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POSITIVE DEFINITE KERNELS AND HILBERT C*-MODULES

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A theory of positive definite kernels in the context of Hilbert C*-modules is presented. Applications are given, including a representation of a Hilbert C*-module as a concrete space of operators and a construction of the exterior tensor product of two Hilbert C*-modules.

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1. Introduction

There are a number of results in the theory of C*-algebras and the unitary representation theory of groups concerned with various kinds of dilations. A unified approach to such problems can be taken via the concept of a Kolmogorov decomposition for a positive definite kernel [1]. The idea to write this paper came from a reading of E. C. Lance's recent book [2], where a dilation theorem for completely positive maps in the context of Hilbert C*-modules is derived by means of a certain tensor product construction. It seemed possible that the scalar theory of positive definite kernels would generalise to the Hilbert C*-module context and that this could then be used to derive a more "natural" proof of the dilation theorem (as given in the scalar case in [1]). The purpose of this paper is, therefore, to present a generalised theory of positive definite kernels in the Hilbert C*-module context. Further justification for such a theory is provided by other applications we give below, where we use it to represent Hilbert C*-modules as concrete spaces of operators and also to construct the exterior tensor product of Hilbert C*-modules. An advantage of our construction of the latter is that we do not need to invoke the stabilisation theorem of Kasparov, as *is* done in the standard construction [2].

It turns out that much of the scalar theory of positive definite kernels goes over to the context of Hilbert C*-modules straightforwardly, although one has to be rather careful at certain points concerning the existence of adjoints for the linear maps under consideration. There are some important differences from the scalar case nevertheless – for instance, the proof of Theorem 2.4 below differs from its scalar analogue because a certain relevant Banach space may not admit a predual. It would be possible to shorten this paper by omitting those parts of proofs that are parallel to the scalar case. However, it seemed preferable to give a self-contained account, partly because the scalar theory of positive definite kernels and Kolmogorov decompositions appears not to be as well known as it deserves to be and partly because such a full account illustrates clearly the elegance of the proofs and shows how easy and natural some Hilbert C*-module results are if one uses the approach adopted here.

2. Positive definite kernels and Hilbert C*-modules

We begin by recalling the definition of positive definiteness.

If S is a non-empty set, a map k from $S \times S$ to a C*-algebra A is said to be a *positive definite kernel* if, for every positive integer n and for all $s_1, \ldots, s_n \in S$, the matrix $(k(s_i, s_j))$ in $M_n(A)$ is positive.

It follows immediately from the definition that $k(s,s) \geq 0$ and that $k(s,t)^* = k(t,s)$, for all $s, t \in S$.

Remark 2.1. Let A be a C*-algebra. It is well known (and easy to prove) that an element (a_{ij}) of $M_n(A)$ is positive if and only if it is a sum of matrices of the form $(a_i^* a_j)$, for some elements $a_1, \ldots a_n$ in A; equivalently [1, p. 31], (a_{ij}) is positive if and only if the sum $\sum_{i,j=1}^n a_i^* a_{ij} a_j$ is positive in A for all elements $a_1, \ldots, a_n$ belonging to A.

It follows that a kernel $k: S \times S \to A$ is positive definite if and only if for all $s_1, \ldots, s_n \in S$ and $a_1, \ldots, a_n \in A$, the sum $\sum_{i,j=1}^n a_i^* k(s_i, s_j) a_j$ is positive in A.

We shall use these observations below.

Example 2.2. Let A and B be C*-algebras. A linear map $\rho : A \to B$ is said to be *completely positive* if, for every positive integer n, the inflation $M_n(A) \to M_n(B)$, $(a_{ij}) \mapsto (\rho a_{ij})$, is positive. Equivalently, ρ is completely positive if and only if the kernel $k : A \times A \to B$, $(a_1, a_2) \mapsto p(a_1^* a_2)$, is positive definite. The equivalence follows easily from the fact that a positive matrix (a_{ij}) of $M_n(A)$ is a sum of matrices of the form $(a_i^* a_j)$, where $a_1, \ldots, a_n \in A$, as observed in Remark 2.1.

As we shall be studying positive definite kernels and completely positive maps in the context of Hilbert C*-module theory, we recall now some basic terminology, notation and results of that theory. (We refer the reader to [2] for details and examples. However, we mention in passing that the importance of Hilbert C*-modules arises out of their applications to Morita equivalence, *KK*-theory and C*-algebraic quantum group theory.)

Let A be a C*-algebra and E a linear space that is right A-module. A pair consisting of E and a map $\langle . , . \rangle$ from $E \times E$ to A is called an *inner-product A-module* if the map is linear in the second variable, conjugate-linear in the first, and satisfies the following conditions for all $x, y \in E$ and all $a \in A$:

(1) $\langle x, ya \rangle = \langle x, y \rangle a$;

(2) $\langle x, y \rangle^* = \langle y, x \rangle$;

(3) $\langle x, x \rangle \geq 0$ and if $\langle x, x \rangle = 0$, then $x = 0$.

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If $\langle .\, , .\rangle$ satisfies all these requirements except possibly for the second part of Condition (3), it is called a *semi-inner product* on E. A version of the Cauchy–Schwarz inequality for semi-inner products holds, namely,

$$\langle y, x\rangle\langle x, y\rangle \leq \|\langle x, x\rangle\|\langle y, y\rangle \qquad (x, y \in E). \tag{1}$$

A *Hilbert C*-module over A*, or *Hilbert A-module*, is an inner product A-module for which the associated norm, $x \mapsto \|\langle x, x\rangle\|^{1/2}$, is complete.

If E, F are Hilbert A-modules, a map $V : E \to F$ is *adjointable* if there exists a map $W : F \to E$ such that $\langle Vx, y\rangle = \langle x, Wy\rangle$ for all $x \in E$ and $y \in F$. Automatically, V is then bounded and A-linear, that is, it is linear and $V(xa) = V(x)a$ for all $x \in E$ and $a \in A$. Moreover, W is unique and is denoted by V^*. The Banach space of all adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$ and $\mathcal{L}(E)$ denotes the C*-algebra $\mathcal{L}(E, E)$.

A map $U : E \to F$ is a *unitary* if it is adjointable and $U^*U = 1$ and $UU^* = 1$. In this case U is isometric, surjective and A-linear. Conversely, if U has these properties, it is a unitary [**2**, Theorem 3.5]. If a unitary mapping from E onto F exists, then E and F are said to be *unitarily equivalent*.

Before turning now to the theory of positive definite kernels, we need a few more items of notation that will be used frequently in the sequel.

We write $B(H, K)$ for the Banach space of all bounded linear operators from H to K, where H and K are Banach spaces, and we write $B(H)$ for the algebra $B(H, H)$.

If $(x, y) \mapsto xy$ is a bilinear map on the product $H \times K$ with values in a Banach space L, and if S and T are subsets of H and K respectively, we denote by ST the closed linear span in L of all products xy, where $x \in S$ and $y \in T$.

We denote by $[S]$ the closed linear span of S.

Let A be a C*-algebra. If V is an arbitrary map from a non-empty set S to $\mathcal{L}(E, E_V)$, where E and E_V are Hilbert A-modules, then the kernel k, defined by setting $k(s, t) = V(s)^*V(t)$, is positive definite (k has values in $\mathcal{L}(E)$). The map V will be called a *Kolmogorov decomposition* for k. If the (scalar) linear span of the set $\cup_{s\in S} V(s)E$ is dense in E_V, then V will be said to be *minimal*.

Every positive definite kernel k with values in $\mathcal{L}(E)$ has an essentially unique minimal Kolmogorov decomposition. This is the content of the following result. The proof is modeled on the scalar (Hilbert space) case, see [**1**]. However, some care about adjointability of maps is required at certain points and a somewhat different approach is taken to demonstrating the properties of the inner product of the space constructed.

Theorem 2.3. *Let E be a Hilbert C*-module over a C*-algebra A. Let S be a non-empty set and k a positive definite map from $S \times S$ to $\mathcal{L}(E)$. Then there exists a minimal Kolmogorov decomposition for k. Moreover, if $V : S \to \mathcal{L}(E, E_V)$ and $W : S \to \mathcal{L}(E, E_W)$ are any two such minimal Kolmogorov decompositions, then there exists a unique unitary $U : E_V \to E_W$ such that $UV(s) = W(s)$, for all $s \in S$.*

Proof. If $f: S \to E$ has finite support, define $kf: S \to E$ by setting $kf(s) = \sum_{t \in S} k(s, t)f(t)$ and denote by E_V^0 the set of all these maps kf. When endowed with the pointwise-defined operations, E_V^0 is a right module over A. Moreover, we may endow E_V^0 with a semi-inner product by setting

$$\langle kf, kg \rangle = \sum_{s,t \in S} \langle k(s, t)f(t), g(s) \rangle.$$

(Positivity is given by positive definiteness of k.) In fact, we actually have an inner product. For, by the Cauchy–Schwarz inequality (1), if $\langle kf, kf \rangle = 0$, then $\langle kf, kg \rangle = \sum_{s \in S} \langle kf(s), g(s) \rangle = 0$, for any map $g: S \to E$ of finite support. If $x \in E$ and $t \in S$, define the map x_t from S to E by setting $x_t(s) = 0$ if $s \neq t$ and by setting $x_t(t) = x$. Then with $g = x_t$, we get $\langle kf(t), x \rangle = \sum_{s \in S} \langle kf(s), x_t(s) \rangle = 0$. Hence, $kf(t) = 0$, for all $t \in S$, so $kf = 0$.

Thus, E_V^0 is an inner product A-module. We complete it to get a Hilbert A-module that we denote by E_V.

If $s \in S$, define $V(s): E \to E_V$ by setting $V(s)x = k(x_s)$. We show that $V(s) \in \mathcal{L}(E, E_V)$, that is, $V(s)$ is adjointable: Obviously, $V(s)$ is A-linear. Also, it is bounded, since $\|V(s)x\|^2 = \|\langle k(x_s), k(x_s) \rangle\| = \|\langle k(s, s)x, x \rangle\| \leq \|k(s, s)\| \|x\|^2$, and therefore, $\|V(s)\| \leq \|k(s, s)\|^{1/2}$. Define $T: E_V^0 \to E$ by setting $T(kf) = (kf)(s)$. Direct computation shows that

$$\langle x, T(kf) \rangle = \langle V(s)x, kf \rangle \tag{2}$$

and therefore,

$$\|T(kf)\| = \sup_{\|x\| \leq 1} \|\langle x, T(kf) \rangle\| = \sup_{\|x\| \leq 1} \|\langle V(s)x, kf \rangle\| \leq \|V(s)\| \|k(f)\|.$$

Hence, $\|T\| \leq \|V(s)\|$. Now extend T to a bounded linear operator from E_V to E. It follows from Equation (2) that $\langle x, Tg \rangle = \langle V(s)x, g \rangle$ for all $x \in E$ and $g \in E_V$. Hence, $V(s)$ is adjointable, with adjoint $V(s)^* = T$. Moreover, if $s, t \in S$ and $x, y \in E$, then $\langle V(s)^* V(t)x, y \rangle = \langle k(x_t), k(y_s) \rangle = \langle k(s, t)x, y \rangle$, so $V(s)^* V(t) = k(s, t)$. Hence, the map, $V: S \mapsto V(s)$, is a Kolmogorov decomposition for k.

If f is a map from S to E of finite support, then it can be written as a sum $f = f_1 + \cdots + f_n$, where $f_i = (x^i)_{s_i}$, for some vectors $x^i \in E$ and elements $s_i \in S$. Since $k(f_i) = V(s_i)x^i$ and $k(f) = \sum_{i=1}^n k(f_i)$, the linear span of the set $\cup_s V(s)E$ contains $k(f)$. Hence, $E_V = [\cup_s V(s)E]$ and V is minimal.

Suppose now that $V: S \to \mathcal{L}(E, E_V)$ and $W: S \to \mathcal{L}(E, E_W)$ are any two minimal Kolmogorov decompositions for k. If $s_1, \ldots, s_n$ belong to S and $x_1, \ldots, x_n$ belong to E, then

$$\left\|\sum_{i=1}^{n} V(s_i)x_i\right\|^2 = \left\|\left\langle \sum_{i=1}^{n} V(s_i)x_i, \sum_{j=1}^{n} V(s_j)x_j\right\rangle\right\|$$
$$= \left\|\sum_{i,j=1}^{n}\langle k(s_j, s_i)x_i, x_j\rangle\right\| = \left\|\sum_{i=1}^{n} W(s_i)x_i\right\|^2.$$

Hence, there is a well-defined isometry from a dense linear subspace of E_V to E_W that maps $V(s)x$ to $W(s)x$. We extend this to get an isometry U from E_V to E_W. We may define similarly an isometry U' from E_W to E_V mapping $W(s)x$ to $V(s)x$. Clearly, U' and U are inverse to each other. Since $\langle UV(s)x, W(t)y\rangle = \langle W(s)x, W(t)y\rangle = \langle k(t,s)x, y\rangle = \langle V(s)x, V(t)y\rangle = \langle V(s)x, U'W(t)y\rangle$, we have $\langle Uf, g\rangle = \langle f, U'g\rangle$ for all $f \in E_V$ and $g \in E_W$. Hence, U is adjointable with $U^* = U' = U^{-1}$. Thus, U is a unitary. Also, $UV(s) = W(s)$, for all $s \in S$. □

As observed earlier (in Example 2.2), a completely positive map determines a positive definite kernel. We use this now to derive a dilation theorem, part of whose proof is parallel to the derivation of Theorem 2.13 in **[1]**. First, we need some definitions.

If E and F are Hilbert A-modules, the *strict* topology on $\mathcal{L}(E, F)$ is the one given by the seminorms

$$V \mapsto \|V_x\| \quad (x \in E), \qquad V \mapsto \|V^*y\| \quad (y \in F).$$

The closed 0-centred ball of $\mathcal{L}(E, F)$ of any finite radius is complete relative to the strict topology.

If A and B are C*-algebras, and E is a Hilbert A-module, a completely positive map $\rho : B \to \mathcal{L}(E)$ is said to be *strict* **[2**, p. 49] if, for some approximate unit (e_i) of B, the net $(\rho(e_i))$ satisfies the Cauchy condition for the strict topology in $\mathcal{L}(E)$. (If B is unital, ρ is automatically strict.)

Theorem 2.4. *Let A and B be C*-algebras, let E be a Hilbert A-module and let $\rho : B \to \mathcal{L}(E)$ be a strict completely positive map. Then there exists a Hilbert A-module E_π, a $*$-homomorphism $\pi : B \to \mathcal{L}(E_\pi)$ and an element $W \in \mathcal{L}(E, E_\pi)$ such that $\rho(b) = W^*\pi(b)W$, for all $b \in B$. Moreover, $[\pi(B)WE] = E_\pi$.*

Proof. Since the kernel $k : (b_1, b_2) \mapsto \rho(b_1^*b_2)$ is positive definite, it has a minimal Kolmogorov decomposition $V : B \to \mathcal{L}(E, E_V)$, by Theorem 2.3. Moreover, V is linear. For, if $b_1, b_2, c \in B$ and $\lambda \in \mathbf{C}$, then

$$V(b_1 + \lambda b_2)^* V(c) = \rho((b_1 + \lambda b_2)^* c) = \rho(b_1^* c) + \bar{\lambda}\rho(b_2^* c)$$
$$= V(b_1)^* V(c) + \bar{\lambda} V(b_2)^* V(c) = (V(b_1) + \lambda V(b_2))^* V(c);$$

hence, since $[\cup_{c\in B} V(c)E] = E_V$, we have $V(b_1 + \lambda b_2) = V(b_1) + \lambda V(b_2)$.

If u is a unitary element of $\tilde{B} = B + \mathbf{C}1$, the unitisation of B, and $b, c \in B$, then $V(ub)^* V(uc) = \rho(b^*u^*uc) = \rho(b^*c) = V(b)^* V(c)$, so the map, $c \mapsto V(uc)$, is a minimal

Kolmogorov decomposition for k. Hence, there exists a unitary $\pi(u) \in \mathcal{L}(E_V)$ such that $\pi(u)V(c) = V(uc)$ $(c \in B)$. If b is a linear combination of unitaries of $\tilde{B}$, say $b = \sum_{i=1}^n \lambda_i u_i$, then $(\sum_{i=1}^n \lambda_i \pi(u_i))V(c) = V((\sum_{i=1}^n \lambda_i u_i)c) = V(bc)$. Using this and again using the fact that $E_V = [\cup_{c \in B} V(c)E]$, it follows that we may define $\pi(b) = \sum_{i=1}^n \lambda_i \pi(u_i)$, independent of the decomposition of b into a linear combination of unitaries. Thus, $\pi(b)V(c) = V(bc)$, and it follows easily that $\pi : B \to \mathcal{L}(E_V)$, $b \mapsto \pi(b)$, is a $*$-homomorphism. Set $E_\pi = E_V$.

Now let (e_i) be an approximate unit of B for which $(\rho(e_i))$ is a Cauchy net relative to the strict topology of $\mathcal{L}(E)$. We show that $(V(e_i))$ is a Cauchy net for the strict topology of $\mathcal{L}(E, E_\pi)$: First, observe that it is bounded. For, if $b \in B$, then

$$\|V(b)\|^2 = \|V(b)^* V(b)\| = \|\rho(b^* b)\| \leq \|\rho\| \|b\|^2. \tag{3}$$

Since $V(e_i)^* V(b) = \rho(e_i b)$ and since $e_i b \to b$ in norm, the set $(V(e_i)^* V(b)x)$ is convergent in E for all $x \in E$. Hence, $(V(e_i)^* y)$ is convergent for all y in the linear span of $\cup_b V(b)E$. Using boundedness of the net $(V(e_i)^*)$ and density of the linear span of $\cup_b V(b)E$ in E_π, it follows that $(V(e_i)^* y)$ is convergent for all $y \in E_\pi$. Now let $x \in E$ and suppose that $e_j \leq e_i$. Then

$$\begin{aligned} \|V(e_i)x - V(e_j)x\|^2 &= \|\langle x, (V(e_i) - V(e_j))^*(V(e_i) - V(e_j))x\rangle\| \\ &= \|\langle x, \rho((e_i - e_j)^2)x\rangle\| \leq \|\langle x, \rho(e_i - e_j)x\rangle\|. \end{aligned}$$

It follows that $(V(e_i)x)$ is a Cauchy net in E_π since $(\rho(e_i))$ forms a Cauchy net for the strict topology. Hence, $(V(e_i))$ is a Cauchy net for the strict topology in the closed 0-centred ball of radius $\|\rho\|^{1/2}$ in $\mathcal{L}(E, E_\pi)$ and therefore, by completeness, it is convergent in that topology to some element $W \in \mathcal{L}(E, E_\pi)$.

If $b \in B$ and $x \in E$, then $\pi(b)Wx = \lim \pi(b)V(e_i)x = \lim V(be_i)x = V(b)x$, since V is continuous. Therefore, $\pi(b)W = V(b)$. Since $[\cup_b V(b)E] = E_\pi$, it follows that $[\pi(B)WE] = E_\pi$. Finally, for any element $x \in E$, we have $W^*\pi(b)Wx = W^* V(b)x = \lim V(e_i)^* V(b)x = \lim \rho(e_i b)x = \rho(b)x$, so $W^*\pi(b)W = \rho(b)$. □

3. Concrete Hilbert C*-modules

It is an important fundamental result that every C*-algebra has a faithful representation as a concrete algebra of operators. We are now going to show that an analogous result holds for Hilbert C*-modules. First, however, we must make an appropriate definition.

Let H and K be Hilbert spaces and let A be a concrete C*-algebra of operators acting on H. Let E be a closed linear subspace of $B(H, K)$ and suppose that the following two conditions are satisfied:

(1) If $x \in E$ and a $a \in A$, then $xa \in E$;

(2) If $x, y \in E$, then $x^* y \in A$.

Endowed with the multiplication $(x, a) \mapsto xa$ (the product is just operator composition), E becomes a right A-module. Setting $\langle x, y \rangle = x^* y$ makes E into a Hilbert A-module. (The induced norm is the operator norm.)

We call E a *concrete* Hilbert C*-module.

The following result states that all Hilbert C*-modules can be represented as concrete ones. A classical (Hilbert-space) Kolmogorov decomposition enters into the proof.

Theorem 3.1. *Let A be a C*-algebra and let E be a Hilbert A-module. Then there exists a faithful representation π of A on a Hilbert space H and an isometric, linear isomorphism U from E onto a concrete Hilbert $\pi(A)$-module F of operators from H to a Hilbert space K such that*

$$\langle U(x), U(y) \rangle = \pi(\langle x, y \rangle) \qquad \textit{and} \qquad U(xa) = U(x)\pi(a)$$

for all $x, y \in E$ and $a \in A$.

Proof. Let (H, π) be any faithful representation of A. Then the kernel, $k: E \times E \to B(H)$, $(x, y) \mapsto \pi(\langle x, y \rangle)$, is positive definite. To see this, suppose $x_1, \ldots, x_n \in E$. Then, by Remark 2.1, the matrix $(\langle x_i, x_j \rangle) \in M_n(A)$ is positive, since, if $a_1, \ldots, a_n \in A$, then $\sum_{i,j=1}^n a_j^* \langle x_j, x_i \rangle a_i = \left\langle \sum_{j=1}^n x_j a_j, \sum_{i=1}^n x_i a_i \right\rangle \geq 0$. Hence, the matrix $(\pi\langle x_i, x_j \rangle)$ is positive in $M_n(B(H))$.

Since k is positive definite, it admits a (classical) Kolmogorov decomposition $U: E \to B(H, K)$, where K is some Hilbert space. Using the fact that $U(x)^* U(y) = \pi(\langle x, y \rangle)$ for all $x, y \in E$, one easily verifies that U is linear and isometric and that $U(xa) = U(x)\pi(a)$ for all $x \in E$ and $a \in A$. Setting $F = U(E)$, it follows that F is a closed linear subspace of $B(H, K)$ for which $F\pi(A) \subseteq F$ and $F^* F \subseteq \pi(A)$. Hence, F is a concrete Hilbert $\pi(A)$-module. This proves the theorem. □

We give an application of this representation to the construction of the exterior tensor product of two Hilbert C*-modules. As remarked by Lance [2, p. 34], the usual construction is hard: it uses the Kasparov stabilisation theorem [2, p. 62]. However, our construction is quite straightforward, using the preceding theorem.

We write $E \otimes_{\text{alg}} F$ for the algebraic tensor product of two linear spaces and $H \otimes K$ for the Hilbert space tensor product of two Hilbert spaces.

Theorem 3.2. *Suppose that B and C are C*-algebras and that E and F are Hilbert C*-modules over B and C, respectively. Suppose also that A is the minimal C*-tensor product of B and C. Then there exists a Hilbert C*-module G over A containing $E \otimes_{\text{alg}} F$ as a dense linear subspace such that for all $x, x' \in E$ and $y, y' \in F$, we have $\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \otimes \langle y, y' \rangle$ and for all $b \in B$ and $c \in C$, we have $(x \otimes y)(b \otimes c) = xb \otimes yc$. Moreover, G is unique up to unitary equivalence: If G and G' are two Hilbert A-modules satisfying these conditions, then there is a unique unitary U from G onto G' which is the identity map when restricted to $E \otimes_{\text{alg}} F$.*

Proof. The uniqueness of G is almost obvious. We show only its existence. Using Theorem 3.1, it is easily seen that we may suppose that B, C are concrete C*-algebras acting on Hilbert spaces H and K, respectively and that E and F are concrete Hilbert C*-modules; thus, they are closed linear subspaces of $B(H, H')$ and $B(K, K')$, respectively, for some Hilbert spaces H' and K'. Also, we regard A as a concrete C*-algebra acting on the Hilbert space tensor product $H \otimes K$.

We can identify $E \otimes_{\text{alg}} F$ as a linear subspace of $B(H \otimes K, H' \otimes K')$ by identifying the elementary tensor $x \otimes y$ with the operator that maps $\eta \otimes \xi$ onto $x(\eta) \otimes y(\xi)$, where $\eta \in H$ and $\xi \in K$. We now define G to be the closure in $B(H \otimes K, H' \otimes K')$ of $E \otimes_{\text{alg}} F$. We have $GA \subseteq G$, since $(x \otimes y)(b \otimes c) = xb \otimes yc$ for all $x \in E$, $y \in F$, $b \in B$ and $c \in C$. Also, $G^*G \subseteq A$, since $(x_1 \otimes y_1)^*(x_2 \otimes y_2) = x_1^*x_2 \otimes y_1^*y_2$ for all $x_1, x_2 \in E$ and $y_1, y_2 \in F$. Hence, G is a concrete Hilbert A-module satisfying the conditions of the theorem. □

The module G is the *exterior tensor product* of E and F.

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