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A NOTE ON THE NUMBER OF SOLUTIONS OF TERNARY PURELY EXPONENTIAL DIOPHANTINE EQUATIONS

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Abstract

Let a, b, c be fixed coprime positive integers with $\min\{a, b, c\} > 1$. We discuss the conjecture that the equation $a^x + b^y = c^z$ has at most one positive integer solution (x, y, z) with $\min\{x, y, z\} > 1$, which is far from solved. For any odd positive integer r with $r > 1$, let $f(r) = (-1)^{(r-1)/2}$ and $2^{g(r)} \,\|\, r - (-1)^{(r-1)/2}$. We prove that if one of the following conditions is satisfied, then the conjecture is true: (i) $c = 2$; (ii) a, b and c are distinct primes; (iii) $a = 2$ and either $f(b) \neq f(c)$, or $f(b) = f(c)$ and $g(b) \neq g(c)$.

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1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let a, b, c be fixed coprime positive integers with $\min\{a, b, c\} > 1$. We assume without loss of generality that a, b and c are not perfect powers. The purely exponential Diophantine equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N} \tag{1.1}$$

has been studied deeply (see [17] for a survey of the results). In 1933, Mahler [18] used his p-adic analogue of the Diophantine approximation method of Thue–Siegel to prove that (1.1) has only finitely many solutions (x, y, z), but his method is ineffective. Let $N(a, b, c)$ denote the number of solutions (x, y, z) of (1.1). An effective upper bound for $N(a, b, c)$ was first given by Gel'fond [7], using his new method in transcendental number theory. Subsequently, as a straightforward consequence of an upper bound for the number of solutions of binary S-unit equations due to Beukers and Schlickewei [2], the bound was improved to $N(a, b, c) \leq 2^{36}$. More accurate upper bounds for $N(a, b, c)$ have been obtained under certain conditions:

(i) if $2 \nmid c$, then $N(a, b, c) \leq 2$ (Scott and Styer [21]);
(ii) if $\max\{a, b, c\} > 5 \times 10^{27}$, then $N(a, b, c) \leq 3$ (Hu and Le [10]);

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(iii) if $2 \mid c$ and $\max\{a, b, c\} > 10^{62}$, then $N(a, b, c) \le 2$ (Hu and Le [11]);
(iv) if $2 \mid c$ and $a < b < 10^{62}$, then $N(a, b, c) \le 2$, except for $N(3, 5, 2) = 3$ (Miyazaki and Pink [19]).

Nevertheless, deeper problems about the number of solutions of (1.1) remain unresolved. Let $N'(a, b, c)$ denote the number of solutions (x, y, z) of (1.1) with $\min\{x, y, z\} > 1$. In this paper, we will discuss the following conjecture.

Conjecture 1.1 (Terai–Jeśmanowicz conjecture). For any triple (a, b, c) of positive integers with $\min\{a, b, c\} > 1$, we have $N'(a, b, c) \le 1$.

This conjecture contains the famous Jeśmanowicz conjecture concerning Pythagorean triples (see [12]) and its original form was put forward by Terai [22]. It is related to the generalised Fermat conjecture (see Problems B19 and D2 of [8]) and seems very difficult. In 2015, Hu and Le [9] gave a general criterion to judge whether Conjecture 1.1 is true, but this criterion is difficult to apply because it involves some unsolved problems such as the existence of Wieferich primes (see Problem A3 of [8]).

We now discuss Conjecture 1.1 for $2 \in \{a, b, c\}$ using a different approach from the one in [9]. First, by means of the results of [14, 20], we can directly prove the following result.

Theorem 1.2. *If $c = 2$ or a, b and c are distinct primes, then Conjecture 1.1 is true.*

For any odd positive integer r with $r > 1$, we define $f(r) = (-1)^{(r-1)/2}$ and $2^{g(r)} \,\|\, r - (-1)^{(r-1)/2}$. Obviously, $f(r) \in \{-1, 1\}$ and $g(r)$ is a positive integer with $g(r) \ge 2$. Using a combination of various methods including Baker's method and known results on exponential Diophantine equations, we prove the following result.

Theorem 1.3. *If $a = 2$ and either $f(b) \ne f(c)$, or $f(b) = f(c)$ and $g(b) \ne g(c)$, then Conjecture 1.1 is true.*

2. Preliminaries

For any positive integer s, let $\mathrm{ord}_2(s)$ denote the order of 2 in s, namely, $2^{\mathrm{ord}_2(s)} \,\|\, s$.

Lemma 2.1. *For any positive integers r and s such that $r > 1$, $2 \nmid r$ and $2 \mid s$, we have*

$$\mathrm{ord}_2(r^s - 1) = g(r) + \mathrm{ord}_2(s). \tag{2.1}$$

Proof. Since $2 \mid s$ and $r = 2^{g(r)} r_1 + f(r)$, where r_1 is an odd positive integer,

$$\begin{aligned} r^s - 1 &= (2^{g(r)} r_1 + f(r))^s - 1 \\ &= ((f(r))^s - 1) + 2^{g(r)} r_1 s (f(r))^{s-1} + \sum_{i=2}^{s} \binom{s}{i} (2^{g(r)} r_1)^i (f(r))^{s-i} \\ &= 2^{g(r)} r_1 s f(r) + \sum_{i=2}^{s} \binom{s}{i} (2^{g(r)} r_1 f(r))^i. \end{aligned} \tag{2.2}$$

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Further, since $g(r) \geq 2$ and $2 \nmid r_1 f(r)$, we see that $2^{g(r)+\mathrm{ord}_2(s)} \parallel 2^{g(r)} r_1 s f(r)$ and

$$\binom{s}{i}(2^{g(r)} r_1 f(r))^i \equiv 2^{g(r)} s \binom{s-1}{i-1} \frac{2^{g(r)(i-1)}}{i} (r_1 f(r))^i \equiv 0 \pmod{2^{g(r)+\mathrm{ord}_2(s)+1}}, \quad i \geq 2.$$

Therefore, by (2.2), we obtain (2.1). □

For any real number α, let $\log \alpha$ denote the natural logarithm of α.

LEMMA 2.2. *Let α_1, α_2, β_1, β_2 be positive integers with* $\min\{\alpha_1, \alpha_2\} > 1$. *Further, let* $\Lambda = \beta_1 \log \alpha_1 - \beta_2 \log \alpha_2$. *If $\Lambda \neq 0$, then*

$$\log |\Lambda| > -25.2(\log \alpha_1)(\log \alpha_2)\Big(\max\Big\{10, 0.38 + \log\Big(\frac{\beta_1}{\log \alpha_2} + \frac{\beta_2}{\log \alpha_1}\Big)\Big\}\Big)^2.$$

PROOF. This is the special case of [13, Corollary 2] for $m = 10$. □

LEMMA 2.3. *Let α_1, α_2 be odd integers with* $\min\{|\alpha_1|, |\alpha_2|\} \geq 3$ *and let β_1, β_2 be positive integers. Further, let* $\Lambda' = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}$. *If $\Lambda' \neq 0$ and* $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod 4$*, then*

$$\begin{aligned} \mathrm{ord}_2(|\Lambda'|) &< 19.55(\log |\alpha_1|)(\log |\alpha_2|) \\ &\quad \times \Big(\max\Big\{12 \log 2, 0.4 + \log(2 \log 2) + \log\Big(\frac{\beta_1}{\log |\alpha_2|} + \frac{\beta_2}{\log |\alpha_1|}\Big)\Big\}\Big)^2. \end{aligned}$$

PROOF. This is the special case of [4, Theorem 2] for $p = 2$, $y_1 = y_2 = 1$, $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod 4$, $g = 1$ and $E = 2$. □

LEMMA 2.4 [6, 16]. *The equation*

$$X^2 + 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \ \gcd(X, Y) = 1, \ n > 2$$

has only the solutions $(X, Y, m, n) = (5, 3, 1, 3), (7, 3, 5, 4)$ *and* $(11, 5, 2, 3)$.

LEMMA 2.5 [1, Theorem 8.4]. *The equation*

$$X^2 - 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \ \gcd(X, Y) = 1, \ m > 1, n > 2$$

has only the solution $(X, Y, m, n) = (71, 17, 7, 3)$.

LEMMA 2.6 [5]. *If $a = 2$ and b and c are distinct odd primes with* $\max\{b, c\} < 100$*, then $N'(a, b, c) \leq 1$.*

LEMMA 2.7 [21]. *If $2 \nmid c$, then $N(a, b, c) \leq 2$.*

LEMMA 2.8 [20, Theorem 6]. *$N'(a, b, 2) \leq 1$.*

According to Theorems 1, 2 and 3 of [14] and the proof of them, we can obtain the following lemma.

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LEMMA 2.9. *Let p and q be fixed odd primes with $p \neq q$.*

(i) *The equation*

$$2^x + p^y = q^z, \quad x, y, z \in \mathbb{N} \tag{2.3}$$

has at most one solution (x, y, z) with $2 \mid y$ and this solution has $z = 1$, except for $(p, q, x, y, z) = (3, 5, 4, 2, 2)$, $(5, 3, 1, 2, 3)$, $(7, 3, 5, 2, 4)$ and $(11, 5, 2, 2, 3)$.

(ii) *Equation (2.3) has at most one solution (x, y, z) with $2 \mid x$ and $2 \nmid y$.*

(iii) *Equation (2.3) has at most one solution (x, y, z) with $2 \nmid xy$.*

REMARK 2.10. The reference [14] is written in Chinese and the proof of the theorems mentioned is rather complicated. In the present case, this lemma can be easily obtained using the tools in [3, 15, 24].

3. Further lemmas on (1.1) for $a = 2$

Let $a = 2$ and let b and c be fixed coprime odd positive integers with $\min\{b, c\} \geq 3$. In this section, we give some results on the solutions (x, y, z) of the equation

$$2^x + b^y = c^z, \quad x, y, z \in \mathbb{N}, \ \min\{x, y, z\} > 1. \tag{3.1}$$

LEMMA 3.1.

(i) *If $f(b) \neq f(c)$, then (3.1) has no solutions (x, y, z) with $2 \nmid yz$.*

(ii) *If $f(b) = f(c)$ and $g(b) \neq g(c)$, then all the solutions (x, y, z) of (3.1) with $2 \nmid yz$ satisfy $x = \min\{g(b), g(c)\}$.*

PROOF. Let (x, y, z) be a solution of (3.1) with $2 \nmid yz$.

(i) Note that $f(b) \neq f(c)$ is equivalent to $b \not\equiv c \pmod 4$. Since $2 \nmid yz$, by (3.1), we have $2^x = c^z - b^y \equiv 2 \pmod 4$. This means $x = 1$, which contradicts $x > 1$. Therefore, we obtain the conclusion (i).

(ii) Since $2 \nmid bc$, we may write

$$b = 2^{g(b)} b_1 + f(b), \quad c = 2^{g(c)} c_1 + f(c), \quad b_1, c_1 \in \mathbb{N}, \ 2 \nmid b_1 c_1. \tag{3.2}$$

Assume that $f(b) = f(c)$. Since $2 \nmid yz$ and $f(b) = f(c) \in \{-1, 1\}$, we see from (3.1) and (3.2) that

$$\begin{aligned} 2^x &= (2^{g(c)} c_1 + f(c))^z - (2^{g(b)} b_1 + f(b))^y \\ &= \sum_{i=1}^{\max\{y,z\}} \left(\binom{z}{i} (2^{g(c)} c_1)^i (f(c))^{z-i} - \binom{y}{i} (2^{g(b)} b_1)^i (f(b))^{y-i} \right) \\ &= (2^{g(c)} c_1 z - 2^{g(b)} b_1 y) + \sum_{i=2}^{\max\{y,z\}} \left(\binom{z}{i} (2^{g(c)} c_1)^i - \binom{y}{i} (2^{g(b)} b_1)^i \right) (f(c))^{i+1}. \end{aligned} \tag{3.3}$$

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When $g(b) \neq g(c)$, since $2 \nmid b_1c_1yz$, we have

$$2^{\min\{g(b),g(c)\}} \parallel 2^{g(c)}c_1z - 2^{g(b)}b_1y$$

and

$$\sum_{i=2}^{\max\{y,z\}} \left(\binom{z}{i}(2^{g(c)}c_1)^i - \binom{y}{i}(2^{g(b)}b_1)^i\right)(f(c))^{i+1} \equiv 0 \pmod{2^{\min\{g(b),g(c)\}+1}}.$$

Since $\min\{g(b), g(c)\} \geq 2$, we see from (3.3) that $x = \min\{g(b), g(c)\}$ and the conclusion (ii) is obtained. □

LEMMA 3.2. *All solutions* (x, y, z) *of* (3.1) *with* $\max\{b, c\} > 100$ *satisfy*

$$x < 16460(\log b)(\log c),\ \ y < 14261 \log c,\ \ z < 14261 \log b \quad \text{if } 2^x < c^{0.8z}, \tag{3.4}$$

and

$$x < 1784(\log b)(\log c),\ \ y < 1236 \log c,\ \ z < 1236 \log b \quad \text{if } 2^x > c^{0.8z}. \tag{3.5}$$

PROOF. We first consider the case $2^x < c^{0.8z}$. By (3.1) with $\max\{b, c\} > 100$, we have $b^y > 2^x$ and

$$\begin{aligned} z\log c = \log(b^y + 2^x) &= y\log b + \log\left(1 + \frac{2^x}{b^y}\right) \\ &< y\log b + \frac{2^x}{b^y} = y\log b + \frac{2^{x+1}}{2b^y} < y\log b + \frac{2c^{0.8z}}{c^z} = y\log b + \frac{2}{c^{z/5}}. \end{aligned} \tag{3.6}$$

Let $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (c, b, z, y)$ and $\Lambda = z\log c - y\log b$. By (3.6), $0 < \Lambda < 2/c^{z/5}$ and

$$\log 2 - \log|\Lambda| > \frac{z}{5}\log c. \tag{3.7}$$

Since $\min\{b, c\} \geq 3$, using Lemma 2.2, we have

$$\log|\Lambda| > -25.2(\log c)(\log b)\left(\max\left\{10, 0.38 + \log\left(\frac{z}{\log b} + \frac{y}{\log c}\right)\right\}\right)^2. \tag{3.8}$$

When $10 \geq 0.38 + \log(z/\log b + y/\log c)$, by (3.7) and (3.8), we have

$$\log 2 + 2520(\log c)(\log b) > \frac{z}{5}\log c,$$

which gives

$$z < 12601 \log b. \tag{3.9}$$

When $10 < 0.38 + \log(z/\log b + y/\log c)$, by (3.6), (3.7) and (3.8), we have

$$\begin{aligned} &\log 2 + 25.2(\log c)(\log b)\left(0.38 + \log\left(\frac{2z}{\log b}\right)\right)^2 \\ &\quad > \log 2 + 25.2(\log c)(\log b)\left(0.38 + \log\left(\frac{z}{\log b} + \frac{y}{\log c}\right)\right)^2 > \frac{z}{5}\log c, \end{aligned}$$

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which gives

$$\frac{5\log 2}{(\log b)(\log c)} + 126\Big(0.38 + \log 2 + \log\Big(\frac{z}{\log b}\Big)\Big)^2 > \frac{z}{\log b}. \tag{3.10}$$

Since $(\log b)(\log c) > (\log 3)(\log 100)$, we can calculate from (3.10) that z satisfies

$$z < 14261\log b. \tag{3.11}$$

Therefore, since $y\log b < z\log c$ and $x\log 2 < (4z\log c)/5$, by (3.9) and (3.11), we obtain (3.4).

Next, we consider the case $2^x > c^{0.8z}$. Let $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (cf(c), bf(b), z, y)$ and $\Lambda' = (cf(c))^z - (bf(b))^y$. Since $x \geq 2$, by (3.1) and (3.2), we have $|\Lambda'| = 2^x$, that is, $\mathrm{ord}_2(|\Lambda'|) = x$. Since $\min\{|cf(c)|, |b(f(b)|\} = \min\{b, c\} \geq 3$ and $cf(c) \equiv bf(b) \equiv 1 \pmod 4$, by Lemma 2.3,

$$x < 19.55(\log c)(\log b)\Big(\max\Big\{12\log 2, 0.4 + \log(2\log 2) + \log\Big(\frac{z}{\log b} + \frac{y}{\log c}\Big)\Big\}\Big)^2. \tag{3.12}$$

However, since $2^x > c^{0.8z}$,

$$x > \frac{0.8z\log c}{\log 2}. \tag{3.13}$$

The combination of (3.12) and (3.13) yields

$$z < 16.94(\log b)\Big(\max\Big\{12\log 2, 0.4 + \log(2\log 2) + \log\Big(\frac{z}{\log b} + \frac{y}{\log c}\Big)\Big\}\Big)^2. \tag{3.14}$$

When $12\log 2 \geq 0.4 + \log(2\log 2) + \log(z/\log b + y/\log c)$, by (3.14), we get $z < 1172\log b$. When $12\log 2 < 0.4 + \log(2\log 2) + \log(z/\log b + y/\log c)$, by (3.6) and (3.14), we have

$$\begin{aligned}\frac{z}{\log b} &< 16.94\Big(0.4 + \log(2\log 2) + \log\Big(\frac{z}{\log b} + \frac{y}{\log c}\Big)\Big)^2\\ &< 16.94\Big(0.4 + \log(2\log 2) + \log\Big(\frac{2z}{\log b}\Big)\Big)^2,\end{aligned}$$

whence

$$z < 1236\log b. \tag{3.15}$$

Hence, if $2^x > c^{0.8z}$, then all the solutions (x, y, z) of (3.1) satisfy (3.15). Therefore, since $y\log b < z\log c$ and $x\log 2 < z\log c$, by (3.15), we obtain (3.5). □

LEMMA 3.3. *Assume that* $x = \min\{g(b), g(c)\} \leq 23$. *Then, all solutions* (x, y, z) *of* (3.1) *satisfy*

$$y < 2530\log c, \quad z < 2530\log b.$$

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PROOF. The proof proceeds along the same lines as that of the first half of Lemma 3.2. Since $2^x \le 2^{g(b)} \le b+1 < b^y$, we see from (3.1) that

$$0 < \Lambda := z\log c - y\log b = \log\Big(1+\frac{2^x}{b^y}\Big) < \frac{2^x}{b^y} < \frac{2^{x+1}}{c^z}, \tag{3.16}$$

which, together with the assumption $x \le 23$, implies that

$$24\log 2 - \log|\Lambda| > z\log c. \tag{3.17}$$

We know by Lemma 2.2 that (3.8) holds.

When $10 \ge 0.38 + \log(z/\log b + y/\log c)$, by (3.8) and (3.17),

$$24\log 2 + 2520(\log c)(\log b) > z\log c,$$

whence

$$z < 2530\log b. \tag{3.18}$$

When $10 < 0.38 + \log(z/\log b + y/\log c)$, by (3.8) and (3.17),

$$24\log 2 + 25.2(\log c)(\log b)\Big(0.38 + \log\Big(\frac{2z}{\log b}\Big)\Big)^2 > z\log c,$$

which together with $(\log b)(\log c) \ge (\log 3)(\log 5)$ yields

$$z < 1879\log b. \tag{3.19}$$

The inequalities in the lemma now follow from (3.18), (3.19) and $y\log b < z\log c$. □

LEMMA 3.4. *If (x,y,z) is a solution of (3.1) with $2 \nmid yz$, then* $\gcd(y,z) = 1$.

PROOF. Let $d = \gcd(y,z)$. Then we have

$$y = dY, \quad z = dZ, \quad Y, Z \in \mathbb{N}. \tag{3.20}$$

By (3.1) and (3.20),

$$2^x = c^z - b^y = (c^Z)^d - (b^Y)^d = (c^Z - b^Y)(c^{Z(d-1)} + \cdots + b^{Y(d-1)}). \tag{3.21}$$

Since $2 \nmid yz$, we have $2 \nmid d$. Since $2 \nmid bc$, if $d > 1$, then $c^{Z(d-1)} + \cdots + b^{Y(d-1)}$ is an odd positive integer greater than 1 contradicting (3.21). So we must have $d = 1$. □

By Lemmas 2.4 and 2.5, we can directly obtain the next two lemmas.

LEMMA 3.5. *Equation (3.1) has only the solution* $(b,c,x,y,z) = (11,5,2,2,3)$ *satisfying* $2 \mid y$ *and* $2 \nmid z$.

LEMMA 3.6. *Equation (3.1) has only the solution* $(b,c,x,y,z) = (17,71,7,3,2)$ *satisfying* $2 \nmid y$ *and* $2 \mid z$.

LEMMA 3.7. *Equation (3.1) has only the solutions*

$$(b,c,x,y,z) = (7,3,5,2,4) \tag{3.22}$$

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and

$$(b, c, x, y, z) = (2^t - 1, 2^t + 1, t + 2, 2, 2), \quad t \in \mathbb{N},\ t \geq 2 \tag{3.23}$$

satisfying $2 \mid y$ *and* $2 \mid z$.

PROOF. Let (x, y, z) be a solution of (3.1) with $2 \mid y$ and $2 \mid z$. Then, $2^x = c^z - b^y = (c^{z/2} + b^{y/2})(c^{z/2} - b^{y/2})$. Further, since $\gcd(c^{z/2} + b^{y/2}, c^{z/2} - b^{y/2}) = 2$,

$$c^{z/2} + b^{y/2} = 2^{x-1}, \quad c^{z/2} - b^{y/2} = 2,$$

which gives

$$c^{z/2} = 2^{x-2} + 1, \quad b^{y/2} = 2^{x-2} - 1. \tag{3.24}$$

Since $b > 1$, we see from the second equality of (3.24) that $y/2$ is odd. If $y/2 > 1$, then $2^{x-2} = b^{y/2} + 1 = (b + 1)(b^{y/2-1} - b^{y/2-2} + \cdots - b + 1)$, where $b^{y/2-1} - b^{y/2-2} + \cdots - b + 1$ is an odd positive integer greater than 1, a contradiction. So we have

$$\frac{y}{2} = 1, \quad b = 2^{x-2} - 1, \quad x \geq 4. \tag{3.25}$$

Similarly, if $z/2$ is odd, then from the first equality of (3.24),

$$\frac{z}{2} = 1, \quad c = 2^{x-2} + 1. \tag{3.26}$$

Hence, by (3.25) and (3.26), we obtain (3.23).

If $z/2$ is even, then $2^{x-2} = c^{z/2} - 1 = (c^{z/4} + 1)(c^{z/4} - 1)$, whence

$$c^{z/4} + 1 = 2^{x-3}, \quad c^{z/4} - 1 = 2. \tag{3.27}$$

Therefore, by (3.25) and (3.27), we obtain (3.22). The lemma is proved. □

Here and below, we assume that (x_1, y_1, z_1) and (x_2, y_2, z_2) are two distinct solutions of (3.1). We can further assume without loss of generality that $x_1 \leq x_2$.

LEMMA 3.8. *We have* $2 \nmid y_1 y_2 z_1 z_2$.

PROOF. By Lemmas 3.5, 3.6 and 3.7, if $2 \mid y_1 y_2 z_1 z_2$, then

$$(b, c) \in \{(11, 5), (17, 71), (7, 3), (2^t - 1, 2^t + 1)\}, \quad t \in \mathbb{N},\ t \geq 2. \tag{3.28}$$

However, by Lemma 2.6, we can eliminate the cases $(b, c) = (11, 5)$, $(17, 71)$ and $(7, 3)$. Alternatively, by Lemma 2.7, if $(a, b, c) = (2, 2^t - 1, 2^t + 1)$, then (1.1) has only two solutions $(x, y, z) = (1, 1, 1)$ and $(t + 2, 2, 2)$. Therefore, we can eliminate the cases $(b, c) = (2^t - 1, 2^t + 1)(t = 2, 3, \ldots)$ in (3.28). Thus, the lemma is proved. □

LEMMA 3.9. *We have* $y_1 z_2 \neq y_2 z_1$.

PROOF. By Lemmas 3.4 and 3.8, $\gcd(y_1, z_1) = \gcd(y_2, z_2) = 1$. Hence, if $y_1 z_2 = y_2 z_1$, then $y_1 \mid y_2$ and $y_2 \mid y_1$. This implies that $y_1 = y_2$, $z_1 = z_2$ and $(x_1, y_1, z_1) = (x_2, y_2, z_2)$, a contradiction. The lemma is proved. □

LEMMA 3.10. *If* $\max\{b, c\} > 8 \times 10^6$, *then* $2^{x_1} < c^{0.8z_1}$.

PROOF. By (3.1), $b^{y_1} \equiv c^{z_1} \pmod{2^{x_1}}$ and $b^{y_2} \equiv c^{z_2} \pmod{2^{x_2}}$. Since $x_1 \le x_2$, we get $b^{y_1y_2} \equiv c^{z_1y_2} \equiv c^{z_2y_1} \pmod{2^{x_1}}$ and $c^{z_1z_2} \equiv b^{y_1z_2} \equiv b^{y_2z_1} \pmod{2^{x_1}}$. Consequently, $b^{|y_1z_2-y_2z_1|} \equiv c^{|y_1z_2-y_2z_1|} \equiv 1 \pmod{2^{x_1}}$. Let $m = \max\{b, c\}$. We have

$$m^{|y_1z_2-y_2z_1|} \equiv 1 \pmod{2^{x_1}}. \tag{3.29}$$

By Lemmas 3.8 and 3.9, $|y_1z_2 - y_2z_1|$ is an even positive integer. Since $2 \nmid m$, by Lemma 2.1,

$$2^{g(m)+\mathrm{ord}_2|y_1z_2-y_2z_1|} \parallel m^{|y_1z_2-y_2z_1|} - 1. \tag{3.30}$$

Hence, by (3.29) and (3.30),

$$2^{g(m)+\mathrm{ord}_2|y_1z_2-y_2z_1|} \ge 2^{x_1}. \tag{3.31}$$

Further, since $2^{g(m)} \le m + 1$ and $2^{\mathrm{ord}_2|y_1z_2-y_2z_1|} \le |y_1z_2 - y_2z_1|$, by (3.31),

$$(m + 1)|y_1z_2 - y_2z_1| \ge 2^{x_1}. \tag{3.32}$$

Furthermore, by Lemma 3.2, if $2^{x_1} > c^{0.8z_1}$, then

$$|y_1z_2 - y_2z_1| < \max\{y_1z_2, y_2z_1\} < 14261 \times 1236\,(\log b)(\log c) < (4199 \log m)^2. \tag{3.33}$$

Hence, by (3.32) and (3.33),

$$(4199 \log m)^2(m + 1) > 2^{x_1}. \tag{3.34}$$

Recall that $c^{z_1} > b^{y_1}$, $2 \nmid y_1z_1$ and $\min\{y_1, z_1\} \ge 3$. We have $c^{z_1} \ge m^3$. Therefore, if $2^{x_1} > c^{0.8z_1}$, then from (3.34), we get

$$(4199 \log m)^2(m + 1) > m^{2.4},$$

whence $m < 8 \times 10^6$. Thus, if $m > 8 \times 10^6$, then $2^{x_1} < c^{0.8z_1}$. □

4. Proof of Theorem 1.2

Obviously, by Lemma 2.8, the theorem holds for $c = 2$. Moreover, in case a, b and c are distinct primes, we only have to consider $(a, b, c) = (2, p, q)$, where p and q are odd primes with $p \ne q$. Then, (1.1) can be rewritten as (2.3). Further, by Lemma 2.6, the theorem holds for $(p, q) = (3, 5),\ (5, 3),\ (7, 3)$ and $(11, 5)$.

We now assume that $N'(2, p, q) > 1$. It follows that (2.3) has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) with $\min\{x_j, y_j, z_j\} > 1$ for $j = 1, 2$. Since we have excluded the cases $(p, q) = (3, 5),\ (5, 3),\ (7, 3)$ and $(11, 5)$, by Lemma 2.9, we can assume without loss of generality that $2 \mid x_1$, $2 \nmid x_2$ and $2 \nmid y_1y_2$. Then

$$2^{x_1} + p^{y_1} = q^{z_1}, \quad 2^{x_2} + p^{y_2} = q^{z_2}. \tag{4.1}$$

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If $p \neq 3$, since $2 \nmid y_1 y_2$ and $p^{y_1} \equiv p^{y_2} \equiv p \pmod 3$, then from (4.1),

$$q^{z_1} \equiv 1 + p \pmod 3, \quad q^{z_2} \equiv 2 + p \pmod 3. \tag{4.2}$$

However, since $p \neq 3$, $1 + p \not\equiv 2 + p \pmod 3$ and $3 \mid (1+p)(2+p)$, (4.2) is false.

If $p = 3$, by (4.1), then we have $q^{z_1} \equiv 1 \pmod 3$ and $q^{z_2} \equiv 2 \pmod 3$, whence $q \equiv 2 \pmod 3$, $2 \mid z_1$ and $2 \nmid z_2$. Hence, by the first equality of (4.1),

$$q^{z_1/2} + 2^{x_1/2} = 3^{y_1}, \quad q^{z_1/2} - 2^{x_1/2} = 1. \tag{4.3}$$

Eliminating $q^{z_1/2}$ from (4.3), we have

$$2^{x_1/2+1} = 3^{y_1} - 1. \tag{4.4}$$

However, since $x_1/2 + 1 \geq 2$ and $2 \nmid y_1$, we get from (4.4) that $0 \equiv 2^{x_1/2+1} \equiv 3^{y_1} - 1 \equiv 3 - 1 \equiv 2 \pmod 4$, a contradiction. Thus, the theorem is proved.

5. Proof of Theorem 1.3

To show Theorem 1.3, we need the following lemma.

LEMMA 5.1. *If* $a = 2$, $f(b) = f(c)$, $g(b) \neq g(c)$ *and* $\max\{b, c\} > 8.4 \times 10^6$, *then Conjecture 1.1 is true.*

PROOF. Assume that $a = 2$, $f(b) = f(c)$, $g(b) \neq g(c)$, $\max\{b, c\} > 8.4 \times 10^6$ and $N'(a, b, c) > 1$. Then, by the conclusions of Lemma 3.8 and of Lemma 3.1(ii), (3.1) has two distinct solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) with

$$x_1 = x_2. \tag{5.1}$$

Since $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$, by (5.1), we may assume without loss of generality that

$$y_1 < y_2, \quad z_1 < z_2. \tag{5.2}$$

Let

$$\Lambda_j = z_j \log c - y_j \log b, \quad j = 1, 2. \tag{5.3}$$

We see from (3.6) and (5.3) that

$$0 < \Lambda_j = \log\left(1 + \frac{2^{x_j}}{b^{y_j}}\right), \quad j = 1, 2. \tag{5.4}$$

Further, by (5.1), (5.2) and (5.4),

$$0 < \Lambda_2 < \Lambda_1 = \log\left(1 + \frac{2^{x_1}}{b^{y_1}}\right). \tag{5.5}$$

Furthermore, by Lemma 3.10, we have $2^{x_1} < c^{0.8 z_1}$. Hence, by (3.6) and (5.5),

$$0 < \Lambda_2 < \Lambda_1 < \frac{2}{c^{z_1/5}}. \tag{5.6}$$

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However, by Lemmas 3.8 and 3.9, $|y_1z_2 - y_2z_1|$ is an even positive integer. So,

$$|y_1z_2 - y_2z_1| \geq 2. \tag{5.7}$$

By (5.3),

$$\begin{aligned}|y_1z_2 - y_2z_1| &= \frac{1}{\log c}|y_1(z_2 \log c) - y_2(z_1 \log c)|\\ &= \frac{1}{\log c}|y_1(y_2 \log b + \Lambda_2) - y_2(y_1 \log b + \Lambda_1)|\\ &= \frac{1}{\log c}|y_1\Lambda_2 - y_2\Lambda_1|. \end{aligned}\tag{5.8}$$

Since $y_1\Lambda_2 > 0$ and $y_2\Lambda_1 > 0$ by (5.4), we see from (5.8) that

$$|y_1z_2 - y_2z_1| < \max\Big\{\frac{y_1\Lambda_2}{\log c}, \frac{y_2\Lambda_1}{\log c}\Big\}. \tag{5.9}$$

Further, by (5.2) and Lemma 3.2,

$$\frac{y_1}{\log c} < \frac{y_2}{\log c} < 14261. \tag{5.10}$$

Hence, by (5.6), (5.7), (5.9) and (5.10),

$$2 \leq |y_1z_2 - y_2z_1| < \max\{14261\Lambda_2, 14261\Lambda_1\} = 14261\Lambda_1 < \frac{2 \times 14261}{c^{z_1/5}},$$

whence we obtain

$$c^{z_1/5} < 14261. \tag{5.11}$$

However, since $\max\{b, c\} > 8.4 \times 10^6$ and $c^{z_1} \geq (\max\{b, c\})^3$, (5.11) is false. Thus, we have $N'(a, b, c) \leq 1$ if $a = 2$, $f(b) = f(c)$, $g(b) \neq g(c)$ and $\max\{b, c\} > 8.4 \times 10^6$. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Obviously, by the conclusion of Lemma 3.1(i), Conjecture 1.1 is true if $a = 2$ and $f(b) \neq f(c)$. We now assume that $a = 2$, $f(b) = f(c)$, $g(b) \neq g(c)$ and $N'(a, b, c) > 1$. Moreover, by Lemma 5.1, we may assume that

$$\max\{b, c\} < 8.4 \times 10^6. \tag{5.12}$$

Then, by the conclusion of Lemma 3.1(ii), (3.1) has two distinct solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) with

$$x_1 = x_2 = \min\{g(b), g(c)\}, \tag{5.13}$$

and we may assume that $y_1 < y_2$ and $z_1 < z_2$. Further, by (5.12), we have $2^{g(b)} \leq b + 1 \leq 8.4 \times 10^6$, which together with (5.13) implies that

$$x_1 = x_2 = \min\{g(b), g(c)\} \leq 23. \tag{5.14}$$

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It follows from Lemma 3.3 that

$$\frac{y_1}{\log c} < \frac{y_2}{\log c} < 2530, \qquad \frac{z_1}{\log b} < \frac{z_2}{\log b} < 2530. \tag{5.15}$$

Furthermore, by (3.16) and (5.14),

$$0 < \Lambda_1 := z_1 \log c - y_1 \log b < \frac{2^{x_1+1}}{c^{z_1}} \leq \frac{2^{24}}{c^{z_1}}. \tag{5.16}$$

Thus, by the same argument as the proof of Lemma 5.1, we see from (5.15) and (5.16) that

$$2 \leq |y_1 z_2 - y_2 z_1| < 2530\Lambda_1 < \frac{2^{24} \times 2530}{c^{z_1}},$$

whence we obtain

$$b^{y_1} < c^{z_1} < 2^{23} \times 2530 < 2.123 \times 10^{10}.$$

Consequently, it only remains to show that (3.1) has no solutions if

$$3 \leq b \leq 2767,\ 3 \leq c \leq 2767,\ 2 \leq x_1 \leq 23,\ 3 \leq y_1 \leq 21,\ 3 \leq z_1 \leq 21$$

with $2 \nmid bcy_1z_1$ (by Lemma 3.8). We checked that the above claim is true by a simple program in PARI/GP [23] with precision 100. Indeed, the result showed that for any c, x_1, y_1, z_1 in the above ranges, the fractional part of $(c^{z_1} - 2^{x_1})^{1/y_1}$ is greater than 10^{-6}. The computation time was within 1 minute. Thus, the theorem is proved. □

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