Proceedings of the Edinburgh Mathematical Society (2024) **67**, 617–642
doi:10.1017/S0013091524000178

MINIMAL MONOIDS GENERATING VARIETIES WITH COMPLEX SUBVARIETY LATTICES

SERGEY V. GUSEV

Department of Mathematics, Mechanics and Computer Science, Institute of Natural Sciences and Mathematics, Ural Federal University, Ekaterinburg, Russia
(sergey.gusb@gmail.com)

(Received 20 February 2023)

Abstract A variety is *finitely universal* if its lattice of subvarieties contains an isomorphic copy of every finite lattice. We show that the 6-element Brandt monoid generates a finitely universal variety of monoids and, by the previous results, it is the smallest generator for a monoid variety with this property. It is also deduced that the join of two Cross varieties of monoids can be finitely universal. In particular, we exhibit a finitely universal variety of monoids with uncountably many subvarieties which is the join of two Cross varieties of monoids whose lattices of subvarieties are the 6-element and the 7-element chains, respectively.

Keywords: monoid; variety; lattice of varieties; finitely universal variety; Brandt monoid

2020 Mathematics subject classification: Primary 20M07
Secondary 08B15

1. Introduction

A *variety* is a class of algebras of a fixed type that is closed under the formation of homomorphic images, subalgebras and arbitrary direct products. A variety is *finitely based* if it can be defined by a finite set of identities, otherwise, it is *non-finitely based*. A variety is *finitely generated* if it is generated by a finite algebra. A variety is *small* if it contains only finitely many subvarieties. A finitely generated, finitely based, small variety of algebras is called a *Cross variety*. Cross varieties have been heavily investigated for many years. For classical algebras such as groups [24], associative rings [17, 23] and Lie rings [1], every finite member generates a Cross variety. However, this result is not true for arbitrary algebras. In general, the variety $\mathbb{V}$ generated by a finite algebra can be non-Cross in several ways, for instance, $\mathbb{V}$ can be non-finitely based, the lattice $\mathfrak{L}(\mathbb{V})$ of subvarieties of $\mathbb{V}$ can be infinite or even uncountable, and $\mathbb{V}$ can be *finitely universal* in the sense that $\mathfrak{L}(\mathbb{V})$ contains an isomorphic of every finite lattice.

Examples of finitely universal varieties of semigroups have been known since the early 1970s [3], and the smallest semigroup generating such a variety is of order four [18];

see Section 12 in the survey [32] for more information. For a long time, however, it was unknown if finitely universal varieties of monoids exist [13, Question 6.3]. The first examples of finitely universal varieties of monoids have recently been found [6]; in fact, there also exist finitely universal varieties that are finitely generated, but an explicit smallest example have not been found; see Section 4 in the very recent survey [7] for more details. Unlike semigroups, the variety generated by any monoid of order five or less is not finitely universal [8, 21]. This naturally leads to the following problem.

Problem 1 (see [7, Problem 4.7]). *Is there a monoid of order six that generates a finitely universal variety of monoids?*

The 6-element Brandt monoid:

$$B_2^1 := \langle a, b \mid aba = a,\, bab = b,\, aa = bb = 0\rangle \cup \{1\},$$

is one of the most famous finite monoids. It can be represented as the matrix semigroup:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Brandt monoid B_2^1 is perhaps the most ubiquitous harbinger of complex behaviour in all finite semigroups. In particular, B_2^1 has no finite basis for its identities [25] and is one of the four smallest semigroups with this property [22]. It generates a monoid variety with uncountably many subvarieties [13, 15] and, moreover, it is the smallest generator for a monoid variety with uncountably many subvarieties [8, 21].

The 6-element monoid:

$$A_2^1 := \langle a, b \mid aba = a,\, bab = b,\, aa = 0,\, bb = b\rangle \cup \{1\},$$

is one more of the most famous 6-element monoids. It can be represented as the matrix semigroup:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is well known that A_2^1 generates a variety properly containing that generated by B_2^1. The monoid A_2^1 as well as the 6-element Brandt monoid B_2^1 plays a critical role in the theory of semigroup varieties. So, the following question is of fundamental interest.

Problem 2 ([6, Question 6.2]). *Which, if any, of the monoids B_2^1 and A_2^1 generates a finitely universal variety?*

Problems 1 and 2 are addressed in the present article. We exhibit a finitely universal monoid variety $\mathbb{C}$ and show that $\mathbb{C}$ is contained in the variety generated by the Brandt monoid B_2^1. Problems 1 and 2 are thus completely solved.

The new finitely universal variety $\mathbb{C}$ allows us to construct examples of two small varieties of monoids with an incredibly complex join resulting in solving the following problem.

Problem 3 ([6, Question 6.4]).

(i) *Are there varieties of monoids* $\mathbb{V}_1$ *and* $\mathbb{V}_2$ *that are not finitely universal such that the join* $\mathbb{V}_1 \vee \mathbb{V}_2$ *is finitely universal?*

(ii) *Are there small varieties of monoids* $\mathbb{V}_1$ *and* $\mathbb{V}_2$ *such that the join* $\mathbb{V}_1 \vee \mathbb{V}_2$ *is finitely universal?*

Remark 1. Problem 3(i) has an affirmative answer within the context of varieties of semigroups, that is, there are two semigroup varieties that are not finitely universal such that their join is finitely universal. However, one of these varieties is not small, so that they do not provide an affirmative answer to Problem 3(ii) within the context of varieties of semigroups; see Section 6.3 in [6] for more details.

In fact, we not only provide an affirmative solution to Problem 3, but establish a much stronger counterintuitive result. Namely, we prove that there are two Cross varieties of monoids, whose lattices of subvarieties are the 6-element and the 7-element chains, respectively, such that the join of these two varieties is finitely universal and contains uncountably many subvarieties. Moreover, we construct infinitely examples of finitely universal varieties with uncountably many subvarieties which are the join of two Cross varieties.

The article consists of five sections. Background information and some basic results are first given in § 2. In § 3, we introduce the variety $\mathbb{C}$, which, as we show in § 4, turns out to be finitely universal (Theorem 1). Then we formulate our main results announced above (Theorems 2 and 3) and deduce them from Theorem 1. The technical core of the article is § 4; it is devoted to the proof of Theorem 1. We prove Theorem 1 by showing that the lattice $\mathfrak{Eq}(\mathsf{A})$ of equivalence relations on every sufficiently large finite set A is anti-isomorphic to some subinterval of the lattice $\mathfrak{L}(\mathbb{C})$ of subvarieties. In view of the well-known theorem of Pudlák and Tůma [29] stating that every finite lattice is embeddable in a lattice of equivalence relations on a finite set, Theorem 1 thus holds. The article ends with some open problems in § 5.

2. Preliminaries

Acquaintance with rudiments of universal algebra is assumed of the reader. Refer to the monograph [4] for more information.

2.1. Words, identities and deduction

Let $\mathscr{X}^*$ denote the free monoid over a countably infinite alphabet $\mathscr{X}$. Elements of $\mathscr{X}$ are called *variables* and elements of $\mathscr{X}^*$ are called *words*. The *content* of a word $\mathbf{w}$, that is, the set of all variables occurring in $\mathbf{w}$ is denoted by $\mathsf{con}(\mathbf{w})$. For a word $\mathbf{w}$ and a variable x, let $\mathsf{occ}_x(\mathbf{w})$ denote the number of occurrences of x in $\mathbf{w}$. A variable x is called *simple* [*multiple*] *in a word* $\mathbf{w}$ if $\mathsf{occ}_x(\mathbf{w}) = 1$ [respectively, $\mathsf{occ}_x(\mathbf{w}) > 1$]. The set of all simple [multiple] variables of a word $\mathbf{w}$ is denoted by $\mathsf{sim}(\mathbf{w})$ [respectively, $\mathsf{mul}(\mathbf{w})$]. A non-empty word $\mathbf{w}$ is called *linear* if $\mathsf{con}(\mathbf{w}) = \mathsf{sim}(\mathbf{w})$. For any $\mathscr{A} \subseteq \mathsf{con}(\mathbf{w})$, let $\mathbf{w}(\mathscr{A})$ denote the word obtained by applying the substitution that fixes the variables in

$\mathscr{A}$ and assigns the empty word 1 to all other variables. Further, for any $\mathscr{A} \subseteq \mathsf{con}(\mathbf{w})$, let $\mathbf{w}_{\mathscr{A}} := \mathbf{w}(\mathsf{con}(\mathbf{w}) \setminus \mathscr{A})$. The expression ${}_{i\mathbf{w}}x$ means the ith occurrence of a variable x in a word $\mathbf{w}$. If the ith occurrence of x precedes the jth occurrence of y in a word $\mathbf{w}$, then we write $({}_{i\mathbf{w}}x) < ({}_{j\mathbf{w}}y)$.

An identity is written as $\mathbf{u} \approx \mathbf{v}$, where $\mathbf{u}, \mathbf{v} \in \mathscr{X}^*$; it is *nontrivial* if $\mathbf{u} \neq \mathbf{v}$. A variety $\mathbb{V}$ *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$, if for any monoid $M \in \mathbb{V}$ and any substitution $\varphi\colon \mathscr{X} \to M$, the equality $\varphi(\mathbf{u}) = \varphi(\mathbf{v})$ holds in M. An identity $\mathbf{u} \approx \mathbf{v}$ is *directly deducible* from an identity $\mathbf{s} \approx \mathbf{t}$ if there exist some words $\mathbf{a}, \mathbf{b} \in \mathscr{X}^*$ and substitution $\varphi\colon \mathscr{X} \to \mathscr{X}^*$ such that $\{\mathbf{u}, \mathbf{v}\} = \big\{\mathbf{a}\varphi(\mathbf{s})\mathbf{b}, \mathbf{a}\varphi(\mathbf{t})\mathbf{b}\big\}$. A nontrivial identity $\mathbf{u} \approx \mathbf{v}$ is *deducible* from a set Σ of identities if there exists some finite sequence $\mathbf{u} = \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m = \mathbf{v}$ of distinct words such that each identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ is directly deducible from some identity in Σ.

Proposition 1 (Birkhoff's Completeness Theorem for Equational Logic; see [4, Theorem II.14.19]). *Let $\mathbb{V}$ be the variety defined by some set Σ of identities. Then $\mathbb{V}$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$ if and only if $\mathbf{u} \approx \mathbf{v}$ is deducible from Σ.*

Two sets of identities Σ_1 and Σ_2 are *equivalent* (within a variety $\mathbb{V}$) if Σ_1 and Σ_2 define the same variety (within $\mathbb{V}$).

2.2. Factor monoids

A word $\mathbf{u}$ is a *factor* of a word $\mathbf{w}$ if $\mathbf{w} = \mathbf{v}'\mathbf{u}\mathbf{v}''$ for some $\mathbf{v}', \mathbf{v}'' \in \mathscr{X}^*$. For any set $\mathscr{W}$ of words, the *factor monoid of $\mathscr{W}$*, denoted by $M(\mathscr{W})$, is the monoid that consists of all factors of words in $\mathscr{W}$ and a zero element 0, with multiplication $\cdot$ given by:

$$\mathbf{u} \cdot \mathbf{v} := \begin{cases} \mathbf{u}\mathbf{v} & \text{if } \mathbf{u}\mathbf{v} \text{ is a factor of } \mathbf{w} \in \mathscr{W}, \\ 0 & \text{otherwise;} \end{cases}$$

the empty word $\emptyset$ more conveniently written as 1, is the identity element of $M(\mathscr{W})$. A word $\mathbf{w}$ is an *isoterm* for a variety $\mathbb{V}$ if $\mathbb{V}$ violates any nontrivial identity of the form $\mathbf{w} \approx \mathbf{w}'$. Given any set $\mathscr{W}$ of words, let $\mathbb{M}(\mathscr{W})$ denote the variety generated by the factor monoid $M(\mathscr{W})$. One advantage in working with factor monoids is the relative ease of checking if a variety $\mathbb{M}(\mathscr{W})$ is contained in some given variety.

Lemma 1. ([12, Lemma 3.3]). *For any variety $\mathbb{V}$ and any set $\mathscr{W}$ of words, the inclusion $\mathbb{M}(\mathscr{W}) \subseteq \mathbb{V}$ holds if and only if any word in $\mathscr{W}$ is an isoterm for $\mathbb{V}$.*

3. Main results

3.1. The variety $\mathbb{C}$

Here, we introduce the variety $\mathbb{C}$, which, as we prove in § 4, is finitely universal. All other finitely universal varieties in this article contain it. We need some notation. We denote by S_k the symmetric group on the set $\{1, 2, \ldots, k\}$. As usual, S_k^n denote the nth direct power of S_k. If $\xi \in S_k^n$, then we denote by ξ_i the ith component of ξ. For any $n \geq 2$

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

and $\xi \in S_2^n$, we define the word:

$$\mathbf{w}_\xi := \mathbf{p}\left(\prod_{i=1}^{n} a_i\right) a \left(\prod_{i=1}^{n} x_{1\xi_i}^{(i)} x_{2\xi_i}^{(i)}\right) b \left(\prod_{i=1}^{n} b_i\right) \mathbf{qr},$$

where

$$\mathbf{p} := \left(\prod_{i=1}^{n} z_i t_i\right)\left(\prod_{i=1}^{n} z_i' t_i'\right)\left(\prod_{i=1}^{n} z_i'' t_i''\right), \tag{1}$$

$$\mathbf{q} := \left(\prod_{i=0}^{n} s_i y_i\right) t, \tag{2}$$

$$\mathbf{r} := b y_0 \left(\prod_{i=1}^{n} x_1^{(i)}\, z_i a_i z_i' b_i z_i''\, x_2^{(i)}\, y_i\right) a. \tag{3}$$

For any $n \geq 2$, we denote by $\mathscr{W}_n$ the set of all words of the form $\mathbf{w}_\xi$ with $\xi \in S_2^n$. Evidently, $|\mathscr{W}_n| = |S_2^n| = 2^n$.

Theorem 1. *The variety $\mathbb{C} := \mathbb{M}(\{\mathscr{W}_n \mid n \geq 2\})$ is finitely universal.*

Remark 2. Recall that a variety is *periodic* if it satisfies the identity $x^{m+k} \approx x^m$ for some $m, k \geq 1$; in this case, the number m is the *index* of the variety. Varieties of index 1 are *completely regular*, that is, consist of unions of groups. The lattice of subvarieties of every completely regular variety of semigroups and, therefore, monoids is modular and moreover, Arguesian; this fundamental result was established in three different ways by Pastijn [26, 27] and Petrich and Reilly [28] (see also Section 5.3 in the survey [7]). Thus, varieties of index 1 are not finitely universal. For each $m \geq 3$, an example of a finitely universal variety of index m was found in [6]. As for varieties of index 2, a finitely universal example was unknown so far; see [6, Question 6.1] or [7, Question 4.10]. It is easy to see that the variety $\mathbb{C}$ satisfies the identity $x^2 \approx x^3$ and so is of index 2. Thus, Theorem 1 provides an example of a finitely universal variety of monoids of index 2.

The proof of Theorem 1 is given in § 4. For the rest of § 3, we discuss our main results and show how to deduce them from Theorem 1.

3.2. The join of two Cross varieties

Let $\mathbb{N}$ denote the variety defined by the identities:

$$x^2 \approx x^3,\ x^2y \approx yx^2,\ xyxzx \approx x^2yz,\ xzxyty \approx xzyxty,\ xzytxy \approx xzytyx. \tag{4}$$

It is verified in [5, Theorem 1.1] that the lattice $\mathfrak{L}(\mathbb{M}(xzytxy) \vee \mathbb{N})$ is as shown in Fig. 1, where $\mathbb{T}$ is the variety of all trivial monoids and the interval $[\mathbb{M}(xzytxy), \mathbb{M}(xzytxy) \vee \mathbb{N}]$ contains uncountably many varieties. In particular, the lattices $\mathfrak{L}(\mathbb{M}(xzytxy))$ and $\mathfrak{L}(\mathbb{N})$

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

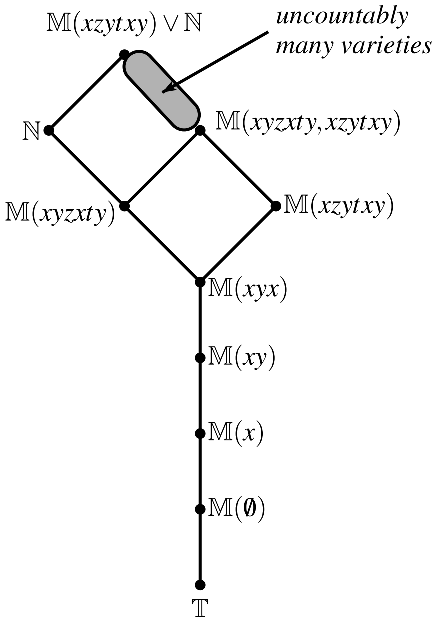

Figure 1. The lattice $\mathfrak{L}(\mathbb{M}(xzytxy) \vee \mathbb{N})$.

are the 6-element and the 7-element chains, respectively. The following counterintuitive result provides a complete solution to Problem 3.

Theorem 2. *There are two Cross varieties of monoids such that the join of these varieties is finitely universal and contains uncountably many subvarieties. Namely, the varieties* $\mathbb{M}(xzytxy)$ *and* $\mathbb{N}$ *satisfy this property.*

The proof of Theorem 2 requires one intermediate result.

Lemma 2. *Let* $\mathbf{u} \approx \mathbf{v}$ *be an identity of* $\mathbb{M}(xzytxy)$. *Suppose that* $\mathbf{u} \in \mathscr{W}_n$. *Then* $\mathbf{v} = \mathbf{p}\mathbf{v}'\mathbf{q}\mathbf{r}$, *where the words* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$ *are defined by the equalities (1), (2) and (3), respectively, while* $\mathbf{v}'$ *is a linear word with* $\mathsf{con}(\mathbf{v}') = \{a, a_i, b, b_i, x_1^{(i)}, x_2^{(i)} \mid 1 \le i \le n\}$.

Proof. In view of Lemma 3.1 in [10], $\mathbf{v} = \mathbf{p}\mathbf{v}'\mathbf{q}\mathbf{r}'$, where the words $\mathbf{p}$ and $\mathbf{q}$ are defined by the equalities (1) and (2), respectively, while $\mathbf{v}'$ and $\mathbf{r}'$ are linear words with $\mathsf{con}(\mathbf{v}') = \{a, a_i, b, b_i, x_1^{(i)}, x_2^{(i)} \mid 1 \le i \le n\}$ and $\mathsf{con}(\mathbf{r}) = \mathsf{con}(\mathbf{r}')$. Since $\mathbf{u}(b, s_0, t, y_0) = bs_0y_0tby_0$ and the identity $\mathbf{u} \approx \mathbf{v}$ is satisfied by $\mathbb{M}(xzytxy)$, the word $\mathbf{v}(b, s_0, t, y_0)$ must coincide with $bs_0y_0tby_0$. Therefore, $({}_{1\mathbf{r}'}b) < ({}_{1\mathbf{r}'}y_0)$. By a similar argument we can show that all the variables occur in $\mathbf{r}'$ in the same order as in $\mathbf{r}$ and, therefore, $\mathbf{r}' = \mathbf{r}$. □

Proof of Theorem 2. It is shown in the Erratum to [12] that the variety $\mathbb{M}(xzytxy)$ is finitely based. In view of this fact and Fig. 1, $\mathbb{M}(xzytxy)$ is a Cross variety. A finite generator for the variety $\mathbb{N}$ is also exhibited in the Erratum to [12]. Thus, $\mathbb{N}$ is also a Cross variety.

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

Let $n \geq 2$, $\xi \in S_2^n$ and $\mathbf{w}_\xi \approx \mathbf{w}$ be an identity of $\mathbb{M}(xzytxy) \vee \mathbb{N}$. It follows from Lemma 2 that $\mathbf{w} = \mathbf{p}\mathbf{w}'\mathbf{q}\mathbf{r}$, where the words $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are defined by the equalities (1), (2) and (3), respectively, while $\mathbf{w}'$ is a linear word with $\mathsf{con}(\mathbf{w}') = \{a, a_i, b, b_i, x_1^{(i)}, x_2^{(i)} \mid 1 \leq i \leq n\}$. Further, consider arbitrary $x, y \in \mathsf{con}(\mathbf{w}')$ with $({}_{1\mathbf{w}_\xi}x) < ({}_{1\mathbf{w}_\xi}y)$. Then $\mathbf{w}_\xi(x, y, t) = xyt\mathbf{a}$ with $\mathbf{a} \in \{xy, yx\}$. Since $\mathbb{N}$ violates $xyt\mathbf{a} \approx yxt\mathbf{a}$, it follows that $({}_{1\mathbf{w}}x) < ({}_{1\mathbf{w}}y)$. Therefore,

$$\mathbf{w}' = \left(\prod_{i=1}^{n} a_i\right) a \left(\prod_{i=1}^{n} x_{1\xi_i}^{(i)} x_{2\xi_i}^{(i)}\right) b \left(\prod_{i=1}^{n} b_i\right)$$

and so $\mathbf{w} = \mathbf{w}_\xi$. We have proved that every word in $\mathscr{W}_n$ is an isoterm for $\mathbb{M}(xzytxy) \vee \mathbb{N}$. Thus, $\mathbb{C} \subseteq \mathbb{M}(xzytxy) \vee \mathbb{N}$ by Lemma 1. Now Theorem 1 applies, yielding that the variety $\mathbb{M}(xzytxy) \vee \mathbb{N}$ is finitely universal. Finally, $\mathbb{M}(xzytxy) \vee \mathbb{N}$ contains uncountably many subvarieties by [5, Theorem 1.1]. □

Remark 3. Theorem 2 implies that a cover of a Cross variety of monoids can be finitely universal. In contrast, it is unknown whether or not the similar result holds within the context of varieties of semigroups. Although it is known that the class of Cross semigroup varieties is closed under neither joins nor covers [31].

For a monoid K, we denote by $\mathbb{K}$ the variety generated by K. We have the following result on the join $\mathbb{M}(xyx) \vee \mathbb{G}$ for a group G of finite exponent.

Corollary 1. *Let G be a group of finite exponent which does not satisfy the identities:*

$$xyzxy \approx yxzyx \ \textit{and} \ \ xyzyx \approx yxzxy. \tag{5}$$

Then $\mathbb{M}(xyx) \vee \mathbb{G}$ is a non-finitely based finitely universal variety with uncountably many subvarieties.

The proof of Corollary 1 requires one auxiliary result.

Lemma 3. *The variety $\mathbb{M}(xzytxy) \vee \mathbb{N}$ is a subvariety of $\mathbb{M}(xyzxy, xyzyx)$.*

Proof. Obviously, $\mathbb{M}(xzytxy) \subseteq \mathbb{M}(xyzxy, xyzyx)$. Further, it is shown in the proof of Lemma 3.14 in [10] that if a variety contains $\mathbb{M}(xyx)$ but does not contain $\mathbb{N}$, then it satisfies one of the identities $xyzxy \approx yxzxy$, $xyzxy \approx xyzyx$ or $xyzxy \approx yxzyx$. Since these three identities do not hold in $\mathbb{M}(xyzxy, xyzyx)$, it follows that $\mathbb{N} \subseteq \mathbb{M}(xyzxy, xyzyx)$. Therefore, $\mathbb{M}(xzytxy) \vee \mathbb{N}$ is a subvariety of $\mathbb{M}(xyzxy, xyzyx)$. □

Proof of Corollary 1. Since the group G does not satisfy the identities (5), this group is non-abelian, whence $\mathbb{M}(xyx) \vee \mathbb{G}$ is non-finitely based by [19, Theorem 3]. Let $xyzxy \approx \mathbf{v}$ be an identity of $\mathbb{M}(xyx) \vee \mathbb{G}$. Since xyx is an isoterm for $\mathbb{M}(xyx) \vee \mathbb{G}$, we have $\mathbf{v} \in \{xyzxy, xyzyx, yxzxy, yxzyx\}$. The word $\mathbf{v}$ cannot coincide with $yxzyx$ because G violates the identities (5). Let m denote the exponent of G. If $\mathbf{v} = xyzyx$, then G

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

satisfies the identities

$$xy \approx (xy)^{m+1} \approx (xy)^m(yx) \approx yx,$$

contradicting the fact that G is a non-abelian group. If $\mathbf{v} = yxzxy$, then G satisfies the identities:

$$xy \approx (xy)^{m+1} \approx (yx)(xy)^m \approx yx,$$

contradicting the fact that the group G is non-abelian again. Therefore, $\mathbf{v} = xyzxy$. We see that $xyzxy$ is an isoterm for $\mathbb{M}(xyx) \vee \mathbb{G}$. By similar arguments we can show that $xyzyx$ is an isoterm for $\mathbb{M}(xyx) \vee \mathbb{G}$ as well. Now Lemmas 1 and 3 apply, yielding that $\mathbb{M}(xzytxy) \vee \mathbb{N}$ is a subvariety of $\mathbb{M}(xyx) \vee \mathbb{G}$. Hence the variety $\mathbb{M}(xyx) \vee \mathbb{G}$ is finitely universal and contains uncountably many subvarieties by Theorem 2. □

Remark 4. By the theorem of Oates and Powell [24], if G is a finite group, then $\mathbb{G}$ is a Cross variety. Hence Corollary 1 provides plenty of examples of non-finitely based finitely universal varieties of monoids with uncountably many subvarieties which are the join of two Cross varieties. Namely, they are the varieties of the form $\mathbb{M}(xyx) \vee \mathbb{G}$ for any finite group G violated the identities (5). For example, for each prime $p > 2$, consider the dihedral group:

$$D_p := \langle a, b \mid a^p = b^2 = (ab)^2 = 1 \rangle,$$

(the group of symmetries of a regular polygon with p sides). This group does not satisfy the identities (5). Indeed, consider the substitutions $\varphi\colon \mathscr{X} \to D_p$ and $\psi\colon \mathscr{X} \to D_p$ defined by:

$$\varphi(v) := \begin{cases} ab & \text{if } v = x, \\ b & \text{if } v = y, \\ 1 & \text{otherwise}, \end{cases} \quad \text{and} \quad \psi(v) := \begin{cases} a & \text{if } v = x, \\ b & \text{if } v = y, \\ 1 & \text{otherwise}. \end{cases}$$

It is routine to check that:

$$\varphi(xyzxy) = \psi(xyzyx) = a^2 \text{ and } \varphi(yxzyx) = \psi(yxzxy) = a^{p-2}.$$

Since $p > 2$ and p is prime, we have $a^2 \neq a^{p-2}$. Hence D_p violates the identities (5). It is well known that D_p is a minimal non-abelian group of order $2p$; see [11, Section 1.9]. From this it can be easily deduced that the lattice $\mathfrak{L}(\mathbb{D}_p)$ is as shown in Fig. 2 (we denote by $\mathbb{Z}_k$ the variety of all abelian groups of exponent dividing k). Thus, we have a countably infinite series of finitely universal varieties of monoids with uncountably many subvarieties which are the join of two Cross varieties whose lattices of subvarieties are 5-element.

Remark 5. The following is claimed in the proof of Corollary 3.1 in [5]: $\mathbb{M}(xzytxy) \vee \mathbb{N}$ is a subvariety of $\mathbb{M}(xyx) \vee \mathbb{G}$ for any finite non-abelian group G. In fact, this result is

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

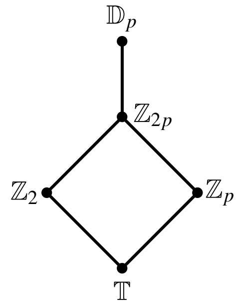

Figure 2. The lattice $\mathfrak{L}(\mathbb{D}_p)$.

wrong in general. For example, it is easy to see that the quaternion group:

$$Q_8 := \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle,$$

satisfies the identities (5), whence $\mathbb{N} \nsubseteq \mathbb{M}(xyx) \vee \mathbb{Q}_8$. As we have shown in the proof of Corollary 1, the discussed result is true whenever G is a finite group violated the identities (5).

3.3. Minimal monoids generating finitely universal varieties

Here we provide a complete solution to Problems 1 and 2. As we have mentioned in the introduction, every monoid of order five or less generates a non-finitely universal variety [8, Proposition 6.9]. Examples of finitely universal varieties generated by 6-element monoids are provided by the following theorem.

Theorem 3. *The 6-element monoids B_2^1 and A_2^1 generate finitely universal varieties.*

Proof. It is easy to show that $xyzxy$ and $xyzyx$ are isoterms for $\mathbb{B}_2^1$; see the proof of Theorem 10 in [25]. This fact and Lemma 1 imply that $\mathbb{M}(xyzxy, xyzyx) \subseteq \mathbb{B}_2^1$. Then $\mathbb{M}(xzytxy) \vee \mathbb{N}$ is a subvariety of $\mathbb{B}_2^1$ by Lemma 3 and, therefore, the variety $\mathbb{B}_2^1$ is finitely universal by Theorem 2. Since $\mathbb{A}_2^1$ properly contains $\mathbb{B}_2^1$, the variety $\mathbb{A}_2^1$ is also finitely universal. □

3.4. Finitely generated finitely based varieties

Recall that a variety is *locally finite* if every finitely generated member of it is finite. More than being just non-finitely based, the varieties $\mathbb{B}_2^1$ and $\mathbb{A}_2^1$ are *inherently non-finitely based* in the sense that every locally finite variety containing it is non-finitely based [30]. However, it is verified in [14, Theorem 3.2] that the variety $\mathbb{M}(xyzxy, xyzyx)$ is defined by the first four identities in (4). Hence Theorem 2 and Lemma 3 imply the following result providing an affirmative answer to Question 6.3 in [6].

Theorem 4. *There is a finitely universal variety of monoids that is both finitely based and finitely generated.*

4. Proof of Theorem 1

We verify Theorem 1 modulo Proposition 2 below and then prove this proposition.

Proof of Theorem 1. The inclusion $\mathbb{M}(\mathscr{W}_n) \subseteq \mathbb{C}$ and Proposition 2 imply that the lattice $\mathfrak{Eq}(\mathscr{W}_n)$ of equivalence relations on the set $\mathscr{W}_n$ is anti-isomorphic to a sublattice of $\mathfrak{L}(\mathbb{C})$. Since $|\mathscr{W}_n| = 2^n$, it is easy to see that, for any $k = 1, 2, \dots, 2^n$, the lattice $\mathfrak{Eq}(\{1, 2, \dots, k\})$ is anti-isomorphic to a sublattice of $\mathfrak{Eq}(\mathscr{W}_n)$. Therefore, the lattice $\mathfrak{L}(\mathbb{C})$ contains an anti-isomorphic copy of every finite lattice of equivalence relations. To complete the proof, it remains to refer to the theorem of Pudlák and Tůma [29], which states that every finite lattice can be embedded in a finite lattice of equivalence relations. The variety $\mathbb{C}$ is thus finitely universal. □

The subvariety of a variety $\mathbb{V}$ defined by a set Σ of identities is denoted by $\mathbb{V}\Sigma$. Given any set $\mathscr{W} \subseteq \mathscr{X}^*$ of words and any equivalence relation $\pi \in \mathfrak{Eq}(\mathscr{W})$, define:

$$\mathsf{Id}(\pi) := \{\mathbf{u} \approx \mathbf{v} \mid (\mathbf{u}, \mathbf{v}) \in \pi,\ \mathbf{u}, \mathbf{v} \in \mathscr{W}\}.$$

For any set A, the universal relation on A is denoted by υ_{A}.

Proposition 2. *For each $n \geq 2$, the lattice $\mathfrak{Eq}(\mathscr{W}_n)$ is anti-isomorphic to the interval $[\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}, \mathbb{M}(\mathscr{W}_n)]$ of the lattice $\mathfrak{L}(\mathbb{M}(\mathscr{W}_n))$.*

The proof of Proposition 2 requires some intermediate results.

Lemma 4. *For each $n \geq 2$, the words xy, xyx, $xyzxty$, $xzytxy$ and $xytzsxzy$ are isoterms for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$.*

Proof. Consider the substitution $\varphi\colon \mathscr{X} \to \mathscr{X}^*$ defined by:

$$\varphi(v) := \begin{cases} x_2^{(1)} & \text{if } v = x, \\ x_1^{(2)} & \text{if } v = y, \\ y_1 & \text{if } v = z, \\ x_2^{(2)}\, x_1^{(3)} x_2^{(3)} x_1^{(4)} x_2^{(4)} \cdots x_1^{(n)} x_2^{(n)}\, b b_1 b_2 \cdots b_n\, s_0 y_0 s_1 & \text{if } v = t, \\ s_2 y_2 s_3 y_3 \cdots s_n y_n\, t b y_0\, x_1^{(1)} z_1 a_1 z_1' b_1 z_1'' & \text{if } v = s, \\ v & \text{otherwise.} \end{cases}$$

Obviously, $\varphi(xytzsxzy)$ is a factor of $\mathbf{w}_\varepsilon$, where ε is the identity element of S_2^n. It follows that a nontrivial identity of the form $xytzsxzy \approx \mathbf{a}$ implies a nontrivial identity $\mathbf{w}_\varepsilon \approx \mathbf{w}$. Therefore, $xytzsxzy$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)$. Further, it is routine to check that $M(xytzsxzy)$ satisfies $\mathbf{w}_\xi \approx \mathbf{w}_\eta$ for any $\xi, \eta \in S_2^n$. Hence $xytzsxzy$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 1. By similar arguments we can show that the words xy, xyx, $xyzxty$, $xzytxy$ are isoterms for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ as well. □

Lemma 5. *Let $n \geq 2$ and $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$. Suppose that $\mathbf{u} \in \mathscr{W}_n$. Then $\mathbf{v} \in \mathscr{W}_n$.*

Proof. According to Lemma 4, the words $xytzsxzy$ and $xzytxy$ are isoterms for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$. Then Lemmas 1 and 2 apply, yielding that $\mathbf{v} = \mathbf{p}\mathbf{v}'\mathbf{q}\mathbf{r}$, where the words $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are defined by the equalities (1), (2) and (3), respectively, while $\mathbf{v}'$ is a linear word with $\mathsf{con}(\mathbf{v}') = \{a, a_i, b, b_i, x_1^{(i)}, x_2^{(i)} \mid 1 \le i \le n\}$. Further, $({}_{1\mathbf{v}}a_i) < ({}_{1\mathbf{v}}a_{i+1})$ for any $i = 1, 2, \dots, n-1$ since $\mathbf{u}(a_i, a_{i+1}, s_i, t, y_i)$ coincides (up to renaming of variables) with the word $xytzsxzy$ which is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$. By a similar argument we can show that:

$$
\begin{aligned}
&({}_{1\mathbf{v}}a_n) < ({}_{1\mathbf{v}}a),\ ({}_{1\mathbf{v}}a) < ({}_{1\mathbf{v}}x_1^{(1)}),\ ({}_{1\mathbf{v}}a) < ({}_{1\mathbf{v}}x_2^{(1)}),\\
&({}_{1\mathbf{v}}x_1^{(i)}) < ({}_{1\mathbf{v}}x_1^{(i+1)}),\ ({}_{1\mathbf{v}}x_1^{(i)}) < ({}_{1\mathbf{v}}x_2^{(i+1)}),\\
&({}_{1\mathbf{v}}x_2^{(i)}) < ({}_{1\mathbf{v}}x_1^{(i+1)}),\ ({}_{1\mathbf{v}}x_2^{(i)}) < ({}_{1\mathbf{v}}x_2^{(i+1)}),\\
&({}_{1\mathbf{v}}x_1^{(n)}) < ({}_{1\mathbf{v}}b),\ ({}_{1\mathbf{v}}x_2^{(n)}) < ({}_{1\mathbf{v}}b),\ ({}_{1\mathbf{v}}b) < ({}_{1\mathbf{v}}b_1),\ ({}_{1\mathbf{v}}b_i) < ({}_{1\mathbf{v}}b_{i+1}),
\end{aligned}
$$

for any $i = 1, 2, \dots, n-1$. It follows that $\mathbf{v} = \mathbf{w}_\eta$ for some $\eta \in S_2^n$ and so $\mathbf{v} \in \mathscr{W}_n$. □

Lemma 6. *Let $n \ge 2$, $\xi, \zeta, \eta \in S_2^n$ and $\mathbf{w} \in \mathscr{X}^*$. Assume that $\mathbf{w}_\zeta = \mathbf{a}\varphi(\mathbf{w}_\xi)\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\varphi(\mathbf{w}_\eta)\mathbf{b}$ for some words $\mathbf{a}, \mathbf{b} \in \mathscr{X}^*$ and substitution $\varphi\colon \mathscr{X} \to \mathscr{X}^*$. If the identity $\mathbf{w}_\zeta \approx \mathbf{w}$ is nontrivial, then φ is the identity map on $\mathsf{con}(\mathbf{w}_\xi)$ and so $\mathbf{a} = \mathbf{b} = 1$ and $(\mathbf{w}_\zeta, \mathbf{w}) = (\mathbf{w}_\xi, \mathbf{w}_\eta)$.*

Proof. Since the identity $\mathbf{w}_\zeta \approx \mathbf{w}$ is nontrivial, Proposition 1 and Lemma 5 imply that $\mathbf{w} = \mathbf{w}_\nu$ for some $\nu \in S_2^n \setminus \{\zeta\}$. Then there is $j \in \{1, 2, \dots, n\}$ such that $(1\nu_j, 2\nu_j) = (2\zeta_j, 1\zeta_j)$. This is only possible when $x_{1\zeta_j}^{(j)} \in \mathsf{con}(\varphi(x_{1\xi_k}^{(k)}))$ and $x_{2\zeta_j}^{(j)} \in \mathsf{con}(\varphi(x_{2\xi_k}^{(k)}))$ for some $k \in \{1, 2, \dots, n\}$ with $(1\xi_k, 2\xi_k) \ne (1\eta_k, 2\eta_k)$. We note that the word $\mathbf{w}_\zeta$ is square-free and every factor of length >1 of $\mathbf{w}_\zeta$ has exactly one occurrence in $\mathbf{w}_\zeta$. It follows that:

(*) $\varphi(c)$ is either the empty word 1 or a variable for any $c \in \mathsf{mul}(\mathbf{w}_\zeta)$.

In view of this fact, $x_{1\zeta_j}^{(j)} = \varphi(x_{1\xi_k}^{(k)})$ and $x_{2\zeta_j}^{(j)} = \varphi(x_{2\xi_k}^{(k)})$. Further, since $({}_{2\mathbf{w}_\zeta}x_1^{(j)}) < ({}_{2\mathbf{w}_\zeta}x_2^{(j)})$ and $({}_{2\mathbf{w}_\xi}x_1^{(k)}) < ({}_{2\mathbf{w}_\xi}x_2^{(k)})$, we have $x_1^{(j)} = \varphi(x_1^{(k)})$ and $x_2^{(j)} = \varphi(x_2^{(k)})$ (this means that $\xi_k = \zeta_j$). Hence

$$\varphi(z_k a_k z_k' b_k z_k'') = z_j a_j z_j' b_j z_j''.$$

It follows from (∗) that $\varphi(a_k) = a_j$, $\varphi(b_k) = b_j$, $\varphi(z_k) = z_j$, $\varphi(z_k') = z_j'$ and $\varphi(z_k'') = z_j''$. Then

$$\varphi\Bigg(\Bigg(\prod_{i=k+1}^{n} a_i\Bigg) a \Bigg(\prod_{i=1}^{n} x_{1\xi_i}^{(i)} x_{2\xi_i}^{(i)}\Bigg) b \Bigg(\prod_{i=1}^{k-1} b_i\Bigg)\Bigg) = \Bigg(\prod_{i=j+1}^{n} a_i\Bigg) a \Bigg(\prod_{i=1}^{n} x_{1\zeta_i}^{(i)} x_{2\zeta_i}^{(i)}\Bigg) b \Bigg(\prod_{i=1}^{j-1} b_i\Bigg).$$

If $k > j+1$, then $\varphi(b_{k-j}) = x_{2\zeta_n}^{(n)}$ and $\varphi(b_{k-j+1}) = b_1$ contradicting the fact that $({}_{2\mathbf{w}_\xi}b_{k-j}) < ({}_{2\mathbf{w}_\xi}b_{k-j+1})$ and $({}_{2\mathbf{w}_\zeta}b_1) < ({}_{2\mathbf{w}_\zeta}x_{2\zeta_n}^{(n)})$. If $k = j+1$, then $\varphi(b) = x_{2\zeta_n}^{(n)}$ and

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

$\varphi(b_1) = b$ contradicting the fact that $({}_{2\mathbf{w}_\xi} b) < ({}_{2\mathbf{w}_\xi} b_1)$ and $({}_{2\mathbf{w}_\zeta} b_1) < ({}_{2\mathbf{w}_\zeta} x^{(n)}_{2\zeta_n})$. Hence $k \le j$. By a similar argument we can show that $j \le k$ and, therefore, $k = j$. Then $\varphi(x^{(i)}_{1\xi_i}) = x^{(i)}_{1\zeta_i}$ and $\varphi(x^{(i)}_{2\xi_i}) = x^{(i)}_{2\zeta_i}$ for any $i = 1, 2, \dots, n$ by $(*)$. Since $({}_{2\mathbf{w}_\zeta} x^{(i)}_1) < ({}_{2\mathbf{w}_\zeta} x^{(i)}_2)$ and $({}_{2\mathbf{w}_\xi} x^{(i)}_1) < ({}_{2\mathbf{w}_\xi} x^{(i)}_2)$, this implies that $\xi = \zeta$ and

$$\varphi(z_i a_i z'_i b_i z''_i) = z_i a_i z'_i b_i z''_i,$$

for any $i = 1, 2, \dots, n$. Now $(*)$ applies again, yielding that $\varphi(a_i) = a_i$, $\varphi(b_i) = b_i$, $\varphi(z_i) = z_i$, $\varphi(z'_i) = z'_i$, $\varphi(z''_i) = z''_i$ for any $i = 1, 2, \dots, n$. It follows that $\varphi(a) = a$, $\varphi(b) = b$ and $\varphi(y_i) = y_i$ for all $i = 1, 2, \dots, n-1$. Hence $\varphi(y_0) = y_0$, $\varphi(y_n) = y_n$ and so $\varphi(t) = t$, $\varphi(t_i) = t_i$, $\varphi(t'_i) = t'_i$, $\varphi(t''_i) = t''_i$, $\varphi(s_i) = s_i$. Thus, φ is the identity map on $\mathsf{con}(\mathbf{w}_\xi)$. Hence $\mathbf{a} = \mathbf{b} = 1$ and so $(\mathbf{w}_\zeta, \mathbf{w}) = (\varphi(\mathbf{w}_\xi), \varphi(\mathbf{w}_\eta)) = (\mathbf{w}_\xi, \mathbf{w}_\eta)$ as required. □

Corollary 2. *Let $n \ge 2$ and $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\pi)\}$ for some $\pi \in \mathfrak{Eq}(\mathscr{W}_n)$. Suppose that $\mathbf{u} \in \mathscr{W}_n$. Then $\mathbf{v} \in \mathscr{W}_n$ and $(\mathbf{u}, \mathbf{v}) \in \pi$.*

Proof. In view of Proposition 1, there is some finite sequence $\mathbf{u} = \mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m = \mathbf{v}$ of distinct words such that each identity $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ is either holds in $\mathbb{M}(\mathscr{W}_n)$ or directly deducible from some identity in $\mathsf{Id}(\pi)$. According to Lemma 5, the word $\mathbf{v}_i$ belongs to $\mathscr{W}_n$ for any $i = 0, 1, \dots, m$. Then Lemma 1 and the fact that the words $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m$ are pairwise distinct imply that $\mathbb{M}(\mathscr{W}_n)$ violates $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ for any $i = 0, 1, \dots, m-1$. Therefore, $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ is directly deducible from some identity in $\mathsf{Id}(\pi)$. Now Lemma 6 applies, yielding that $(\mathbf{v}_i, \mathbf{v}_{i+1}) \in \pi$, whence $(\mathbf{u}, \mathbf{v}) \in \pi$. □

Lemma 7. *Let $n \ge 2$ and $\zeta \in S^n_2$. A word $\mathbf{w}$ is an isoterm for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ if one of the following holds:*

- (i) *$\mathbf{w}$ is obtained from $\mathbf{w}_\zeta$ by replacing some occurrence of a multiple variable with a variable $h \notin \mathsf{con}(\mathbf{w}_\zeta)$;*
- (ii) *$\mathbf{w}$ is obtained from $\mathbf{w}_\zeta$ by replacing some factor of length > 1 with a variable $h \notin \mathsf{con}(\mathbf{w}_\zeta)$;*
- (iii) *$\mathbf{w}$ is a proper factor of $\mathbf{w}_\zeta$.*

Proof. (i) The word $\mathbf{w}$ is obtained from $\mathbf{w}_\zeta$ by replacing some occurrence of a multiple variable c with the variable $h \notin \mathsf{con}(\mathbf{w}_\zeta)$. Clearly, $\psi(\mathbf{w}) = \mathbf{w}_\zeta$, where $\psi\colon \mathscr{X} \to \mathscr{X}^*$ is the substitution defined by:

$$\psi(v) := \begin{cases} c & \text{if } v = h, \\ v & \text{if } v \ne h. \end{cases}$$

Since xy is an isoterm for $\mathbb{M}(\mathscr{W}_n)$ by Lemma 4 and $c, h \in \mathsf{sim}(\mathbf{w})$, it follows that $\mathbf{w}$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)$. Hence, by Proposition 1, if $\mathbf{w}$ is not an isoterm for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$, then some nontrivial identity $\mathbf{w} \approx \mathbf{w}'$ is directly deducible from some identity of the form $\mathbf{w}_\xi \approx \mathbf{w}_\eta$. By symmetry, we may assume that $\mathbf{w} = \mathbf{a}\varphi(\mathbf{w}_\xi)\mathbf{b}$ and

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

$\mathbf{w}' = \mathbf{a}\varphi(\mathbf{w}_\eta)\mathbf{b}$ for some words $\mathbf{a}, \mathbf{b} \in \mathscr{X}^*$ and substitution $\varphi\colon \mathscr{X} \to \mathscr{X}^*$. Hence $\mathbf{w}_\zeta = \psi(\mathbf{w}) = \psi(\mathbf{a})\psi(\varphi(\mathbf{w}_\xi))\psi(\mathbf{b})$. Then $\psi(\mathbf{w}') \neq \psi(\mathbf{w})$ because xy is an isoterm for the variety defined by the identity $\mathbf{w}_\xi \approx \mathbf{w}_\eta$ by Lemma 4, $c, h \in \mathsf{sim}(\mathbf{w})$ and the identity $\mathbf{w} \approx \mathbf{w}'$ is nontrivial. Now Lemma 6 applies, yielding that the substitution $\psi\varphi$ is the identity map on $\mathsf{con}(\mathbf{w}_\xi)$ and so $\psi(\mathbf{a}) = \psi(\mathbf{b}) = 1$ and $\mathbf{w}_\zeta = \mathbf{w}_\xi$. Then $\mathbf{a} = \mathbf{b} = 1$ by the definition of the substitution ψ. Thus, $\mathbf{w} = \varphi(\mathbf{w}_\xi)$. Since $c \in \mathsf{sim}(\mathbf{w})$, there is $c' \in \mathsf{sim}(\mathbf{w}_\xi)$ such that $c \in \mathsf{con}(\varphi(c'))$. Clearly, $\psi(c) = c$. Hence $c \in \mathsf{con}(\psi(\varphi(c')))$. Since the substitution $\psi\varphi$ is the identity map on $\mathsf{con}(\mathbf{w}_\xi)$, we have $c = c'$ contradicting the fact that $c \in \mathsf{mul}(\mathbf{w}_\zeta) = \mathsf{mul}(\mathbf{w}_\xi)$ and $c' \in \mathsf{sim}(\mathbf{w}_\xi)$.

(ii) The word $\mathbf{w}$ is obtained from $\mathbf{w}_\zeta$ by replacing some factor cd with the variable $h \notin \mathsf{con}(\mathbf{w}_\zeta)$. Since every factor of length >1 of $\mathbf{w}_\zeta$ contains a multiple variable, we may assume without any loss that $c \in \mathsf{mul}(\mathbf{w}_\zeta)$. Then the word $\psi(\mathbf{w})$ is obtained from $\mathbf{w}_\zeta$ by replacing an occurrence of c with the variable h, where $\psi\colon \mathscr{X} \to \mathscr{X}^*$ is the substitution defined by:

$$\psi(v) := \begin{cases} hd & \text{if } v = h, \\ v & \text{if } v \neq h. \end{cases}$$

By Part (i), the word $\psi(\mathbf{w})$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$. Hence $\mathbf{w}$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ as well.

(iii) Let $\mathbf{w}_1$ and $\mathbf{w}_2$ denote words obtained from $\mathbf{w}_\zeta$ by replacing the variables ${}_{1\mathbf{w}_\zeta}z_1$ and ${}_{2\mathbf{w}_\zeta}a$ with the variable h, respectively. Clearly, if some proper factor of $\mathbf{w}_\zeta$ is not an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$, then at least one of the words $\mathbf{w}_1$ or $\mathbf{w}_2$ is not an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ as well. Thus, $\mathbf{w}$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Part (i). $\square$

Lemma 8. *Let $n \geq 2$ and $\mathbf{u}$ be a word such that $\mathbf{u}_{\mathscr{C}} = \mathbf{w}_\zeta$ for some $\zeta \in S_2^n$ and $\mathscr{C} \subseteq \mathsf{con}(\mathbf{u})$. Assume that the following three claims hold:*

(a) *every factor of length >1 of $\mathbf{u}$ has exactly one occurrence in $\mathbf{u}$;*
(b) *there are no simple variables between ${}_{1\mathbf{u}}a_1$ and ${}_{1\mathbf{u}}b_n$ and between ${}_{2\mathbf{u}}b$ and ${}_{2\mathbf{u}}a$ in $\mathbf{u}$;*
(c) *for some $c \in \mathscr{C}$, either $({}_{1\mathbf{u}}a_1) < ({}_{1\mathbf{u}}c) < ({}_{1\mathbf{u}}b_n)$ or $({}_{2\mathbf{u}}b) < ({}_{2\mathbf{u}}c) < ({}_{2\mathbf{u}}a)$.*

If $\mathbf{u} \approx \mathbf{v}$ is a nontrivial identity directly deducible from some identity of the form $\mathbf{w}_\xi \approx \mathbf{w}_\eta$ with $\xi, \eta \in S_2^n$, then $\mathbf{v}_{\mathscr{C}} = \mathbf{w}_\zeta$.

Proof. By symmetry, we may assume that $\mathbf{u} = \mathbf{a}\varphi(\mathbf{w}_\xi)\mathbf{b}$ and $\mathbf{v} = \mathbf{a}\varphi(\mathbf{w}_\eta)\mathbf{b}$ for some words $\mathbf{a}, \mathbf{b} \in \mathscr{X}^*$ and substitution $\varphi\colon \mathscr{X} \to \mathscr{X}^*$. Then $\mathbf{w}_\zeta = \psi(\mathbf{u}) = \psi(\mathbf{a})\psi(\varphi(\mathbf{w}_\xi))\psi(\mathbf{b})$, where $\psi\colon \mathscr{X} \to \mathscr{X}^*$ is the substitution defined by:

$$\psi(v) := \begin{cases} 1 & \text{if } v \in \mathscr{C}, \\ v & \text{if } v \notin \mathscr{C}. \end{cases}$$

Arguing by contradiction, suppose that $\mathbf{v}_{\mathscr{C}} = \psi(\mathbf{v}) \neq \mathbf{w}_\zeta$. Then, by Lemma 6, the substitution $\psi\varphi$ is the identity map on $\mathsf{con}(\mathbf{w}_\xi)$ and so $\psi(\mathbf{a}) = \psi(\mathbf{b}) = 1$ and $\mathbf{w}_\zeta = \mathbf{w}_\xi$. Hence $\mathsf{con}(\mathbf{ab}) \subseteq \mathscr{C}$. Assume that $({}_{1\mathbf{u}}a_1) < ({}_{1\mathbf{u}}c) < ({}_{1\mathbf{u}}b_n)$ for some $c \in \mathscr{C}$. Then there

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

is $c' \in \mathsf{con}(\mathbf{w}_\xi)$ such that φ maps some occurrence of c' to a factor of $\mathbf{u}$ containing ${}_{1\mathbf{u}}c$. The fact that $\psi(\varphi(c')) = c'$ implies that $\varphi(c')$ is a word of length >1. By the condition of the lemma, the word $\mathbf{u}$ may contain at most one occurrence of the factor $\varphi(c')$. This only possible when $c' \in \mathsf{sim}(\mathbf{w}_\xi)$. Since there are no simple variables between ${}_{1\mathbf{u}}a_1$ and ${}_{1\mathbf{u}}b_n$ in $\mathbf{u}$, it follows that $\mathsf{con}(\varphi(c'))$ must contain either a_1 or b_n contradicting $\psi(\varphi(c')) = c'$. Therefore, $\mathbf{v}_{\mathscr{C}} = \psi(\mathbf{v}) = \mathbf{w}_\zeta$. By a similar argument we can show that if $({}_{2\mathbf{u}}b) < ({}_{2\mathbf{u}}c) < ({}_{2\mathbf{u}}a)$ for some $c \in \mathscr{C}$, then $\mathbf{v}_{\mathscr{C}} = \psi(\mathbf{v}) = \mathbf{w}_\zeta$. □

A *block* of a word $\mathbf{w}$ is a maximal factor of $\mathbf{w}$ that does not contain any simple variables in $\mathbf{w}$. A word $\mathbf{w}$ is called *block-linear* if every block of $\mathbf{w}$ is a linear word.

Lemma 9. *Let $n \geq 2$ and $\mathbf{u}$ be a block-linear word such that $\mathbf{u}_{\{c,h\}} = \mathbf{w}_\zeta$ for some $\zeta \in S_2^n$ and $c, h \in \mathscr{X}$ with $h \in \mathsf{sim}(\mathbf{u})$ and $\mathsf{occ}_c(\mathbf{u}) = 2$. Assume that, for some $x, y \in \mathsf{mul}(\mathbf{w}_\zeta)$ and i, j with $\{i,j\} = \{1,2\}$, the word ${}_{i\mathbf{u}}x\ {}_{i\mathbf{u}}c\ {}_{i\mathbf{u}}y$ is a factor of $\mathbf{u}$, while the word ${}_{j\mathbf{u}}c$ forms a block of $\mathbf{u}$. If $\mathbf{u} \approx \mathbf{v}$ is a nontrivial identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$, then $\mathbf{v}_{\{c,h\}} = \mathbf{w}_\zeta$.*

Proof. In view of Proposition 1, there is some finite sequence $\mathbf{u} = \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m = \mathbf{v}$ of distinct words such that each identity $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ either holds in $\mathbb{M}(\mathscr{W}_n)$ or is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$. We will use induction on m.

Induction base: $m=1$. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ holds in $\mathbb{M}(\mathscr{W}_n)$, then the required claim follows from Lemma 1. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$, then the condition of the lemma implies that the conditions (a), (b) and (c) of Lemma 8 holds. So, we can apply Lemma 8, yielding that $\mathbf{v}_{\{c,h\}} = \mathbf{w}_\zeta$.

Induction step: $m>1$. First, notice that, as in the induction base, $(\mathbf{v}_1)_{\{c,h\}} = \mathbf{w}_\zeta$ by Lemmas 1 and 8. Since xyx is an isoterm for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, the word ${}_{j\mathbf{u}}c$ forms a block of $\mathbf{v}_1$. Hence ${}_{j\mathbf{v}_1}x$ and ${}_{j\mathbf{v}_1}y$ do not lie in the block of $\mathbf{v}_1$ containing ${}_{j\mathbf{v}_1}c$. Then, since $xyzxty$ and $xzytxy$ are isoterms for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, $({}_{i\mathbf{v}_1}x) < ({}_{i\mathbf{v}_1}c) < ({}_{i\mathbf{v}_1}y)$ and so the word ${}_{i\mathbf{v}_1}x\ {}_{i\mathbf{v}_1}c\ {}_{i\mathbf{v}_1}y$ is a factor of $\mathbf{v}_1$. Thus, we can apply the induction assumption, yielding that $\mathbf{v}_{\{c,h\}} = \mathbf{w}_\zeta$ as required. □

Lemma 10. *Let $n \geq 2$ and $\mathbf{u}$ be a block-linear word such that $\mathbf{u}_c = \mathbf{w}_\zeta$ for some $\zeta \in S_2^n$ and $c \in \mathsf{mul}(\mathbf{u})$ with $\mathsf{occ}_c(\mathbf{u}) = 2$. Assume that, for some $x, y \in \mathsf{mul}(\mathbf{w}_\zeta)$, the word ${}_{2\mathbf{u}}x\ {}_{2\mathbf{u}}c\ {}_{2\mathbf{u}}y$ is a factor of $\mathbf{u}$ and one of the following holds:*

(i) *$x \neq b$, $y \neq a$ and ${}_{1\mathbf{u}}c$ is not adjacent to ${}_{1\mathbf{u}}x$ and ${}_{1\mathbf{u}}y$ in $\mathbf{u}$;*
(ii) *$x = b$, the variables ${}_{1\mathbf{u}}c$ and ${}_{1\mathbf{u}}x$ lie in different blocks of $\mathbf{u}$ and ${}_{1\mathbf{u}}c$ is not adjacent to ${}_{1\mathbf{u}}y$ in $\mathbf{u}$;*
(iii) *$y = a$, the variables ${}_{1\mathbf{u}}c$ and ${}_{1\mathbf{u}}y$ lie in different blocks of $\mathbf{u}$ and ${}_{1\mathbf{u}}c$ is not adjacent to ${}_{1\mathbf{u}}x$ in $\mathbf{u}$.*

If $\mathbf{u} \approx \mathbf{v}$ is a nontrivial identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$, then $\mathbf{v}_c = \mathbf{w}_\zeta$.

Proof. In view of Proposition 1, there is some finite sequence $\mathbf{u} = \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m = \mathbf{v}$ of distinct words such that each identity $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ either holds in $\mathbb{M}(\mathscr{W}_n)$ or is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$. We will use induction on m.

Induction base: $m=1$. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ holds in $\mathbb{M}(\mathscr{W}_n)$, then the required claim follows from Lemma 1. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ is directly deducible from some identity

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$, then the condition of the lemma implies that every factor of length >1 of $\mathbf{u}$ has exactly one occurrence in $\mathbf{u}$ and $({}_{2\mathbf{u}}b) < ({}_{2\mathbf{u}}c) < ({}_{2\mathbf{u}}a)$. Then we can apply Lemma 8, yielding that $\mathbf{v}_c = \mathbf{w}_\zeta$.

Induction step: $m>1$. First, notice that, as in the induction base, $(\mathbf{v}_1)_c = \mathbf{w}_\zeta$ by Lemmas 1 and 8. By symmetry, it suffices to verify only Parts (i) and (ii). The proof of Part (ii) is very similar to the proof of Part (i) but a bit simpler and we omit it. So, we assume below that (i) holds.

By symmetry, we may assume without any loss that $x \in \{a, a_i, b, b_i, x_1^{(i)}, x_2^{(i)} \mid 1 \le i \le n\}$ and $y \in \{y_0, y_i, z_i, z_i', z_i'' \mid 1 \le i \le n\}$. Then the variables ${}_{1\mathbf{u}}c$ and ${}_{1\mathbf{u}}y$ do not lie in the same block of $\mathbf{u}$ because these variables are not adjacent to each other in $\mathbf{u}$. The variables ${}_{1\mathbf{u}}c$ and ${}_{2\mathbf{u}}c$ also do not lie in the same block of the word $\mathbf{u}$ because this word is block-linear. Since $xzytxy$ and so $xzytyx$ are isoterms for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4 and $\mathsf{occ}_c(\mathbf{u}) = 2$, this implies that $\mathsf{occ}_c(\mathbf{v}_1) = 2$ and $({}_{2\mathbf{v}_1}c) < ({}_{2\mathbf{v}_1}y)$. Further, if ${}_{1\mathbf{u}}c$ and ${}_{1\mathbf{u}}x$ do not lie in the same block of $\mathbf{u}$, then $({}_{2\mathbf{v}_1}x) < ({}_{2\mathbf{v}_1}c)$ and so ${}_{2\mathbf{v}_1}x\ {}_{2\mathbf{v}_1}c\ {}_{2\mathbf{v}_1}y$ is a factor of $\mathbf{v}_1$. If ${}_{1\mathbf{u}}c$ and ${}_{1\mathbf{u}}x$ lie in the same block of $\mathbf{u}$, then $({}_{2\mathbf{v}_1}z) < ({}_{2\mathbf{v}_1}c)$, where $z \in \{y_0, y_i, z_i, z_i', z_i'' \mid 1 \le i \le n\}$ is the variable such that ${}_{2\mathbf{w}_\zeta}z\ {}_{2\mathbf{w}_\zeta}x$ is a factor of $\mathbf{w}_\zeta$, and, therefore, either ${}_{2\mathbf{v}_1}x\ {}_{2\mathbf{v}_1}c\ {}_{2\mathbf{v}_1}y$ or ${}_{2\mathbf{v}_1}z\ {}_{2\mathbf{v}_1}c\ {}_{2\mathbf{v}_1}x$ is a factor of $\mathbf{v}_1$.

Suppose that ${}_{1\mathbf{v}_1}c$ is adjacent to ${}_{1\mathbf{v}_1}x$. If $\mathbf{u} \approx \mathbf{v}_1$ holds in $\mathbb{M}(\mathscr{W}_n)$, then the word $(\mathbf{v}_1)_x$ coincides (up to renaming of variables) with $\mathbf{w}_\zeta$ and $\mathbf{u}_x \ne (\mathbf{v}_1)_x$ contradicting the fact that $\mathbf{w}_\zeta$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)$. Therefore, $\mathbf{u} \approx \mathbf{v}_1$ is directly deducible from some identity $\mathbf{w}_\xi \approx \mathbf{w}_\eta$ in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$. Then $(\mathbf{v}_1')_c = \mathbf{w}_\zeta \ne \mathbf{u}_c'$, where $\mathbf{u}' := \psi(\mathbf{u})$, $\mathbf{v}_1' := \psi(\mathbf{v}_1)$ and $\psi\colon \mathscr{X} \to \mathscr{X}^*$ is the substitution defined by:

$$\psi(v) := \begin{cases} x & \text{if } v = c, \\ c & \text{if } v = x, \\ v & \text{otherwise.} \end{cases}$$

According to Lemma 5, there is $\nu \in S_2^n \setminus \{\zeta\}$ such that $\mathbf{u}_c' = \mathbf{w}_\nu$. In particular, ${}_{1\mathbf{u}'}x$ and ${}_{1\mathbf{u}'}c$ lie in the same block of $\mathbf{u}'$. Evidently, $\mathbf{u}' \approx \mathbf{v}_1'$ is directly deducible from $\mathbf{w}_\xi \approx \mathbf{w}_\eta$, the word ${}_{2\mathbf{u}'}z\ {}_{2\mathbf{u}'}c\ {}_{2\mathbf{u}'}x$ is a factor of $\mathbf{u}'$, and ${}_{1\mathbf{u}'}c$ is not adjacent to ${}_{1\mathbf{u}'}x$ and ${}_{1\mathbf{u}'}z$ in $\mathbf{u}'$. Then $(\mathbf{v}_1')_c = \mathbf{w}_\nu$ by Lemma 8 contradicting the fact that $\zeta \ne \nu$. Thus, ${}_{1\mathbf{v}_1}c$ is not adjacent to ${}_{1\mathbf{v}_1}x$ in $\mathbf{v}_1$ in any case.

Further, since xyx is an isoterm for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4 and ${}_{1\mathbf{u}}y$ and ${}_{1\mathbf{u}}c$ do not lie in the same block of $\mathbf{u}$, the variables ${}_{1\mathbf{v}_1}y$ and ${}_{1\mathbf{v}_1}c$ cannot lie in the same block of $\mathbf{v}_1$. Hence ${}_{1\mathbf{v}_1}c$ is not adjacent to ${}_{1\mathbf{v}_1}y$ in $\mathbf{v}_1$. So, if ${}_{2\mathbf{v}_1}x\ {}_{2\mathbf{v}_1}c\ {}_{2\mathbf{v}_1}y$ is a factor of $\mathbf{v}_1$, then we can apply the induction assumption, yielding that $\mathbf{v}_c = \mathbf{w}_\zeta$. If ${}_{2\mathbf{v}_1}z\ {}_{2\mathbf{v}_1}c\ {}_{2\mathbf{v}_1}x$ is a factor of $\mathbf{v}_1$, then ${}_{1\mathbf{v}_1}c$ and ${}_{1\mathbf{v}_1}x$ must lie in the same block of $\mathbf{v}_1$ because $xzytxy$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$. In this case, ${}_{1\mathbf{v}_1}c$ and ${}_{1\mathbf{v}_1}z$ lie in different blocks of $\mathbf{v}_1$ and so ${}_{1\mathbf{v}_1}c$ is not adjacent to ${}_{1\mathbf{v}_1}z$ in $\mathbf{v}_1$. Therefore, we can apply the induction assumption again, yielding that $\mathbf{v}_c = \mathbf{w}_\zeta$ as required. □

Lemma 11. *Let $n \ge 2$ and $\mathbf{u}$ be a block-linear word such that $\mathbf{u}_c = \mathbf{w}_\zeta$ for some $\zeta \in S_2^n$ and $c \in \mathsf{mul}(\mathbf{u})$ with $\mathsf{occ}_c(\mathbf{u}) = 2$. Assume that, for some $x, y \in \mathsf{mul}(\mathbf{u}_\zeta)$, the word ${}_{1\mathbf{u}}x\ {}_{1\mathbf{u}}c\ {}_{1\mathbf{u}}y$ is a factor of $\mathbf{u}$, while ${}_{2\mathbf{u}}c$ is not adjacent to ${}_{2\mathbf{u}}x$ and ${}_{2\mathbf{u}}y$ in $\mathbf{u}$. If $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$, then $\mathbf{v}_c = \mathbf{w}_\zeta$.*

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

Proof. Evidently, $x, y \in \{a, a_i, b, b_i, x_1^{(i)}, x_2^{(i)} \mid 1 \le i \le n\}$. If $({_{2\mathbf{u}}}y_0) < ({_{2\mathbf{u}}}c) < ({_{2\mathbf{u}}}y_n)$, then the required claim follows from Lemma 10(i). So, since the word $\mathbf{u}$ is block-linear, it remains to consider the case when one of the words ${_{2\mathbf{u}}}c\,{_{2\mathbf{u}}}b$, ${_{2\mathbf{u}}}b\,{_{2\mathbf{u}}}c$, ${_{2\mathbf{u}}}c\,{_{2\mathbf{u}}}a$, ${_{2\mathbf{u}}}a\,{_{2\mathbf{u}}}c$, ${_{2\mathbf{u}}}c\,{_{1\mathbf{u}}}y_j$ or ${_{1\mathbf{u}}}y_j\,{_{2\mathbf{u}}}c$ with $j \in \{0, 1, \dots, n\}$ is a factor of $\mathbf{u}$.

In view of Proposition 1, there is some finite sequence $\mathbf{u} = \mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m = \mathbf{v}$ of distinct words such that each identity $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ either holds in $\mathbb{M}(\mathscr{W}_n)$ or is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$. We will use induction on m.

Induction base: $m = 1$. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ holds in $\mathbb{M}(\mathscr{W}_n)$, then the required claim follows from Lemma 1. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$, then the condition of the lemma implies that every factor of length > 1 of $\mathbf{u}$ has exactly one occurrence in $\mathbf{u}$ and $({_{1\mathbf{u}}}a_1) < ({_{1\mathbf{u}}}c) < ({_{1\mathbf{u}}}b_n)$. Then we can apply Lemma 8, yielding that $\mathbf{v}_c = \mathbf{w}_\zeta$.

Induction step: $m > 1$. First, notice that, as in the induction base, $(\mathbf{v}_1)_c = \mathbf{w}_\zeta$ by Lemmas 1 and 8. If either ${_{2\mathbf{u}}}c\,{_{1\mathbf{u}}}y_j$ or ${_{1\mathbf{u}}}y_j\,{_{2\mathbf{u}}}c$ is a factor of $\mathbf{u}$ for some $j \in \{0, 1, \dots, n\}$, then $\mathbf{u}(c, s_j, t, x) = xcs_jctx$ and $\mathbf{u}(c, s_j, t, y) = cys_jcty$. Since the word $xyzxty$ and so the word $yxzxty$ are isoterms for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, this implies that $\mathbf{v}_1(c, s_j, t, x) = xcs_jctx$ and $\mathbf{v}_1(c, s_j, t, y) = cys_jcty$. If either ${_{2\mathbf{u}}}c\,{_{2\mathbf{u}}}b$ or ${_{2\mathbf{u}}}b\,{_{2\mathbf{u}}}c$ is a factor of $\mathbf{u}$, then $x \ne b$, $y \ne b$ and so $\mathbf{u}(c, s_0, t, x, y_0) = xcs_0y_0tcy_0x$ and $\mathbf{u}(c, s_0, t, y, y_0) = cys_0y_0tcy_0y$. Since $xytzsxzy$ and so $yxtzsxzy$ are isoterms for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, this implies that $\mathbf{v}_1(c, s_0, t, x, y_0) = xcs_0y_0tcy_0x$ and $\mathbf{v}_1(c, s_0, t, y, y_0) = cys_0y_0tcy_0y$. By a similar argument we can show that if one of the words ${_{2\mathbf{u}}}c\,{_{2\mathbf{u}}}a$ or ${_{2\mathbf{u}}}a\,{_{2\mathbf{u}}}c$ is a factor of the word $\mathbf{u}$, then $\mathbf{v}_1(c, s_n, t, x, y_n) = xcs_ny_ntxy_nc$ and $\mathbf{v}_1(c, s_n, t, y, y_n) = cys_ny_ntyy_nc$. Thus, we have proved that $\mathsf{occ}_c(\mathbf{v}_1) = 2$, the word ${_{1\mathbf{v}_1}}x\,{_{1\mathbf{v}_1}}c\,{_{1\mathbf{v}_1}}y$ is a factor $\mathbf{v}_1$, while ${_{2\mathbf{v}_1}}c$ is not adjacent to ${_{2\mathbf{v}_1}}x$ and ${_{2\mathbf{v}_1}}y$ in $\mathbf{v}_1$. So, we can apply the induction assumption, yielding that $\mathbf{v}_c = \mathbf{w}_\zeta$. □

Corollary 3. *Let $n \ge 2$ and $\mathbf{u}$ be a word such that $\mathbf{u}_h = \mathbf{w}_\zeta$ for some $\zeta \in S_2^n$ and $h \in \mathsf{sim}(\mathbf{u})$. Assume that h is adjacent to two different multiple variables of $\mathbf{u}$. If $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$, then $\mathbf{v}_h = \mathbf{w}_\zeta$.*

Proof. Obviously, there is $j \in \{0, 1, \dots, n\}$ such that ${_{2\mathbf{u}}}y_j$ is not adjacent to ${_{1\mathbf{u}}}h$ in $\mathbf{u}$. Then Lemmas 10 and 11 imply that $(\varphi(\mathbf{v}))_h = (\varphi(\mathbf{u}))_h = \mathbf{w}_\zeta$, where $\varphi\colon \mathscr{X} \to \mathscr{X}^*$ is the substitution given by:

$$\varphi(v) := \begin{cases} s_jh & \text{if } v = s_j, \\ v & \text{if } v \ne s_j. \end{cases}$$

It remains to note that $\mathbf{v}_h = (\varphi(\mathbf{v}))_h$. □

Lemma 12. *Let $n \ge 2$ and $\mathbf{u}$ be a word such that $\mathbf{u}_{\{c_1, c_2\}} = \mathbf{w}_\zeta$ for some $\zeta \in S_2^n$ and $c_1, c_2 \in \mathsf{mul}(\mathbf{u})$ with $\mathsf{occ}_{c_1}(\mathbf{u}) = \mathsf{occ}_{c_2}(\mathbf{u}) = 2$. Assume that, for some $x, y \in \mathsf{mul}(\mathbf{w}_\zeta)$, the word ${_{2\mathbf{u}}}x\,{_{2\mathbf{u}}}c_1\,{_{2\mathbf{u}}}c_2\,{_{2\mathbf{u}}}y$ is a factor of $\mathbf{u}$, while ${_{1\mathbf{u}}}c_1$ and ${_{1\mathbf{u}}}c_2$ lie in the same blocks as ${_{1\mathbf{u}}}y$ and ${_{1\mathbf{u}}}x$ in $\mathbf{u}$, respectively. If $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$, then $\mathbf{v}_{\{c_1, c_2\}} = \mathbf{w}_\zeta$.*

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

Proof. In view of Proposition 1, there is some finite sequence $\mathbf{u} = \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m = \mathbf{v}$ of distinct words such that each identity $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ either holds in $\mathbb{M}(\mathscr{W}_n)$ or is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$. We will use induction on m.

Induction base: $m = 1$. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ holds in $\mathbb{M}(\mathscr{W}_n)$, then the required claim follows from Lemma 1. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$, then the condition of the lemma implies that every factor of length >1 of $\mathbf{u}$ has exactly one occurrence in $\mathbf{u}$ and $({}_{2\mathbf{u}}b) < ({}_{2\mathbf{u}}c_1) < ({}_{2\mathbf{u}}a)$. Then we can apply Lemma 8, yielding that $\mathbf{v}_{\{c_1,c_2\}} = \mathbf{w}_\zeta$.

Induction step: $m > 1$. First, notice that, as in the induction base, $(\mathbf{v}_1)_{\{c_1,c_2\}} = \mathbf{w}_\zeta$ by Lemmas 1 and 8. Since xyx is an isoterm for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, ${}_{1\mathbf{v}_1}c_1$ and ${}_{1\mathbf{v}_1}c_2$ lie in the same blocks as ${}_{1\mathbf{v}_1}y$ and ${}_{1\mathbf{v}_1}x$ in $\mathbf{v}_1$, respectively, and $\mathsf{occ}_{c_1}(\mathbf{v}_1) = \mathsf{occ}_{c_2}(\mathbf{v}_1) = 2$. Further, ${}_{1\mathbf{u}}y$ and ${}_{1\mathbf{u}}x$ lie in different blocks of $\mathbf{u}$. Hence, since $xzytxy$ is an isoterm for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, the word ${}_{2\mathbf{v}_1}x\; {}_{2\mathbf{v}_1}c_1\; {}_{2\mathbf{v}_1}c_2\; {}_{2\mathbf{v}_1}y$ must be a factor of $\mathbf{v}_1$. Thus, we can apply the induction assumption, yielding that $\mathbf{v}_{\{c_1,c_2\}} = \mathbf{w}_\zeta$ as required. □

Lemma 13. *Let $n \geq 2$ and $\mathbf{u}$ be a word such that $\mathbf{u}_{\{c_1,c_2\}} = \mathbf{w}_\zeta$ for some $\zeta \in S_2^n$ and $c_1, c_2 \in \mathsf{mul}(\mathbf{u})$ with $\mathsf{occ}_{c_1}(\mathbf{u}) = \mathsf{occ}_{c_2}(\mathbf{u}) = 2$. Assume that, for some $x, y \in \mathsf{mul}(\mathbf{w}_\zeta)$, the word ${}_{1\mathbf{u}}x\; {}_{1\mathbf{u}}c_1\; {}_{1\mathbf{u}}c_2\; {}_{1\mathbf{u}}y$ is a factor of $\mathbf{u}$, while ${}_{2\mathbf{u}}c_1$ and ${}_{2\mathbf{u}}c_2$ are adjacent to ${}_{2\mathbf{u}}y$ and ${}_{2\mathbf{u}}x$ in $\mathbf{u}$, respectively. If $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$, then $\mathbf{v}_{\{c_1,c_2\}} = \mathbf{w}_\zeta$.*

Proof. In view of Proposition 1, there is some finite sequence $\mathbf{u} = \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m = \mathbf{v}$ of distinct words such that each identity $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ either holds in $\mathbb{M}(\mathscr{W}_n)$ or is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$. We will use induction on m.

Induction base: $m = 1$. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ holds in $\mathbb{M}(\mathscr{W}_n)$, then the required claim follows from Lemma 1. If $\mathbf{u} = \mathbf{v}_0 \approx \mathbf{v}_1 = \mathbf{v}$ is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$, then the condition of the lemma implies that every factor of length >1 of $\mathbf{u}$ has exactly one occurrence in $\mathbf{u}$ and $({}_{1\mathbf{u}}a_1) < ({}_{1\mathbf{u}}c_1) < ({}_{1\mathbf{u}}b_n)$. Then we can apply Lemma 8, yielding that $\mathbf{v}_{\{c_1,c_2\}} = \mathbf{w}_\zeta$.

Induction step: $m > 1$. First, notice that, as in the induction base, $(\mathbf{v}_1)_{\{c_1,c_2\}} = \mathbf{w}_\zeta$ by Lemmas 1 and 8. Since $xzytxy$ is an isoterm for the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, it is easy to show that $\mathsf{occ}_{c_1}(\mathbf{v}_1) = \mathsf{occ}_{c_2}(\mathbf{v}_1) = 2$ and the variables ${}_{2\mathbf{v}_1}c_1$ and ${}_{2\mathbf{v}_1}c_2$ are adjacent to ${}_{2\mathbf{v}_1}y$ and ${}_{2\mathbf{v}_1}x$ in $\mathbf{v}_1$, respectively. Assume first that $\{x, y\} = \{x_1^{(k)}, x_2^{(k)}\}$ for some $k \in \{1, 2, \ldots, n\}$. Evidently, the words $\mathbf{u}_{\{x,c_1\}}$ and $\mathbf{u}_{\{c_2,y\}}$ coincide (up to renaming of variables) with $\mathbf{w}_\zeta$, while $\mathbf{u}_{\{x_1,x_2\}}$ coincides (up to renaming of variables) with $\mathbf{w}_{\bar\zeta}$, where $\bar\zeta := (\zeta_1, \ldots, \zeta_{k-1}, \zeta_k^2, \zeta_{k+1}, \ldots, \zeta_n)$. Hence if $\mathbf{u} \approx \mathbf{v}_1$ holds in $\mathbb{M}(\mathscr{W}_n)$, then ${}_{1\mathbf{v}_1}x\; {}_{1\mathbf{v}_1}c_1\; {}_{1\mathbf{v}_1}c_2\; {}_{1\mathbf{v}_1}y$ is a factor of $\mathbf{v}_1$ by Lemma 1; if $\mathbf{u} \approx \mathbf{v}_1$ is directly deducible from some identity in $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$, then we apply Lemma 8 three times, yielding that $({}_{1\mathbf{v}_1}x) < ({}_{1\mathbf{v}_1}c_1) < ({}_{1\mathbf{v}_1}c_2) < ({}_{1\mathbf{v}_1}y)$ and so ${}_{1\mathbf{v}_1}x\; {}_{1\mathbf{v}_1}c_1\; {}_{1\mathbf{v}_1}c_2\; {}_{1\mathbf{v}_1}y$ is a factor of $\mathbf{v}_1$ again. Assume now that $\{x, y\} \neq \{x_1^{(k)}, x_2^{(k)}\}$ for all $k = 1, 2, \ldots, n$. In this case, there exists $j \in \{0, 1, \ldots, n\}$ such that ${}_{2\mathbf{u}}y_j$ lies between ${}_{2\mathbf{u}}x$ and ${}_{2\mathbf{u}}y$ in $\mathbf{u}$. Then the words $\mathbf{u}(c_1, s_j, t, x, y_j)$, $\mathbf{u}(c_1, c_2, s_j, t, y_j)$ and $\mathbf{u}(c_2, s_j, t, y, y_j)$ coincide (up to renaming of variables) with either $xysztxzy$ or $yxsztxzy$. Since the latter two words are isoterms for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, we have $\mathbf{u}(c_1, s_j, t, x, y_j) = \mathbf{v}_1(c_1, s_j, t, x, y_j)$, $\mathbf{u}(c_1, c_2, s_j, t, y_j) = \mathbf{v}_1(c_1, c_2, s_j, t, y_j)$ and $\mathbf{u}(c_2, s_j, t, y, y_j) = \mathbf{v}_1(c_2, s_j, t, y, y_j)$. It follows that $({}_{1\mathbf{v}_1}x) < ({}_{1\mathbf{v}_1}c_1) < ({}_{1\mathbf{v}_1}c_2) < ({}_{1\mathbf{v}_1}y)$. We see that ${}_{1\mathbf{v}_1}x\; {}_{1\mathbf{v}_1}c_1\; {}_{1\mathbf{v}_1}c_2\; {}_{1\mathbf{v}_1}y$ is a factor of

$\mathbf{v}_1$ in any case. Thus, we can apply the induction assumption, yielding that $\mathbf{v}_{\{c_1,c_2\}} = \mathbf{w}_\zeta$ as required. □

For any $n, m, k \geq 1$ and $\rho \in S_{n+m+k}$, we define the words:

$$\mathbf{c}_{n,m,k}[\rho] := \biggl(\prod_{i=1}^{n} z_i t_i\biggr) xyt \biggl(\prod_{i=n+1}^{n+m} z_i t_i\biggr) x \biggl(\prod_{i=1}^{n+m+k} z_{i\rho}\biggr) y \biggl(\prod_{i=n+m+1}^{n+m+k} t_i z_i\biggr),$$
$$\mathbf{c}'_{n,m,k}[\rho] := \biggl(\prod_{i=1}^{n} z_i t_i\biggr) yxt \biggl(\prod_{i=n+1}^{n+m} z_i t_i\biggr) x \biggl(\prod_{i=1}^{n+m+k} z_{i\rho}\biggr) y \biggl(\prod_{i=n+m+1}^{n+m+k} t_i z_i\biggr).$$

Let $\mathbb{O}$ denote the variety defined by the first four identities in (4) together with all the identities of the form:

$$\mathbf{c}_{n,m,k}[\rho] \approx \mathbf{c}'_{n,m,k}[\rho],$$

with $n, m, k \geq 1$ and $\rho \in S_{n+m+k}$. An *island* of a word $\mathbf{w}$ is a maximal factor of $\mathbf{w}$ that consists of only the second occurrences of variables whose first occurrences lie in the same block of $\mathbf{w}$. The next statement follows from the dual to Lemma 3.12 in [10].

Lemma 14. *If* $\mathbf{w} := \mathbf{p}\ {}_{2\mathbf{w}}x\ {}_{2\mathbf{w}}y\ \mathbf{q}$ *and the variables* ${}_{2\mathbf{w}}x$ *and* ${}_{2\mathbf{w}}y$ *lie in the same island of* $\mathbf{w}$*, then* $\mathbb{O}$ *satisfies the identity* $\mathbf{w} \approx \mathbf{p}\, yx\, \mathbf{q}$.

If xy is an isoterm for a variety $\mathbb{V}$ and $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbb{V}$, then it is easy to see that either $\mathsf{con}(\mathbf{u}) = \mathsf{mul}(\mathbf{u}) = \mathsf{con}(\mathbf{v}) = \mathsf{mul}(\mathbf{v})$ or $\mathbf{u} \approx \mathbf{v}$ is of the form:

$$\mathbf{u}_0 \biggl(\prod_{i=1}^{m} t_i \mathbf{u}_i\biggr) \approx \mathbf{v}_0 \biggl(\prod_{i=1}^{m} t_i \mathbf{v}_i\biggr), \tag{6}$$

for some $m \geq 1$, where $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v}) = \{t_1, t_2, \ldots, t_m\}$. For each $i = 0, 1, \ldots, m$, we say the blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ are *corresponding*. An identity $\mathbf{u} \approx \mathbf{v}$ of the form (6) with $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v}) = \{t_1, t_2, \ldots, t_m\}$ is *linear-balanced* if, for any $i = 0, 1, \ldots, m$, the corresponding blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ are linear words depending on the same variables. A linear-balanced identity $\mathbf{u} \approx \mathbf{v}$ is *reduced* if all corresponding blocks of $\mathbf{u}$ and $\mathbf{v}$ are of the forms $\mathbf{ac}$ and $\mathbf{bc}$, where $\mathbf{a}$ and $\mathbf{b}$ consist of the first occurrences of variables in $\mathbf{u}$ and $\mathbf{v}$, respectively, while $\mathbf{c}$ consists of the second occurrences of variables in both $\mathbf{u}$ and $\mathbf{v}$. Evidently, if $\mathbf{u} \approx \mathbf{v}$ is a reduced identity, then every variable occurs in both $\mathbf{u}$ and $\mathbf{v}$ at most twice.

Lemma 15. *Each variety in the interval* $[\mathbb{M}(xzytxy), \mathbb{O}]$ *may be defined within* $\mathbb{O}$ *by a set of reduced identities.*

Proof. We need to show that an arbitrary identity $\mathbf{u} \approx \mathbf{v}$ of $\mathbb{M}(xzytxy)$ is equivalent within $\mathbb{O}$ to a reduced identity. Let

$\mathscr{A} := \{x \in \mathsf{mul}(\mathbf{u}) \mid \mathbf{u}(xyx) = xyx$ for some $y \in \mathsf{sim}(\mathbf{u})\}$ and

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

$\mathscr{B} := \mathsf{mul}(\mathbf{u}) \setminus \mathscr{A} = \{x \in \mathsf{mul}(\mathbf{u}) \mid \mathsf{occ}_x(\mathbf{u}) > 2 \text{ or } {}_{1\mathbf{u}}x \text{ and } {}_{2\mathbf{u}}x \text{ lie in the same block of } \mathbf{u}\}.$

Since the word xyx is an isoterm for $\mathbb{M}(xzytxy)$, it is routine to show that:

$\mathscr{A} = \{x \in \mathsf{mul}(\mathbf{v}) \mid \mathbf{v}(xyx) = xyx \text{ for some } y \in \mathsf{sim}(\mathbf{v})\}$ and
$\mathscr{B} = \mathsf{mul}(\mathbf{v}) \setminus \mathscr{A} = \{x \in \mathsf{mul}(\mathbf{v}) \mid \mathsf{occ}_x(\mathbf{v}) > 2 \text{ or } {}_{1\mathbf{v}}x \text{ and } {}_{2\mathbf{v}}x \text{ lie in the same block of } \mathbf{v}\}.$

Let $\mathscr{B} = \{b_1, b_2, \ldots, b_r\}$. Arguments similar to those of the proof of Lemma 3.11 in [10] imply that the identity:

$$\left(\prod_{i=1}^{n} z_i t_i\right) x \left(\prod_{i=1}^{n+m} z_{i\rho}\right) x \left(\prod_{i=n+1}^{n+m} t_i z_i\right) \approx \left(\prod_{i=1}^{n} z_i t_i\right) x^2 \left(\prod_{i=1}^{n+m} z_{i\rho}\right) \left(\prod_{i=n+1}^{n+m} t_i z_i\right),$$

is satisfied by $\mathbb{O}$ for any $n, m \geq 1$ and $\rho \in S_{n+m}$. Then, by Lemma 4.5 in [9], the variety $\mathbb{O}$ satisfies the identities $\mathbf{u} \approx b_1^2 \cdots b_r^2 \mathbf{u}_{\mathscr{B}}$ and $\mathbf{v} \approx b_1^2 \cdots b_r^2 \mathbf{v}_{\mathscr{B}}$. Hence $\mathbb{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbb{O}\{\mathbf{u}_{\mathscr{B}} \approx \mathbf{v}_{\mathscr{B}}\}$. The identity $\mathbf{u}_{\mathscr{B}} \approx \mathbf{v}_{\mathscr{B}}$ is linear-balanced and every variable occurs in $\mathbf{u}_{\mathscr{B}}$ and $\mathbf{v}_{\mathscr{B}}$ at most twice. Further, the fourth identity in (4) allows us to swap the first and the second occurrences of two multiple variables whenever these occurrences are adjacent to each other. In view of this fact, the variety $\mathbb{O}$ satisfies the identities $\mathbf{u}_{\mathscr{B}} \approx \mathbf{w}_1$ and $\mathbf{v}_{\mathscr{B}} \approx \mathbf{w}_2$ for some words $\mathbf{w}_1$ and $\mathbf{w}_2$ such that each block of $\mathbf{w}_1$ or $\mathbf{w}_2$ is a product of two words consisting of the first and the second occurrences of variables, respectively. This means that $\mathbf{w}_i = \mathbf{a}_0^{(i)} \mathbf{c}_0^{(i)} t_1 \mathbf{a}_1^{(i)} \mathbf{c}_1^{(i)} \cdots t_k \mathbf{a}_k^{(i)} \mathbf{c}_k^{(i)}$, where $\mathsf{sim}(\mathbf{w}_i) = \{t_1, t_2, \ldots, t_k\}$ and, for any $j = 0, 1, \ldots, k$, the word $\mathbf{a}_j^{(i)}$ [respectively, $\mathbf{c}_j^{(i)}$] consists of the first [second] occurrences of variables in $\mathbf{w}_i$. Clearly, $\mathbf{c}_j^{(i)}$ can be represented as a product of some islands $\mathbf{c}_{j1}^{(i)}, \mathbf{c}_{j2}^{(i)}, \ldots, \mathbf{c}_{jr_j^{(i)}}^{(i)}$ of $\mathbf{w}_i$. Since $\mathbf{w}_1 \approx \mathbf{w}_2$ holds in $\mathbb{M}(xzytxy)$, it is easy to deduce from Lemma 1 that $r_j := r_j^{(1)} = r_j^{(2)}$ and $\mathsf{con}(\mathbf{c}_{j\ell}^{(1)}) = \mathsf{con}(\mathbf{c}_{j\ell}^{(2)})$ for any $j = 0, 1, \ldots, k$ and $\ell = 1, 2, \ldots, r_j$. Now Lemma 14 applies, yielding that $\mathbb{O}$ satisfies $\mathbf{w}_1 \approx \mathbf{w}_1'$, where $\mathbf{w}_1' := \mathbf{a}_0^{(1)} \mathbf{c}_0^{(2)} t_1 \mathbf{a}_1^{(1)} \mathbf{c}_1^{(2)} \cdots t_k \mathbf{a}_k^{(1)} \mathbf{c}_k^{(2)}$. Clearly, the identity $\mathbf{w}_1' \approx \mathbf{w}_2$ is reduced and $\mathbb{O}\{\mathbf{u} \approx \mathbf{v}\} = \mathbb{O}\{\mathbf{w}_1' \approx \mathbf{w}_2\}$ as required. □

We call an identity $\mathbf{w} \approx \mathbf{w}'$ 1-*invertible* if $\mathbf{w} = \mathbf{a}\, xy\, \mathbf{b}$ and $\mathbf{w}' = \mathbf{a}\, yx\, \mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathscr{X}^*$ and $x, y \in \mathsf{con}(\mathbf{ab})$. Let $k > 1$. An identity $\mathbf{w} \approx \mathbf{w}'$ is called k-*invertible* if there is a sequence of words $\mathbf{w} = \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_k = \mathbf{w}'$ such that the identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ is 1-invertible for each $i = 0, 1, \ldots, k-1$ and k is the least number with such a property. For convenience, we will call the trivial identity 0-*invertible*.

For the rest of this section, the mapping $\Phi \colon \mathfrak{Eq}(\mathscr{W}_n) \to [\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}, \mathbb{M}(\mathscr{W}_n)]$ given by:

$$\Phi(\pi) := \mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\pi)\},$$

is shown to be an anti-isomorphism. The proof of Proposition 2 is thus complete.

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

The mapping Φ is injective

Suppose that $\Phi(\pi) = \Phi(\rho)$ for some $\pi, \rho \in \mathfrak{Eq}(\mathscr{W}_n)$, so that $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\pi)\} = \mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\rho)\}$. If $(\mathbf{u}, \mathbf{v}) \in \rho$, then the variety $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\pi)\}$ satisfies the identity $\mathbf{u} \approx \mathbf{v}$, whence $(\mathbf{u}, \mathbf{v}) \in \pi$ by Corollary 2. Therefore the inclusion $\rho \subseteq \pi$ holds; the reverse inclusion $\rho \supseteq \pi$ holds by a symmetrical argument, thus $\pi = \rho$.

The mapping Φ is surjective

It suffices to show that for any variety $\mathbb{V}$ from the interval $[\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}, \mathbb{M}(\mathscr{W}_n)]$, there exists some $\pi \in \mathfrak{Eq}(\mathscr{W}_n)$ such that $\Phi(\pi) = \mathbb{V}$. Since $\Phi(\varepsilon_{\mathscr{W}_n}) = \mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\varepsilon_{\mathscr{W}_n})\} = \mathbb{M}(\mathscr{W}_n)$, where $\varepsilon_{\mathscr{W}_n}$ is the equality relation on $\mathscr{W}_n$, suppose that $\mathbb{V} \neq \mathbb{M}(\mathscr{W}_n)$. Then there exists a nontrivial set Σ of identities such that $\mathbb{V} = \mathbb{M}(\mathscr{W}_n)\{\Sigma\}$; by the inclusions $\mathbb{M}(xzytxy) \subseteq \mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\} \subseteq \mathbb{M}(\mathscr{W}_n) \subseteq \mathbb{O}$ and Lemma 15, the identities in Σ can be chosen to be reduced. It is shown below that any identity $\mathbf{u} \approx \mathbf{v}$ in Σ is equivalent within $\mathbb{M}(\mathscr{W}_n)$ to a subset of $\mathsf{Id}(\upsilon_{\mathscr{W}_n})$. By Lemma 2.2 in [6], there exists some $\pi \in \mathfrak{Eq}(\mathscr{W}_n)$ such that $\mathbb{V} = \mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\pi)\}$, so that $\Phi(\pi) = \mathbb{V}$ as required.

Since the identity $\mathbf{u} \approx \mathbf{v}$ is reduced (and so linear-balanced), this identity is r-invertible for some $r \geq 0$. We will use induction by r.

Induction base: $r = 0$. Then $\mathbf{u} = \mathbf{v}$, whence $\mathbb{M}(\mathscr{W}_n)\{\mathbf{u} \approx \mathbf{v}\} = \mathbb{M}(\mathscr{W}_n)\{\emptyset\}$.

Induction step: $r > 0$. If $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{M}(\mathscr{W}_n)$, then $\mathbb{M}(\mathscr{W}_n)\{\mathbf{u} \approx \mathbf{v}\} = \mathbb{M}(\mathscr{W}_n)\{\emptyset\}$. So, we may further assume that $\mathbf{u} \approx \mathbf{v}$ is violated by $\mathbb{M}(\mathscr{W}_n)$. Then there is a substitution $\psi\colon \mathscr{X} \to \mathbb{M}(\mathscr{W}_n)$ such that $\psi(\mathbf{u}) \neq \psi(\mathbf{v})$ in $\mathbb{M}(\mathscr{W}_n)$. This is only possible when $\psi(\mathbf{u})$ or $\psi(\mathbf{v})$, say $\psi(\mathbf{u})$, is a non-empty factor of some word $\mathbf{w}_\xi$ in $\mathscr{W}_n$. According to Lemma 7(iii), every proper factor of $\mathbf{w}_\xi$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$. Hence $\mathbf{w}_\xi = \psi(\mathbf{u})$. Clearly, $\psi(\mathbf{v})$ represents a non-empty word, which does not equal to $\psi(\mathbf{u})$. In view of Lemma 5, $\psi(\mathbf{v}) = \mathbf{w}_\eta$ for some $\eta \in S_2^n \setminus \{\xi\}$. Let $\mathscr{V} := \{x \in \mathscr{X} \mid \psi(x) \neq 1\}$. Clearly, $\mathbf{w}_\xi = \psi(\mathbf{u}) = \psi(\mathbf{u}(\mathscr{V}))$ and $\mathbf{w}_\eta = \psi(\mathbf{v}) = \psi(\mathbf{v}(\mathscr{V}))$.

Notice that every factor of length > 1 of $\mathbf{w}_\xi$ has exactly one occurrence in $\mathbf{w}_\xi$. It follows that $\psi(v)$ is a variable for any $v \in \mathscr{V} \cap \mathsf{mul}(\mathbf{u})$. Let us now consider an arbitrary variable $c \in \mathscr{V} \cap \mathsf{sim}(\mathbf{u})$. If $\psi(c)$ is not a variable, then $\psi_c(\mathbf{u}(\mathscr{V}))$ is obtained from $\mathbf{w}_\xi$ by replacing some factor of length > 1 with the variable c, where $\psi_c\colon \mathscr{X} \to \mathscr{X}^*$ is the substitution defined by

$$\psi_c(v) := \begin{cases} \psi(v) & \text{if } v \neq c, \\ c & \text{if } v = c. \end{cases}$$

According to Lemma 7(ii), the word $\psi_c(\mathbf{u}(\mathscr{V}))$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ contradicting the fact that $\psi(\mathbf{u}(\mathscr{V})) \approx \psi(\mathbf{v}(\mathscr{V}))$ is a nontrivial identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$. Therefore, $\psi(c)$ is a variable. Further, if $\psi(c) \in \mathsf{mul}(\mathbf{w}_\xi)$, then $\psi_c(\mathbf{u}(\mathscr{V}))$ is obtained from $\mathbf{w}_\xi$ by replacing some occurrence of the multiple variable $\psi(c)$ with the variable c. In view of Lemma 7(i), the word $\psi_c(\mathbf{u}(\mathscr{V}))$ is an isoterm for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ contradicting the fact that $\psi(\mathbf{u}(\mathscr{V})) \approx \psi(\mathbf{v}(\mathscr{V}))$ is a nontrivial identity of $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\upsilon_{\mathscr{W}_n})\}$ again. Therefore, $\psi(c) \in \mathsf{sim}(\mathbf{w}_\xi)$. Since the identity $\mathbf{w}_\xi \approx \mathbf{w}_\eta$ is reduced, $\psi(\mathbf{v}_1)$ and $\psi(\mathbf{v}_2)$ cannot coincide with each other for distinct $\mathbf{v}_1, \mathbf{v}_2 \in \mathscr{V}$. Therefore, $\mathbf{u}(\mathscr{V})$ and $\mathbf{v}(\mathscr{V})$ coincide

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

(up to renaming of variables) with $\mathbf{w}_\xi$ and $\mathbf{w}_\eta$, respectively. We may assume without any loss that $\mathbf{u}(\mathscr{V}) = \mathbf{w}_\xi$ and $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$.

Let $\mathscr{V}' := \mathscr{V} \cap \mathsf{mul}(\mathbf{u})$. For any $c \in \mathscr{V}'$, let $\check{\mathbf{c}}$ denote the island of $\mathbf{u}$ containing ${}_{2\mathbf{u}}c$. Consider arbitrary $x, y \in \mathscr{V}'$ such that ${}_{2\mathbf{w}_\xi}x \; {}_{2\mathbf{w}_\xi}y$ is a factor of $\mathbf{w}_\xi$. Clearly, ${}_{1\mathbf{u}}x$ and ${}_{1\mathbf{u}}y$ lie in different blocks of $\mathbf{u}$, whence $\mathsf{con}(\check{\mathbf{x}}) \cap \mathsf{con}(\check{\mathbf{y}}) = \emptyset$. Denote by $\mathbf{c}$ the factor of $\mathbf{u}$ lying between the factors $\check{\mathbf{x}}$ and $\check{\mathbf{y}}$. Assume that $\mathbf{c}$ is non-empty. Corollary 3 together with the fact that $\mathbf{u}(\mathscr{V}) = \mathbf{w}_\xi$ and $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$ imply that $\mathsf{con}(\mathbf{c}) \subseteq \mathsf{mul}(\mathbf{u})$. Since the identity $\mathbf{u} \approx \mathbf{v}$ is reduced, this implies that $\mathbf{c}$ consists of the second occurrences of variables in $\mathbf{u}$. Let c_1 denote the first variable of $\mathbf{c}$. The variables ${}_{1\mathbf{u}}c_1$ and ${}_{1\mathbf{u}}x$ do not lie in the same block of $\mathbf{u}$ because ${}_{1\mathbf{u}}c_1$ belongs to the island $\check{\mathbf{x}}$ otherwise. Therefore, there is $h \in \mathsf{sim}(\mathbf{u})$ such that ${}_{1\mathbf{u}}h$ lies between ${}_{1\mathbf{u}}x$ and ${}_{1\mathbf{u}}c_1$ in $\mathbf{u}$. If the variables ${}_{1\mathbf{u}_1}c_1$ and ${}_{1\mathbf{u}_1}x$ lie in the same block of $\mathbf{u}_1 := \mathbf{u}(\mathscr{V} \cup \{c_1\})$, then taking into account Corollary 3 and the fact that $\mathbf{u}(\mathscr{V}) = \mathbf{w}_\xi$ and $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$, we conclude that $h \notin \mathscr{V}$ and, in the word $\mathbf{u}_2 := \mathbf{u}(\mathscr{V} \cup \{h, c_1\})$, the variable ${}_{1\mathbf{u}_2}h$ does not lie between ${}_{1\mathbf{u}_2}a_1$ and ${}_{1\mathbf{u}_2}b_n$. Hence ${}_{1\mathbf{u}_2}c_1$ forms a block of $\mathbf{u}_2$ contradicting Lemma 9. Therefore, ${}_{1\mathbf{u}_1}c_1$ and ${}_{1\mathbf{u}_1}x$ lie in different blocks of $\mathbf{u}_1$. Then ${}_{1\mathbf{u}_1}y$ and ${}_{1\mathbf{u}_1}c_1$ lie in the same block of $\mathbf{u}_1$ by Lemma 10. By similar arguments we can show that if c_2 is the last variable of $\mathbf{c}$, then ${}_{1\mathbf{u}_3}x$ and ${}_{1\mathbf{u}_3}c_2$ lie in the same block of $\mathbf{u}_3 := \mathbf{u}(\mathscr{V} \cup \{c_2\})$ (and so $c_1 \neq c_2$). This implies that the word $\mathbf{u}_4 := \mathbf{u}(\mathscr{V} \cup \{c_1, c_2\})$ contains the factor ${}_{2\mathbf{u}_4}x \; {}_{2\mathbf{u}_4}c_1 \; {}_{2\mathbf{u}_4}c_2 \; {}_{2\mathbf{u}_4}y$, while ${}_{1\mathbf{u}_4}c_1$ and ${}_{1\mathbf{u}_4}c_2$ lie in the same blocks as ${}_{1\mathbf{u}_4}y$ and ${}_{1\mathbf{u}_4}x$ in $\mathbf{u}_4$, respectively. This contradicts Lemma 12 because $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\xi$ and $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$. Therefore, the word $\mathbf{c}$ must be empty. Since the variables x and y are arbitrary, we have proved that the word:

$$\mathbf{r} := \check{\mathbf{b}}\,\check{\mathbf{y}}_0 \left(\prod_{i=1}^{n} \check{\mathbf{x}}_1^{(i)}\, \check{\mathbf{z}}_i \check{\mathbf{a}}_i \check{\mathbf{z}}_i' \check{\mathbf{b}}_i \check{\mathbf{z}}_i''\, \check{\mathbf{x}}_2^{(i)}\, \check{\mathbf{y}}_i \right) \check{\mathbf{a}},$$

forms a factor of $\mathbf{u}$.

Further, for any $c \in \mathscr{V}' \setminus \{a, b\}$, let $\hat{\mathbf{c}}$ denote the minimal factor of $\mathbf{u}$ containing all first occurrences of variables in $\mathsf{con}(\check{\mathbf{c}})$. Consider an arbitrary variable $x \in \mathscr{V}' \setminus \{a, b\}$. Let d denote the last variable of $\hat{\mathbf{x}}$. By the definition of $\hat{\mathbf{x}}$, we have $d \in \mathsf{con}(\check{\mathbf{x}})$. Consider an arbitrary variable $e \in \mathsf{con}(\hat{\mathbf{x}}) \setminus \mathsf{con}(\check{\mathbf{x}})$ such that some occurrence of e lies between ${}_{1\mathbf{u}}x$ and ${}_{1\mathbf{u}}d$ in $\mathbf{u}$. Since the identity $\mathbf{u} \approx \mathbf{v}$ is reduced, this occurrence of e must be the first one in $\mathbf{u}$. By the definition of the island $\check{\mathbf{x}}$, the variable e is multiple in $\mathbf{u}$. Denote by y the variable different from x that is adjacent to the second occurrence of d in the word $\mathbf{u}_5 := \mathbf{u}(\mathscr{V} \cup \{d\})$. Then, since $x \in \mathscr{V}' \setminus \{a, b\}$ and the word $\mathbf{u}$ is block-linear, we have $y \in \mathscr{V}'$ and either ${}_{2\mathbf{u}_5}x \; {}_{2\mathbf{u}_5}d \; {}_{2\mathbf{u}_5}y$ or ${}_{2\mathbf{u}_5}y \; {}_{2\mathbf{u}_5}d \; {}_{2\mathbf{u}_5}x$ is a factor of $\mathbf{u}_5$. Further, since ${}_{1\mathbf{u}_5}x$ and ${}_{1\mathbf{u}_5}y$ lie in different blocks of $\mathbf{u}_5$, the variable ${}_{1\mathbf{u}_5}d$ is not adjacent to the variable ${}_{1\mathbf{u}_5}y$ in $\mathbf{u}_5$ as well. If $e \in \mathscr{V}$, then ${}_{1\mathbf{u}_5}d$ is not adjacent to ${}_{1\mathbf{u}_5}x$ in $\mathbf{u}_5$ because $({}_{1\mathbf{u}}x) < ({}_{1\mathbf{u}}e) < ({}_{1\mathbf{u}}d)$. Then $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\xi$ by Lemma 10(i) contradicting $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$. Therefore, $e \notin \mathscr{V}$. Since the variable e is arbitrary, we have proved that there are no variables in $\mathscr{V}$ lying between ${}_{1\mathbf{u}}x$ and ${}_{1\mathbf{u}}d$ in $\mathbf{u}$.

Suppose that $x = b_n$. In this case, ${}_{2\mathbf{u}}d$ lies between ${}_{2\mathbf{u}}z_n'$ and ${}_{2\mathbf{u}}z_n''$ in $\mathbf{u}$, while ${}_{2\mathbf{u}}e$ does not. It follows from the fact that $\mathbf{u}(\mathscr{V}) = \mathbf{w}_\xi$ and $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$ and Lemma 10(i) that either $({}_{2\mathbf{u}}e) < ({}_{2\mathbf{u}}y_0)$ or $({}_{2\mathbf{u}}y_n) < ({}_{2\mathbf{u}}e)$. It is easy to see that the word $\mathbf{u}_6 := \mathbf{u}((\mathscr{V} \setminus \{x\}) \cup \{d\})$ coincides (up to renaming variables) with $\mathbf{u}(\mathscr{V}) = \mathbf{w}_\xi$. Since the word $xzytxy$ and so the

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

word $xzytyx$ are isoterms for $\mathbb{M}(\mathscr{W}_n)\{\mathsf{ld}(\upsilon_{\mathscr{W}_n})\}$ by Lemma 4, we have $({}_{2\mathbf{v}}z_n') < ({}_{2\mathbf{v}}d) < ({}_{2\mathbf{v}}z_n'')$. Further, since $\mathbf{u}(b_{n-1}, d, s_{n-1}, t, y_{n-1}) = b_{n-1}ds_{n-1}y_{n-1}tb_{n-1}y_{n-1}d$, we can apply Lemma 4 again, yielding that $\mathbf{v}(b_{n-1}, d, s_{n-1}, t, y_{n-1}) = b_{n-1}ds_{n-1}y_{n-1}tb_{n-1}y_{n-1}d$. Hence $({}_{1\mathbf{v}}b_{n-1}) < ({}_{1\mathbf{v}}d)$. It follows that the word $\mathbf{v}((\mathscr{V} \setminus \{x\}) \cup \{d\})$ coincides (up to renaming variables) with $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$. However, since ${}_{1\mathbf{u}_7}b_{n-1}\ {}_{1\mathbf{u}_7}e\ {}_{1\mathbf{u}_7}d$ is a factor of $\mathbf{u}_7 := \mathbf{u}((\mathscr{V} \setminus \{x\}) \cup \{d, e\})$ and either $({}_{2\mathbf{u}_7}e) < ({}_{2\mathbf{u}_7}y_0)$ or $({}_{2\mathbf{u}_7}y_n) < ({}_{2\mathbf{u}_7}e)$, Lemma 11 implies that $\mathbf{v}((\mathscr{V} \setminus \{x\}) \cup \{d\})$ must coincide (up to renaming variables) with $\mathbf{w}_\xi$, a contradiction.

Suppose now that $x \neq b_n$. Denote by z the variable that directly follows ${}_{1\mathbf{w}_\xi}x$ in $\mathbf{w}_\xi$. In view of the above, $({}_{1\mathbf{u}}d) < ({}_{1\mathbf{u}}z)$, whence ${}_{1\mathbf{u}_8}x\ {}_{1\mathbf{u}_8}e\ {}_{1\mathbf{u}_8}z$ is a factor of the word $\mathbf{u}_8 := \mathbf{u}(\mathscr{V} \cup \{e\})$. Clearly, ${}_{2\mathbf{u}_8}e$ is not adjacent to ${}_{2\mathbf{u}_8}x$ in $\mathbf{u}_8$. Then Lemma 11 and the fact that $\mathbf{u}(\mathscr{V}) = \mathbf{w}_\xi$ and $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$ imply that ${}_{2\mathbf{u}_8}e$ is adjacent to ${}_{2\mathbf{u}_8}z$ in $\mathbf{u}_8$ contradicting Lemma 13.

Thus, we have proved that if e is a variable lying between ${}_{1\mathbf{u}}x$ and ${}_{1\mathbf{u}}d$ in $\mathbf{u}$, then $e \in \mathsf{con}(\check{\mathbf{x}})$. By similar arguments we can show that if d' is the first variable of $\hat{\mathbf{x}}$, then every variable lying between ${}_{1\mathbf{u}}d'$ and ${}_{1\mathbf{u}}x$ in $\mathbf{u}$ must belong to $\mathsf{con}(\check{\mathbf{x}})$. Therefore, we have proved that $\mathsf{con}(\hat{\mathbf{c}}) = \mathsf{con}(\check{\mathbf{c}})$ for any $c \in \mathscr{V}' \setminus \{a, b\}$.

Now consider arbitrary $x, y \in \mathscr{V}' \setminus \{a, b\}$ such that ${}_{1\mathbf{w}_\xi}x\ {}_{1\mathbf{w}_\xi}y$ is a factor of $\mathbf{w}_\xi$. Denote by $\mathbf{c}$ the factor of $\mathbf{u}$ lying between the factors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Assume that $\mathbf{c}$ is non-empty. Then we can take $c \in \mathsf{con}(\mathbf{c})$. Corollary 3 and the fact that $\mathbf{u}(\mathscr{V}) = \mathbf{w}_\xi$ and $\mathbf{v}(\mathscr{V}) = \mathbf{w}_\eta$ imply that $c \in \mathsf{mul}(\mathbf{u})$. Clearly, the variable ${}_{2\mathbf{u}}c$ does not occur in the islands $\check{\mathbf{x}}$ and $\check{\mathbf{y}}$ of $\mathbf{u}$. It follows that the second occurrence of c is not adjacent to the second occurrences of x and y in $\mathbf{u}(\mathscr{V} \cup \{c\})$ contradicting Lemma 11. Therefore, the word $\mathbf{c}$ must be empty. Since the variables x and y are arbitrary, we have proved that the word:

$$\mathbf{h} := \left(\prod_{i=1}^{n} \hat{\mathbf{a}}_i\right) \hat{\mathbf{a}} \left(\prod_{i=1}^{n} \hat{\mathbf{x}}_{1\xi_i}^{(i)} \hat{\mathbf{x}}_{2\xi_i}^{(i)}\right) \hat{\mathbf{b}} \left(\prod_{i=1}^{n} \hat{\mathbf{b}}_i\right)$$

forms a factor of $\mathbf{u}$, where $\hat{\mathbf{a}}$ [respectively, $\hat{\mathbf{b}}$] denotes the factor of $\mathbf{u}$ lying between the factors $\hat{\mathbf{a}}_n$ and $\hat{\mathbf{x}}_{1\xi_1}^{(1)}$ [respectively, $\hat{\mathbf{x}}_{2\xi_n}^{(2)}$ and $\hat{\mathbf{b}}_1$]. Evidently, $a \in \mathsf{con}(\hat{\mathbf{a}})$ and $b \in \mathsf{con}(\hat{\mathbf{b}})$.

Consider an arbitrary variable $c \in \mathsf{con}(\hat{\mathbf{a}}) \setminus \mathsf{con}(\check{\mathbf{a}})$. It follows from Corollary 3 that $c \in \mathsf{mul}(\mathbf{u})$. Further, since $\mathsf{con}(\hat{\mathbf{a}}_n) = \mathsf{con}(\check{\mathbf{a}}_n)$ and $\mathsf{con}(\hat{\mathbf{x}}_{1\xi_1}^{(1)}) = \mathsf{con}(\check{\mathbf{x}}_{1\xi_1}^{(1)})$, Lemma 11 implies that the second occurrence of c is adjacent to the second occurrence of a in $\mathbf{u}(\mathscr{V} \cup \{c\})$. By the definition of the island $\check{\mathbf{a}}$, either ${}_{2\mathbf{u}}c$ and ${}_{2\mathbf{u}}a$ lie in different blocks of $\mathbf{u}$ or ${}_{2\mathbf{u}}c$ and ${}_{2\mathbf{u}}a$ lie in the same block of $\mathbf{u}$ but in different islands of this block. If ${}_{2\mathbf{u}}c$ and ${}_{2\mathbf{u}}a$ lie in different blocks of $\mathbf{u}$, then there is $h \in \mathsf{sim}(\mathbf{u})$ such that $({}_{2\mathbf{u}}a) < ({}_{1\mathbf{u}}h) < ({}_{2\mathbf{u}}c)$ contradicting Lemma 9 because the second occurrence of c in the word $\mathbf{u}(\mathscr{V} \cup \{c, h\})$ forms a block in this word. If ${}_{2\mathbf{u}}c$ and ${}_{2\mathbf{u}}a$ lie in the same block of $\mathbf{u}$ but in different islands of this block, then there is $c_1 \in \mathsf{mul}(\mathbf{u})$ such that $({}_{2\mathbf{u}}a) < ({}_{2\mathbf{u}}c_1) < ({}_{2\mathbf{u}}c)$ and ${}_{1\mathbf{u}}c_1$ do not lie in the block of $\mathbf{u}$ containing ${}_{1\mathbf{u}}a$ and ${}_{1\mathbf{u}}c$. Since the identity $\mathbf{u}((\mathscr{V} \setminus \{a\}) \cup \{c\}) \approx \mathbf{v}((\mathscr{V} \setminus \{a\}) \cup \{c\})$ coincides (up to renaming variables) with $\mathbf{w}_\xi \approx \mathbf{w}_\eta$ and ${}_{2\mathbf{u}_9}y_n\ {}_{2\mathbf{u}_9}c_1\ {}_{2\mathbf{u}_9}c$ is a factor of the word $\mathbf{u}_9 := \mathbf{u}((\mathscr{V} \setminus \{a\}) \cup \{c, c_1\})$, Lemma 10(iii) implies that ${}_{1\mathbf{u}_9}c_1$ is adjacent to ${}_{1\mathbf{u}_9}y_n$ in $\mathbf{u}_9$ contradicting Lemma 12. Therefore, $\mathsf{con}(\hat{\mathbf{a}}) \subseteq \mathsf{con}(\check{\mathbf{a}})$. By similar arguments we can show that $\mathsf{con}(\hat{\mathbf{b}}) \subseteq \mathsf{con}(\check{\mathbf{b}})$.

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

In view of the above, there are words $\mathbf{t}$, $\mathbf{t}_i$, $\mathbf{t}'_i$, $\mathbf{t}''_i$ and $\mathbf{s}_i$ such that the word $\mathbf{phqr}$ is a factor of $\mathbf{u}$, where

$$\mathbf{p} := \left(\prod_{i=1}^{n} \hat{\mathbf{z}}_i \mathbf{t}_i\right)\left(\prod_{i=1}^{n} \hat{\mathbf{z}}'_i \mathbf{t}'_i\right)\left(\prod_{i=1}^{n} \hat{\mathbf{z}}''_i \mathbf{t}''_i\right), \quad \mathbf{q} := \left(\prod_{i=0}^{n} \mathbf{s}_i \hat{\mathbf{y}}_i\right)\mathbf{t}.$$

By similar arguments one can show that the word $\mathbf{v}$ contains a factor:

$$\bar{\mathbf{p}} \cdot \left(\prod_{i=1}^{n} \hat{\bar{\mathbf{a}}}_i\right) \hat{\bar{\mathbf{a}}} \left(\prod_{i=1}^{n} \hat{\bar{\mathbf{x}}}^{(i)}_{1\eta_i} \hat{\bar{\mathbf{x}}}^{(i)}_{2\eta_i}\right) \hat{\bar{\mathbf{b}}} \left(\prod_{i=1}^{n} \hat{\bar{\mathbf{b}}}_i\right) \cdot \bar{\mathbf{q}}\bar{\mathbf{r}}$$

with

$$\bar{\mathbf{p}} := \left(\prod_{i=1}^{n} \hat{\bar{\mathbf{z}}}_i \bar{\mathbf{t}}_i\right)\left(\prod_{i=1}^{n} \hat{\bar{\mathbf{z}}}'_i \bar{\mathbf{t}}'_i\right)\left(\prod_{i=1}^{n} \hat{\bar{\mathbf{z}}}''_i \bar{\mathbf{t}}''_i\right),$$

$$\bar{\mathbf{q}} := \left(\prod_{i=0}^{n} \bar{\mathbf{s}}_i \hat{\bar{\mathbf{y}}}_i\right)\bar{\mathbf{t}},$$

$$\bar{\mathbf{r}} := \check{\bar{\mathbf{b}}}\check{\bar{\mathbf{y}}}_0 \left(\prod_{i=1}^{n} \check{\bar{\mathbf{x}}}^{(i)}_1 \check{\bar{\mathbf{z}}}_i \check{\bar{\mathbf{a}}}_i \check{\bar{\mathbf{z}}}'_i \check{\bar{\mathbf{b}}}_i \check{\bar{\mathbf{z}}}''_i \check{\bar{\mathbf{x}}}^{(i)}_2 \check{\bar{\mathbf{y}}}_i\right) \check{\bar{\mathbf{a}}},$$

such that

- for any $c \in \mathscr{V}'$, the word $\check{\bar{\mathbf{c}}}$ is the island of $\mathbf{v}$ containing ${}_{2\mathbf{v}}c$;
- $\mathsf{con}(\hat{\bar{\mathbf{c}}}) = \mathsf{con}(\check{\bar{\mathbf{c}}})$ for any $c \in \mathscr{V}' \setminus \{a, b\}$;
- $c \in \mathsf{con}(\hat{\bar{\mathbf{c}}}) \subseteq \mathsf{con}(\check{\bar{\mathbf{c}}})$ for any $c \in \{a, b\}$.

Since the identity $\mathbf{u} \approx \mathbf{v}$ is reduced, $\mathsf{con}(\check{\mathbf{c}}) = \mathsf{con}(\check{\bar{\mathbf{c}}})$ for any $c \in \mathscr{V}'$. Hence $\mathsf{con}(\hat{\mathbf{c}}) = \mathsf{con}(\hat{\bar{\mathbf{c}}})$ for any $c \in \mathscr{V}' \setminus \{a, b\}$. Further, one can deduce from Lemma 5 that $\mathsf{con}(\hat{\mathbf{c}}) = \mathsf{con}(\hat{\bar{\mathbf{c}}})$ for any $c \in \{a, b\}$.

Clearly, $\mathbf{u} = \mathbf{a} \cdot \mathbf{phqr} \cdot \mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathscr{X}^*$. Define $\mathbf{w} := \mathbf{a} \cdot \mathbf{p}\tilde{\mathbf{h}}\mathbf{qr} \cdot \mathbf{b}$, where

$$\tilde{\mathbf{h}} := \left(\prod_{i=1}^{n} \hat{\mathbf{a}}_i\right) \hat{\mathbf{a}} \left(\prod_{i=1}^{n} \hat{\mathbf{x}}^{(i)}_{1\eta_i} \hat{\mathbf{x}}^{(i)}_{2\eta_i}\right) \hat{\mathbf{b}} \left(\prod_{i=1}^{n} \hat{\mathbf{b}}_i\right).$$

Since $\mathsf{con}(\hat{\bar{\mathbf{c}}}) = \mathsf{con}(\hat{\mathbf{c}})$ for any $c \in \mathscr{V}'$ and the identity $\mathbf{u} \approx \mathbf{v}$ is reduced, the identity $\mathbf{w} \approx \mathbf{v}$ is $(r - r')$-invertible with:

$$r' = \sum_{i \in \{j \mid \xi_j \neq \eta_j\}} |\hat{\mathbf{x}}^{(i)}_{1\xi_i}| \cdot |\hat{\mathbf{x}}^{(i)}_{2\xi_i}| > 0.$$

Clearly, the identity $\mathbf{w} \approx \mathbf{v}$ is reduced. So, we can apply the induction assumption, yielding that $\mathbb{M}(\mathscr{W}_n)\{\mathbf{w} \approx \mathbf{v}\} = \mathbb{M}(\mathscr{W}_n)\{\Psi\}$ for some $\Psi \subseteq \mathsf{Id}(\upsilon_{\mathscr{W}_n})$. Further, $\mathbb{M}(\mathscr{W}_n)\{\mathbf{w}_\xi \approx \mathbf{w}_\eta\}$ satisfies the identities:

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

$$\begin{aligned}
\mathbf{u} &= \mathbf{a}\cdot\mathbf{phqr}\cdot\mathbf{b} \\
&\approx \mathbf{a}\cdot\mathbf{phq\tilde{r}}\cdot\mathbf{b} && \text{by Lemma 14} \\
&\approx \mathbf{a}\cdot\mathbf{p\tilde{h}q\tilde{r}}\cdot\mathbf{b} && \text{by } \mathbf{w}_\xi \approx \mathbf{w}_\eta \\
&\approx \mathbf{a}\cdot\mathbf{p\tilde{h}qr}\cdot\mathbf{b} && \text{by Lemma 14,} \\
&= \mathbf{w},
\end{aligned}$$

where

$$\tilde{\mathbf{r}} := (\check{\mathbf{b}})_{\mathsf{con}(\hat{\mathbf{b}})}\hat{\mathbf{b}}\hat{\mathbf{y}}_0\left(\prod_{i=1}^{n}\hat{\mathbf{x}}_1^{(i)}\,\hat{\mathbf{z}}_i\hat{\mathbf{a}}_i\hat{\mathbf{z}}_i'\hat{\mathbf{b}}_i\hat{\mathbf{z}}_i''\,\hat{\mathbf{x}}_2^{(i)}\,\hat{\mathbf{y}}_i\right)\hat{\mathbf{a}}(\check{\mathbf{a}})_{\mathsf{con}(\hat{\mathbf{a}})}.$$

Since $\mathbf{w}_\xi \approx \mathbf{w}_\eta$ is a consequence of $\mathbf{u} \approx \mathbf{v}$, this implies that:

$$\mathbb{M}(\mathscr{W}_n)\{\mathbf{u}\approx\mathbf{v}\} = \mathbb{M}(\mathscr{W}_n)\{\mathbf{u}\approx\mathbf{w}\approx\mathbf{v}\} = \mathbb{M}(\mathscr{W}_n)\{\mathbf{w}_\xi\approx\mathbf{w}_\eta,\ \Psi\}.$$

The mapping Φ is an anti-isomorphism

Let $\pi,\rho \in \mathfrak{Eq}(\mathscr{W}_n)$. If $\pi \subseteq \rho$, then the inclusion $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\rho)\} \subseteq \mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\pi)\}$ holds, so that $\Phi(\rho) \subseteq \Phi(\pi)$. Conversely, assume the inclusion $\Phi(\rho) \subseteq \Phi(\pi)$, so that $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\rho)\} \subseteq \mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\pi)\}$. Then for any $(\mathbf{u},\mathbf{v}) \in \pi$, the identity $\mathbf{u}\approx\mathbf{v}$ is satisfied by $\mathbb{M}(\mathscr{W}_n)\{\mathsf{Id}(\rho)\}$, whence $(\mathbf{u},\mathbf{v}) \in \rho$ by Corollary 2. Therefore $\pi \subseteq \rho$.

5. Some open problems

5.1. Monoids of order at least six

In the present article, we show that the 6-element monoids A_2^1 and B_2^1 generate finitely universal varieties. We do not know any of 6-element monoids distinct from A_2^1 and B_2^1 generating varieties with this property. Thus, the following question is relevant.

Question 1. Is there a 6-element monoid distinct from A_2^1 and B_2^1 generating a finitely universal variety?

As we have mentioned above, the varieties $\mathbb{B}_2^1$ and $\mathbb{A}_2^1$ are non-finitely based. The following question is still open.

Question 2. What is the least order of a finitely based monoid that generates a finitely universal variety?

It is shown in [20] that every monoid of order six distinct from A_2^1 and B_2^1 generates a finitely based variety. In view of this result, the affirmative answer to Question 1 provides a solution to Question 2.

5.2. Lattice universal varieties

Here we remind an open question from [6] and [7]. It follows from [29] that a variety $\mathbb{V}$ is finitely universal if and only if for all sufficiently large $n \geq 1$, the lattice $\mathfrak{Eq}(\{1,2,\ldots,n\})$

https://doi.org/10.1017/S0013091524000178 Published online by Cambridge University Press

is anti-isomorphic to some sublattice of $\mathfrak{L}(\mathbb{V})$. In the present article, finitely universal varieties $\mathbb{V}$ of monoids are exhibited with the stronger property that for all sufficiently large $n \geq 1$, the lattice $\mathfrak{Eq}(\{1, 2, \ldots, n\})$ is anti-isomorphic to some subinterval of $\mathfrak{L}(\mathbb{V})$. A yet even stronger property that a variety $\mathbb{V}$ can satisfy is when the lattice $\mathfrak{Eq}(\{1, 2, 3, \ldots\})$ is anti-isomorphic to some subinterval of $\mathfrak{L}(\mathbb{V})$; following [32], such a variety is said to be *lattice universal*. Lattice universal varieties of semigroups have been found in [2] and [16]; it is natural to question if a variety of monoids can also satisfy this property.

Question 3. ([6, Question 6.5]; see also [7, Question 4.11b)]). Is there a variety of monoids that is lattice universal?

Notice that, for locally finite varieties the answer to Question 3 is negative. This immediately follows from the following three folkloric facts: the subvariety lattice of an arbitrary locally finite variety is algebraic; the lattice $\mathfrak{Eq}(\{1, 2, 3, \ldots\})$ is not coalgebraic; an interval of an algebraic lattice is again algebraic.

Acknowledgements. The author thanks Edmond W.H. Lee for several comments and suggestions for improving the manuscript and many discussions, and the anonymous referee for providing a careful review and a number of remarks.

Funding Statement. This study was supported by the Ministry of Science and Higher Education of the Russian Federation (project no. FEUZ-2023-0022).

References

(1) Yu. A. Bahturin and A. Yu. Ol'shanskiĭ, *Identical relations in finite Lie rings*, Mat. Sb. (N.S.) **96**(138) (1975), 543–559; Engl. translation: Sb. Math. 25 (1975), 507–523 [in Russian].

(2) S. Burris and S. Nelson, *Embedding the dual of Π_∞ in the lattice of equational classes of semigroups*, Algebra Universalis, **1** (1971), 248–253.

(3) S. Burris and S. Nelson, *Embedding the dual of Π_m in the lattice of equational classes of commutative semigroups*, Proc. Amer. Math. Soc., **30** (1971), 37–39.

(4) S. Burris and H. P. Sankappanavar, A Course in Universal Algebra, Springer Verlag, New York, 1981.

(5) S. V. Gusev, *On the ascending and descending chain conditions in the lattice of monoid varieties*, Siberian Electronic Math. Reports, **16** (2019), 983–997.

(6) S. V. Gusev and E. W. H. Lee, *Varieties of monoids with complex lattices of subvarieties*, Bull. Lond. Math. Soc., **52** (2020), 762–775.

(7) S. V. Gusev, E. W. H. Lee and B. M. Vernikov, *The lattice of varieties of monoids*, Japan. J. Math., **17** (2022), 117–183.

(8) S. V. Gusev, Y. X. Li and W. T. Zhang, *Limit varieties of monoids satisfying a certain identity*, Algebra Colloq., to appear; available at: http://arxiv.org/abs/2107.07120v2.

(9) S. V. Gusev and B. M. Vernikov, *Chain varieties of monoids*, Dissertationes Math., **534** (2018), 1–73.

(10) S. V. Gusev and B. M. Vernikov, *Two weaker variants of congruence permutability for monoid varieties*, Semigroup Forum, **103** (2021), 106–152.

(11) M. Jr. Hall, The Theory of Groups, The Macmillan Company, New York, 1959.

(12) M. Jackson, *Finiteness properties of varieties and the restriction to finite algebras*, Semigroup Forum, **70** (2005), 159–187; *Erratum to "Finiteness properties of varieties and the restriction to finite algebras"*, Semigroup Forum, **96** (2018), 197–198.
(13) M. Jackson and E. W. H. Lee, *Monoid varieties with extreme properties*, Trans. Amer. Math. Soc., **370** (2018), 4785–4812.
(14) M. Jackson and O. Sapir, *Finitely based, finite sets of words*, Int. J. Algebra Comput. **10** (2000), 683–708.
(15) M. Jackson and W. T. Zhang, *From A to B to Z*, Semigroup Forum, **103** (2021), 165–190.
(16) J. Ježek, *Intervals in the lattice of varieties*, Algebra Universalis, **6** (1976), 147–158.
(17) R. L. Kruse, *Identities satisfied by a finite ring*, J. Algebra, **26** (1973), 298–318.
(18) E. W. H. Lee, *Minimal semigroups generating varieties with complex subvariety lattices*, Int. J. Algebra Comput., **17** (2007), 1553–1572.
(19) E. W. H. Lee, *Finite basis problem for the direct product of some J-trivial monoid with groups of finite exponent*, Vestnik of St. Petersburg State University. Series 1. Mathematics. Mechanics. Astronomy, No.4 (2013), 60–64.
(20) E. W. H. Lee and J. R. Li, *Minimal non-finitely based monoids*, *Dissertationes Math.*, **475** (2011), 1–65.
(21) E. W. H. Lee and W. T. Zhang, *The smallest monoid that generates a non-Cross variety*, Xiamen Daxue Xuebao Ziran Kexue Ban, **53** (2014), 1–4 [in Chinese].
(22) E. W. H. Lee and W. T. Zhang, *Finite basis problem for semigroups of order six*, LMS J. Comput. Math., **18** (2015), 1–129.
(23) I. V. L'vov, *Varieties of associative rings*. I, Algebra i Logika, **12** (1973), 269–297 Engl. translation: Algebra and Logic, **12** (1973), 150–167 [in Russian].
(24) S. Oates and M. B. Powell, *Identical relations in finite groups*, J. Algebra, **1** (1964), 11–39.
(25) P. Perkins, *Bases for equational theories of semigroups*, J. Algebra, **11** (1969), 298–314.
(26) F. J. Pastijn, *The lattice of completely regular semigroup varieties*, J. Austral. Math. Soc. Ser. A, **49** (1990), 24–42.
(27) F. J. Pastijn, *Commuting fully invariant congruences on free completely regular semigroups*, Trans. Amer. Math. Soc., **323** (1991), 79–92.
(28) M. Petrich and N. R. Reilly, *The modularity of the lattice of varieties of completely regular semigroups and related representations*, Glasgow Math. J., **32** (1990), 137–152.
(29) P. Pudlák and J. Tůma, *Every finite lattice can be embedded in a finite partition lattice*, Algebra Universalis, **10** (1980), 74–95.
(30) M. V. Sapir, *Problems of Burnside type and the finite basis property in varieties of semigroups*, Izv. Akad. Nauk SSSR Ser. Mat., **51**, No.2 (1987), 319–340 Engl. translation: Math. USSR-Izv., **30**, No.2 (1988), 295–314 [in Russian].
(31) M. V. Sapir, *On Cross semigroup varieties and related questions*, Semigroup Forum, **42** (1991), 345–364.
(32) L. N. Shevrin, B. M. Vernikov and M. V. Volkov, *Lattices of semigroup varieties*, Izv. Vyssh. Uchebn. Zaved. Matem., No.3 (2009), 3–36 [Engl. translation: Russian Math. (Iz. VUZ), **53**, No.3 (2009), 1–28 [in Russian].