mathscr{N}_\mathbb{Q} : \langle v, \alpha\rangle \le 0\}$. Since the curves $C_{01}$ and $C_{02}$ are $G$-invariant, the varieties $X_1$ and $X_4$ are spherical $G$-varieties and the blow-up morphisms $X_4 \to X_1 \to W_4$ can be described by maps of colored fans. The following figure illustrates this.




Here, the elements $u_{31} = -d_1$ and $u_{32} = -d_2$ are the valuations of the $G$-invariant prime divisors $\mathbb{V}(z_{31})$ and $\mathbb{V}(z_{32})$, respectively, while the elements $u_{01} = d_1 - d_2$ and $u_{02} = -d_1 + d_2$ are the valuations of the exceptional divisors $E_{01}$ and $E_{02}$ over $C_{01}$ and $C_{02}$, respectively. In particular, we see that $X_1$ is the fourth line and that $X_4$ is the last line of Hofscheier's list.

The dotted circles in the colored fans of $X_1$ and $X_4$ specify projective ambient toric varieties $Y_1$ and $Y_4$, respectively. From the description of $\Sigma_{\max}$ in Section 10.3, we deduce that $Y_1$ and $Y_4$ are smooth, that $-K_{X_1}$ is ample on $Y_1$ and that $-K_{X_4}$ is ample on $Y_4$. Hence, assumption (2.3) holds.

**11.1.2. $X_2$ of type III.20**

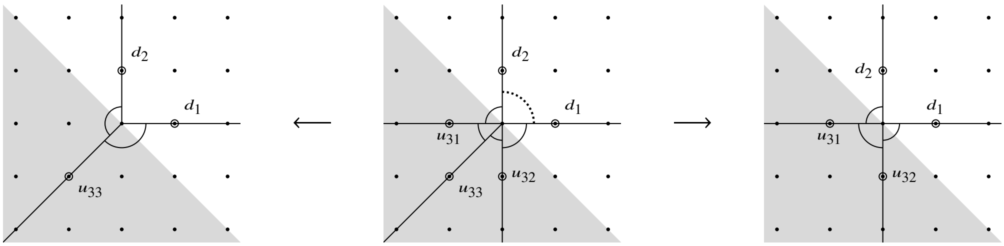
Consider $\mathbb{P}^4_\mathbb{Q}$ with coordinates $(z_{11} : z_{12} : z_{21} : z_{22} : z_{33})$ and the hypersurface $Q = \mathbb{V}(z_{11}z_{12} - z_{21}z_{22} - z_{33}^2) \subset \mathbb{P}^4_\mathbb{Q}$. It contains the lines

$$C_{31} = \mathbb{V}(z_{12}, z_{22}, z_{33}), \quad C_{32} = \mathbb{V}(z_{11}, z_{21}, z_{33}).$$

Let $X_2$ be the blow-up of $Q$ in the lines $C_{31}$ and $C_{32}$. This is a smooth Fano threefold of type III.20. We may define an action of $G = \mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}$ on $Q$ by

$$(A,t)\cdot\left(\begin{pmatrix} z_{11} & z_{22} \\ z_{21} & z_{12}\end{pmatrix}, z_{33}\right) = \left(A\cdot\begin{pmatrix} z_{11} & z_{22} \\ z_{21} & z_{12}\end{pmatrix}\cdot\begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}, z_{33}\right),$$

which turns $Q$ into a spherical variety. Since the lines $C_{31}$ and $C_{32}$ are $G$-invariant, the variety $X_2$ is a spherical $G$-variety. Since $X_2$ is also the blow-up of $W_4$ in the curve $C_{33} = \mathbb{V}(z_{31}, z_{32})$, it has the same birational invariants as $W_4$ and the blow-up morphisms $Q \leftarrow X_2 \rightarrow W_4$ can be described by maps of colored fans as illustrated in the following picture.




In particular, we see that $X_2$ is the fifth line of Hofscheier's list.

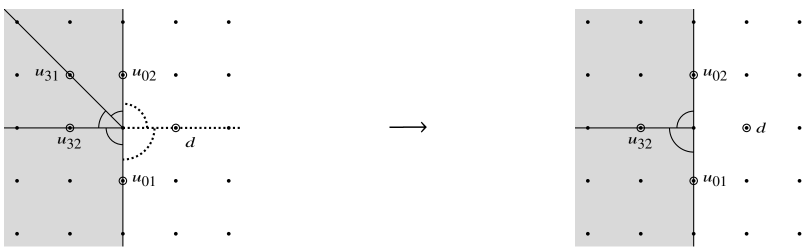
As before, the dotted circle in the colored fan of $X_2$ specifies a projective ambient toric variety $Y_2$, which satisfies (2.3).

**11.1.3. $X_3$ of type IV.8**

Consider $W_3 = \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ with coordinates $(z_{01} : z_{02})$, $(z_{11} : z_{21})$ and $(z_{12} : z_{22})$. This is a smooth Fano threefold of type III.27. Let $C_{31}$ be the curve $\mathbb{V}(z_{02}, z_{11}z_{12} - z_{21}z_{22})$ of tridegree $(0, 1, 1)$ on $W_3$. Let $X_3$ be the blow-up of $W_3$ in $C_{31}$. This is a smooth Fano threefold of type IV.8. We may define an action of $G = \mathrm{SL}_2 \times \mathbb{G}_{\mathrm{m}}$ on $W_3$ by

$$(A, t) \cdot \left(z_{01}, z_{02}, \begin{pmatrix} z_{11} & z_{22} \\ z_{21} & z_{12} \end{pmatrix}\right) = \left(t \cdot z_{01}, z_{02}, A \cdot \begin{pmatrix} z_{11} & z_{22} \\ z_{21} & z_{12} \end{pmatrix}\right),$$

which turns $W_3$ into a spherical variety. Its Luna–Vust description is a follows. The lattice $\mathcal{M}$ has basis $(\alpha, \varepsilon_1)$. We denote the corresponding dual basis of the lattice $\mathcal{N}$ by $(d, \varepsilon_1^*)$. Then there are two colors with the same valuation $d = \frac{1}{2}\alpha^\vee$, and the valuation cone is given by $\mathcal{V} = \{v \in \mathcal{N}_{\mathbb{Q}} : \langle v, \alpha \rangle \le 0\}$. Since the curve $C_{31}$ is $G$-invariant, the variety $X_3$ is a spherical $G$-variety and the blow-up morphism $X_3 \rightarrow W_3$ can be described by the map of colored fans in the figure below.




Here, the elements $u_{01} = -\varepsilon_1^*$ and $u_{02} = \varepsilon_1^*$ are the valuations of the $G$-invariant prime divisors $\mathbb{V}(z_{01})$ and $\mathbb{V}(z_{02})$, respectively, the element $u_{32} = -d$ is the valuation of the $G$-invariant prime divisor $\mathbb{V}(z_{11}z_{12} - z_{21}z_{22})$, and $u_{31} = -d + \varepsilon_1^*$ is the valuation of the exceptional divisor $E_{31}$ over $C_{31}$. This is the penultimate line of Hofscheier's list.

The dotted circles in the colored fan of $X_3$ are meant to specify a projective ambient toric variety $Y_3$, but since there are two colors with the same valuation $d$, the picture is ambiguous. There are three possibilities for which unsupported colored cones could be added to the colored cone of $X_3$ to obtain an ambient toric variety:

1. $(\mathrm{cone}(u_{01}, d), \{D_{11}\})$ and $(\mathrm{cone}(u_{02}, d), \{D_{11}\})$,
2. $(\mathrm{cone}(u_{01}, d), \{D_{12}\})$ and $(\mathrm{cone}(u_{02}, d), \{D_{12}\})$ or
3. $(\mathrm{cone}(u_{01}, d), \{D_{11}, D_{12}\})$ and $(\mathrm{cone}(u_{02}, d), \{D_{11}, D_{12}\})$.

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From the description of $\Sigma_{\max}$ in Section 10.3, we deduce that the ambient toric variety in case (3) is singular. On the other hand, in cases (1) and (2), the ambient toric variety is smooth, and $-K_{X_3}$ not ample but semiample on it. We fix $Y_3$ to be as in case (1), satisfying (2.3).

### *11.2. Cox rings and torsors*

We proceed to compute explicitly the Cox rings $\mathscr{R}(X)$ in the examples from Section 11.1 using Section 10.2 together with [30] since we work over $\mathbb{Q}$ here. To obtain the universal torsor $\mathscr{T} = X_0$, we compute the set $Z_Y$ as in Section 2.2. Moreover, we give simplified expressions for $Z_X = Z_Y \cap \operatorname{Spec} \mathscr{R}(X)$, which can be verified using the equation $\Phi$. Finally the anticanonical class is computed using [17, 4.1 and 4.2] or [2, Proposition 3.3.3.2]. In the case of a spherical variety of semisimple rank one of type $T$ or $N$, this is simply the sum of all $B$-invariant divisors.

#### 11.2.1. Type III.24

We have

$$\mathscr{R}(X_1) = \mathbb{Q}[x_{01}, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32})$$

with $\operatorname{Pic} X_1 \cong \mathbb{Z}^3$, where

$$\begin{aligned}&\deg(x_{01}) = (0,0,1), \quad \deg(x_{11}) = \deg(x_{21}) = (0,1,-1),\\ &\deg(x_{12}) = \deg(x_{22}) = (1,0,0), \quad \deg(x_{31}) = (0,1,0), \quad \deg(x_{32}) = (1,0,-1).\end{aligned}$$

Note that each generator $x_{ij}$ of the Cox ring corresponds to the strict transform of $\mathbb{V}(z_{ij})$ or to the element $u_{ij}$ in Section 11.1.1. The anticanonical class is $-K_{X_1} = (2,2,-1)$. A universal torsor over $X_1$ is

$$\mathscr{T}_1 = \operatorname{Spec} \mathscr{R}(X_1) \setminus Z_{Y_1} = \operatorname{Spec} \mathscr{R}(X_1) \setminus Z_{X_1},$$

where

$$\begin{aligned}Z_{Y_1} &= \mathbb{V}(x_{11}, x_{21}, x_{31}) \cup \mathbb{V}(x_{11}, x_{21}, x_{32}) \cup \mathbb{V}(x_{12}, x_{22}, x_{01}) \cup \mathbb{V}(x_{12}, x_{22}, x_{32}) \cup \mathbb{V}(x_{01}, x_{31}),\\ Z_{X_1} &= \mathbb{V}(x_{11}, x_{21}) \cup \mathbb{V}(x_{12}, x_{22}, x_{32}) \cup \mathbb{V}(x_{01}, x_{31}).\end{aligned}$$

#### 11.2.2. Type III.20

The Cox ring is

$$\mathscr{R}(X_2) = \mathbb{Q}[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{33}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}^2)$$

with $\operatorname{Pic} X_2 \cong \mathbb{Z}^3$, where

$$\begin{aligned}&\deg(x_{11}) = \deg(x_{21}) = (0,1,0), \quad \deg(x_{12}) = \deg(x_{22}) = (1,0,0),\\ &\deg(x_{31}) = (0,1,-1), \quad \deg(x_{32}) = (1,0,-1), \quad \deg(x_{33}) = (0,0,1).\end{aligned}$$

The anticanonical class is $-K_{X_2} = (2,2,-1)$. A universal torsor over $X_2$ is

$$\mathscr{T}_2 = \operatorname{Spec} \mathscr{R}(X_2) \setminus Z_{Y_2} = \operatorname{Spec} \mathscr{R}(X_2) \setminus Z_{X_2},$$

where

$$\begin{aligned}Z_{Y_2} &= \mathbb{V}(x_{11}, x_{21}, x_{31}) \cup \mathbb{V}(x_{11}, x_{21}, x_{33}) \cup \mathbb{V}(x_{12}, x_{22}, x_{32}) \cup \mathbb{V}(x_{12}, x_{22}, x_{33}) \cup \mathbb{V}(x_{31}, x_{32}),\\ Z_{X_2} &= \mathbb{V}(x_{11}, x_{21}, x_{31}) \cup \mathbb{V}(x_{11}, x_{21}, x_{33}) \cup \mathbb{V}(x_{12}, x_{22}, x_{32}) \cup \mathbb{V}(x_{12}, x_{22}, x_{33}) \cup \mathbb{V}(x_{31}, x_{32}).\end{aligned}$$

**11.2.3. Type IV.8**
The Cox ring is

$$\mathscr{R}(X_3) = \mathbb{Q}[x_{01}, x_{02}, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32})$$

with $\operatorname{Pic} X_3 \cong \mathbb{Z}^4$, where

$$\begin{aligned}
&\deg(x_{01}) = (1,0,0,0), \quad \deg(x_{02}) = (1,0,0,-1),\\
&\deg(x_{11}) = \deg(x_{21}) = (0,0,1,0), \quad \deg(x_{12}) = \deg(x_{22}) = (0,1,0,0),\\
&\deg(x_{31}) = (0,0,0,1), \quad \deg(x_{32}) = (0,1,1,-1).
\end{aligned}$$

The anticanonical class is $-K_{X_3} = (2,2,2,-1)$. A universal torsor over $X_3$ is

$$\mathscr{T}_3 = \operatorname{Spec} \mathscr{R}(X_3) \setminus Z_{Y_3} = \operatorname{Spec} \mathscr{R}(X_3) \setminus Z_{X_3},$$

where

$$\begin{aligned}
Z_{Y_3} &= \mathbb{V}(x_{11}, x_{21}, x_{31}) \cup \mathbb{V}(x_{11}, x_{21}, x_{32}) \cup \mathbb{V}(x_{12}, x_{22}) \cup \mathbb{V}(x_{02}, x_{32}) \cup \mathbb{V}(x_{01}, x_{02}) \cup \mathbb{V}(x_{01}, x_{31}),\\
Z_{X_3} &= \mathbb{V}(x_{11}, x_{21}) \cup \mathbb{V}(x_{12}, x_{22}) \cup \mathbb{V}(x_{02}, x_{32}) \cup \mathbb{V}(x_{01}, x_{02}) \cup \mathbb{V}(x_{01}, x_{31}).
\end{aligned}$$

**11.2.4. Type IV.7**
The Cox ring is

$$\mathscr{R}(X_4) = \mathbb{Q}[x_{01}, x_{02}, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32})$$

with $\operatorname{Pic} X_4 \cong \mathbb{Z}^4$, where

$$\begin{aligned}
&\deg(x_{01}) = (0,0,0,1), \quad \deg(x_{02}) = (0,0,1,0),\\
&\deg(x_{11}) = \deg(x_{21}) = (0,1,0,-1), \quad \deg(x_{12}) = \deg(x_{22}) = (1,0,-1,0),\\
&\deg(x_{31}) = (0,1,-1,0), \quad \deg(x_{32}) = (1,0,0,-1).
\end{aligned}$$

The anticanonical class is $-K_{X_4} = (2,2,-1,-1)$. A universal torsor is over $X_4$ is

$$\mathscr{T}_4 = \operatorname{Spec} \mathscr{R}(X_4) \setminus Z_{Y_4} = \operatorname{Spec} \mathscr{R}(X_4) \setminus Z_{X_4},$$

where

$$\begin{aligned}
Z_{Y_4} = {} & \mathbb{V}(x_{11}, x_{21}, x_{01}) \cup \mathbb{V}(x_{11}, x_{21}, x_{31}) \cup \mathbb{V}(x_{11}, x_{21}, x_{32})\\
& \cup \mathbb{V}(x_{12}, x_{22}, x_{02}) \cup \mathbb{V}(x_{12}, x_{22}, x_{31}) \cup \mathbb{V}(x_{12}, x_{22}, x_{32})\\
& \cup \mathbb{V}(x_{02}, x_{32}) \cup \mathbb{V}(x_{01}, x_{02}) \cup \mathbb{V}(x_{01}, x_{31}),\\
Z_{X_4} = {} & \mathbb{V}(x_{11}, x_{21}) \cup \mathbb{V}(x_{12}, x_{22}) \cup \mathbb{V}(x_{02}, x_{32}) \cup \mathbb{V}(x_{01}, x_{02}) \cup \mathbb{V}(x_{01}, x_{31}).
\end{aligned}$$

Note that this is the same variety as $\mathscr{T}_3$ but with a different action of $\mathbb{G}_{\mathrm{m},\mathbb{Q}}^4$.

### *11.3. Counting problems*

Applying Proposition 3.8 to the Cox rings of the previous section gives the following counting problems, in which $U$ is always the subset where all Cox coordinates are nonzero. To lighten the notation, we generally write $\{x, y\}$ to mean $x$ or $y$, and as in the introduction, we write $N_j(B)$ for $N_{X_j,U_j,H_j}(B)$.

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**Corollary 11.1.** (a) *We have*

$$N_1(B) = \frac{1}{8}\#\left\{\mathbf{x} \in \mathbb{Z}^7_{\neq 0} : \begin{array}{l} x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32} = 0, \quad \max |\mathscr{P}_1(\mathbf{x})| \leq B, \\ (x_{11}, x_{21}) = (x_{12}, x_{22}, x_{32}) = (x_{01}, x_{31}) = 1 \end{array}\right\},$$

*where*

$$\mathscr{P}_1(\mathbf{x}) = \left\{\begin{array}{l} x_{31}^2x_{32}^2x_{01}, x_{32}^2x_{01}^3\{x_{11}, x_{21}\}^2, x_{31}^2x_{32}\{x_{12}, x_{22}\}, \\ x_{31}\{x_{11}, x_{21}\}\{x_{12}, x_{22}\}^2, x_{01}\{x_{11}, x_{21}\}^2\{x_{12}, x_{22}\}^2 \end{array}\right\}.$$

(b) *We have*

$$N_2(B) = \frac{1}{8}\#\left\{\mathbf{x} \in \mathbb{Z}^7_{\neq 0} : \begin{array}{l} x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}^2 = 0, \quad \max |\mathscr{P}_2(\mathbf{x})| \leq B, \\ (x_{11}, x_{21}, x_{31}) = (x_{11}, x_{21}, x_{33}) = 1 \\ (x_{12}, x_{22}, x_{32}) = (x_{12}, x_{22}, x_{33}) = (x_{31}, x_{32}) = 1 \end{array}\right\},$$

*where*

$$\mathscr{P}_2(\mathbf{x}) = \left\{\begin{array}{l} x_{32}\{x_{11}, x_{21}\}^2\{x_{12}, x_{22}\}, x_{32}^2x_{33}\{x_{11}, x_{21}\}^2, x_{31}\{x_{11}, x_{21}\}\{x_{12}, x_{22}\}^2, \\ x_{31}^2x_{33}\{x_{12}, x_{22}\}^2, x_{31}^2x_{32}^2x_{33}^3 \end{array}\right\}.$$

(c) *We have*

$$N_3(B) = \frac{1}{16}\#\left\{\mathbf{x} \in \mathbb{Z}^8_{\neq 0} : \begin{array}{l} x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32} = 0, \quad \max |\mathscr{P}_3(\mathbf{x})| \leq B, \\ (x_{11}, x_{21}) = (x_{12}, x_{22}) = (x_{02}, x_{32}) = (x_{01}, x_{02}) = (x_{01}, x_{31}) = 1 \end{array}\right\},$$

*where*

$$\mathscr{P}_3(\mathbf{x}) = \left\{\begin{array}{l} x_{02}^2x_{31}^3x_{32}^2, x_{01}^2x_{31}x_{32}^2, x_{02}^2\{x_{11}, x_{21}\}^2\{x_{12}, x_{22}\}^2x_{31} \\ x_{01}^2\{x_{11}, x_{21}\}\{x_{12}, x_{22}\}x_{32}, x_{01}x_{02}\{x_{11}, x_{21}\}^2\{x_{12}, x_{22}\}^2 \end{array}\right\}.$$

(d) *We have*

$$N_4(B) = \frac{1}{16}\#\left\{\mathbf{x} \in \mathbb{Z}^8_{\neq 0} : \begin{array}{l} x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32} = 0, \quad \max |\mathscr{P}_4(\mathbf{x})| \leq B, \\ (x_{11}, x_{21}) = (x_{12}, x_{22}) = (x_{02}, x_{32}) = (x_{01}, x_{02}) = (x_{01}, x_{31}) = 1 \end{array}\right\},$$

*where*

$$\mathscr{P}_4(\mathbf{x}) = \left\{\begin{array}{l} x_{01}x_{02}x_{31}^2x_{32}^2, x_{01}^2\{x_{11}, x_{21}\}x_{31}x_{32}^2, x_{02}^2\{x_{12}, x_{22}\}x_{31}^2x_{32}, \\ x_{01}^2\{x_{11}, x_{21}\}^2\{x_{12}, x_{22}\}x_{32}, x_{02}^2\{x_{11}, x_{21}\}\{x_{12}, x_{22}\}^2x_{31}, x_{01}x_{02}\{x_{11}, x_{21}\}^2\{x_{12}, x_{22}\}^2 \end{array}\right\}.$$

*Proof.* This is a special case of Proposition 3.8. Note that the coprimality conditions are derived from the expressions for $Z_X$ (instead of $Z_Y$) from Section 11.2. It can be explicitly verified using the equation $\Phi$ that this is correct even over $\mathbb{Z}$ as required here. □

### *11.4. Application: proof of Theorem 1.1*

We now show how to use Theorem 10.1 in practice and complete the proof of Theorem 1.1 for the varieties $X_1, \dots, X_4$.

**11.4.1. The variety $X_4$**

By Corollary 11.1(d), we have $J = 8$ torsor variables $x_{ij}$ with $0 \le i \le 3$, $1 \le j \le 2$ satisfying the equation

$$x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{32} = 0 \tag{11.1}$$

(after changing the signs of $x_{22}, x_{32}$) with $k = 3$ and $h_{ij} = 1$ for $i \ge 1$, $h_{0j} = 0$. In particular, Remark 10.2 applies. We have $N = 17$ height conditions with corresponding exponent matrix

$$\mathscr{A}_1 = \begin{pmatrix} 1 & 2 & 2 & & & 2 & 2 & 2 & 2 & & & & & 1 & 1 & 1 & 1 \\ 1 & & & 2 & 2 & & & & & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ & & 1 & & & & 2 & & 2 & & 1 & & 1 & & & 2 & 2 \\ & & & 1 & & 1 & 1 & & & 2 & 2 & & & 2 & & 2 & \\ & 1 & & & & 2 & & 2 & & 1 & & 1 & & 2 & 2 & & \\ & & & & 1 & & & 1 & 1 & & & 2 & 2 & & 2 & & 2 \\ 2 & 1 & 1 & 2 & 2 & & & & & 1 & 1 & 1 & 1 & & & & \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & & \end{pmatrix} \in \mathbb{R}_{\ge 0}^{8\times 17}, \quad \mathscr{A}_2 = \begin{pmatrix} & & -1 \\ & & -1 \\ 1 & & -1 \\ 1 & & -1 \\ & 1 & -1 \\ & 1 & -1 \\ -1 & -1 & \\ -1 & -1 & \end{pmatrix} \in \mathbb{R}^{8\times 3}.$$

As usual, missing entries indicate zeros. We have $r = 5$ coprimality conditions with

$$S_1 = \{(1,1),(2,1)\}, \quad S_2 = \{(1,2),(2,2)\}, \quad S_3 = \{(0,2),(3,2)\}, \tag{11.2}$$

$$S_4 = \{(0,1),(0,2)\}, \quad S_5 = \{(0,1),(3,1)\}. \tag{11.2}$$

We choose

$$\tau^{(2)} = (\underbrace{1,\dots,1}_{J_0}, \tfrac{2}{3},\dots,\tfrac{2}{3}), \quad \zeta = (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}). \tag{11.4}$$

(In our case $J_0 = 2$, but we will use the same definition also in other cases later.) Using a computer algebra system, we confirm $C_2(\tau^{(2)})$, $C_2((1 - h_{ij}/3)_{ij})$, and with $c_2 = 3$, we find

$$\dim(\mathscr{H} \cap \mathscr{P}) = 3, \quad \dim(\mathscr{H} \cap \mathscr{P}_{ij}) = 2 \text{ for all } (i,j),$$

confirming (7.35). We have now checked all assumptions of Theorem 10.1.

We show in Appendix A how to derive Hypothesis 7.2 without computer help and how to compute the Peyre constant in explicit algebraic terms.

**11.4.2. The variety $X_3$**

This is very similar to the previous case, so we can be brief. By Corollary 11.1(c), we have the same torsor variables as in the previous application satisfying (11.1). The corresponding exponent matrix is given by

$$\mathscr{A}_1 = \begin{pmatrix} & 2 & & & & & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & & 2 & 2 & 2 & 2 & & & & & 1 & 1 & 1 & 1 \\ & & & 2 & & 2 & & 1 & & 1 & & 2 & & 2 \\ & & 2 & 2 & & & 1 & 1 & & & 2 & 2 & & \\ & & 2 & & 2 & & 1 & & 1 & & 2 & & 2 & \\ & & & & 2 & 2 & & & 1 & 1 & & & 2 & 2 \\ 3 & 1 & 1 & 1 & 1 & 1 & & & & & & & & \\ 2 & 2 & & & & & 1 & 1 & 1 & 1 & & & & \end{pmatrix} \in \mathbb{R}_{\ge 0}^{8\times 14}.$$

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We choose $\tau^{(2)}$ and $\zeta$ as before and confirm (7.35) in the same way with

$$\dim(\mathscr{H} \cap \mathscr{P}) = 3, \quad \dim(\mathscr{H} \cap \mathscr{P}_{ij}) = 1 \text{ for } (i, j) = (0, 1) \text{ and } \dim(\mathscr{H} \cap \mathscr{P}_{ij}) = 2 \text{ otherwise.}$$

### 11.4.3. The variety $X_1$

Again, the computations are a minor variation on the previous two cases. By Corollary 11.1(a), the height matrix is

$$\mathscr{A}_1 = \begin{pmatrix} 1 & 3 & 3 & & & & & & & 1 & 1 & 1 & 1 \\ & 2 & & & & 1 & 1 & & & 2 & 2 & & \\ & & & & 1 & & 2 & & 2 & & 2 & & 2 \\ & & 2 & & & & & 1 & 1 & & & 2 & 2 \\ & & & 1 & & 2 & & 2 & & 2 & & 2 & \\ 2 & & & 2 & 2 & 1 & 1 & 1 & 1 & & & & \\ 2 & 2 & 2 & 1 & 1 & & & & & & & & \end{pmatrix} \in \mathbb{R}^{7\times 13}_{\geq 0}.$$

We make the same choice (11.4) for $\tau^{(2)}$ and $\zeta$ and confirm (7.35) with $c_2 = 2$ and

$$\dim(\mathscr{H} \cap \mathscr{P}) = 2, \quad \dim(\mathscr{H} \cap \mathscr{P}_{ij}) = 0 \text{ for } (i, j) = (1, 2), (2, 2), (3, 1), \ \dim(\mathscr{H} \cap \mathscr{P}_{ij}) = 1 \text{ otherwise.}$$

### 11.4.4. The variety $X_2$

This case has some new features, as the torsor equation has a slightly different shape. By Corollary 11.1(b), we have $J_0 = 0$ and $J = 7$ torsor variables satisfying the more complicated torsor equation

$$x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{32}x_{33}^2 = 0.$$

The height matrix is given by

$$\mathscr{A}_1 = \begin{pmatrix} 2 & 2 & & & 2 & & 1 & 1 & & & & & \\ & 1 & & 1 & & & & 2 & & 2 & & 2 & \\ & & 2 & 2 & & 2 & & & 1 & 1 & & & \\ 1 & & 1 & & & & 2 & & 2 & & 2 & & \\ & & & & & & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & & & & & & & 2 \\ & & & & 1 & 1 & & & & & 1 & 1 & 3 \end{pmatrix} \in \mathbb{R}^{7\times 13}_{\geq 0}, \quad \mathscr{A}_2 = \begin{pmatrix} 1 & & -1 \\ 1 & & -1 \\ & 1 & -1 \\ & 1 & -1 \\ -1 & -1 & \\ -1 & -1 & \\ -2 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{7\times 3}.$$

Proposition 5.2 ensures the validity of Hypothesis 5.1 with $\lambda = 1/45,000$. We have $r = 5$ coprimality conditions

$$S_1 = \{(1, 1), (2, 1), (3, 1)\}, \quad S_2 = \{(1, 1), (2, 1), (3, 3)\}, \quad S_3 = \{(1, 2), (2, 2), (3, 2)\},$$
$$S_4 = \{(1, 2), (2, 2), (3, 3)\}, \quad S_5 = \{(3, 1), (3, 2)\}.$$

We see that (7.24) holds. We choose

$$\tau^{(2)} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1)$$

satisfying (7.18) and confirm $C_2(\tau^{(2)})$, $C_2((1 - h_{ij}/3)_{ij})$. Finally, we note that $c_2 = 2$ and compute[2]

$$\begin{aligned} \dim(\mathscr{H} \cap \mathscr{P}) &= 2, \\ \dim(\mathscr{H} \cap \mathscr{P}_{ij}) &= \begin{cases} 1, & (i, j) = (3, 1), (3, 2), (3, 3), \\ 0, & \text{otherwise}, \end{cases} \\ \dim(\mathscr{H} \cap \mathscr{P}(1/44800, \pi)) &= -1 \end{aligned}$$

[2]Dimension $-1$ indicates that the set is empty.

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for the vector $(1 - h_{ij}/3)_{ij}$, and

$$\dim(\mathscr{H} \cap \mathscr{P}) = 0,$$
$$\dim(\mathscr{H} \cap \mathscr{P}_{ij}) = \begin{cases} 0, & (i,j) = (3,1), (3,2), \\ -1, & \text{otherwise}, \end{cases}$$
$$\dim(\mathscr{H} \cap \mathscr{P}(1/44800, \pi)) = -1$$

for the vector $\tau^{(2)}$. This confirms (7.35).

## 12. Higher-dimensional examples

### *12.1. Geometry*

Consider $G = \mathrm{SL}_2 \times \mathbb{G}_m^r$ and, for $i = 1, \dots, r$, let $\varepsilon_i \in \mathfrak{X}(B)$ be a primitive character of $\mathbb{G}_\mathrm{m}$ composed with the natural inclusion $\mathfrak{X}(\mathbb{G}_\mathrm{m}) \to \mathfrak{X}(B)$ into the $i$-th factor $\mathbb{G}_m$ of $G$. Let $T_{\mathrm{SL}_2} \subset \mathrm{SL}_2$ be a maximal torus, and let $\chi : T_{\mathrm{SL}_2} \to \mathbb{G}_m$ be a primitive character. We consider the subgroup

$$H = \{(\lambda, \chi(\lambda), 1, \dots, 1) : \lambda \in T_{\mathrm{SL}_2}\} \subset G.$$

Then $G/H$ is a spherical homogeneous space of semisimple rank one and type $T$. The lattice $\mathscr{M}$ has basis $(\frac{1}{2}\alpha + \varepsilon_1, \frac{1}{2}\alpha - \varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$. We denote the corresponding dual basis of the lattice $\mathscr{N}$ by $(d_1, d_2, e_3, \dots, e_{r+1})$. There are two colors $D_{11}$ and $D_{12}$ with valuations $d_1$ and $d_2$, respectively. The valuation cone is given by $\mathscr{V} = \{v \in \mathscr{N}_\mathbb{Q} : \langle v, \alpha \rangle \leq 0\}$.

#### 12.1.1. The fourfold $X_5$

Let $r = 2$, and consider the polytope in $\mathscr{N}_\mathbb{Q}$ spanned by the vectors

$$d_1 = (1,0,0), \quad d_2 = (0,1,0), \quad u_{31} = (0,-1,0), \quad u_{32} = (-1,0,0),$$
$$u_{33} = (-1,0,-1), \quad u_{01} = (1,-1,1), \quad u_{02} = (1,-1,0), \quad u_{03} = (-1,1,0).$$

The colored spanning fan of this polytope, as defined in [36, Remark 2.6], contains the following maximal colored cones:

$$(\mathrm{cone}(d_1, d_2, u_{33}), \{D_{11}, D_{12}\}), \quad (\mathrm{cone}(d_1, u_{02}, u_{33}), \{D_{11}\}), \quad (\mathrm{cone}(d_2, u_{03}, u_{33}), \{D_{12}\}),$$
$$(\mathrm{cone}(u_{01}, u_{02}, u_{31}), \emptyset), \quad (\mathrm{cone}(u_{01}, u_{03}, u_{32}), \emptyset), \quad (\mathrm{cone}(u_{01}, u_{31}, u_{32}), \emptyset),$$
$$(\mathrm{cone}(u_{31}, u_{32}, u_{33}), \emptyset), \quad (\mathrm{cone}(u_{03}, u_{32}, u_{33}), \emptyset), \quad (\mathrm{cone}(u_{02}, u_{31}, u_{33}), \emptyset).$$

It can be verified that each colored cone satisfies the conditions of the smoothness criterion [21, Théorème A]; see also [34, Theorem 1.2]. Let $X_5$ be the spherical embedding of $G/H$ corresponding to this colored fan. Then $X_5$ is a smooth Fano fourfold with Picard number 5.

The unsupported colored spanning fan of the polytope above (i. e., including the unsupported colored cones) specifies a projective ambient toric variety $Y_5$. From the description of $\Sigma_{\max}$ in Section 10.3, we deduce that $Y_5$ is smooth and that $-K_{X_5}$ is ample on $Y_5$; hence (2.3) holds.

#### 12.1.2. The fivefold $X_6$

Let $r = 3$, and consider the polytope in $\mathscr{N}_\mathbb{Q}$ spanned by the vectors

$$d_1 = (1,0,0,0), \quad d_2 = (0,1,0,0), \quad u_{31} = (-1,0,1,0), \quad u_{32} = (-1,-1,1,0),$$
$$u_{01} = (-1,1,-1,-1), \quad u_{02} = (1,-1,0,1), \quad u_{03} = (0,0,-1,0).$$

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The colored spanning fan of this polytope contains the following maximal colored cones:

$$
\begin{aligned}
&(\mathrm{cone}(d_1, d_2, u_{01}, u_{31}), \{D_{11}, D_{12}\}), && (\mathrm{cone}(d_1, d_2, u_{02}, u_{31}), \{D_{11}, D_{12}\}),\\
&(\mathrm{cone}(d_1, u_{01}, u_{31}, u_{32}), \{D_{11}\}), && (\mathrm{cone}(d_1, u_{02}, u_{31}, u_{32}), \{D_{11}\}),\\
&(\mathrm{cone}(d_1, u_{02}, u_{03}, u_{32}), \{D_{11}\}), && (\mathrm{cone}(d_1, u_{01}, u_{03}, u_{32}), \{D_{11}\}),\\
&(\mathrm{cone}(d_2, u_{01}, u_{03}, u_{31}), \{D_{12}\}), && (\mathrm{cone}(d_2, u_{02}, u_{03}, u_{31}), \{D_{12}\}),\\
&(\mathrm{cone}(u_{02}, u_{03}, u_{31}, u_{32}), \emptyset), && (\mathrm{cone}(u_{01}, u_{03}, u_{31}, u_{32}), \emptyset).
\end{aligned}
$$

As in the previous example, we obtain a smooth spherical Fano fivefold $X_6$ with Picard number 3 in a smooth projective ambient toric variety $Y_6$ on which $-K_{X_6}$ is ample.

**12.1.3. The sixfold $X_7$**

Let $r = 4$, and consider the polytope in $\mathcal{N}_{\mathbb{Q}}$ spanned by the vectors

$$
\begin{aligned}
d_1 &= (1,0,0,0,0), & d_2 &= (0,1,0,0,0), & u_{01} &= (0,0,1,0,0), & u_{02} &= (0,0,0,1,0),\\
u_{03} &= (0,0,0,0,1), & u_{31} &= (0,-1,0,0,0), & u_{32} &= (-1,0,0,0,1), & u_{33} &= (-1,0,0,0,0),\\
u_{34} &= (-1,0,-1,-1,-1), & u_{35} &= (-1,-1,-1,-1,-1).
\end{aligned}
$$

As above, we obtain a smooth spherical Fano sixfold $X_7$ with Picard number 5 in a smooth projective ambient toric variety $Y_7$ on which $-K_{X_7}$ is ample.

**12.1.4. The sevenfold $X_8$**

Let $r = 5$, and consider the polytope in $\mathcal{N}_{\mathbb{Q}}$ spanned by the vectors

$$
\begin{aligned}
d_1 &= (1,0,0,0,0,0), & d_2 &= (0,1,0,0,0,0), & u_{01} &= (0,0,1,0,1,0),\\
u_{02} &= (0,0,0,1,0,1), & u_{03} &= (0,0,0,0,0,1), & u_{04} &= (0,0,1,0,0,-1),\\
u_{05} &= (0,0,0,1,0,0), & u_{06} &= (0,0,0,0,1,1), & u_{31} &= (0,-1,0,0,0,0),\\
u_{32} &= (-1,0,-1,-1,-1,-1), & u_{33} &= (-1,-1,0,0,0,0), & u_{34} &= (-1,-1,-1,-1,-1,-1).
\end{aligned}
$$

As above, we obtain a smooth spherical Fano sevenfold $X_8$ with Picard number 6 in a smooth projective ambient toric variety $Y_8$ on which $-K_{X_8}$ is ample.

### *12.2. Cox rings and torsors*

We argue as in Section 11.2.

**12.2.1. The fourfold $X_5$**

The Cox ring is

$$\mathscr{R}(X_5) = \mathbb{Q}[x_{01}, x_{02}, x_{03}, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{33}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33})$$

with $\mathrm{Pic}\, X_5 \cong \mathrm{Cl}\, X_5 \cong \mathbb{Z}^5$, where

$$
\begin{aligned}
&\deg(x_{01}) = \deg(x_{33}) = (1,0,0,0,0),\ \deg(x_{02}) = (0,1,0,1,0),\ \deg(x_{03}) = (0,1,0,0,0),\\
&\deg(x_{11}) = \deg(x_{21}) = (0,0,1,0,0),\ \deg(x_{12}) = \deg(x_{22}) = (0,0,0,0,1),\\
&\deg(x_{31}) = (-1,0,0,-1,1),\ \deg(x_{32}) = (0,0,1,1,0).
\end{aligned}
$$

The anticanonical class is $-K_{X_5} = (1,2,2,1,2)$. A universal torsor over $X_5$ is

$$\mathscr{T}_5 = \mathrm{Spec}\, \mathscr{R}(X_5) \setminus Z_{X_5},$$

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where

$$\begin{aligned}Z_{X_5} = {} & \mathbb{V}(x_{31},x_{11},x_{21}) \cup \mathbb{V}(x_{02},x_{12},x_{22}) \cup \mathbb{V}(x_{12},x_{22},x_{31}) \cup \mathbb{V}(x_{32},x_{11},x_{21}) \\ & \cup \mathbb{V}(x_{31},x_{03}) \cup \mathbb{V}(x_{02},x_{32}) \cup \mathbb{V}(x_{02},x_{03}) \cup \mathbb{V}(x_{33},x_{01}) \cup \mathbb{V}(x_{12},x_{22},x_{32}) \cup \mathbb{V}(x_{03},x_{11},x_{21}).\end{aligned}$$

**12.2.2. The fivefold $X_6$**
The Cox ring is

$$\mathscr{R}(X_6) = \mathbb{Q}[x_{01},x_{02},x_{03},x_{11},x_{12},x_{21},x_{22},x_{31},x_{32}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}^2)$$

with $\operatorname{Pic} X_6 \cong \operatorname{Cl} X_6 \cong \mathbb{Z}^3$, where

$$\begin{aligned}&\deg(x_{01}) = \deg(x_{02}) = (0,0,-1),\ \deg(x_{03}) = (1,0,1),\ \deg(x_{11}) = \deg(x_{21}) = (1,0,0),\\ &\deg(x_{12}) = \deg(x_{22}) = (0,1,0),\ \deg(x_{31}) = (1,-1,0),\ \deg(x_{32}) = (0,1,0).\end{aligned}$$

The anticanonical class is $-K_{X_6} = (3,1,-1)$. A universal torsor over $X_6$ is

$$\mathscr{T}_6 = \operatorname{Spec} \mathscr{R}(X_6) \setminus Z_{X_6},$$

where

$$Z_{X_6} = \mathbb{V}(x_{01},x_{02}) \cup \mathbb{V}(x_{32},x_{12},x_{22}) \cup \mathbb{V}(x_{03},x_{31},x_{11},x_{21}).$$

**12.2.3. The sixfold $X_7$**
The Cox ring is

$$\mathscr{R}(X_7) = \mathbb{Q}[x_{01},x_{02},x_{03},x_{11},x_{12},x_{21},x_{22},x_{31},\dots,x_{35}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}x_{34}x_{35}^2)$$

with $\operatorname{Pic} X_7 \cong \operatorname{Cl} X_7 \cong \mathbb{Z}^5$, where

$$\begin{aligned}&\deg(x_{01}) = \deg(x_{02}) = (-1,-1,0,1,0),\ \deg(x_{03}) = (-2,-1,0,1,0),\\ &\deg(x_{11}) = \deg(x_{21}) = (0,0,0,1,0),\ \deg(x_{12}) = \deg(x_{22}) = (0,0,0,0,1),\\ &\deg(x_{31}) = (1,1,1,-1,1),\ \deg(x_{32}) = (1,0,0,0,0),\ \deg(x_{33}) = (0,1,0,0,0),\\ &\deg(x_{34}) = (0,0,1,0,0),\ \deg(x_{35}) = (-1,-1,-1,1,0).\end{aligned}$$

The anticanonical class is $-K_{X_7} = (-3,-2,1,4,2)$. A universal torsor over $X_7$ is

$$\mathscr{T}_7 = \operatorname{Spec} \mathscr{R}(X_7) \setminus Z_{X_7},$$

where

$$\begin{aligned}Z_{X_7} = {} & \mathbb{V}(x_{01},x_{02},x_{03},x_{34}) \cup \mathbb{V}(x_{01},x_{02},x_{03},x_{35}) \cup \mathbb{V}(x_{01},x_{02},x_{32},x_{34}) \\ & \cup \mathbb{V}(x_{01},x_{02},x_{32},x_{35}) \cup \mathbb{V}(x_{03},x_{33}) \cup \mathbb{V}(x_{11},x_{21},x_{32}) \\ & \cup \mathbb{V}(x_{11},x_{21},x_{33}) \cup \mathbb{V}(x_{12},x_{22},x_{31}) \cup \mathbb{V}(x_{12},x_{22},x_{35}) \cup \mathbb{V}(x_{31},x_{34}).\end{aligned}$$

**12.2.4. The sevenfold $X_8$**
The Cox ring is

$$\mathscr{R}(X_8) = \mathbb{Q}[x_{01},\dots,x_{06},x_{11},x_{12},x_{21},x_{22},x_{31},\dots,x_{34}]/(x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}^2x_{34}^2)$$

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with Pic $X_8 \cong \mathrm{Cl}\, X_8 \cong \mathbb{Z}^6$, where

$$\begin{aligned}
&\deg(x_{01}) = (1,1,0,-1,0,0),\ \deg(x_{02}) = (1,1,-1,0,0,0),\\
&\deg(x_{03}) = \deg(x_{05}) = (0,0,1,0,0,0),\ \deg(x_{04}) = \deg(x_{06}) = (0,0,0,1,0,0),\\
&\deg(x_{11}) = \deg(x_{21}) = (0,0,0,0,1,0),\ \deg(x_{12}) = \deg(x_{22}) = (0,0,0,0,0,1),\\
&\deg(x_{31}) = (0,1,0,0,-1,1),\ \deg(x_{32}) = (0,1,0,0,0,0),\\
&\deg(x_{33}) = (-1,-1,0,0,1,0), \deg(x_{34}) = (1,0,0,0,0,0).
\end{aligned}$$

The anticanonical class is $-K_{X_8} = (2,3,1,1,1,2)$. A universal torsor over $X_8$ is

$$\mathscr{T}_8 = \operatorname{Spec} \mathscr{R}(X_8) \setminus Z_{X_8},$$

where

$$\begin{aligned}
Z_{X_8} = {}&\mathbb{V}(x_{01}, x_{02}, x_{32}) \cup \mathbb{V}(x_{01}, x_{02}, x_{34}) \cup \mathbb{V}(x_{03}, x_{05}) \cup \mathbb{V}(x_{04}, x_{06})\\
&\cup \mathbb{V}(x_{11}, x_{21}, x_{33}) \cup \mathbb{V}(x_{12}, x_{22}, x_{31}) \cup \mathbb{V}(x_{12}, x_{22}, x_{34}) \cup \mathbb{V}(x_{31}, x_{32}).
\end{aligned}$$

### *12.3. Counting problems*

**Corollary 12.1.** (a) *We have*

$$N_5(B) = \frac{1}{32}\#\left\{\mathbf{x} \in \mathbb{Z}^{10}_{\neq 0} : \begin{array}{l} x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33} = 0, \quad \max |\mathscr{P}_5(\mathbf{x})| \leq B \\ (x_{31}, x_{11}, x_{21}) = (x_{02}, x_{12}, x_{22}) = (x_{12}, x_{22}, x_{31}) = 1 \\ (x_{32}, x_{11}, x_{21}) = (x_{31}, x_{03}) = (x_{02}, x_{32}) = 1 \\ (x_{02}, x_{03}) = (x_{33}, x_{01}) = (x_{12}, x_{22}, x_{32}) = (x_{03}, x_{11}, x_{21}) = 1 \end{array}\right\},$$

*with*

$$\mathscr{P}_5(\mathbf{x}) = \left\{\begin{array}{l} \{x_{01}, x_{33}\}^2 x_{02}^2 \{x_{12}, x_{22}\} x_{31} \{x_{11}, x_{21}\}^2, x_{32}\{x_{01}, x_{33}\}^3 x_{02}^2 x_{31}^2 \{x_{11}, x_{21}\}, \\ x_{03}\{x_{01}, x_{33}\} x_{02} \{x_{12}, x_{22}\}^2 \{x_{11}, x_{21}\}^2, x_{03} x_{32}^2 \{x_{01}, x_{33}\}^3 x_{02} x_{31}^2, \\ x_{03}^2 x_{32} \{x_{01}, x_{33}\}\{x_{12}, x_{22}\}^2 \{x_{11}, x_{21}\}, x_{03}^2 x_{32}^2 \{x_{01}, x_{33}\}^2 \{x_{12}, x_{22}\} x_{31} \end{array}\right\}.$$

(b) *We have*

$$N_6(B) = \frac{1}{8}\#\left\{\mathbf{x} \in \mathbb{Z}^{9}_{\neq 0} : \begin{array}{l} x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}^2 = 0, \quad \max |\mathscr{P}_6(\mathbf{x})| \leq B \\ (x_{01}, x_{02}) = (x_{32}, x_{12}, x_{22}) = (x_{03}, x_{31}, x_{11}, x_{21}) = 1 \end{array}\right\},$$

*with*

$$\mathscr{P}_6(\mathbf{x}) = \left\{\begin{array}{l} \{x_{01}, x_{02}\}\{x_{12}, x_{22}, x_{32}\}^4 x_{31}^3, \{x_{01}, x_{02}\}\{x_{11}, x_{21}\}^3 \{x_{12}, x_{22}, x_{32}\}, \\ \{x_{01}, x_{02}\}^4 x_{03}^3 \{x_{12}, x_{22}, x_{32}\} \end{array}\right\}.$$

(c) *We have*

$$N_7(B) = \frac{1}{32}\#\left\{\mathbf{x} \in \mathbb{Z}^{12}_{\neq 0} : \begin{array}{l} x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}x_{34}x_{35}^2 = 0, \quad \max |\mathscr{P}_7(\mathbf{x})| \leq B \\ (x_{01}, x_{02}, x_{03}, x_{34}) = (x_{01}, x_{02}, x_{03}, x_{35}) = (x_{01}, x_{02}, x_{32}, x_{34}) = 1 \\ (x_{01}, x_{02}, x_{32}, x_{35}) = (x_{03}, x_{33}) = (x_{11}, x_{21}, x_{32}) = 1 \\ (x_{11}, x_{21}, x_{33}) = (x_{12}, x_{22}, x_{31}) = (x_{12}, x_{22}, x_{35}) = (x_{31}, x_{34}) = 1 \end{array}\right\},$$

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*with*

$$\mathscr{P}_7(\mathbf{x}) = \left\{\begin{array}{l} x_{31}^2x_{32}x_{33}^2x_{34}^5x_{35}^6,\ \{x_{12},x_{22}\}^2x_{32}x_{33}^2x_{34}^5x_{35}^4,\ \{x_{11},x_{21}\}x_{31}^2x_{33}x_{34}^4x_{35}^5,\\ \{x_{11},x_{21}\}\{x_{12},x_{22}\}^2x_{33}x_{34}^4x_{35}^3,\ x_{03}\{x_{11},x_{21}\}^2x_{31}^2x_{34}^2x_{35}^3,\\ x_{03}\{x_{11},x_{21}\}^2\{x_{12},x_{22}\}^2x_{34}^2x_{35},\ x_{03}^2\{x_{11},x_{21}\}^2\{x_{12},x_{22}\}^2x_{32}x_{34},\\ x_{03}^3\{x_{11},x_{21}\}^2x_{31}^2x_{32}^2x_{35},\ x_{03}^3\{x_{11},x_{21}\}^2\{x_{12},x_{22}\}x_{31}x_{32}^2,\ x_{03}^4\{x_{12},x_{22}\}^2x_{32}^5x_{33}^2x_{34},\\ x_{03}^5x_{31}^2x_{32}^6x_{33}^2x_{35},\ x_{03}^5\{x_{12},x_{22}\}x_{31}x_{32}^6x_{33}^2,\ \{x_{01},x_{02}\}x_{03}\{x_{11},x_{21}\}^2\{x_{12},x_{22}\}^2x_{34},\\ \{x_{01},x_{02}\}^2x_{03}\{x_{11},x_{21}\}^2x_{31}^2x_{35},\ \{x_{01},x_{02}\}^2x_{03}\{x_{11},x_{21}\}^2\{x_{12},x_{22}\}x_{31},\\ \{x_{01},x_{02}\}^3\{x_{11},x_{21}\}\{x_{12},x_{22}\}^2x_{33}x_{34},\ \{x_{01},x_{02}\}^4\{x_{12},x_{22}\}^2x_{32}x_{33}^2x_{34},\\ \{x_{01},x_{02}\}^4\{x_{11},x_{21}\}x_{31}^2x_{33}x_{35},\ \{x_{01},x_{02}\}^4\{x_{11},x_{21}\}\{x_{12},x_{22}\}x_{31}x_{33},\\ \{x_{01},x_{02}\}^5x_{31}^2x_{32}x_{33}^2x_{35},\ \{x_{01},x_{02}\}^5\{x_{12},x_{22}\}x_{31}x_{32}x_{33}^2 \end{array}\right\}.$$

(d) *We have*

$$N_8(B) = \frac{1}{64}\#\left\{\mathbf{x}\in\mathbb{Z}_{\neq 0}^{14} : \begin{array}{c} x_{11}x_{12}-x_{21}x_{22}-x_{31}x_{32}x_{33}^2x_{34}^2=0, \quad \max|\mathscr{P}_8(\mathbf{x})|\le B\\ (x_{01},x_{02},x_{32})=(x_{01},x_{02},x_{34})=(x_{03},x_{05})=(x_{04},x_{06})=1\\ (x_{11},x_{21},x_{33})=(x_{12},x_{22},x_{31})=(x_{12},x_{22},x_{34})=(x_{31},x_{32})=1 \end{array}\right\},$$

*where* $\mathscr{P}_8(\mathbf{x})$ *is*

$$\left\{\begin{array}{l} \{x_{03},x_{05}\}\{x_{04},x_{06}\}x_{31}^2x_{32}^4x_{33}^3x_{34}^5,\ \{x_{03},x_{05}\}\{x_{04},x_{06}\}\{x_{12},x_{22}\}^2x_{32}^4x_{33}x_{34}^3,\\ \{x_{03},x_{05}\}\{x_{04},x_{06}\}\{x_{11},x_{21}\}\{x_{12},x_{22}\}^2x_{32}^3x_{34}^2,\ \{x_{03},x_{05}\}\{x_{04},x_{06}\}\{x_{11},x_{21}\}^3x_{31}^2x_{32}x_{34}^2,\\ x_{02}\{x_{03},x_{05}\}^2\{x_{04},x_{06}\}\{x_{11},x_{21}\}^3x_{31}^2x_{34},\ x_{02}^2\{x_{03},x_{05}\}^3\{x_{04},x_{06}\}\{x_{11},x_{21}\}\{x_{12},x_{22}\}^2x_{32},\\ x_{02}^2\{x_{03},x_{05}\}^3\{x_{04},x_{06}\}\{x_{11},x_{21}\}^2\{x_{12},x_{22}\}x_{31},\ x_{02}^3\{x_{03},x_{05}\}^4\{x_{04},x_{06}\}\{x_{12},x_{22}\}^2x_{32}x_{33},\\ x_{02}^4\{x_{03},x_{05}\}^5\{x_{04},x_{06}\}x_{31}^2x_{33}^3x_{34},\ x_{02}^4\{x_{03},x_{05}\}^5\{x_{04},x_{06}\}\{x_{12},x_{22}\}x_{31}x_{33}^2,\\ x_{01}\{x_{03},x_{05}\}\{x_{04},x_{06}\}^2\{x_{11},x_{21}\}^3x_{31}^2x_{34},\ x_{01}^2\{x_{03},x_{05}\}\{x_{04},x_{06}\}^3\{x_{11},x_{21}\}\{x_{12},x_{22}\}^2x_{32},\\ x_{01}^2\{x_{03},x_{05}\}\{x_{04},x_{06}\}^3\{x_{11},x_{21}\}^2\{x_{12},x_{22}\}x_{31},\ x_{01}^3\{x_{03},x_{05}\}\{x_{04},x_{06}\}^4\{x_{12},x_{22}\}^2x_{32}x_{33},\\ x_{01}^4\{x_{03},x_{05}\}\{x_{04},x_{06}\}^5x_{31}^2x_{33}^3x_{34},\ x_{01}^4\{x_{03},x_{05}\}\{x_{04},x_{06}\}^5\{x_{12},x_{22}\}x_{31}x_{33}^2 \end{array}\right\}.$$

*Proof.* This is analogous to Corollary 11.1. □

### *12.4. Application: proof of Theorem 1.2*

All cases can be proved exactly as in Section 11.4.

#### **12.4.1. The variety $X_5$**

By Corollary 12.1(a), we have $J = 10$ torsor variables $x_{ij}$ satisfying the equation

$$x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{32}x_{33} = 0.$$

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We have $N = 34$ height conditions with corresponding exponent matrix

$$\mathscr{A}_1 = \left(\begin{array}{cccccccccccccccccccccccccccccccccc}
&&&&&&&&&&&&&&&&&1&1&1&1&1&1&1&1&2&2&2&2&2&2&3&3&3 \\
&&&&&&1&1&1&1&1&2&2&2&2&2&2&&&&&1&1&1&1&&&2&2&2&2&1&2&2 \\
2&2&2&2&2&2&1&1&1&1&1&&&&&&&2&2&2&2&1&1&1&1&2&2&&&&&1&& \\
&&&&1&1&&&&2&2&&&&1&2&2&&&1&1&&&2&2&&&&&2&2&&&1 \\
&&1&2&&2&&&2&&2&&&1&&&1&&2&&2&&2&&2&&1&&1&&1&&& \\
&1&&1&&&&2&2&&&1&2&2&&&&1&1&&&2&2&&&&&2&2&&&&1& \\
1&2&&&2&&&2&&2&&&1&&&1&&2&&2&&2&&2&&1&&1&&1&&&& \\
1&&1&&&&2&&&&&2&1&1&2&1&1&&&&&&&&&1&1&1&1&1&1&2&2&2 \\
2&1&2&1&1&1&2&&&&&1&&&1&&&1&1&1&1&&&&&2&2&&&&&2&1&1 \\
2&1&2&1&1&1&3&1&1&1&1&3&2&2&3&2&2&&&&&&&&&&&&&&&&&
\end{array}\right), \quad \mathscr{A}_2 = \left(\begin{array}{ccc}
&&-1 \\
&&-1 \\
&&-1 \\
1&&-1 \\
1&&-1 \\
&1&-1 \\
&1&-1 \\
-1&-1& \\
-1&-1& \\
-1&-1&
\end{array}\right).$$

Proposition 5.2 gives us $\lambda = 1/34300$. We have $r = 10$ coprimality conditions, and we see immediately in this and all other cases that (7.24) holds. We choose

$$\tau^{(2)} = (1, 1, 1, \tfrac{2}{3}, \dots, \tfrac{2}{3}) = (1 - h_{ij}/3)_{ij}.$$

We verify $C_2(\tau^{(2)})$ and $C_2((1 - h_{ij}/3)_{ij})$ and compute and confirm (7.35) by

$$\dim(\mathscr{H} \cap \mathscr{P}) = 4, \quad \dim(\mathscr{H} \cap \mathscr{P}_{ij}) = 3, \quad \dim(\mathscr{H} \cap \mathscr{P}(1/34300, \pi)) = 0.$$

**12.4.2. The variety $X_6$**

By Corollary 12.1(b), we have $J = 9$ torsor variables $x_{ij}$ satisfying the equation

$$x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{32}^2 = 0.$$

We have $N = 24$ height conditions with corresponding exponent matrix

$$\mathscr{A}_1 = \left(\begin{array}{cccccccccccccccccccccccc}
&&&&&&&&&&&&1&1&1&1&1&1&1&1&1&4&4&4 \\
1&1&1&1&1&1&1&1&1&4&4&4&&&&&&&&&&&& \\
&&&&&&&&&3&3&3&&&&&&&&&&3&3&3 \\
&&&&&&3&3&3&&&&&&&&&&3&3&3&&& \\
&&&&1&4&&&1&&&1&&&&&1&4&&&1&&&1 \\
&&3&3&3&&&&&&&&&&3&3&3&&&&&&& \\
&4&&1&&&&1&&&1&&&4&&1&&&&1&&&1& \\
3&3&&&&3&&&&&&&3&3&&&&3&&&&&& \\
4&&1&&&&1&&&1&&&4&&1&&&&1&&&1&&
\end{array}\right), \quad \mathscr{A}_2 = \left(\begin{array}{ccc}
&&-1 \\
&&-1 \\
&&-1 \\
1&&-1 \\
1&&-1 \\
&1&-1 \\
&1&-1 \\
-1&-1& \\
-2&-2&1
\end{array}\right).$$

Proposition 5.2 yields $\lambda = 1/34300$. We choose

$$\tau^{(2)} = (1, 1, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1)$$

satisfying (7.18). We verify $C_2(\tau^{(2)})$ and $C_2((1 - h_{ij}/3)_{ij})$ and compute

$$\begin{aligned}
&\dim(\mathscr{H} \cap \mathscr{P}) = 2, \\
&\dim(\mathscr{H} \cap \mathscr{P}_{ij}) = -1, (i, j) = (1, 1), (2, 1), \quad \dim(\mathscr{H} \cap \mathscr{P}_{ij}) = 1 \text{ otherwise}, \\
&\dim(\mathscr{H} \cap \mathscr{P}(1/34300, \pi)) = -1 \text{ for all } \pi
\end{aligned}$$

for the vector $(1 - h_{ij}/3)_{ij}$ and

$$\begin{aligned}
&\dim(\mathscr{H} \cap \mathscr{P}) = 1, \\
&\dim(\mathscr{H} \cap \mathscr{P}_{ij}) = \begin{cases} 1, & (i, j) = (3, 1), \\ 0, & (i, j) = (0, 1), (0, 2), (0, 3), \\ -1, & \text{otherwise}, \end{cases} \\
&\dim(\mathscr{H} \cap \mathscr{P}(1/34300, \pi)) = -1 \text{ for all } \pi
\end{aligned}$$

for the vector $\tau^{(2)}$. This confirms (7.35).

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**12.4.3. The variety $X_7$**
By Corollary 12.1(c), we have $J = 12$ torsor variables $x_{ij}$ satisfying the equation

$$x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{32}x_{33}x_{34}x_{35}^2 = 0.$$

We have $N = 80$ height conditions; the corresponding matrix $\mathscr{A}_1$ is

$$\left(\begin{array}{*{80}{c}}
&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&1&1&1&1&2&2&2&2&2&2&3&3&3&3&4&4&4&4&4&4&4&4&5&5&5 \\
&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&1&1&1&1&2&2&2&2&2&2&3&3&3&3&4&4&4&4&4&4&4&4&5&5&5&&&&&&&&&&&&&&&&&&&&&&&&& \\
&&&&&&&&&1&1&1&1&1&1&2&2&2&2&3&3&3&3&3&3&4&4&5&5&5&1&1&1&1&1&1&1&1&1&1&&&&&&&&&&&&&&&&1&1&1&1&1&1&1&1&1&1&&&&&&&&&&&&&&& \\
&&&&&&1&1&1&&&&2&2&2&&&2&2&&&&2&2&2&&&&&&&&2&2&&&&2&2&2&&&1&1&&&&&&1&1&1&&&&&&2&2&&&&2&2&2&&&1&1&&&&&&1&1&1&&& \\
&&&&2&2&&&2&&&2&&&2&&2&&2&&&1&&&1&&2&&&1&&2&&2&&&1&&&1&&2&&2&&&&1&2&&&1&&&1&&2&&2&&&1&&&1&&2&&2&&&1&2&&&1&&&1 \\
&&1&1&&1&&&&2&2&2&&&&2&2&&&2&2&2&&&&&&&&&2&2&&&2&2&2&&&&1&1&&&&1&1&1&&&&&&&&2&2&&&2&2&2&&&&1&1&&&&1&1&1&&&&&&& \\
&2&&2&&&&2&&&2&&&2&&2&&2&&&1&&&1&&2&&&1&&2&&2&&&1&&&1&&2&&2&&2&&1&&&&1&&&1&&2&&2&&&1&&&1&&2&&2&&2&&1&&&&1&&&1& \\
2&&2&&&&2&&&2&&&2&&&&&&&2&1&1&2&1&1&&&2&1&1&&&&&2&1&1&2&1&1&&&&&&2&1&1&&2&1&1&2&1&1&&&&&2&1&1&2&1&1&&&&&&2&1&1&&2&1&1&2&1&1 \\
1&1&&&1&&&&&&&&&&&1&1&1&1&2&2&2&2&2&2&5&5&6&6&6&&&&&&&&&&&&&&&1&&&&1&&&&1&1&1&&&&&&&&&&&&&&1&&&&1&&&&1&1&1 \\
2&2&1&1&2&1&1&1&1&&&&&&&&&&&&&&&&&2&2&2&2&2&&&&&&&&&&&1&1&1&1&2&1&1&1&2&1&1&1&2&2&2&&&&&&&&&&&&&&&1&1&1&1&2&1&1&1&2&1&1&1&2&2&2 \\
5&5&4&4&5&4&4&4&4&2&2&2&2&2&2&1&1&1&1&&&&&&&1&1&&&&1&1&1&1&&&&&&&1&1&1&1&1&&&&1&&&&&&&1&1&1&1&&&&&&&1&1&1&1&1&&&&1&&&&&& \\
6&4&5&3&4&3&5&3&3&3&1&1&3&1&1&&&&&1&&&1&&&&&1&&&&&&&1&&&1&&&&&&&&1&&&&1&&&1&&&&&&&1&&&1&&&&&&&&1&&&&1&&&1&&
\end{array}\right),$$

Proposition 5.2 yields $\lambda = 1/70,000$. We choose

$$\tau^{(2)} = \left(1, 1, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}, 1\right)$$

satisfying (7.18). We verify $C_2(\tau^{(2)})$ and $C_2((1 - h_{ij}/3)_{ij})$ and compute

$$\begin{aligned}
&\dim(\mathscr{H} \cap \mathscr{P}) = 4, \\
&\dim(\mathscr{H} \cap \mathscr{P}_{ij}) = \begin{cases} 1, & (i, j) = (0, 1), (0, 2), \\ 0, & (i, j) = (1, 1), (2, 1), \\ 2, & (i, j) = (1, 2), (2, 2), \\ 3, & \text{otherwise}, \end{cases} \\
&\dim(\mathscr{H} \cap \mathscr{P}(1/70,000, \pi)) = -1 \text{ for all } \pi
\end{aligned}$$

for the vector $(1 - h_{ij}/3)_{ij}$ and

$$\begin{aligned}
&\dim(\mathscr{H} \cap \mathscr{P}) = 0, \\
&\dim(\mathscr{H} \cap \mathscr{P}_{ij}) = \begin{cases} 0, & (i, j) = (3, 1), (3, 2), (3, 3), (3, 4), \\ -1, & \text{otherwise}, \end{cases} \\
&\dim(\mathscr{H} \cap \mathscr{P}(1/70,000, \pi)) = -1 \text{ for all } \pi
\end{aligned}$$

for the vector $\tau^{(2)}$. This confirms (7.35).

**12.4.4. The variety $X_8$**
By Corollary 12.1(d), we have $J = 14$ torsor variables $x_{ij}$ with $0 \leq i \leq 3$, $J_0 = 6$, $J_1 = J_2 = 2$, $J_3 = 4$ satisfying the equation

$$x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{32}x_{33}^2x_{34}^2 = 0$$

with $k = 3$. We have $N = 156$ height conditions; it is straightforward to extract the corresponding matrices $\mathscr{A}_1, \mathscr{A}_2$ from Corollary 12.1(d), which we do not spell out for obvious space reasons. Proposition 5.2 yields $\lambda = 1/70,000$. We choose

$$\tau^{(2)} = \left(1, 1, 1, 1, 1, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1, 1\right)$$

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satisfying (7.18). We verify $C_2(\tau^{(2)})$ and $C_2((1-h_{ij}/3)_{ij})$ and compute

$$\begin{aligned}
&\dim(\mathscr{H}\cap\mathscr{P})=5,\\
&\dim(\mathscr{H}\cap\mathscr{P}_{ij})=\begin{cases}0, & (i,j)=(1,1),(2,1)\\ 2, & (i,j)=(1,2),(2,2),\\ 4, & \text{otherwise},\end{cases}\\
&\dim(\mathscr{H}\cap\mathscr{P}(1/34300,\pi))=-1 \text{ for all } \pi
\end{aligned}$$

for the vector $(1-h_{ij}/3)_{ij}$ and

$$\begin{aligned}
&\dim(\mathscr{H}\cap\mathscr{P})=3,\\
&\dim(\mathscr{H}\cap\mathscr{P}_{ij})=\begin{cases}-1, & (i,j)=(1,1),(1,2),(2,1),(2,2),\\ 0, & (i,j)=(3,4)\\ 3, & (i,j)=(3,1),(3,2),\\ 2, & \text{otherwise},\end{cases}\\
&\dim(\mathscr{H}\cap\mathscr{P}(1/70,000,\pi))=-1 \text{ for all } \pi
\end{aligned}$$

for the vector $\tau^{(2)}$. This confirms (7.35).

## 13. A singular example

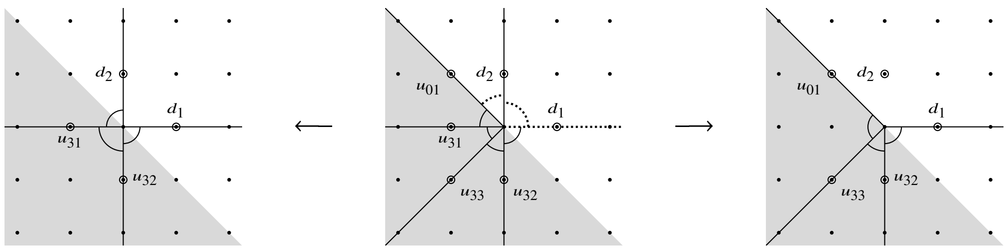
As in Section 11.1.1, we consider the spherical $G$-variety $W_4=\mathbb{V}(z_{11}z_{12}-z_{21}z_{22}-z_{31}z_{32})\subset\mathbb{P}^2_{\mathbb{Q}}\times\mathbb{P}^2_{\mathbb{Q}}$. Let $\widetilde{X}^\dagger\to W_4$ be the blow-up in the two disjoint $G$-invariant curves

$$C_{01}=\mathbb{V}(z_{12},z_{22},z_{31})=\mathbb{V}(z_{31})\times\{(0:0:1)\},\quad C_{33}=\mathbb{V}(z_{31},z_{32}).$$

The anticanonical divisor $-K_{\widetilde{X}^\dagger}$ is not ample but semiample. Moreover,

$$H^1(\widetilde{X}^\dagger,\mathscr{O}_{\widetilde{X}^\dagger})=H^2(\widetilde{X}^\dagger,\mathscr{O}_{\widetilde{X}^\dagger})=0$$

since $\widetilde{X}^\dagger$ is smooth and rational. Hence, $\widetilde{X}^\dagger$ is an almost Fano variety. We obtain an anticanonical contraction $\pi\colon\widetilde{X}^\dagger\to X^\dagger$. Here, $X^\dagger$ is a singular Fano variety with desingularization $\widetilde{X}^\dagger$. The sequence of morphisms $W_4\leftarrow\widetilde{X}^\dagger\to X^\dagger$ corresponds to the following sequence of maps of colored fans.




We denote by $E_{31}$ the $G$-invariant exceptional divisor contracted by $\pi$. The singular locus of $X^\dagger$ is $\pi(E_{31})$. The dotted circles in the colored fan of $\widetilde{X}^\dagger$ specify a smooth projective ambient toric variety $Y^\dagger$ such that $-K_{\widetilde{X}^\dagger}$ is ample on $Y^\dagger$.

In the same way as before, a universal torsor of $\widetilde{X}^\dagger$ can be obtained. The straightforward computations are omitted. This leads to the following counting problem.

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**Corollary 13.1.** *We have*

$$N^\dagger(B) = \frac{1}{16}\#\left\{\mathbf{x}\in\mathbb{Z}^8_{\neq 0} : \begin{array}{c} x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}^2 = 0, \quad \max|\mathscr{P}^\dagger(\mathbf{x})| \le B \\ (x_{11},x_{21},x_{33}) = (x_{11},x_{21},x_{31}) = (x_{01},x_{11},x_{21}) = 1 \\ (x_{12},x_{22}) = (x_{01},x_{32}) = (x_{01},x_{33}) = (x_{31},x_{32}) = 1 \end{array}\right\},$$

*with*

$$\mathscr{P}^\dagger(\mathbf{x}) = \begin{Bmatrix} x_{01}x_{31}^2x_{32}^2x_{33}^3, \{x_{11},x_{21}\}x_{31}x_{32}^2x_{33}^2, \{x_{11},x_{21}\}^2\{x_{12},x_{22}\}x_{32}, \\ x_{01}^3\{x_{12},x_{22}\}^2x_{31}^2x_{33}, x_{01}^2\{x_{11},x_{21}\}\{x_{12},x_{22}\}^2x_{31} \end{Bmatrix}.$$

By the same type of computations as before, one concludes Theorem 1.3 from Corollary 13.1 and Theorem 10.1 applied to the almost Fano variety $\widetilde{X}^\dagger$.

## A. Some explicit computations

We return to the variety $X_4$ discussed in Section 11.4.1 and explain how to obtain Hypothesis 7.2 by 'bare hands' and how to compute Peyre's constant explicitly. We use $X_4$ as a showcase, the computations are similar (and similarly uninspiring) in the other cases.

Recall from (7.22) and (11.4) that for Hypothesis 7.2, we need to show

$$\sum_{\mathbf{X}}{}^{*}(X_{01}X_{02}(X_{11}X_{12}X_{21}X_{22}X_{31}X_{32})^{2/3})^\alpha \ll B^\alpha(\log B)^2(1+\log H) \tag{A.1}$$

for fixed $0<\alpha<1$, where each $X_{ij}$ is restricted to a power of 2 and subject to

$$\min(X_{ij}) \le H \quad \text{and} \quad \prod_{ij} X_{ij}^{\alpha_{ij}^\nu} \le B.$$

By symmetry, we can assume without loss of generality that

$$X_{12}\ge X_{22}, \quad X_{21}\ge X_{11}.$$

The columns $\nu = 4,5$ and $\nu = 2,3$ of in the matrix $\mathscr{A}_1$ yield

$$X_{31}X_{12}\max(X_{31}X_{32}, X_{12}X_{21})X_{02}^2 \le B, \quad X_{32}X_{21}\max(X_{31}X_{32}, X_{12}X_{21})X_{01}^2 \le B, \tag{A.2}$$

respectively. Let us first assume that $\min(X_{ij}) \asymp \min(X_{11}, X_{22}, X_{31}, X_{32})$, that is, $X_{01}, X_{02}$ are not the smallest parameters. Summing over $X_{01}, X_{02}$, we bound the $\mathbf{X}$-sum in (A.1) by

$$\sum_{\mathbf{X}}\left(\frac{B(X_{11}X_{12}X_{21}X_{22}X_{31}X_{32})^{2/3}}{(X_{12}X_{21}X_{31}X_{32})^{1/2}\max(X_{31}X_{32},X_{12}X_{21})}\right)^\alpha \le \sum_{\mathbf{X}}\left(\frac{B(X_{31}X_{32})^{1/6}(X_{21}X_{22})^{2/3}}{\max(X_{31}X_{32},X_{12}X_{21})^{5/6}}\right)^\alpha.$$

Here and in similar situations, the precise summation conditions on $\mathbf{X}$ and the variables involved will always be clear from the context. Suppose that the minimum is taken at $X_{11}$ or $X_{22}$. We glue together the variables $X_{31}X_{32} = X_3$, say, where $X_3$ runs over powers of 2 with multiplicity $O(\log B)$. Summing over $X_3$, the $\mathbf{X}$-sum becomes

$$\log B \sum_{\substack{X_{22}\le X_{12}\le B \\ X_{11}\le X_{21}\le B \\ \min(X_{11},X_{22})\le H}} \left(\frac{B(X_{22}X_{11})^{2/3}}{(X_{12}X_{21})^{2/3}}\right)^\alpha \ll B^\alpha(\log B)^2(1+\log H).$$

If the minimum is taken at $X_{31}$ or $X_{32}$, there are only $O(1+\log H)$ possibilities for the value of $X_3$, and we can argue in the same way.

Finally, we treat the case where the minimum is taken at $X_{01}$ or $X_{02}$. Without loss of generality (by symmetry), assume $X_{01} \le X_{02}$. We use (A.2) to sum over $X_{02}$ and then sum over $X_{11} \le X_{21}$ and $X_{22} \le X_{12}$. In this way, we bound the $\mathbf{X}$-sum in (A.1) by

$$\sum_{\mathbf{X}} \Big( \frac{B^{1/2} X_{01} (X_{11}X_{12}X_{21}X_{22}X_{31}X_{32})^{2/3}}{(X_{31}X_{12} \max(X_{31}X_{32}, X_{12}X_{21}))^{1/2}} \Big)^{\alpha} \ll \sum_{\mathbf{X}} \Big( \frac{B^{1/2} X_{01} (X_{12}^2 X_{21}^2 X_{31}X_{32})^{2/3}}{(X_{31}X_{12} \max(X_{31}X_{32}, X_{12}X_{21}))^{1/2}} \Big)^{\alpha},$$

where the sum is restricted to $X_{01}, X_{12}, X_{21}, X_{31}, X_{32}$ powers of 2 satisfying $X_{01} \le H$ and the second bound in (A.2). We now distinguish two cases. If $X_{31}X_{32} \ge X_{12}X_{21}$, we sum over $X_{12} \le X_{31}X_{32}/X_{21}$, getting

$$\sum_{\substack{X_{01} \le H \\ X_{32}^2 X_{21} X_{31} X_{01}^2 \le B}} \Big( B^{1/2} X_{01} X_{32} (X_{31}X_{21})^{1/2} \Big)^{\alpha} \ll \sum_{X_{01} \le H, X_{21}, X_{31} \le B} B^{\alpha} \ll B^{\alpha} (\log B)^2 (1+\log H).$$

If $X_{31}X_{32} \le X_{12}X_{21}$, we sum over $X_{31} \le X_{12}X_{21}/X_{32}$ instead, obtaining the same result.

Now, we compute the Peyre constant. We start with the computation of the Euler product $c_{\mathrm{fin}}$. By (11.2), (8.11) and (8.14), we have

$$\gamma = ([g_4, g_5], [g_3, g_4], g_1, g_2, g_1, g_2, g_5, g_3]) \in \mathbb{N}^8, \quad \gamma^* = (g_1g_2, g_1g_2, g_3g_5) \in \mathbb{N}^3.$$

A simple computation (cf. Lemma 5.4) shows

$$\mathscr{E}_{\mathbf{b}} = \sum_{q=1}^{\infty} q^{-6} \sum_{a \bmod q}^{*} \prod_{i=1}^{3} \Big( \sum_{x,y \bmod q} e\Big(\frac{a}{q} b_i xy\Big)\Big) = \sum_{q=1}^{\infty} \frac{\phi(q)(q,b_1)(q,b_2)(q,b_3)}{q^3}$$

for $\mathbf{b} \in \mathbb{N}^3$ so that

$$c_{\mathrm{fin}} = \sum_{\mathbf{g} \in \mathbb{N}^5} \frac{\mu(\mathbf{g})}{g_1^2 g_2^2 g_3 g_5 [g_4, g_5][g_3, g_4]} \sum_{q=1}^{\infty} \frac{\phi(q)(q, g_1g_2)^2 (q, g_3g_5)}{q^3}.$$

We expand this into an Euler product, and by brute force computation one verifies

$$c_{\mathrm{fin}} = \prod_p \Big(1 - \frac{1}{p}\Big)^4 \Big(1 + \frac{1}{p}\Big)\Big(1 + \frac{3}{p} + \frac{1}{p^2}\Big).$$

In order to compute $c^*$ and $c_\infty$, we follow the argument in Section 8.5. We can take the rows 3, 4, 5, 6 (i.e., corresponding to $(ij) = (11), (12), (21), (22)$) of $(\mathscr{A}_1\ \mathscr{A}_2)$ as $Z_1, \dots, Z_4$ in (8.23) so that

$$\begin{aligned} y_1 &= w_{11} = s_3 + 2s_7 + 2s_9 + s_{11} + s_{13} + 2s_{16} + 2s_{17} + z_1 - 1, \\ y_2 &= w_{12} = s_4 + s_6 + s_7 + 2s_{10} + 2s_{11} + 2s_{14} + 2s_{16} + z_1 - 1, \\ y_3 &= w_{21} = s_2 + 2s_6 + 2s_8 + s_{10} + s_{12} + 2s_{14} + 2s_{15} + z_2 - 1, \\ y_4 &= w_{22} = s_5 + s_8 + s_9 + 2s_{12} + 2s_{13} + 2s_{15} + 2s_{17} + z_2 - 1, \\ y_5 &= s_1 + \cdots + s_{17} - 1. \end{aligned}$$

An explicit choice for a vector $\sigma$ satisfying (7.6) is, for instance,

$$\sigma = (\tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{1}{12}, \tfrac{1}{12}, \tfrac{1}{18}) \in \mathbb{R}^{17}_{>0}.$$

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The linear forms $\mathscr{L}_\iota(\mathbf{y})$ in (8.27) containing the entries of the matrix $\mathscr{B} \in \mathbb{R}^{4\times 5}$ are given by

$$w_{31} = y_5 + y_3 - y_2 + y_1 - y_4, \quad w_{32} = y_5 - y_3 + y_2 - y_1 + y_4,$$
$$w_{01} = 2y_5 - y_2 - y_4, \quad w_{02} = 2y_5 - y_3 - y_1.$$

By contour shifts as in Section 8.5 or by the explicit formula (8.34), we compute

$$c^* = \frac{1}{3!}\cdot\frac{1}{12}.$$

To compute $c_\infty$, we need to choose a matrix $\mathscr{C}$ as in (8.25), that is, variables $y_6, \dots, y_{17}$ as functions of $\mathbf{s}$. A simple possible choice is $y_\nu = s_\nu$, $6 \le \nu \le 17$ (Jacobi-determinant $-1$). In these variables, we have

$$\begin{aligned}\Big(\prod_{\nu=1}^{17} s_\nu\Big)\Big|_{y_1=\dots=y_5=0} = \Big(\prod_{\nu=6}^{17} y_\nu\Big)&(2(y_6+\dots+y_{13}) + 3(y_{14}+y_{15}+y_{16}+y_{17}) - 3 + 2z_1 + 2z_2)\\ &\times(2y_6+2y_8+y_{10}+y_{12}+2y_{14}+2y_{15}+z_2-1)(2y_7+2y_9+y_{11}+y_{13}+2y_{16}+2y_{17}+z_1-1)\\ &\times(y_6+y_7+2y_{10}+2y_{11}+2y_{14}+2y_{16}+z_1-1)(y_8+y_9+2y_{12}+2y_{13}+2y_{15}+2y_{17}+z_2-1).\end{aligned}$$

For fixed $z_1, z_2$, the integrand is a rational function in $y_6, \dots, y_{17}$, and we simply shift each contour to $+\infty$ or $-\infty$ (again it does not matter which direction we choose) and pick up the poles. After a long computation (or a quick application of a computer algebra system), we obtain

$$c_\infty = \frac{2^8}{\pi}\int_{(1/3)}^{(2)} \mathscr{K}(z_1)\mathscr{K}(z_2)\mathscr{K}(z_3)\frac{2(3-z_3^2)}{(z_1-1)^2(z_2-1)^2(z_3-1)^2}\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(2\pi \mathrm{i})^2},$$

with $\mathscr{K}(z) = \Gamma(z)\cos(\pi z/2)$, $z_3 = 1 - z_1 - z_2$. Let us define

$$\mathtt{K}(z) = \frac{\Gamma(z)\cos(\pi z/2)}{(z-1)^2}, \quad \mathtt{K}^*(z) = \frac{2\Gamma(z)\cos(\pi z/2)(3-z^2)}{(z-1)^2},$$

and let us denote by

$$\check{\mathtt{K}}(x) = \int_{(1/3)} \mathtt{K}(z)x^{-z}\frac{\mathrm{d}z}{2\pi \mathrm{i}}, \quad x > 0,$$

and similarly by $\check{\mathtt{K}}^*$ the corresponding inverse Mellin transforms. By [40, 6.246], we have $\check{\mathtt{K}}(x) = \mathrm{Si}(x)/x$, where $\mathrm{Si}(x) = \int_0^x \sin t\, dt/t$ is the integral sine. To deal with convergence issues, let

$$\mathscr{C} = (-10 - i\infty, -10 - i] \cup [-10 - i, 1/3] \cup [1/3, -10 + i] \cup [-10 + i, -10 + i\infty).$$

Then

$$\begin{aligned}\frac{\pi}{2^8}c_\infty &= \int_{(1/3)}^{(2)} \mathtt{K}(z_1)\mathtt{K}(z_2)\mathtt{K}^*(1-z_1-z_2)\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(2\pi i)^2} = \int_{(1/3)}^{(2)} \mathtt{K}(z_1)\mathtt{K}(1-z_1-z_2)\mathtt{K}^*(z_2)\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(2\pi i)^2}\\ &= \int_0^\infty \check{\mathtt{K}}(x)\int_{(1/3)} \mathtt{K}(z_1)x^{-z_1}\frac{\mathrm{d}z_1}{2\pi \mathrm{i}}\int_{\mathscr{C}} \mathtt{K}^*(z_2)x^{-z_2}\frac{\mathrm{d}z_2}{2\pi \mathrm{i}}\,\mathrm{d}x = \int_0^\infty \check{\mathtt{K}}(x)^2\int_{\mathscr{C}} \mathtt{K}^*(z_2)x^{-z_2}\frac{\mathrm{d}z_2}{2\pi \mathrm{i}}\,\mathrm{d}x.\end{aligned}$$

The $z_2$-integral is also an inverse Mellin transform, but in order to avoid convergence issues, we compute it directly by shifting the contour to the far left and collect the poles. Comparing power series (cf. [40, 8.232, 8.253]), we obtain

$$\int_{\mathscr{C}} \mathrm{K}^*(z) x^{-z} \frac{\mathrm{d}z}{(2\pi \mathrm{i})} = \frac{4\mathrm{Si}\, x + 4\sin x - 2x\cos x}{x}.$$

For this and related expressions appearing in the computation of the Peyre constant of the varieties $X_1, \dots, X_4$, the following lemma can be used. Let

$$\mathbf{F}(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) \mathrm{d}t.$$

**Lemma A.1.** *We have*

$$\int_0^\infty \left(\frac{\mathrm{Si}\, x}{x}\right)^3 \mathrm{d}x = \frac{33}{32}\pi - \frac{1}{32}\pi^3, \qquad \int_0^\infty \left(\frac{\mathrm{Si}\, x}{x}\right)^2 \frac{\sin x}{x} \mathrm{d}x = \frac{1}{4}\pi + \frac{\pi}{48}(21 - \pi^2),$$
$$\int_0^\infty \left(\frac{\mathrm{Si}\, x}{x}\right)^2 \cos(x)\, \mathrm{d}x = \frac{\pi(12 - \pi^2)}{24}.$$

*Moreover,*

$$\int_0^\infty \frac{(\mathrm{Si}\, x)^2}{x^2} \left(\frac{\pi}{2x}\right)^{1/2} \mathbf{F}\left(\left(\frac{2x}{\pi}\right)^{1/2}\right) \mathrm{d}x = -\frac{\pi^3}{72} + \pi\left(\frac{59}{54} - \frac{4}{9}\log 2\right),$$
$$\int_0^\infty \frac{(\mathrm{Si}\, x)\sin x}{x^2} \left(\frac{\pi}{2x}\right)^{1/2} \mathbf{F}\left(\left(\frac{2x}{\pi}\right)^{1/2}\right) \mathrm{d}x = \frac{\pi}{36}(25 - 12\log 2).$$

*Proof.* The first integral is computed in [6, Theorem 3]. To compute the second, we observe that

$$\int_0^\infty \left(\frac{\mathrm{Si}(x)}{x}\right)^2 \frac{\sin(x)}{x} \mathrm{d}x = \int_0^1 \int_0^1 \int_0^\infty \frac{\sin(x)\sin(tx)\sin(sx)}{x^3} \mathrm{d}x \frac{\mathrm{d}t\, \mathrm{d}s}{ts}.$$

By the residue theorem, it is readily seen that the inner integral equals

$$\frac{\pi}{16}((s+t+1)^2 - (s+t-1)^2 \mathrm{sgn}(s+t-1) - (s-t+1)^2 \mathrm{sgn}(s-t+1) - (t-s+1)^2 \mathrm{sgn}(t-s+1))$$
$$= \frac{\pi}{16}\begin{cases} -2 + 4s - 2s^2 + 4t + 4st - 2t^2, & s+t \geq 1 \\ 8st, & s+t \leq 1 \end{cases}$$

for $0 \leq s, t \leq 1$, and a straightforward computation gives the desired result. Similarly, one computes the other integrals. □

The previous lemma confirms the evaluation

$$c_\infty = 32(47 - \pi^2).$$

## B. Final remarks

Here, we show that $X_3, \dots, X_8, X^\dagger, \widetilde{X}^\dagger$ do not belong to any of the families of varieties described in the introduction for which Manin's conjecture is already known. Whether or not $X_1$, $X_2$ are biequivariant compactifications of a unipotent group is not obvious to us, but it is not hard to see that they are certainly neither horospherical nor equivariant compactifications of $\mathbb{G}_\mathrm{a}^d$ nor wonderful compactification of a semisimple group of adjoint type.

**Proposition B.1.** *None of the varieties $X_3, \dots, X_8, X^\dagger, \widetilde{X}^\dagger$ is isomorphic to a biequivariant compactification of a unipotent group.*

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***Table B.1.*** *Flag varieties of simple groups and of dimension up to* 6.

| root system | parabolic subgroup | $\dim G/P$ | $\mathscr{L}$ | $\operatorname{rk}\operatorname{Cl} G/P$ | remark |
|---|---|---|---|---|---|
| $A_1$ | $\alpha_1$ | 1 | (2) | 1 | toric |
| $A_2$ | $\alpha_1$ | 2 | (3) | 1 | toric |
| $A_2$ | $\alpha_1, \alpha_2$ | 3 | (3, 3) | 2 | |
| $A_3$ | $\alpha_1$ | 3 | (4) | 1 | toric |
| $A_3$ | $\alpha_2$ | 4 | (6) | 1 | |
| $A_3$ | $\alpha_1, \alpha_2$ | 5 | (4, 6) | 2 | |
| $A_3$ | $\alpha_1, \alpha_3$ | 5 | (4, 4) | 2 | |
| $A_3$ | $\alpha_1, \alpha_2, \alpha_3$ | 6 | (4, 4, 6) | 3 | |
| $A_4$ | $\alpha_1$ | 4 | (5) | 1 | toric |
| $A_4$ | $\alpha_2$ | 6 | (10) | 1 | |
| $A_5$ | $\alpha_1$ | 5 | (6) | 1 | toric |
| $A_6$ | $\alpha_1$ | 6 | (7) | 1 | toric |
| $B_2$ | $\alpha_1$ | 3 | (5) | 1 | |
| $B_2$ | $\alpha_2$ | 3 | (4) | 1 | toric |
| $B_2$ | $\alpha_1, \alpha_2$ | 4 | (4, 5) | 2 | |
| $B_3$ | $\alpha_1$ | 5 | (7) | 1 | |
| $B_3$ | $\alpha_3$ | 6 | (8) | 1 | |
| $C_3$ | $\alpha_1$ | 5 | (6) | 1 | toric |
| $C_3$ | $\alpha_3$ | 6 | (14) | 1 | |
| $D_4$ | $\alpha_1$ | 6 | (8) | 1 | |
| $G_2$ | $\alpha_1$ | 5 | (7) | 1 | |
| $G_2$ | $\alpha_2$ | 5 | (14) | 1 | |
| $G_2$ | $\alpha_1, \alpha_2$ | 6 | (7, 14) | 2 | |

*Proof.* By [22, Proposition 1.1], the effective cone of every equivariant compactification of $\mathbb{G}_a^3$ is simplicial. More generally, by [67, Proposition 7.2], the same is true for biequivariant compactifications of unipotent groups. However, the effective cones of $X_3, \dots, X_8, X^\dagger, \widetilde{X}^\dagger$ are not simplicial. □

**Proposition B.2.** *Neither $X_1$ nor $X_2$ is isomorphic to an equivariant compactification of* $\mathbb{G}_a^3$.

*Proof.* By [46], only the first two entries of Table 11.1 are equivariant compactifications of $\mathbb{G}_a^3$. □

**Proposition B.3.** *None of the varieties $X_1, \dots, X_8, X^\dagger, \widetilde{X}^\dagger$ is isomorphic to a wonderful compactification of a semisimple group of adjoint type or to a wonderful variety covered by [39, Corollary 1.5].*

*Proof.* Over $\overline{\mathbb{Q}}$, the only wonderful variety of dimension 3 and Picard rank 3 is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; see, for instance, [12]. Hence, $X_1$ and $X_2$ are not wonderful varieties.

Moreover, by [18, Example 2.3.5], the effective cone of a wonderful compactification of a semisimple group of adjoint type is simplicial. Similarly, by [39, Section 3.3], the effective cone of a wonderful variety covered by [39, Corollary 1.5] is simplicial. Hence, the result for $X_3, \dots, X_8, X^\dagger, \widetilde{X}^\dagger$ follows as in Proposition B.1. □

**Proposition B.4.** *None of the varieties $X_1, \dots, X_8, X^\dagger, \widetilde{X}^\dagger$ is isomorphic to a horospherical variety.*

*Proof.* By [44, §6] and [12], the varieties in Table 11.1 are not horospherical; hence, $X_1, \dots, X_4$ are not horospherical.

Now, let $X$ be a complete horospherical $G$-variety. After possibly removing a set of codimension at least 2, we obtain a surjective $G$-equivariant morphism $X \longrightarrow G/P$, where $P \subseteq G$ is a parabolic subgroup and the fiber $Y$ is a toric variety. The fan of $Y$ is obtained from the colored fan of $X$ by ignoring the colors. For details, we refer to [3, Section 2]. The generators of the effective cone $\operatorname{Eff} G/P$ are a basis of the divisor class group $\operatorname{Cl} G/P$. Moreover, we have $\mathscr{R}(X) = \mathscr{R}(G/P)[X_1, \dots, X_r]$, where

$$r = \operatorname{rk}\operatorname{Cl} X - \operatorname{rk}\operatorname{Cl} G/P + \dim X - \dim G/P = \text{the number of rays in the fan of } Y;$$

this follows from [18, Theorem 4.3.2]. See also [33, Theorem 3.8].

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***Table B.2.*** *Nontoric flag varieties of dimension up to* 6.

| root system | parabolic subgroup | $\dim G/P$ | $\mathscr{Z}$ | $\operatorname{rk}\operatorname{Cl} G/P$ | $r_{X_6}$ | $r_{X_7}$ | $r_{X_8}$ |
|---|---|---|---|---|---|---|---|
| $A_2$ | $\alpha_1, \alpha_2$ | 3 | (3, 3) | 2 | 3 | 6 | 8 |
| $B_2$ | $\alpha_1$ | 3 | (5) | 1 | 4 | 7 | 9 |
| $A_3$ | $\alpha_2$ | 4 | (6) | 1 | 3 | 6 | 8 |
| $B_2$ | $\alpha_1, \alpha_2$ | 4 | (4, 5) | 2 | 2 | 5 | 7 |
| $A_2 \times A_1$ | $\alpha_1, \alpha_2, \beta_1$ | 4 | (2, 3, 3) | 3 | 1 | 4 | 6 |
| $B_2 \times A_1$ | $\alpha_1, \beta_1$ | 4 | (2, 5) | 2 | 2 | 5 | 7 |
| $A_3$ | $\alpha_1, \alpha_2$ | 5 | (4, 6) | 2 | | 1 | 3 |
| $A_3$ | $\alpha_1, \alpha_3$ | 5 | (4, 4) | 2 | | 1 | 3 |
| $B_3$ | $\alpha_1$ | 5 | (7) | 1 | | 0 | 2 |
| $G_2$ | $\alpha_1$ | 5 | (7) | 1 | | 0 | 2 |
| $G_2$ | $\alpha_2$ | 5 | (14) | 1 | | 0 | 2 |
| $A_3 \times A_1$ | $\alpha_2, \beta_1$ | 5 | (2, 6) | 2 | | 1 | 3 |
| $B_2 \times A_1$ | $\alpha_1, \alpha_2, \beta_1$ | 5 | (2, 4, 5) | 3 | | −1 | 0 |
| $A_2 \times A_2$ | $\alpha_1, \alpha_2, \beta_1$ | 5 | (3, 3, 3) | 3 | | −1 | 0 |
| $B_2 \times A_2$ | $\alpha_1, \beta_2$ | 5 | (3, 5) | 2 | | 1 | 3 |
| $A_2 \times A_1 \times A_1$ | $\alpha_1, \alpha_2, \beta_1, \gamma_1$ | 5 | (2, 2, 3, 3) | 4 | | −2 | −1 |
| $B_2 \times A_1 \times A_1$ | $\alpha_1, \beta_1, \gamma_1$ | 5 | (2, 2, 5) | 3 | | −1 | 0 |
| $A_3$ | $\alpha_1, \alpha_2, \alpha_3$ | 6 | (4, 4, 6) | 3 | | | 4 |
| $A_4$ | $\alpha_2$ | 6 | (10) | 1 | | | 6 |
| $B_3$ | $\alpha_3$ | 6 | (8) | 1 | | | 6 |
| $C_3$ | $\alpha_3$ | 6 | (14) | 1 | | | 6 |
| $D_4$ | $\alpha_1$ | 6 | (8) | 1 | | | 6 |
| $G_2$ | $\alpha_1, \alpha_2$ | 6 | (7, 14) | 2 | | | 5 |
| $A_3 \times A_1$ | $\alpha_1, \alpha_2, \beta_1$ | 6 | (2, 4, 6) | 3 | | | 4 |
| $A_3 \times A_1$ | $\alpha_1, \alpha_3, \beta_1$ | 6 | (2, 4, 4) | 3 | | | 4 |
| $B_3 \times A_1$ | $\alpha_1, \beta_1$ | 6 | (2, 7) | 2 | | | 5 |
| $G_2 \times A_1$ | $\alpha_1, \beta_1$ | 6 | (2, 7) | 2 | | | 5 |
| $G_2 \times A_1$ | $\alpha_2, \beta_1$ | 6 | (2, 14) | 2 | | | 5 |
| $A_3 \times A_2$ | $\alpha_2, \beta_1$ | 6 | (3, 6) | 2 | | | 5 |
| $B_2 \times A_2$ | $\alpha_1, \alpha_2, \beta_1$ | 6 | (3, 4, 5) | 3 | | | 4 |
| $A_3 \times A_1 \times A_1$ | $\alpha_2, \beta_1, \gamma_1$ | 6 | (2, 2, 6) | 3 | | | 4 |
| $B_2 \times A_1 \times A_1$ | $\alpha_1, \alpha_2, \beta_1, \gamma_1$ | 6 | (2, 2, 4, 5) | 4 | | | 3 |
| $A_2 \times A_2$ | $\alpha_1, \alpha_2, \beta_1, \beta_2$ | 6 | (3, 3, 3, 3) | 4 | | | 3 |
| $A_2 \times B_2$ | $\alpha_1, \alpha_2, \beta_1$ | 6 | (3, 3, 5) | 3 | | | 4 |
| $B_2 \times B_2$ | $\alpha_1, \beta_1$ | 6 | (5, 5) | 2 | | | 5 |
| $A_2 \times A_3$ | $\alpha_1, \alpha_2, \beta_1$ | 6 | (3, 3, 4) | 3 | | | 4 |
| $A_2 \times B_2$ | $\alpha_1, \alpha_2, \beta_2$ | 6 | (3, 3) | 3 | | | 4 |
| $B_2 \times A_3$ | $\alpha_1, \beta_1$ | 6 | (4, 4, 5) | 2 | | | 5 |
| $B_2 \times B_2$ | $\alpha_1, \beta_2$ | 6 | (4, 5) | 2 | | | 5 |
| $A_2 \times A_2 \times A_1$ | $\alpha_1, \alpha_2, \beta_1, \gamma_1$ | 6 | (2, 3, 3, 3) | 4 | | | 3 |
| $B_2 \times A_2 \times A_1$ | $\alpha_1, \beta_1, \gamma_1$ | 6 | (2, 3, 5) | 3 | | | 4 |
| $A_2 \times A_1 \times A_1 \times A_1$ | $\alpha_1, \alpha_2, \beta_1, \gamma_1, \delta_1$ | 6 | (2, 2, 2, 3, 3) | 5 | | | 2 |
| $B_2 \times A_1 \times A_1 \times A_1$ | $\alpha_1, \beta_1, \gamma_1, \delta_1$ | 6 | (2, 2, 2, 6) | 4 | | | 3 |

Table B.2 contains the data of all nontoric flag varieties $G/P$ required here. It can be computed from Table B.1 by forming products. The parabolic subgroup $P$ is described by the complement of the subset of the simple roots used in [69, Theorem 8.4.3]. It follows that the set of colors of $G/P$ is in bijection with the subset of simple roots given in the tables; see [58, after Définition 2.6]. By [18, Proposition 4.1.1], the rank of $\operatorname{Cl} G/P$ is the number of colors. The dimension of $G/P$ can be deduced, for instance, by [71, p. 9]. For simple $G$, it follows from [37, Proposition 6.1] that $G/P$ is toric if and only if the Dynkin diagram of $G$ marked with the subset of simple roots given in the tables appears in [57, Lemme 2.13]. The meaning of $\mathscr{Z}$ will be explained below.

First, assume that $X^\dagger$ or $\widetilde{X}^\dagger$ is isomorphic to $X$. Then we have $\dim X = 3$. Recall that the effective cones of $X^\dagger$ and $\widetilde{X}^\dagger$ are not simplicial. Since the effective cone of any flag variety is simplicial, we deduce $\dim G/P \leq 2$. It follows that $G/P$ is isomorphic to a toric variety, and hence the same is true for $X$. But according to Section 13, the Cox rings of $X^\dagger$ and $\widetilde{X}^\dagger$ are not polynomial rings, a contradiction.

Next, assume that $X_5$ is isomorphic to $X$. Then we have $\dim X = 4$. As before, we obtain $\dim G/P \leq 3$ from the fact that the effective cone of $X_5$ is not simplicial and $\dim G/P \geq 3$ from the fact that the variety $X_5$ is not isomorphic to a toric one. Hence, we have $\dim G/P = 3$, and therefore, $\operatorname{rk}\operatorname{Cl} G/P \leq 3$. Moreover, we have $\dim Y = 1$ and therefore $r \leq 2$. We obtain $\operatorname{rk}\operatorname{Cl} X \leq 4$, a contradiction to $\operatorname{rk}\operatorname{Cl} X_5 = 5$.

Next, assume that $X_6$ is isomorphic to $X$. Then we have $\dim X = 5$. Let $\mathscr{Z}(X_6)$ be the ordered tuple of the dimensions of the homogeneous parts of the Cox ring $\mathscr{R}(X_6)$ for the generators of the effective cone of $X_6$. According to Section 12.2.2, we have

$$\mathscr{Z}(X_6) = (1, 1, 2, 3).$$

As in the previous cases, we obtain $3 \leq \dim G/P \leq 4$. The possible values for $\mathscr{Z}(G/P)$ and $r = r_{X_6}$ are given in Table B.2 (the toric cases are excluded). The values for $\mathscr{Z}(G/P)$ are computed using the Weyl dimension formula; see, for instance, [47, Corollary 24.3]. We have a natural surjective map $\phi\colon \operatorname{Cl} G/P \times \mathbb{Z}^r \to \operatorname{Cl} X$ compatible with the $\operatorname{Cl} X$-grading and the finer $\operatorname{Cl} G/P \times \mathbb{Z}^r$-grading of $\mathscr{R}(X)$. It maps the cone $\operatorname{Eff} G/P \times \mathbb{Z}^r_{\geq 0}$ generated by $\operatorname{Eff} G/P$ and the degrees of $X_1, \ldots, X_r$ onto $\operatorname{Eff} X$. Moreover, we have $(\operatorname{Eff} G/P \times \mathbb{Z}^r_{\geq 0}) \cap \ker \phi = \{0\}$. It follows that every element of $\mathscr{Z}(X_6)$ is a sum where the summands are taken from the elements of $\mathscr{Z}(R/P)$ and from $r_{X_6}$ times the summand 1 and each summand may be used at most once in total. This is impossible for all cases in Table B.2. The same argument works for $X_8$, which satisfies

$$\mathscr{Z}(X_8) = (1, 1, 1, 1, 1, 1, 2, 2)$$

according to Section 12.2.4.

Finally, assume that $X_7$ is isomorphic to $X$. According to Section 12.2.3, we have

$$\mathscr{Z}(X_7) = (1, 1, 1, 1, 1, 1).$$

It follows that there exists an isomorphism

$$\begin{aligned} \mathscr{R}(X_7) &\to \mathscr{R}(G/P)[X_1, \ldots, X_r], \\ (x_{03}, x_{31}, x_{32}, x_{33}, x_{34}, x_{35}) &\mapsto (X_1, X_2, X_3, X_4, X_5, X_6). \end{aligned}$$

After dividing out the ideal $(x_{03}, x_{31}, x_{32}, x_{33}, x_{34}, x_{35})$, we obtain an isomorphism

$$\mathbb{Q}[x_{01}, x_{02}, x_{11}, x_{12}, x_{21}, x_{22}]/(x_{11}x_{12} - x_{21}x_{22}) \to \mathscr{R}(G/P)[X_7, \ldots, X_r].$$

This is a contradiction since the second ring is factorial by [2, Proposition 1.4.1.5(i)], while the first ring is not. □

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