GLOBAL ATTRACTIVITY OF THE PERIODIC KOLMOGOROV SYSTEM

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Abstract

We study a periodic Kolmogorov model with m predators and n prey. By means of the comparison theorem and a Liapunov function, a set of easily verifiable sufficient conditions that guarantee the existence, uniqueness and global attractivity of the positive periodic solution is obtained. Finally, some suitable applications are given to illustrate that the conditions of the main theorem are feasible.

1. Introduction

The Kolmogorov system is a rudimentary model in mathematical ecology and has been extensively investigated and developed (see [8, 9] and references therein). But most of the literature requires that the systems be representable as autonomous differential equations. If we consider the effects of environmental factors, the assumption of periodicity of the parameters is both realistic and important.

Our main purpose in this paper is to study the asymptotic behaviour of the general periodic Kolmogorov predator-prey system. Moreover, competition among predator species and among prey species is simultaneously considered, that is, we will investigate the following nonautonomous Kolmogorov predator-prey system of differential equations:

$$\begin{cases} \dot{x}_i(t) = x_i(t) F_i(t, x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t)), & i = 1, \dots, n, \\ \dot{y}_j(t) = y_i(t) G_j(t, x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t)), & j = 1, \dots, m, \end{cases} \quad (1.1)$$

where $x_i(t)$ denotes the density of species X_i at time t and $y_j(t)$ denotes the density of species Y_j at time t. To describe system (1.1), we give the following assumptions.

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(H1) $F_i : R \times R_+^{n+m} \to R$, is continuous and periodic with respect to t, that is, there exists a positive constant T, such that $F_i(t + T, \cdot, \ldots, \cdot) = F_i(t, \cdot, \ldots, \cdot)$; $G_j : R \times R_+^{n+m} \to R$, is continuous and periodic with respect to t, that is, for the above positive constant T, $G_j(t + T, \cdot, \ldots, \cdot) = G_j(t, \cdot, \ldots, \cdot)$. Moreover,

$$F_i(t, 0, \ldots, 0) > 0, \quad G_j(t, 0, \ldots, 0) < 0, \quad i = 1, \ldots, n, \; j = 1, \ldots, m.$$

(H2) $F_i(t, x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $G_j(t, x_1, \ldots, x_n, y_1, \ldots, y_m)$ are continuously differentiable with respect to $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in R_+^{n+m}$. Moreover,

$$\frac{\partial F_i}{\partial x_s} < 0, \quad \frac{\partial F_i}{\partial y_l} < 0, \quad \frac{\partial G_j}{\partial x_s} > 0, \quad \frac{\partial G_j}{\partial y_l} < 0,$$
$$(i = 1, \ldots, n, \; j = 1, \ldots, m, \; s = 1, \ldots, n, \; l = 1, \ldots, m),$$

for $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in E$, where E is an arbitrary bounded set of R_+^{n+m}.

(H3) There exist positive constants $B_i, i = 1, \ldots, n$, such that

$$F_i(t, 0, \ldots, 0, B_i, 0, \ldots, 0, 0, \ldots, 0) < 0, \quad i = 1, \ldots, n.$$

Moreover, for arbitrary bounded positive constants $K_1, \ldots, K_n$, there exist positive constants $D_j = D_j(K_1, \ldots, K_n) > 0, j = 1, \ldots, m$, such that

$$G_j(t, K_1, \ldots, K_n, 0, \ldots, 0, D_j, 0, \ldots, 0) < 0, \quad j = 1, \ldots, m.$$

Wu and Zhao obtained the globally attractive almost periodic solution in the almost competition Kolmogorov system in [12]. Yang and Xu obtained a globally attractive periodic solution in the periodic Volterra-Lotka predator-prey system in [13]. In this paper, we will study the periodic Kolmogorov predator-prey system. As an application, we will consider some ecological systems.

The structure of this paper is as follows: in Section 2, we investigate the persistence of system (1.1) and derive a persistence result. In Section 3 we obtain sufficient conditions for the uniqueness and global attractivity of the periodic solution of (1.1). Finally, in Section 4 we give some suitable applications to illustrate that the conditions of the main theorem are feasible.

2. Persistence

Throughout this paper,

$$u(t, u_0) = (x_1(t, u_0), \ldots, x_n(t, u_0), y_1(t, u_0), \ldots, y_m(t, u_0))^T$$

denotes the solution of (1.1) with initial condition

$$u_0 = (x_{10}, \ldots, x_{n0}, y_{10}, \ldots, y_{m0})^T \in \text{int}(R_+^{n+m}).$$

The following lemmas are required for the derivation of a persistence result, which follows directly from [12, Theorem 4.2].

LEMMA 2.1. *If for some $i \in \{1, 2, \ldots, n\}$,*

$$\int_0^T F_i(t, 0, \ldots, 0)\, dt > 0, \qquad (P1)$$

then the equation $dx_i/dt = x_i F_i(t, 0, \ldots, 0, x_i, 0, \ldots, 0)$ admits a globally asymptotically stable positive T-periodic solution $\bar{x}_i^*(t)$ in $\mathrm{int}(R_+)$.

LEMMA 2.2. *If (P1) holds for all $i = 1, \ldots, n$, and for some $j \in \{1, 2, \ldots, m\}$,*

$$\int_0^T G_j(t, \bar{x}_1^*(t), \ldots, \bar{x}_n^*(t), 0, \ldots, 0)\, dt > 0, \qquad (P2)$$

then the equation $dy_j/dt = y_j G_j(t, \bar{x}_1^*, \ldots, \bar{x}_n^*, 0, \ldots, 0, y_j, 0, \ldots, 0)$ admits a globally asymptotically stable positive T-periodic solution $\bar{y}_j^*(t)$ in $\mathrm{int}(R_+)$.

LEMMA 2.3. *If (P1) holds for all $i = 1, \ldots, n$, and (P2) holds for all $j = 1, \ldots, m$, and for some $i \in \{1, \ldots, n\}$,*

$$\int_0^T F_i(t, \bar{x}_1^*(t), \ldots, \bar{x}_{i-1}^*(t), 0, \bar{x}_{i+1}^*(t), \ldots, \bar{x}_n^*(t), \bar{y}_1^*(t), \ldots, \bar{y}_m^*(t))\, dt > 0, \quad (P3)$$

then the equation $dx_i/dt = x_i F_i(t, \bar{x}_1^*, \ldots, \bar{x}_{i-1}^*, x_i, \bar{x}_{i+1}^*, \ldots, \bar{x}_n^*, \bar{y}_1^*, \ldots, \bar{y}_m^*)$ admits a globally asymptotically stable positive T-periodic solution $\underline{x}_i^*(t)$ in $\mathrm{int}(R_+)$.

LEMMA 2.4. *If (P1), (P3) hold for all $i = 1, \ldots, n$, and (P2) holds for all $j = 1, \ldots, m$, and for some $j \in \{1, \ldots, m\}$,*

$$\int_0^T G_j(t, \underline{x}_1^*(t), \ldots, \underline{x}_n^*(t), \bar{y}_1^*(t), \ldots, \bar{y}_{j-1}^*(t), 0, \bar{y}_{j+1}^*(t), \cdots, \bar{y}_m^*(t))\, dt > 0, \quad (P4)$$

then the equation $dy_j/dt = y_j G_j(t, \underline{x}_1^*, \ldots, \underline{x}_n^*, \bar{y}_1^*, \ldots, \bar{y}_{j-1}^*, y_j, \bar{y}_{j+1}^*, \ldots, \bar{y}_m^*)$ admits a globally asymptotically stable positive T-periodic solution $\underline{y}_j^*(t)$ in $\mathrm{int}(R_+)$.

We can now obtain our main result of the section.

THEOREM 2.5. *If (H1)–(H3) hold, (P1), (P3) hold for all $i = 1, \ldots, n$, and (P2), (P4) hold for all $j = 1, \ldots, m$, then system (1.1) is persistent, that is, there exist constants M and η, $M > \eta > 0$, such that for arbitrary*

$$u_0 = (x_{10}, \ldots, x_{n0}, y_{10}, \ldots, y_{m0})^T \in \mathrm{int}(R_+^{n+m}),$$

there exists a $T(u_0) > 0$, such that the solution $u(t, u_0)$ of (1.1) satisfies

$$\eta < x_i(t, u_0) < M, \quad \eta < y_j(t, u_0) < M,$$

for all $t > T(u_0)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

PROOF. We prove the theorem in two steps.

Step (1). **Solutions exist globally on $[0, \infty)$.**

For any $u_0 = (x_{10}, \ldots, x_{n0}, y_{10}, \ldots, y_{m0}) \in \text{int}(R_+^{n+m})$, let $I^+(u_0) = [0, \beta(u_0))$ be the maximal interval of existence of $u(t, u_0)$. Then $x_i(t, u_0) > 0, y_j(t, u_0) > 0$ for $1 \leq i \leq n, 1 \leq j \leq m, t \in [0, \beta(u_0))$. By the assumption (H2), $x_i(t, u_0)$ satisfies

$$\frac{dx_i(t)}{dt} \leq x_i(t) F_i(t, 0, \ldots, 0, x_i(t), 0, \ldots, 0).$$

Therefore, by the comparison theorem, $0 < x_i(t) \leq \bar{x}_i(t), t \in I^+(u_0)$, where $\bar{x}_i(t)$ is the unique solution of

$$\frac{dx_i(t)}{dt} = x_i(t) F_i(t, 0, \ldots, 0, x_i(t), 0, \ldots, 0)$$

with $\bar{x}_i(0) = x_{i0}$. Since $\bar{x}_i(t)$ exists globally on $[0, \infty)$, $x_i(t)$ exists globally on $[0, \infty)$. Moreover, for arbitrary $\delta > 0$, there exists a $t_0 > 0$, such that $0 < x_i(t) < \bar{x}_i^*(t) + \delta$, for all $t \geq t_0$.

By the assumption (H2), $y_j(t)$ satisfies

$$\frac{dy_j(t)}{dt} \leq y_j(t) G_j(t, K_1, \ldots, K_n, 0, \ldots, 0, y_j(t), 0, \cdots, 0),$$

where $K_i = \max\{\sup_{t \in [0, \infty)}[\bar{x}_i^* + \delta], \max_{t \in [0, t_0]} x_i(t)\}$. Therefore, by the comparison theorem, $0 < y_j(t) \leq \bar{y}_j(t), t \in I^+(u_0)$, where $\bar{y}_j(t)$ is the unique solution of

$$\frac{dy_j(t)}{dt} = y_j(t) G_j(t, K_1, \ldots, K_n, 0, \ldots, 0, y_j(t), 0, \ldots, 0),$$

with $\bar{y}_j(0) = y_{j0}$. Since $\bar{y}_j(t)$ exists globally on $[0, \infty)$, $y_j(t)$ exists globally on $[0, \infty)$. Therefore $\beta(u_0) = \infty$ and $I^+(u_0) = [0, \infty)$.

Step (2). **Uniform persistence.**

By the first step, we know that for arbitrary $\delta > 0$, there exists a $t_0 > 0$, such that $0 < x_i(t) < \bar{x}_i^* + \delta, 1 \leq i \leq n$ for $t \geq t_0$. Now we prove that $y_j(t)$ is bounded for $1 \leq j \leq m$. Let $v_i(t) = x_i(t + t_0), v(t) = (v_1(t), \ldots, v_n(t)), v^*(t) = (\bar{x}_1^*(t + t_0), \ldots, \bar{x}_n^*(t + t_0))$ and $z_j(t) = y_j(t + t_0)$. Then

$$\frac{dz_j(t)}{dt} = z_j(t) G_j(t + t_0, v_1(t), \ldots, v_n(t), z_1(t), \ldots, z_m(t))$$

$$\leq z_j(t) G_j(t + t_0, v_1(t), \ldots, v_n(t), 0, \ldots, 0, z_j(t), 0, \ldots, 0)$$

$$\leq z_j(t) G_j(t + t_0, \bar{x}_1^*(t + t_0) + \delta, \ldots, \bar{x}_n^*(t + t_0) + \delta, 0, \ldots, 0, z_j(t), 0, \ldots, 0).$$

Therefore, by the comparison theorem, $0 < z_j(t) \leq w_j(t)$, where $w_j(t)$ is the solution of

$$\frac{dw_j(t)}{dt} = w_j(t) G_j(t + t_0, \bar{x}_1^*(t + t_0) + \delta, \ldots, \bar{x}_n^*(t + t_0) + \delta,$$

$$0, \ldots, 0, w_j(t), 0, \ldots, 0) \tag{2.1}$$

with $w_i(0) = y_j(t_0) > 0$. Choose $\delta_0 > 0$ small enough so that

$$\int_0^T G_j(t, \bar{x}_1^*(t) + \delta, \dots, \bar{x}_n^*(t) + \delta, 0, \dots, 0) \, dt > 0$$

as $0 < \delta < \delta_0$. Then (2.1) admits a unique globally asymptotically stable positive periodic solution $\bar{w}_j^*(t)$ in $int(R_+)$. For arbitrary $\varepsilon > 0$, there exists a $t_1' = t(y_j(0)) > 0$, such that $0 < w_j(t) < \bar{w}_j^*(t) + \varepsilon/2$ for $t \geq t_1'$.

Considering the system

$$\frac{d\bar{z}_j(t)}{dt} = \bar{z}_j(t) G_j(t + t_0, \bar{x}_1^*(t + t_0), \dots, \bar{x}_n^*(t + t_0), 0, \dots, 0, \bar{z}_j(t), 0, \dots, 0), \quad (2.2)$$

then $\lim_{t \to +\infty}(\bar{z}_j(t) - \bar{y}_j^*(t + t_0)) = 0$, where $\bar{z}_j(t)$ is a solution of (2.2) with $\bar{z}_j(0) = y(t_0) > 0$. Since the solution of (2.1) is continuous with respect to parameter δ, $\bar{w}_j^*(t)$ and $\bar{y}_j^*(t + t_0)$ are periodic solutions of (2.1) and (2.2) respectively. Hence, for the above ε, there exists a δ_1, $0 < \delta_1 < \delta_0$, such that $|\bar{y}_j^*(t + t_0) - \bar{w}_j^*(t)| < \varepsilon/2$ for $0 < \delta < \delta_1$. Therefore

$$0 < z_j(t) \leq w_j(t) < \bar{w}^*(t) + \varepsilon/2 < \bar{y}_j^*(t + t_0) + \varepsilon, \quad t \geq t_1', \ \delta < \delta_1.$$

Accordingly, $0 < y_j(t) < \bar{y}_j^*(t) + \varepsilon, t \geq t_1 = t_1' + t_0$.

Let $u_i(t) = x_i(t + t_1)$, $i = 1, \dots, n$. Then $u_i(t)$ satisfies the inequality

$$\frac{du_i(t)}{dt} \geq u_i(t) F_i\Big(t + t_1, \bar{x}_1^*(t + t_1) + \delta, \dots, \bar{x}_{i-1}^*(t + t_1) + \delta, u_i(t),$$
$$\bar{x}_{i+1}^*(t + t_1) + \delta, \dots, \bar{x}_n^*(t + t_1) + \delta,$$
$$\bar{y}_1^*(t + t_1) + \varepsilon, \dots, \bar{y}_m^*(t + t_1) + \varepsilon\Big).$$

Therefore, by the comparison theorem, $u_i(t) \geq \underline{u}_i(t)$, $t \geq 0$, $1 \leq i \leq n$, where $\underline{u}_i(t)$ is the unique solution of

$$\frac{du_i(t)}{dt} = u_i(t) F_i\Big(t + t_1, \bar{x}_1^*(t + t_1) + \delta, \dots, \bar{x}_{i-1}^*(t + t_1) + \delta, u_i(t),$$
$$\bar{x}_{i+1}^*(t + t_1) + \delta, \dots, \bar{x}_n^*(t + t_1) + \delta,$$
$$\bar{y}_1^*(t + t_1) + \varepsilon, \dots, \bar{y}_m^*(t + t_1) + \varepsilon\Big) \quad (2.3)$$

with $\underline{u}_i(0) = u_i(0) = x_i(t_1) > 0$. Define

$$\bar{F}_i(t + t_1, \delta, \varepsilon) = F_i\Big(t + t_1, \bar{x}_1^*(t + t_1) + \delta, \dots, \bar{x}_{i-1}^*(t + t_1) + \delta, 0,$$
$$\bar{x}_{i+1}^*(t + t_1) + \delta, \dots, \bar{x}_n^*(t + t_1) + \delta,$$
$$\bar{y}_1^*(t + t_1) + \varepsilon, \dots, \bar{y}_m^*(t + t_1) + \varepsilon\Big).$$

There exist positive constants ε_1 and $\delta_2, \delta_2 < \delta_1$, such that $\int_0^T \bar{F}_i(\xi + t_1, \delta, \varepsilon)\, d\xi > 0$, when $0 < \delta < \delta_2, 0 < \varepsilon < \varepsilon_1, i = 1, \ldots, n$. Therefore we have

$$\lim_{t \to \infty} (\underline{u}_i(t) - u_i^*(t)) = 0, \tag{2.4}$$

where $u_i^*(t)$ is the globally asymptotically stable positive periodic solution of (2.3). By (2.4), we know that for arbitrary $\gamma < \inf_{t \in R_+} \underline{u}_i^*(t)$, $1 \leq i \leq n$, there exists $t_2' = t(x_i(0)) > 0$, such that $\underline{u}_i(t) > u_i^*(t) - \gamma/2, t > t_1, 1 \leq i \leq n$. Considering the equation

$$\begin{aligned}
\frac{dw_i(t)}{dt} = w_i(t) F_i\big(t + t_1, \bar{x}_1^*(t + t_1), \ldots, &\bar{x}_{i-1}^*(t + t_1), w_i(t), \\
&\bar{x}_{i+1}^*(t + t_1), \ldots, \bar{x}_n^*(t + t_1), \bar{y}_1^*(t + t_1), \ldots, \bar{y}_m^*(t + t_1)\big), \quad (2.5)
\end{aligned}$$

we know that (2.5) admits a unique positive periodic solution $\underline{x}_i^*(t + t_1)$, which is globally asymptotically stable in $\operatorname{int}(R_+)$. This holds since the solution of (2.3) is continuous with respect to parameters δ and ε, and $u_i^*(t)$ and $\underline{x}_i^*(t + t_1)$ are positive periodic solutions of (2.3) and (2.5), respectively. For the above γ, there exist constants $\varepsilon_2, \delta_3, 0 < \delta_3 < \delta_2, 0 < \varepsilon_2 < \varepsilon_1$, such that

$$|\underline{x}_i^*(t + t_1) - u_i^*(t)| < \gamma/2, \quad 0 < \delta < \delta_3, \ 0 < \varepsilon < \varepsilon_2.$$

Hence, for $0 < \delta < \delta_3, 0 < \varepsilon < \varepsilon_2$ and $t \geq t_2'$,

$$u_i(t) \geq \underline{u}_i(t) > u_i^*(t) - \gamma/2 > \underline{x}_i^*(t + t_1) - \gamma.$$

Therefore, letting $t_2 = t_2' + t_1$, we obtain

$$x_i(t) \geq \underline{x}_i^*(t) - \gamma, \quad t \geq t_2 > 0, \ 1 \leq i \leq n. \tag{2.6}$$

Similarly, for arbitrary $0 < \nu < \inf_{t \in R} \underline{y}_j^*(t)$, there exists $t_3 > t_2$, such that

$$y_j(t) \geq \underline{y}_j^*(t) - \nu, \quad t \geq t_3 > 0, \ 1 \leq j \leq m. \tag{2.7}$$

By the above proof, we know that system (1.1) is persistent, which completes our proof.

3. Global attractivity

We first give an assumption and then introduce our main theorem.

(H4)   There exist continuously differentiable bounded functions $\beta_i : (0, \infty) \to (0, \infty)$, $\alpha_j : (0, \infty) \to (0, \infty)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and a positive T-periodic function $b(t) : R_+ \to R_+$, such that

$$
\frac{\partial \beta_i(x_i) F_i(t, x, y)}{\partial x_i} + \left[\sum_{l \neq i}^{n} \beta_l(x_l) \left| \frac{\partial F_l(t, x, y)}{\partial x_i} \right| + \sum_{s=1}^{m} \alpha_s(y_s) \left| \frac{\partial G_s(t, x, y)}{\partial x_i} \right| \right]
$$
$$
\leq -b(t),
$$
$$
\frac{\partial \alpha_j(y_j) G_j(t, x, y)}{\partial y_j} + \left[\sum_{l=1}^{n} \beta_l(x_l) \left| \frac{\partial F_l(t, x, y)}{\partial y_j} \right| + \sum_{s \neq j}^{m} \alpha_s(y_s) \left| \frac{\partial G_s(t, x, y)}{\partial y_j} \right| \right]
$$
$$
\leq -b(t)
$$

for all $t \in R_+$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$, $x_i \in [\eta, M]$, $y_j \in [\eta, M]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

THEOREM 3.1. *If* (H4) *and the conditions of Theorem* 2.5 *hold, then* (1.1) *admits a globally attractive positive T-periodic solution $u^*(t) \in \mathrm{int}(R_+^{n+m})$, that is, for any $u_0 \in \mathrm{int}(R_+^{n+m})$, the solution $u(t)$ of* (1.1) *with $u(0) = u_0$ satisfies $\lim_{t \to \infty}(u(t) - u^*(t)) = 0$.*

PROOF.   By Theorem 2.5, we know that system (1.1) is persistent, that is, there exist $M > \eta > 0$, for any $u_0 = (x_{10}, \ldots, x_{n0}, y_{10}, \ldots, y_{m0})^T \in \mathrm{int}(R_+^{n+m})$, there exists $T_0 = T(u_0) > 0$, such that the positive solution $u(t, u_0)$ of system (1.1) satisfies

$$
\eta < x_i(t, u_0) < M, \quad \eta < y_j(t, u_0) < M,
$$

for $t > T_0$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. By [11], we know that system (1.1) admits a positive T-periodic solution $u^*(t) = (x_1^*(t), \ldots, x_n^*(t), y_1^*(t), \ldots, y_m^*(t))^T$.
   Define

$$
V(t) = V(u(t + T_0), u^*(t + T_0))
$$
$$
= \sum_{i=1}^{n} \left| \int_{x_i(t+T_0)}^{x_i^*(t+T_0)} \frac{\beta_i(w)}{w} \, dw \right| + \sum_{j=1}^{m} \left| \int_{y_j(t+T_0)}^{y_j^*(t+T_0)} \frac{\alpha_j(w)}{w} \, dw \right|.
$$

For any $w \in [\eta, M]$, we have $\beta_i(w)/w > 0$ and $\alpha_j(w)/w > 0$. Hence there exist positive constants D_1 and D_2, such that

$$
D_1 \left[\sum_{i=1}^{n} \left| x_i^*(t + T_0) - x_i(t + T_0) \right| + \sum_{j=1}^{m} \left| y_j^*(t + T_0) - y_j(t + T_0) \right| \right]
$$
$$
\leq V(t) \leq D_2 \left[\sum_{i=1}^{n} \left| x_i^*(t + T_0) - x_i(t + T_0) \right| + \sum_{i=1}^{m} \left| y_j^*(t + T_0) - y_j(t + T_0) \right| \right].
$$

Using the notation $t_+ = t + T_0$ and letting $\eta(x_i^*, x_i) = sx_i^*(t_+) + (1-s)x_i(t_+)$, we have

$$
\begin{aligned}
\frac{d}{dt} &\int_{x_i(t_+)}^{x_i^*(t_+)} \frac{\beta_i(w)}{w} \, dw \\
&= \beta_i(x_i^*(t_+)) F_i(t_+, u^*(t_+)) - \beta_i(x_i(t_+)) F_i(t_+, u(t_+)) \\
&= \int_0^1 \frac{d}{ds} [\beta_i(\eta(x_i^*, x_i)) F_i(t_+, \eta(u^*, u))] \, ds \\
&= \int_0^1 \left[\sum_{j=1}^n \frac{\partial(\beta_i(\eta(x_i^*, x_i)) F_i(t_+, \eta(u^*, u)))}{\partial x_j} (x_j^*(t_+) - x_j(t_+)) \right. \\
&\quad \left. + \sum_{j=1}^m \frac{\partial(\beta_i(\eta(x_i^*, x_i)) F_i(t_+, \eta(u^*, u)))}{\partial y_j} (y_j^*(t_+) - y_j(t_+)) \right] ds, \\
\frac{d}{dt} &\int_{y_j(t_+)}^{y_j^*(t_+)} \frac{\alpha_j(w)}{w} \, dw \\
&= \alpha_j(y_j^*(t_+)) G_j(t_+, u^*(t_+)) - \alpha_j(y_j(t_+)) G_j(t_+, u(t_+)) \\
&= \int_0^1 \frac{d}{ds} [\alpha_j(\eta(y_j^*, y_j)) G_j(t_+, \eta(u^*, u))] \, ds \\
&= \int_0^1 \left[\sum_{r=1}^n \frac{\partial(\alpha_j(\eta(y_j^*, y_j)) G_j(t_+, \eta(u^*, u)))}{\partial x_r} (x_r^*(t_+) - x_r(t_+)) \right. \\
&\quad \left. + \sum_{r=1}^m \frac{\partial(\alpha_j(\eta(y_j^*, y_j)) G_j(t_+, \eta(u^*, u)))}{\partial y_r} (y_r^*(t_+) - y_r(t_+)) \right] ds.
\end{aligned}
$$

Therefore, calculating the upper right derivative $D^+ V(t)$ of $V(t)$ along the solutions of (1.1), we get

$$
\begin{aligned}
D^+ V(t) \leq \int_0^1 &\left\{ \sum_{i=1}^n \left[\frac{\partial \beta_i(\eta(x_i^*, x_i)) F_i(t_+, \eta(u^*, u))}{\partial x_i} \right. \right. \\
&+ \left(\sum_{j \neq i}^n \beta_j(\eta(x_j^*, x_j)) \left| \frac{\partial F_j(t_+, \eta(u^*, u))}{\partial x_i} \right| \right. \\
&+ \left. \left. \left. \sum_{j=1}^m \alpha_j(\eta(y_j^*, y_j)) \left| \frac{\partial G_j(t_+, \eta(u^*, u))}{\partial x_i} \right| \right) \right] |x_i^*(t_+) - x_i(t_+)| \right\} ds \\
&+ \int_0^1 \left\{ \sum_{j=1}^m \left[\frac{\partial \alpha_j(\eta(y_j^*, y_j)) G_j(t_+, \eta(u^*, u))}{\partial y_j} \right. \right.
\end{aligned}
$$

$$+ \left(\sum_{r=1}^{n} \beta_r(\eta(x_r^*, x_r)) \left| \frac{\partial F_r(t_+, \eta(u^*, u))}{\partial y_j} \right| \right.$$

$$+ \left. \sum_{\substack{r=j}}^{m} \alpha_r(\eta(y_r^*, y_r)) \left| \frac{\partial G_r(t_+, \eta(u^*, u))}{\partial y_j} \right| \right) \Bigg] |y_j^*(t_+) - y_j(t_+)| \Bigg\} ds$$

$$\leq -b(t_+) \left(\sum_{i=1}^{n} |x_i^*(t_+) - x_i(t_+)| + \sum_{j=1}^{m} |y_j^*(t_+) - y_j(t_+)| \right) \leq -\frac{b(t_+)}{D_2} V(t).$$

Therefore, by the comparison theorem, we get

$$V(t) \leq V(u^*(T_0), u(T_0)) \, e^{-\int_0^t (b(s+T_0)/D_2) \, ds},$$

since $b(t)$ is a positive periodic function. This implies that $\lim_{t\to\infty} V(t) = 0$. Accordingly, we have

$$\lim_{t\to\infty} |x_i(t) - x_i^*(t)| = 0, \quad i = 1, \dots, n,$$

and

$$\lim_{t\to\infty} |y_j(t) - y_j^*(t)| = 0, \quad j = 1, \dots, m.$$

This completes the proof of Theorem 3.1.

4. Applications

In this section, some suitable applications are given to illustrate that the conditions of Theorem 3.1 are feasible.

We first consider the system

$$\begin{cases} \dot{x}_i(t) = x_i(t) \left[b_i(t) - \sum_{s=1}^{n} a_{is}(t)x_s(t) - \sum_{s=1}^{m} c_{is}(t)y_s(t) \right], \\ \dot{y}_j(t) = y_j(t) \left[-r_j(t) + \sum_{s=1}^{n} d_{js}(t)x_s(t) - \sum_{s=1}^{m} e_{js}(t)y_s(t) \right], \end{cases} \quad (4.1)$$

where $b_i(t)$, $r_j(t)$, $a_{ij}(t)$, $c_{ij}(t)$, $d_{ij}(t)$, $e_{ij}(t)$ are positive continuous T-periodic functions with respect to t. There exists a positive constant $\eta_0 > 0$, such that $\min_{t\in R^+}\{b_i(t), a_{ii}(t), e_{jj}(t)\} > \eta_0$.

Letting $\beta_i(\cdot) = 1$, $\alpha_j(\cdot) = 1$, $i = 1, \dots, n$, $j = 1, \dots, m$ in the proof of Theorem 3.1, we obtain the following theorem.

THEOREM 4.1. *If system (4.1) is persistent, and there exists a continuous positive periodic function $b(t)$, with $\int_0^T b(s) \, ds > 0$ such that*

$$\begin{cases} a_{ii}(t) - \left[\sum_{s\neq i}^{n} a_{si}(t) + \sum_{k=1}^{m} d_{si}(t) \right] > b(t), \\ e_{jj}(t) - \left[\sum_{s=1}^{n} c_{sj}(t) + \sum_{s\neq j}^{m} e_{sj}(t) \right] > b(t), \end{cases}$$

for all $i = 1, \dots, n, j = 1, \dots, m$, *then there exists a unique strictly positive periodic solution in the system* (4.1) *which is globally attracting.*

In the following, we consider a simple system

$$\begin{cases} \dot{x}_1(t) = x_1(t)(r_1 - a_{11}x_1(t) - a_{12}x_2(t)) \\ \dot{x}_i(t) = x_i(t)(-r_i + a_{i,i-1}x_{i-1}(t) - a_{ii}x_i(t) - a_{i,i+1}x_{i+1}(t)), \quad 2 \le i \le n-1, \quad (4.2) \\ \dot{x}_n(t) = x_n(t)(-r_n + a_{n,n-1}x_{n-1}(t) - a_{nn}x_n(t)), \end{cases}$$

where $r_i, a_{ij}, 1 \le i \le n, 1 \le j \le n$, are all positive constants.

Define $D_1 = r_1$, $\Delta_i = \prod_{j=1}^{j=i} a_{jj}$,

$$D_i = \begin{vmatrix} a_{11} & 0 & \cdots & 0 & r_1 \\ -a_{21} & a_{22} & \cdots & 0 & -r_2 \\ 0 & -a_{32} & \ddots & 0 & -r_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{i,i-1} & -r_i \end{vmatrix}, \quad i = 2, \dots, n, \qquad (4.3)$$

and also define $E_1 = r_1 - a_{12}D_2/\Delta_2$, $a_{n,n+1} = 0$ and, for $i = 2, \dots, n$,

$$E_i = \begin{vmatrix} a_{11} & 0 & \cdots & 0 & r_1 - a_{12}D_2/\Delta_2 \\ -a_{21} & a_{22} & \cdots & 0 & -r_2 - a_{23}D_3/\Delta_3 \\ 0 & -a_{32} & \ddots & 0 & -r_3 - a_{34}D_4/\Delta_4 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{i,i-1} & -r_i - a_{i,i+1}D_{i+1}/\Delta_{i+1} \end{vmatrix}. \qquad (4.4)$$

We then have the following theorem

THEOREM 4.2. *If* $D_i > 0$ *and* $E_i > 0$, *for* $i = 1, \dots, n$, *and*

$$a_{11} > a_{21}, \quad a_{nn} > a_{n-1,n}, \qquad (4.5)$$

$$a_{ii} > a_{i-1,i} + a_{i+1,i}, \quad \text{for } i = 2, \dots, n-1, \qquad (4.6)$$

then the positive steady state of system (4.2) *is globally attracting in* $\text{int}(R_+^n)$.

PROOF. In fact

$$\bar{x}_i^* = \frac{D_i}{\Delta_i} > 0, \quad \text{for } i = 1, \dots, n,$$

$$\underline{x}_i^* = \frac{E_i}{\Delta_i} > 0, \quad \text{for } i = 1, \dots, n-1.$$

By Theorem 2.5, we know that system (4.2) is persistent. Finally, by (4.5) and (4.6), we know that assumption (H4) of Theorem 3.1 holds, which completes the proof.

In particular, we consider

$$\begin{cases} \dot{x}(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y(t)) \\ \dot{y}(t) = y(t)(-r_2 + a_{21}x(t) - a_{22}y(t)), \end{cases} \tag{4.7}$$

where $r_i, a_{ij}, i, j = 1, 2$ are all positive constants.

COROLLARY 4.3 ([9]). If $a_{11}a_{22} - a_{12}a_{21} > 0$, then the positive steady state of system (4.7) is globally attracting.

PROOF. In fact, $r_1 a_{21} - r_2 a_{11} > 0$ is a necessary condition for system (4.7) if there exists a positive steady state in the system since

$$\bar{x}^* = \frac{r_1}{a_{11}} > 0, \qquad \underline{x}^* = \frac{(a_{11}a_{22} - a_{12}a_{21})r_1 + a_{11}a_{12}r_2}{a_{11}^2 a_{22}} > 0,$$

$$\bar{y}^* = \frac{r_1 a_{21} - r_2 a_{11}}{a_{11}a_{22}} > 0, \qquad \underline{y}^* = \frac{(a_{11}a_{22} - a_{12}a_{21})(r_1 a_{21} - r_2 a_{11})}{a_{11}^2 a_{22}^2} > 0.$$

Hence, by Theorem 2.5, we know that system (4.7) is persistent.

Choose constants c_1 and d_1 satisfying

$$\frac{a_{22}}{a_{12}} > \frac{d_1}{c_1} > \frac{a_{21}}{a_{11}}.$$

Let $\alpha(\cdot) = c_1, \beta(\cdot) = d_1$ all be constant, then we have

$$\begin{cases} a_{11}d_1 - a_{21}c_1 > 0 \\ a_{22}c_1 - a_{12}d_1 > 0. \end{cases}$$

Therefore assumption (H4) of Theorem 3.1 holds. This completes the proof of Corollary 4.3.

Finally, we consider a Michaelis-Menten type functional response predator-prey system

$$\begin{cases} \dot{S}(t) = S(t)\left(1 - \dfrac{S(t)}{K} - \dfrac{mx(t)}{a + S(t)}\right), \\ \dot{x}(t) = x(t)\left(\dfrac{mS(t)}{a + S(t)} - d - x(t)\right), \end{cases} \tag{4.8}$$

where a, d, K and m are all positive constants. We suppose that $K < a$ and $d < m$.

Letting $d\tau = ad(a + S)\, dt, \bar{x} = x(a - K)/K, \bar{S} = S(a - K)/adK$, replace $d\tau$, $\bar{S}$ and $\bar{x}$ by dt, S and x, respectively. Then we have

$$\begin{cases} \dot{S}(t) = S(t)(r_1 - S(t) - a_1 S^2(t) - m_1 x(t)), \\ \dot{x}(t) = x(t)(-1 + a_2 S(t) - x(t)S(t) - m_2 x(t)), \end{cases} \tag{4.9}$$

where $r_1 = 1/d$, $a_1 = adK/(a - K)^2$, $a_2 = K(m - d)/(a - K)$, $m_1 = m(a - K)/adK$ and $m_2 = (a - K)/dK$.

Letting $\bar{S}^* = 2r_1/(1 + \sqrt{4a_1 r_1 + 1})$, $\bar{x}^* = (a_2 \bar{S}^* - 1)/(m_2 + \bar{S}^*)$, $r_2 = r_1 - m_1 \bar{x}^*$, $\underline{S}^* = 2r_2/(1 + \sqrt{4a_1 r_2 + 1})$ and $\underline{x}^* = (a_2 \underline{S}^* - 1)/(m_2 + \underline{S}^*)$, by Theorems 2.5 and 3.1, we obtain the following theorem.

THEOREM 4.4. *If there is only one positive steady state in system* (4.9), $\underline{S}^* > 0$, $\bar{x}^* > 0$, $\underline{x}^* > 0$ *and* $a_2 m_1 < m_2$, *then the positive steady state is globally attracting.*

If $r_1 = 15$, $a_1 = 1/4$, $a_2 = 1/5$, $m_1 = 2$ and $m_2 = 4$, by simple computations we know that the equations

$$\begin{cases} 15 - S - S^2/4 - 2x = 0, \\ -1 + S/5 - Sx - 4x = 0 \end{cases}$$

have only one positive solution. Moreover, we obtain

$$\bar{S}^* = 6, \quad \bar{x}^* = \frac{1}{50}, \quad \underline{S}^* = \frac{748}{25 + 5\sqrt{399}} > \frac{29}{5}, \quad \underline{x}^* > \frac{2}{125}.$$

The conditions of Theorem 4.4 then hold.

Acknowledgement

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