8
Chiral symmetry

In this chapter we pay attention to a very important aspect of QCD and the Standard Model: chiral symmetry. It is a symmetry that is natural in the continuum but it poses special problems for regularizations, including the lattice. We review first some aspects of chiral symmetry in QCD, then discuss chiral aspects of QCD on a lattice, and finally give a brief introduction to chiral gauge theories, of which the Standard Model is an example.

8.1 Chiral symmetry and effective action in QCD

Consider the mass terms in the QCD action,

$$S_{\rm mass} = -\int d^4x\, \bar{\psi} m \psi, \tag{8.1}$$

where $m = \mathrm{diag}\,(m_u, m_d, m_s, \cdots)$ is the diagonal matrix of mass parameters. We know that the first three quarks u, d and s have relatively small masses compared with a typical hadronic scale such as the Regge slope $(\alpha')^{-1/2} \approx 1100$ MeV or the string tension $\sqrt{\sigma} \approx 400$ MeV. (Recall that $m_{ud} = 4.4$ MeV and $m_s \approx 90$ MeV in section 7.5.) Suppose we set the first $n_{\rm f}$ quark-mass parameters to zero. In that case the QCD action has $U(n_{\rm f}) \times U(n_{\rm f})$ symmetry, loosely called chiral symmetry, in which the left- and right-handed components of the Dirac field are subjected to independent flavor transformations $V_{\rm L,R} \in U(n_{\rm f})$,

$$\begin{aligned} \psi &\to V\psi, \quad V = V_{\rm L} P_{\rm L} + V_{\rm R} P_{\rm R}, \\ \bar{\psi} &\to \bar{\psi}\bar{V}, \quad \bar{V} = V_{\rm L}^\dagger P_{\rm R} + V_{\rm R}^\dagger P_{\rm L} = \beta V^\dagger \beta, \\ P_{\rm L} &= \tfrac{1}{2}(1-\gamma_5), \quad P_{\rm R} = \tfrac{1}{2}(1+\gamma_5). \end{aligned} \tag{8.2}$$

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Here $V_{\rm L,R}$ act only on the first $n_{\rm f}$ flavor indices of the quark field. The matrix $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma_1\gamma_2\gamma_3\gamma_4$ has the properties $\gamma_5^\dagger = \gamma_5$, $\gamma_5^2 = 1$ and it anticommutes with the γ_μ, i.e. $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$. The $P_{\rm L,R}$ are orthogonal projectors, $P_{\rm L}^2 = P_{\rm L}$, $P_{\rm R}^2 = P_{\rm R}$, $P_{\rm L}P_{\rm R} = 0$, $P_{\rm L} + P_{\rm R} = 1$. Because of these properties the derivative terms in the action are invariant,

$$\bar\psi\gamma^\mu D_\mu\psi \to \bar\psi\bar V\gamma^\mu V D_\mu\psi = \bar\psi\gamma^\mu(V_{\rm L}^\dagger V_{\rm L} + V_{\rm R}^\dagger V_{\rm R})D_\mu\psi = \bar\psi\gamma^\mu D_\mu\psi. \quad (8.3)$$

The mass terms transform as

$$\bar\psi m\psi \to \psi\bar V m V\psi = \bar\psi(V_{\rm R}^\dagger m V_{\rm L} P_{\rm L} + V_{\rm L}^\dagger m V_{\rm R} P_{\rm R})\psi, \quad (8.4)$$

so they break the symmetry. A flavor-symmetric mass term has $m \propto 1\!\!1$. Such a mass term is invariant under *flavor transformations*, for which $V_{\rm L} = V_{\rm R}$. However, it is not invariant under transformations with $V_{\rm L} \neq V_{\rm R}$. A special case of these are *chiral transformations* in the narrow sense,[1] for which $V_{\rm L} = V_{\rm R}^\dagger$. For $m = 0$ in the $n_{\rm f} \times n_{\rm f}$ subspace the action is invariant under chiral $U(n_{\rm f}) \times U(n_{\rm f})$ transformations.

In the quantum theory the $U(n_{\rm f}) \times U(n_{\rm f})$ symmetry is reduced to $SU(n_{\rm f}) \times SU(n_{\rm f}) \times U(1)$ by so-called *anomalies* (this will be reviewed in section 8.4). Here $U(1)$ is the group of ordinary (Abelian) phase transformations $\psi \to e^{i\omega}\,\psi$, $\bar\psi \to e^{-i\omega}\,\bar\psi$. Furthermore, the dynamics is such that the $SU(n_{\rm f}) \times SU(n_{\rm f})$ symmetry is spontaneously broken.

An informative way to exhibit the physics of this situation is by using an effective action. We have met already in chapter 3 the $O(4)$ model for pions (which can be extended to include nucleons, cf. problem (i)). This illustrates the case $n_{\rm f} = 2$ (the group $SO(4)$ is equivalent to $SU(2) \times SU(2)/Z_2$, cf. (D.19) in appendix D). One introduces effective fields ϕ which transform in the same way as the quark bilinear scalar fields $\bar\psi_g\psi_f$ and pseudoscalar fields $\bar\psi_g i\gamma_5\psi_f$, $f, g = 1, \ldots, n_{\rm f}$. We start with

$$\Phi_{fg} \equiv \bar\psi_g P_{\rm L}\psi_f, \quad (8.5)$$

which transforms as

$$\Phi_{fg} \to (V_{\rm L})_{ff'}(V_{\rm R}^\dagger)_{g'g}\,\Phi_{f'g'}, \quad (8.6)$$

or, in matrix notation,

$$\Phi \to V_{\rm L}\Phi V_{\rm R}^\dagger. \quad (8.7)$$

The other possibility leads to $\Phi^\dagger$:

$$\bar\psi_g P_{\rm R}\psi_f = (\psi_f^+ P_{\rm R}\,\beta\psi_g)^* = (\bar\psi_f P_{\rm L}\psi_g)^* = (\Phi_{gf})^* \quad (8.8)$$

$$\equiv (\Phi^\dagger)_{fg}. \quad (8.9)$$

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Under parity Φ and $\Phi^\dagger$ are interchanged,

$$\Phi(x^0, \mathbf{x}) \xrightarrow{P} \Phi^\dagger(x^0, -\mathbf{x}). \tag{8.10}$$

Ignoring the symmetry breaking due to anomalies, the effective action for the *effective field* $\phi \leftrightarrow \Phi$ has the same chiral transformation properties as the QCD action. We shall first examine the form of this effective action and derive some consequences, and later take into account that anomalies reduce the $U(n_{\rm f})\times U(n_{\rm f})$ symmetry to $SU(n_{\rm f})\times SU(n_{\rm f})\times U(1)$.

For $m = 0$ we want an invariant action. The combination $\mathrm{Tr}[(\phi\phi^\dagger)^k]$ is invariant under (8.7). An invariant action is given by

$$S = -\int d^4x \, \mathrm{Tr}(F_2 \partial_\mu \phi^\dagger F_1 \partial^\mu \phi + G), \tag{8.11}$$

where $F_{1,2}$ and G have the forms

$$F_1 = \sum_k f_{1k} (\phi\phi^\dagger)^k, \quad F_2 = \sum_k f_{2k} (\phi^\dagger \phi)^k, \tag{8.12}$$

$$G = \sum_k g_k (\phi\phi^\dagger)^k. \tag{8.13}$$

Reality of the action requires the coefficients f_{1k}, f_{2k} and g_k to be real. Invariance under parity requires

$$f_{1k} = f_{2k}. \tag{8.14}$$

The action might also contain terms of the type

$$\mathrm{Tr}[(\phi\phi^\dagger)^k] \, \mathrm{Tr}[(\phi\phi^\dagger)^l]. \tag{8.15}$$

At this point we assume such terms to be absent and come back to them later.

There may also be higher derivative terms. Their systematic inclusion is part of *chiral perturbation theory*, see e.g. [19]. For slowly varying fields, which is all we need for describing physics on the low-energy–momentum scale, we may assume such higher derivative terms to be negligible.

The classical ground state will be characterized by $\partial_\mu \phi = 0$ and correspond to a minimum of $\mathrm{Tr}\, G$. Let $\lambda_1, \ldots, \lambda_{n_{\rm f}}$ be the eigenvalues of the Hermitian matrix $\phi\phi^\dagger$. Then

$$\mathrm{Tr}\, G = \sum_k g_k (\lambda_1^k + \cdots + \lambda_{n_{\rm f}}^k). \tag{8.16}$$

A stationary point of $\mathrm{Tr}\, G$ has to satisfy

$$0 = \frac{\partial}{\partial \lambda_j} \mathrm{Tr}\, G = \sum_k g_k k \lambda_j^{k-1}, \tag{8.17}$$

which is the same equation for each j. Hence, the solution is

$$\lambda_1 = \cdots = \lambda_{n_\mathrm{f}} \equiv \lambda, \quad \phi\phi^\dagger = \lambda \mathbb{1}. \tag{8.18}$$

Since $\phi\phi^\dagger \geq 0$ (i.e. all eigenvalues are ≥ 0), $\lambda \geq 0$.

We shall now assume that $\lambda \neq 0$ at the minimum of $\mathrm{Tr}\, G$. The symmetry is then spontaneously broken, because a non-zero ϕ in the ground state is not invariant under $U(n_\mathrm{f}) \times U(n_\mathrm{f})$ transformations. It is helpful to use a generalized polar decomposition for ϕ,

$$\phi = HU, \quad H = H^\dagger, \quad U^\dagger = U^{-1}. \tag{8.19}$$

The H and U can be found as follows: H can be calculated from $H = \pm\sqrt{\phi\phi^\dagger}$ and then $U = H^{-1}\phi$. In the ground state $H = \pm\sqrt{\lambda}\mathbb{1}$. The degeneracy of the ground state is described by U, which is an element of $U(n_\mathrm{f})$. It transforms as $U \to V_\mathrm{L} U V_\mathrm{R}^\dagger$. Without loss of generality we may assume that $U = \mathbb{1}$ and $H = -\sqrt{\lambda}\mathbb{1}$ (the minus sign becomes natural on taking into account the explicit symmetry breaking due to the quark masses). This exhibits clearly the residual degeneracy of the ground state: it is invariant under the diagonal $U(n)$ subgroup, for which $V_\mathrm{L} = V_\mathrm{R}$. The pattern of spontaneous symmetry breaking is

$$U(n_\mathrm{f}) \times U(n_\mathrm{f}) \to U(n_\mathrm{f}). \tag{8.20}$$

The variables of U (e.g. using the exponential parameterization) are analogous to angular variables for the $O(n)$-vector field φ_α in the $O(n)$ model. We expect these to correspond to Nambu–Goldstone bosons.

Let us linearize the effective action about the ground state, writing $H = -v + h$, $U = \exp(i\alpha)$,

$$\phi = (-v + h)\bigl(1 + i\alpha - \tfrac{1}{2}\alpha^2 + \cdots\bigr), \quad v = \sqrt{\lambda} > 0 \tag{8.21}$$

(from now on we no longer distinguish between 1 and $\mathbb{1}$). We keep only terms up to second order in h and α. Since $\partial_\mu \phi$ is of first order, we may replace ϕ by v in $F_{1,2}$ in (8.12),

$$F_1 = F_2 \equiv F, \quad \text{for } \phi = -v, \tag{8.22}$$

and obtain

$$S = -\int d^4x\, \mathrm{Tr}\,(F^2 v^2\, \partial_\mu\alpha\partial^\mu\alpha + F^2 \partial_\mu h \partial^\mu h + r\, h^2) + \cdots. \tag{8.23}$$

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The $\cdots$ also include the ground-state value of S. The coefficient r follows from

$$\mathrm{Tr}\, G = \sum_k g_k v^{2k}\, \mathrm{Tr}\, [(-1 + h/v)^{2k}] \tag{8.24}$$

$$= \sum_k g_k v^{2k} [n_{\mathrm{f}} + k(2k-1) v^{-2}\, \mathrm{Tr}\, h^2 + \cdots], \tag{8.25}$$

where the term linear in h vanishes because $\mathrm{Tr}\, G$ is stationary at $h = 0$. Since we expand around a minimum of $\mathrm{Tr}\, G$, the coefficient

$$r = \sum_k g_k k(2k-1) v^{2k-2} \tag{8.26}$$

is positive. The form (8.23) shows that the α fields have zero mass parameter – they correspond to n_{f}^2 Nambu–Goldstone bosons. The h fields have a mass given by

$$m_h^2 = r/F^2. \tag{8.27}$$

At this point we shall make the useful approximation of ‘freezing’ the ‘radial’ degrees of freedom H to their ground-state value $H = -v$, or $h = 0$. This approximation is justified when m_h is sufficiently large compared with the momenta of interest (cf. problem (i) for numbers) and it simplifies the derivations to follow. Thus we get

$$\phi(x) = -v\, U(x), \tag{8.28}$$

$$S = -\int d^4x\, \frac{f^2}{4}\, \mathrm{Tr}\, (\partial_\mu U^\dagger \partial^\mu U), \quad f^2 = 4F^2 v^2, \tag{8.29}$$

where we omitted a constant term.

We now comment on the terms of the form (8.15). When these are included the uniqueness of the form (8.18) is no longer compelling and other solutions with $\lambda_j \neq \lambda_k$ are also possible. This depends on the details of the action. However, arguments based on the large-n_{c} behavior of the generalization of QCD to an $SU(n_{\mathrm{c}})$ gauge theory suggest that terms of the form (8.15) are subdominant [100]. The ground-state solution of the complete effective action including terms of the form (8.15) is still expected to have the symmetric form (8.18), and the symmetry-breaking pattern is still expected to be $U(n_{\mathrm{f}}) \times U(n_{\mathrm{f}}) \to U(n_{\mathrm{f}})$.

The quark-mass terms in the QCD action explicitly break chiral symmetry. They have the form of an external source coupled to quark bilinears,

$$S_{\rm mass} = -\int d^4x\, \bar{\psi} m(P_{\rm L} + P_{\rm R})\psi \tag{8.30}$$

$$= \int d^4x\, {\rm Tr}\,(J^\dagger \Phi + \Phi^\dagger J), \quad J = -m. \tag{8.31}$$

Hence, we can absorb the quark-mass terms in the coupling to such an external source. The total effective action including this source has the form $\ln Z(J) = S(\phi) + \int d^4x\, {\rm Tr}\,(J^\dagger\phi + \phi^\dagger J)$, where ϕ is again the effective field. Setting $J = -m$ thus leads to an addition ΔS in the effective action

$$\Delta S = -\int d^4x\, {\rm Tr}\,[m(\phi + \phi^\dagger)]. \tag{8.32}$$

Expanding† $\phi = -vU = -v\exp(i\alpha)$ to second order in α gives

$$\Delta S = -\int d^4x\, v\, {\rm Tr}\,(m\alpha^2) + \cdots = -\int d^4x\, v \sum_{fg} m_f\, \alpha_{fg}\, \alpha_{gf} + \cdots. \tag{8.33}$$

Since U is unitary, $\alpha_{gf} = \alpha^*_{fg}$. Taking α_{fg} with $f \leq g$ as independent variables leads to

$$\Delta S = -\int d^4x\, v \left[\sum_{f<g} (m_f + m_g)\alpha_{fg}\, \alpha^*_{fg} + \sum_f m_f\, \alpha^2_{ff} + \cdots\right]. \tag{8.34}$$

Similarly, expanding the gradient term (8.29) gives

$$S = -\int d^4x\, \frac{f^2}{4}\left(2\sum_{f<g} \partial_\mu \alpha^*_{fg}\, \partial^\mu \alpha_{fg} + \sum_f \partial_\mu \alpha_{ff}\, \partial^\mu \alpha_{ff} + \cdots\right). \tag{8.35}$$

As expected from the $O(4)$ model, ΔS gives a mass to the Goldstone bosons, which for small m is linear in m,

$$m^2_{fg} = B(m_f + m_g), \quad B = 2v/f^2. \tag{8.36}$$

In the next section we shall confront these mass relations with experiment.

By coupling the effective action to the electroweak gauge fields it can be shown that the constant f determines the leptonic decays

† We neglect here the effect of the quark masses on the ground-state value of ϕ.

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of the pseudoscalar mesons. It is known as the pion decay constant, $f = f_\pi \approx 93$ MeV. This constant also determines the size of the s-wave pi–pi scattering lengths in good agreement with experiment.

To end this section we note a relation between the pion decay constant f, the unrenormalized *chiral condensate* $\langle\bar\psi\psi\rangle = \sum_f \langle\bar\psi_f\psi_f\rangle = 2\sum_f\langle\phi_{ff}\rangle = -2n_{\rm f}v$, and the wave-function renormalization constant Z of the pseudoscalar fields $P_{fg} \equiv i(\phi^\dagger - \phi)_{fg} \leftrightarrow \bar\psi_g i\gamma_5\psi_f$. The constant Z can be read off from $S = -\int d^4x\, Z^{-1}\sum_{f<g}\partial_\mu P^*_{fg}\partial^\mu P_{fg} + \cdots$, using $U = -\phi/v$ and (8.29) and (8.35): $Z^{-1} = f^2/8v^2$. Hence, f is given by the renormalized chiral condensate $\langle\bar\psi\psi\rangle/\sqrt{Z}$,

$$f = \frac{2\sqrt{2}v}{\sqrt{Z}} = \frac{-\sqrt{2}\langle\bar\psi\psi\rangle}{n_{\rm f}\sqrt{Z}}. \tag{8.37}$$

8.2 Pseudoscalar masses and the $U(1)$ problem

The candidate Nambu–Goldstone (NG) bosons and their masses are

$$\begin{aligned}
\pi^\pm\!: &\quad m^2_{\pi^+} = m^2_{ud} = 0.0195\ \mathrm{GeV}^2\\
K^\pm\!: &\quad m^2_{K^+} = m^2_{us} = 0.244\ \mathrm{GeV}^2\\
K^0, \bar K^0\!: &\quad m^2_{K^0} = m^2_{ds} = 0.248\ \mathrm{GeV}^2\\
\pi^0\!: &\quad m^2_{\pi^0} = 0.0182\ \mathrm{GeV}^2\\
\eta\!: &\quad m^2_{\eta} = 0.301\ \mathrm{GeV}^2\\
\eta'\!: &\quad m^2_{\eta'} = 0.917\ \mathrm{GeV}^2
\end{aligned} \tag{8.38}$$

For the unequal-flavor particles ($f \neq g$) we have indicated the quark labels. For the neutral π^0, η and η' the quark assignment turns out to be less straightforward.

Consider two light flavors, $n_{\rm f} = 2$. The mass formula (8.36) with $f = u, d$ and $g = u, d$ predicts four NG bosons in this case. The obvious candidates are $\pi^\pm$, π^0 and η, with

$$m^2_{\pi^+} = m^2_{ud} = B(m_u + m_d). \tag{8.39}$$

According to (8.36), the other eigenstates are $\bar uu$ and $\bar dd$. If we try to assign $\pi^0 \leftrightarrow \bar uu$, $\eta \leftrightarrow \bar dd$, the relation

$$m^2_{ud} = \tfrac{1}{2}(m^2_{uu} + m^2_{dd}) \tag{8.40}$$

cannot be fulfilled at all. If we assume that $m_u \approx m_d$ and π^0 is an equal mixture of $\bar uu$ and $\bar dd$ to get $m^2_{\pi^0} \approx m^2_{\pi^+}$, the orthogonal combination of

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$\bar{u}u$ and $\bar{d}d$ should have approximately the same mass as π^0: the η does not fit in.

Consider next three light flavors, $n = 3$. The mass formulas now predict nine NG bosons. We find

$$\frac{m_u + m_d}{m_u + m_s} = \frac{m_{\pi^+}^2}{m_{K^+}^2} \equiv R_1, \quad \frac{m_u + m_s}{m_d + m_s} = \frac{m_{K^+}^2}{m_{K^0}^2} \equiv R_2, \tag{8.41}$$

and from this

$$\frac{m_s}{m_u} = \frac{R_2(R_1 - 1)}{1 - R_2 - R_1 R_2} = 31, \quad \frac{m_s}{m_d} = \frac{R_2}{1 - R_2 + m_u/m_s} = 20. \tag{8.42}$$

Hence $m_u : m_d : m_s \approx 1 : 1.5 : 30$. The effective action furthermore predicts particles with masses

$$m_{uu}^2 = \frac{2m_u}{m_u + m_d} m_{\pi^+}^2 = 0.0155 \text{ GeV}^2, \tag{8.43}$$

$$m_{dd}^2 = \frac{2m_d}{m_u + m_d} m_{\pi^+}^2 = 0.0235 \text{ GeV}^2, \tag{8.44}$$

$$m_{ss}^2 = \frac{2m_s}{m_u + m_s} m_{K^+}^2 = 0.473 \text{ GeV}^2. \tag{8.45}$$

The candidates π^0, η and η' do not fit into the $n = 3$ formulas either. The effective action obtained so far must be wrong.

This is an aspect of the notorious $U(1)$ problem. The problem is the chiral $U(1)$ invariance contained in $U(n_\mathrm{f}) \times U(n_\mathrm{f})$. These are the transformations of the type $V_\mathrm{L} = V_\mathrm{R}^\dagger = \exp(i\omega)\,\mathbb{1}$, or more generally, transformations $V_\mathrm{L} = V_\mathrm{R}^\dagger$ with $\det V_\mathrm{L} \neq 1$. We know that this invariance of the classical QCD action is broken in the quantum theory by 'anomalies': QCD has only approximate $SU(n_\mathrm{f}) \times SU(n_\mathrm{f})$ chiral symmetry, plus the flavor $U(1)$ symmetry $V_\mathrm{L} = V_\mathrm{R} = \exp(i\omega)\,\mathbb{1}$ corresponding to quark-number conservation.

The resolution of the $U(1)$ problem through 'anomalies' turned out to be a difficult but very interesting task. Here we shall simply add terms to the action that break the chiral $U(1)$ symmetry and see what this implies for the mass formulas. We need to introduce terms of the type $\det U$, which is invariant under $SU(n_\mathrm{f}) \times SU(n_\mathrm{f})$ but not under $U(n_\mathrm{f}) \times U(n_\mathrm{f})$:

$$\det U \to \det(V_\mathrm{L} U V_\mathrm{R}^\dagger) = \det(U)\, \det(V_\mathrm{L} V_\mathrm{R}^\dagger) = \det U, \tag{8.46}$$

for $V_{\mathrm{L,R}} \in SU(n)$. A term like

$$\Delta' S = \int d^4x\, c\, (\det U + \det U^\dagger) \tag{8.47}$$

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would do, but considerations of the large-n_c behavior of 'n_c-color QCD' suggest using instead the form

$$\Delta' S = \int d^4x\, c\, (\mathrm{Tr}\, \ln U - \mathrm{Tr}\, \ln U^\dagger)^2. \tag{8.48}$$

In fact, $c \propto 1/n_c$. (Both choices for $\Delta' S$ lead to the same form of the mass matrix for the neutral pseudoscalar mesons to be derived below.) Writing $U = \exp(i\alpha)$ gives

$$\Delta' S = -\int dx\, 4c \left(\sum_f \alpha_{ff} \right)^2. \tag{8.49}$$

Hence, the masses of α_{fg}, $f < g$, are unaffected, but the α_{ff} modes are now coupled by a mass matrix of the form

$$m^2_{ff,gg} = 2B m_f \delta_{fg} + \lambda, \quad \lambda = 16c/f^2, \tag{8.50}$$

or

$$m^2 = 2B \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} + \lambda \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{8.51}$$

We shall treat the quark-mass term as a perturbation to the λ term. For $m_f = 0$ we have the eigenvectors and eigenvalues

$$\phi_0 = \frac{1}{\sqrt{3}}(1,1,1), \quad m^2 = 3\lambda, \tag{8.52}$$

$$\phi_3 = \frac{1}{\sqrt{2}}(1,-1,0), \quad m^2 = 0, \tag{8.53}$$

$$\phi_8 = \frac{1}{\sqrt{6}}(1,1,-2), \quad m^2 = 0. \tag{8.54}$$

Using $m_{u,d,s}$ as a perturbation (in the way familiar from quantum mechanics) leads to the following mass formulas:

$$m^2_{\eta'} = 3\lambda + B(\tfrac{2}{3} m_u + \tfrac{2}{3} m_d + \tfrac{2}{3} m_s), \tag{8.55}$$

$$m^2_{\pi^0} = B(m_u + m_d), \tag{8.56}$$

$$m^2_{\eta} = B(\tfrac{1}{3} m_u + \tfrac{1}{3} m_d + \tfrac{4}{3} m_s), \tag{8.57}$$

which hold for the mass ratios (8.42) up to tiny corrections. The eigenvectors are also interesting, but here we merely mention that π^0 and η are mainly ϕ_3 and ϕ_8, whereas the η' is predominantly ϕ_0. From (8.55) we can determine the chiral $U(1)$ breaking strength λ,

$$3\lambda = m^2_{\eta'} - \tfrac{1}{2}(m^2_{\pi^0} + m^2_{\eta}) = 3(0.252)\ \mathrm{GeV}^2. \tag{8.58}$$

The mass terms in the effective action depend on four parameters, Bm_u, Bm_d, Bm_s and λ. Hence we have two predictions for the five pseudoscalar masses:

$$m_{\pi^0}^2 = m_{\pi^+}^2, \tag{8.59}$$

$$m_\eta^2 = \tfrac{1}{6}(m_{uu}^2 + m_{dd}^2) + \tfrac{2}{3}m_{ss}^2 = 0.322\ \mathrm{GeV}^2, \tag{8.60}$$

which agree reasonably well with experiment. It should be kept in mind that electromagnetic corrections, which affect in particular the electrically charged particles, are neglected.

In the early days the near equality of m_{π^0} and m_{π^+} was interpreted as an aspect of approximate flavor symmetry, $m_u \approx m_d$. Now we know that m_d is substantially larger than m_u and that the approximate flavor symmetry is due to approximate chiral symmetry, $m_{u,d} \ll \sqrt{\sigma}$, the spontaneous-symmetry-breaking pattern $U(n_\mathrm{f}) \times U(n_\mathrm{f}) \to U(n_\mathrm{f})_\mathrm{flavor}$, and the flavor-singlet character of the chiral-anomaly term $\Delta' S$.

8.3 Chiral anomalies

The Noether argument tells us that to each continuous symmetry of the action corresponds a 'conserved current' j^μ, $\partial_\mu j^\mu = 0$, and a conserved 'charge' $Q = \int d^3x\, j^0(x)$, $\partial_0 Q = 0$. This is true in the classical theory but not necessarily in the quantum theory, which needs more specification than merely giving the action, such as the precise definition of the path integral. In case the quantum analog of j^μ is not conserved, one speaks of an *anomaly* $\mathcal{A} \equiv \partial_\mu j^\mu$. In four space–time dimensions $\mathcal{A}$ is typically $\propto \epsilon^{\kappa\lambda\mu\nu}\,\mathrm{Tr}\,(G_{\kappa\lambda}G_{\mu\nu})$, where $G_{\mu\nu}$ is a gauge-field tensor. Relations like $\partial_\mu j^\mu = \mathcal{A}$ can be found in perturbation theory by studying correlation functions of j^μ and $\mathcal{A}$ with other fields.

Chiral anomalies correspond to diagrams of the type shown in figure 8.1, and related diagrams, in which one vertex corresponds to a (polar) vector current, $\bar\psi i\gamma^\mu\psi$, or an axial vector current, $\bar\psi i\gamma^\mu\gamma_5\psi$, and the other two vertices to gauge fields. There must be an odd number of γ_5's in the trace over the Dirac indices ($\mathrm{Tr}\,(\gamma_5\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu) = 4i\epsilon_{\kappa\lambda\mu\nu}$), hence the name 'chiral anomalies'. These γ_5 may come from the gauge-field vertices or from the current.

In QCD there is no γ_5 associated with the gauge-field vertices and only axial vector currents can have an anomaly. In the Euclidean formulation

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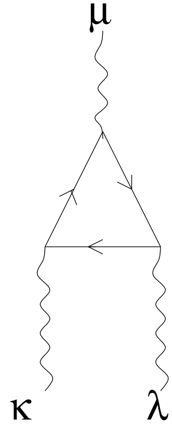

Fig. 8.1. Triangle diagram in which chiral anomalies show up.

their divergence reads†

$$\partial_\mu(\bar{\psi}_f i\gamma_\mu\gamma_5\psi_g) = (m_f + m_g)\bar{\psi}_f i\gamma_5\psi_g + \delta_{fg}\, 2iq, \tag{8.61}$$

$$q = \frac{g^2}{32\pi^2}\,\epsilon_{\kappa\lambda\mu\nu}\,\mathrm{Tr}\,(G_{\kappa\lambda}G_{\mu\nu}). \tag{8.62}$$

For zero quark masses the right-hand side of (8.61) is the anomaly. The vector currents have no such anomaly. Their divergence reads

$$\partial_\mu(\bar{\psi}_f i\gamma_\mu\psi_g) = i(m_f - m_g)\bar{\psi}_f\psi_g, \tag{8.63}$$

which is zero in the symmetry limit $m_f = m_g$, hence also in the chiral limit $m_f = m_g = 0$. The right-hand sides of the divergence equations (8.61) and (8.63) are zero for the currents corresponding to $SU(n_\mathrm{f}) \times SU(n_\mathrm{f})$ symmetry, obtained by contraction of $\bar{\psi}_f i\gamma^\mu P_{\mathrm{L,R}}\psi_g$ with the $n_\mathrm{f}^2 - 1$ flavor $SU(n_\mathrm{f})$ generators $(\lambda_k)_{fg}/2$, $\mathrm{Tr}\,\lambda_k = 0$. Hence, the anomaly in (8.61) breaks only chiral $U(1)$ invariance corresponding to $\lambda_0 \propto \mathbb{1}$ with $\partial_\mu \sum_f \bar{\psi}_f i\gamma_\mu\gamma_5\psi_f = 2n_\mathrm{f} iq$.

The quantity q is called the topological charge density. Continuum gauge fields on topologically non-trivial manifolds (such as the torus T^4 which corresponds to periodic boundary conditions) fall into so-called Chern classes characterized by an integer, the Pontryagin index or

† The gauge fields are normalized here according to $S = -\int d^4x\, G^k_{\mu\nu}G^k_{\mu\nu}/4 + \cdots$ with $G^k_{\mu\nu} = \partial_\mu G^k_\nu - \partial_\nu G^k_\mu + g f_{klm} G^l_\mu G^m_\nu$.

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topological charge $Q_{\rm top}$:

$$Q_{\rm top} = \int d^4x\, q(x). \tag{8.64}$$

An important example of configurations with topological charge is given by superpositions of (anti)instantons. The latter are solutions of the Euclidean field equations (hence they are saddle points in the path integral) with localized action density, non-perturbative action $S = 8\pi^2/g^2$ and topological charge ± 1. In this context we mention also the Atiyah–Singer index theorem:

$$Q_{\rm top} = n_+ - n_-, \tag{8.65}$$

where $n_\pm$ are the numbers of zero modes (eigenvectors with zero eigenvalue) of the Dirac operator $\gamma_\mu D_\mu$ with chirality $\gamma_5 = \pm 1$ (cf. problem (iii)).

The significance of all this for our pseudoscalar particle mass spectrum is that the phenomenologically required chiral $U(1)$ breaking is present indeed in *quantum* chromodynamics, provided that gauge-field configurations with topological charge density give sufficiently important contributions to the path integral. The analysis of this is complicated [101] but fortunately there is a simple approximate formula which expresses the effect of the chiral anomaly on the neutral pseudoscalar masses, the Witten–Veneziano formula [102, 103]:

$$\lambda \approx \frac{1}{2f_\pi^2}\, \chi_{\rm top}, \qquad \text{no quarks.} \tag{8.66}$$

Here λ is the $U(1)$-breaking mass term introduced in (8.50) and $\chi_{\rm top}$ is the *topological susceptibility*,

$$\chi_{\rm top} = \int d^4x\, \langle q(x)q(0)\rangle. \tag{8.67}$$

Note that in (8.66) $\chi_{\rm top}$ is to be computed in the pure gauge theory without quarks, although it can of course also be evaluated in the full theory with dynamical fermions. From (8.58) we have $\chi_{\rm top} \approx (180\,{\rm MeV})^4$.

8.4 Chiral symmetry and the lattice

With Wilson's fermion method chiral symmetry is explicitly broken by two large mass terms $\propto M$ and r/a in the action. With staggered fermions there are not even any flavor indices to act on with chiral

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transformations (cf. (6.67)). So we have a problem translating the continuum lore in the previous section to the lattice using these fermion formulations. As will be mentioned at the end of this section, this problem can be avoided or at least ameliorated with formulations of the 'Ginsparg–Wilson variety', but an introduction in terms of Wilson fermions is instructive and this will be the focus of our immediate attention.

Let us first derive the Noether currents of chiral symmetry. Consider the fermion part of the action,

$$\begin{aligned} S_{\mathrm{F}} = &\sum_{x\mu f} \tfrac{1}{2}\left[\bar{\psi}_{fx}(r-\gamma_\mu)U_{\mu x}\psi_{fx+\hat{\mu}} + \bar{\psi}_{fx+\hat{\mu}}(r+\gamma_\mu)U^\dagger_{\mu x}\psi_{fx}\right] \\ &- \sum_{xf} M_f \bar{\psi}_{fx}\psi_{fx}, \end{aligned} \tag{8.68}$$

where we have explicitly indicated the flavor index f in addition to x. We make a variation of ψ and $\bar{\psi}$ that looks like a chiral transformation,

$$\psi'_{fx} = V_{fgx}\,\psi_{gx}, \quad \bar{\psi}'_{fx} = \bar{\psi}_{gx}\,\bar{V}_{gxf}, \tag{8.69}$$

in which V has been generalized to depend on the space–time point x:

$$V_{fgx} = \delta_{fg} + i\omega^{\mathrm{L}}_{fgx}P_{\mathrm{L}} + i\omega^{\mathrm{R}}_{fgx}P_{\mathrm{R}} + O(\omega^2) \tag{8.70}$$

$$\equiv \delta_{fg} + i\omega^{\mathrm{V}}_{fgx} + i\omega^{\mathrm{A}}_{fgx}\gamma_5 + \cdots, \tag{8.71}$$

$$\bar{V}_{fgx} = \delta_{fg} - i\omega^{\mathrm{V}}_{fgx} + i\omega^{\mathrm{A}}_{fgx}\gamma_5 + \cdots, \tag{8.72}$$

where $\omega_{fg} = \omega^*_{gf}$ for L, R, V and A. The variation of the action can be written for infinitesimal ω's as

$$\begin{aligned} \delta S_{\mathrm{F}} &= S_{\mathrm{F}}(\psi', \bar{\psi}') - S_{\mathrm{F}}(\psi, \bar{\psi}) \\ &= -\sum_x \Big[V^\mu_{fgx}\partial_\mu\omega^{\mathrm{V}}_{fgx} + A^\mu_{fgx}\partial_\mu\omega^{\mathrm{A}}_{fgx} \\ &\qquad + D^{\mathrm{V}}_{fgx}\omega^{\mathrm{V}}_{fgx} + D^{\mathrm{A}}_{fgx}\omega^{\mathrm{A}}_{fgx} + O(\omega^2)\Big] \end{aligned} \tag{8.73}$$

$$= \sum_x \Big[(\partial'_\mu V^\mu_{fgx} - D^{\mathrm{V}}_{fgx})\omega^{\mathrm{V}}_{fgx} + (\partial'_\mu A^\mu_{fgx} - D^{\mathrm{A}}_{fgx})\omega^{\mathrm{A}}_{fgx}\Big]. \tag{8.74}$$

We recall that ∂_μ and ∂'_μ denote the forward and backward lattice derivatives, $\partial_\mu\omega_x = \omega_{x+\hat{\mu}} - \omega_x$ and $\partial'_\mu\omega_x = \omega_x - \omega_{x-\hat{\mu}}$. In (8.73), the terms without derivatives of ω are due to symmetry breaking, while the terms containing $\partial_\mu\omega$ are a consequence of the fact that ω depends on x – they serve to identify the vector (V^μ) and axial-vector (A^μ) currents. The classical Noether argument can be given as follows: if ψ and $\bar{\psi}$

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satisfy the equations of motion, the action is stationary, $\delta S_{\mathrm{F}} = 0$, and consequently

$$\partial'_\mu V^\mu_{fg} = D^{\mathrm{V}}_{fg}, \quad \partial'_\mu A^\mu_{fg} = D^{\mathrm{A}}_{fg}. \tag{8.75}$$

Explicitly we have

$$V^\mu_{fgx} = \frac{1}{2}\big[\bar\psi_{fx} i(\gamma_\mu - r)U_{\mu x}\psi_{gx+\hat\mu} + \bar\psi_{fx+\hat\mu} i(\gamma_\mu + r)U^\dagger_{\mu x}\psi_{gx}\big], \tag{8.76}$$

$$A^\mu_{fgx} = \frac{1}{2}\big[\bar\psi_{fx} i\gamma_\mu\gamma_5 U_{\mu x}\psi_{gx+\hat\mu} + \bar\psi_{fx+\hat\mu} i\gamma_\mu\gamma_5 U^\dagger_{\mu x}\psi_{gx}\big], \tag{8.77}$$

$$D^{\mathrm{V}}_{fgx} = i(M_f - M_g)\bar\psi_{fx}\psi_{gx}, \tag{8.78}$$

$$\begin{aligned} D^{\mathrm{A}}_{fgx} = {} & (M_f + M_g)\bar\psi_{fx} i\gamma_5\psi_{gx} \\ & - \frac{r}{2}\sum_\mu \big[\bar\psi_{fx} i\gamma_5 (U_{\mu x}\psi_{gx+\hat\mu} + U^\dagger_{\mu x-\hat\mu}\psi_{gx-\hat\mu}) \\ & \qquad + (\bar\psi_{fx+\hat\mu} U^\dagger_{\mu x} + \bar\psi_{fx-\hat\mu} U_{\mu x-\hat\mu}) i\gamma_5\psi_{gx}\big]. \end{aligned} \tag{8.79}$$

We see that, in the flavor-symmetry limit $M_f = M_g = M$, the vector-current divergence $D^{\mathrm{V}} = 0$. For the axial-vector divergence the story is more subtle: we can set all mass parameters M_f and r to zero, in which case $D^{\mathrm{A}} = 0$, but then we get back the species doublers, which is not Wilson's method. To get chiral symmetry without fermion doubling, we have to take the continuum limit. In the classical continuum limit we expect D^{A}_{fg} to be proportional to the quark masses because then the mass terms in the action reduce to $\int d^4x\, \bar\psi m\psi$, by construction (recall (6.58)):[2]

$$D^{\mathrm{A}}_{fg}(x) = (m_f + m_g)\bar\psi_f(x) i\gamma_5\psi_g(x) + O(a). \tag{8.80}$$

Hence, the *classical* D^{A} vanishes in the chiral limit, which is 'Noether's theorem' for Wilson fermions.

In the quantum theory the fields become operators. Their correlation functions can be obtained with the path integral. Consider the expectation value of an arbitrary set of fields $\phi_1 \cdots \phi_n \equiv F$, composed of the fermion fields and/or gauge fields,

$$\langle F\rangle = \frac{1}{Z}\int D\bar\psi D\psi\, DU\, e^S\, F, \quad Z = \int D\bar\psi D\psi\, DU\, e^S, \tag{8.81}$$

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and let us make the transformation of variables (8.69). A transformation of variables cannot change the integrals, so $Z' = Z$ and $\langle F\rangle' = \langle F\rangle$. However, by following how the path-integral measure and the integrant transform we can derive useful relations, called Ward–Takahashi identities. The path-integral measure is *invariant*,

$$D\bar{\psi}' D\psi' \equiv \prod_{x\alpha f} d\bar{\psi}'_{x\alpha f}\, d\psi'_{x\alpha f} = \prod_{x\alpha f} d\bar{\psi}_{x\alpha f}\, d\psi_{x\alpha f} (\det V_x \det \bar{V}_x)^{-1}$$
$$= D\bar{\psi} D\psi, \tag{8.82}$$

because $\det V \det \bar{V} = \det(V\bar{V}) = \det(V_{\rm L}V_{\rm R}^\dagger P_{\rm L} + V_{\rm R}V_{\rm L}^\dagger P_{\rm R}) = \det(V_{\rm L}V_{\rm R}^\dagger) \times \det(V_{\rm R}V_{\rm L}^\dagger) = \det(V_{\rm L}V_{\rm R}^\dagger V_{\rm R}V_{\rm L}^\dagger) = \det \mathbb{1} = 1$. On the other hand, the change in the action is given in (8.74) and the fields in F may also change, $F' = F + \sum_{fgx} \omega^{\rm A}_{fgx} \partial F/\partial^{\rm A}_{fgx} + \cdots + A \to V$. So we get the identity, e.g. for a chiral transformation,

$$0 = \frac{\partial}{\partial\omega^{\rm A}_{fgx}} \langle F\rangle' = \frac{\partial}{\partial\omega^{\rm A}_{fgx}} \left(\frac{1}{Z'} \int D\bar{\psi}'\, D\psi'\, DU\, e^{S'}\, F'\right)$$
$$= \frac{1}{Z} \int D\bar{\psi} D\psi\, DU\, \frac{\partial}{\partial\omega^{\rm A}_{fgx}} \left(e^{S'}\, F'\right) = \left\langle \frac{\partial S}{\partial\omega^{\rm A}_{fgx}}\, F + \frac{\partial F}{\partial\omega^{\rm A}_{fgx}} \right\rangle$$
$$= \left\langle (\partial'_\mu A^\mu_{fgx} - D^{\rm A}_{fgx})\, F + \frac{\partial F}{\partial\omega^{\rm A}_{fgx}} \right\rangle. \tag{8.83}$$

The content of such relations may be studied in perturbation theory. To one-loop order this can be done in the way seen in section 3.4 and the problems in section 6.6. A crucial example is the case $F = G_{\kappa x} G_{\lambda y}$, for which $\partial F/\partial\omega^{\rm A}_{fg} = 0$ since F consists only of gluon fields, which leads to triangle-diagram contributions of the type shown in figure 8.1. A calculation [70] shows that, for this case, $D^{\rm A}_{fg} \to (m_f + m_g)\bar{\psi}_f i\gamma_5 \psi_g + \delta_{fg}\, 2iq$ in the continuum limit. The topological-charge-density contribution is due to the Wilson mass term and the coefficient of q is formally $\propto r$, but actually independent of r, provided that it is non-zero.

Another example is $F = \psi_{fx}\bar{\psi}_{gy}$, which leads to the conclusion that, for this case, $D^{\rm A}_{fg} \to (m_f + m_g)\kappa_{\rm P}\bar{\psi}_f i\gamma_5\psi_g - (\kappa_{\rm A} - 1)\partial_\mu A^\mu_{fg}$, where $\kappa_{\rm P}$ and $\kappa_{\rm A}$ are finite renormalization constants of order g^2 (cf. [70, 109, 104]). The topological charge density does not contribute here in this order because it is already of order g^2.

At one-loop order we get the same contributions as those found in continuum perturbation theory because the bare vertex functions reduce to the continuum ones (in the balls around the origin of the loop-momentum integration) in the classical continuum limit. There are also

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differing contributions, which are, however, only contact terms.† These translate into the finite renormalization constants κ.‡ The anomaly is a singlet under flavor transformations (i.e. $\propto \delta_{fg}$) because the r-mass term is a flavor singlet. Note that, by a global finite chiral transformation, we could transform $r\delta_{fg} \to r(\bar{V}V)_{fg}$, implying that $M_{\rm c} \propto \bar{V}V$. However, this is merely a change of reference frame and the physics cannot depend on it. The quark masses have to be identified as the mismatch between M and $M_{\rm c}$.

The above examples show the phenomenon of operator mixing: operators (fields) with the same quantum numbers tend to go over into linear combinations of each other in the continuum limit (the scaling region). Such mixing is restricted by the symmetries of the model and there is more mixing on the lattice than there is in the continuum because there is less symmetry on the lattice. The κ's above are due to the chiral-symmetry breaking of the Wilson mass term at non-zero lattice spacing. On general grounds of scaling and universality one assumes these results to be qualitatively valid also non-perturbatively. One introduces renormalized field combinations that are finite as $a \to 0$ that satisfy some standard normalization conditions. Before writing these down, let us introduce a lattice field that reduces to the topological charge density q in the classical continuum limit. There are many of course, as usual, e.g. the one introduced in [105],

$$q_x = -\frac{1}{32\pi^2}\left[\sum_{\kappa\lambda\mu\nu} \epsilon_{\kappa\lambda\mu\nu} \mathrm{Tr}\,(U_{\kappa\lambda x} U_{\mu\nu x})\right]_{\rm symmetrized}, \tag{8.84}$$

where the symmetrization is such that q_x transform as a scalar under lattice rotations. Denoting the renormalized fields by a 'bar', they can be written as [104]

$$\bar{A}^{\mu}_{fg} = \kappa_{\rm A} A^{\mu}_{fg} + \delta_{fg}(Z_{\rm A} - 1)\kappa_{\rm A}\frac{1}{2n_{\rm f}}\sum_f \partial'_\mu A^{\mu}_{ff}, \tag{8.85}$$

$$\begin{aligned}\bar{D}^{\rm A}_{fg} = {} & D^{\rm A}_{fg} + (\kappa_{\rm A} - 1)\partial'_\mu A^{\mu}_{fg} \\ & + \delta_{fg}(Z_{\rm A} - 1)\kappa_{\rm A}\frac{1}{2n_{\rm f}}\sum_f \partial'_\mu A^{\mu}_{ff},\end{aligned} \tag{8.86}$$

† Recall that contact corresponds in momentum space to polynomials in the momenta, of degree less than or equal to the mass dimension of the vertex function under consideration.

‡ In the literature these κ's are often denoted by Z, which notation we have reserved for renormalizations diverging when $a \to 0$.

$$\partial'_\mu \bar{A}^\mu_{fg} = \bar{D}^{\mathrm{A}}_{fg}, \tag{8.87}$$

$$\bar{q} = \kappa_q q - i(Z_{\mathrm{A}} - 1)\kappa_{\mathrm{A}} \frac{1}{2n_{\mathrm{f}}} \sum_f \partial'_\mu A^\mu_{ff}, \tag{8.88}$$

where Z_{A} is a diverging renormalization constant of order g^4. The operator subtractions $\propto (Z_{\mathrm{A}} - 1)$ are suggested by analysis at two-loop order in the continuum [106]. In the quenched approximation $Z_{\mathrm{A}} = 1$. In the scaling region

$$\bar{D}^{\mathrm{A}}_{fg} = (m_f + m_g)\kappa_{\mathrm{P}} \bar{\psi}_f i\gamma_5 \psi_g + \delta_{fg}\, 2i\bar{q} + O(a), \tag{8.89}$$

with $m_f = M_f - M_{\mathrm{c}}$. Similar analysis of Ward–Takahashi identities shows that the vector currents V^μ_{fg} need no finite renormalization, $\bar{V}^\mu_{fg} = V^\mu_{fg}$, $\kappa_{\mathrm{V}} = 1$. The reason is that they are conserved if $m_f = m_g$ even for $a \neq 0$.

The implications of the lattice Ward–Takahashi identities can of course be studied also non-perturbatively. As a first step one can use only external gauge fields with $F = 1$ and test the index theorem (8.65), using topologically non-trivial gauge fields transcribed from the continuum to the lattice [107, 108]. Adding dynamical gauge fields, we can then also use the Ward–Takahashi identities to determine the renormalization constants κ in the quenched approximation [104, 70, 109]. The computation of the topological susceptibility turns out to be complicated by the fact that $\langle \bar{q}_x \bar{q}_y \rangle$ has divergent contact terms that severely influence the value of $\sum_x \langle \bar{q}_x \bar{q}_0 \rangle$. One can try to subtract this contribution,

$$\chi_{\mathrm{top}} = \frac{1}{V} \sum_{xy} \langle \bar{q}_x \bar{q}_y \rangle_U - \text{contact contribution} \tag{8.90}$$

(assuming periodic boundary conditions, space–time volume $V \to \infty$), but it is hard to define it unambiguously [110]. In practice it appears to work well [111]. By 'cooling' the gauge fields after they have been generated by a Monte Carlo process this problem can be reduced further (see e.g. [112] and also [108]).

A different approach to the topological susceptibility is to accept that the configurations in the path integral are inherently not smooth functions of space–time and to avoid defining a topological integer from a collection of wildly fluctuating lattice variables. Instead, one can return to the physical role played by χ_{top} and derive the Witten–Veneziano formula entirely within the lattice formulation. This can be done by studying the pseudoscalar meson contribution in the $\langle \bar{A}^\mu \bar{A}^\nu \rangle$

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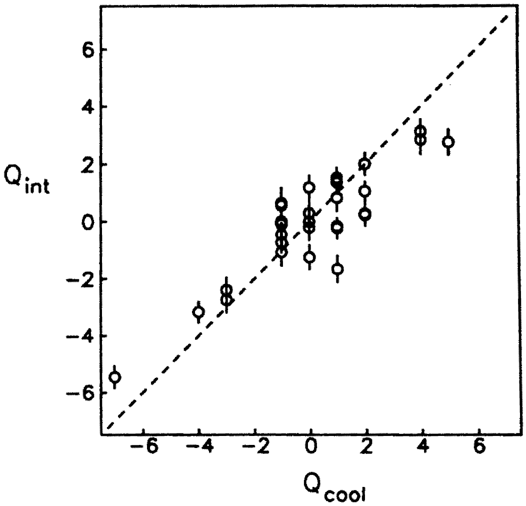

Fig. 8.2. Correlation between 'fermionic' and 'cooling' topological charge assignments for 32 $SU(3)$ gauge-field configurations at $\beta = 6.0$. From [117].

and $\langle \bar{D}^{\rm A} \bar{D}^{\rm A} \rangle$ correlators. The analysis is subtle [104] but results in the simple formula

$$\chi_{\rm top} = \frac{\kappa_{\rm P}^2 m_f^2}{V} \, \langle {\rm Tr}\,[\gamma_5 S_{ff}(U)]\, {\rm Tr}\,[\gamma_5 S_{ff}(U)] \rangle_{\rm U}\,. \tag{8.91}$$

Here $S_{ff}(U)$ is the fermion propagator in the gauge field U and the trace is over all non-flavor indices (x, a and α). The large-$n_{\rm c}$ limit is not taken in this derivation, only the quenched approximation. From this, the formula in terms of $\bar{q}$ can be understood from (8.87), (8.89), and $\sum_x \partial'_\mu \bar{A}^\mu = 0$ for periodic boundary conditions. A derivation for staggered fermions can also be given [113]. The limit $m_f \to 0$ is needed in order to avoid divergences (this limit must be carefully controlled by taking m_f at the lower end of a scaling window that extends to zero as $a \to 0$).

In the two-dimensional $U(1)$ model the properties of (8.91) have been studied and compared with the index theorem as well as with definitions of $\chi_{\rm top}$ in terms of the gauge field only [114, 115]. The staggered form was explored in numerical $SU(3)$ simulations [116, 117]. Figure 8.2 shows that the individual topological charges obtained with this 'fermionic method' are at $\beta = 6/g^2 = 6.0$ already quite correlated to the charges obtained with the cooling method. This is expected to improve at higher β but at lower β the gauge fields are too 'rough' on the lattice scale for notions of topology to make sense (also, the staggered-fermion renormalization factor κ_{P} becomes uncannily very large [116]). The resulting $\chi_{\mathrm{top}} \approx (154\pm17\ \mathrm{MeV})^4$ seems a bit low compared with the experimental value following from the Witten–Veneziano formula $(180\ \mathrm{MeV})^4$, but this may be due to the somewhat low value $a^{-1} = 1900$ MeV used for conversion to physical units. Using $a^{-1} = 2216$ MeV inferred from the values 1934 MeV ($\beta = 5.9$) and 2540 MeV ($\beta = 6.1$) recorded in [98] would give $\chi_{\mathrm{top}} \approx (180 \pm 20\ \mathrm{MeV})^4$.

By contracting the currents with the $n_{\mathrm{f}}^2 - 1$ $SU(n_{\mathrm{f}})$ generators $\lambda^k_{fg}/2$, we can form the left- and right-handed currents $j^{\mathrm{L,R}}_{\mu k} = (\bar{V}^{\mu}_{fg} \pm \bar{A}^{\mu}_{fg}) \times(\lambda_k)_{fg}/4$. According to (8.78) and (8.89), these currents and the $U(1)$ vector current $V_\mu = \sum_f V^{\mu}_{ff}$ are conserved in the limit $m_f \to 0$. Further Ward–Takahashi identities can be derived to fix renormalization constants and ensure that the currents satisfy 'current algebra' [118]. The corresponding charges would then satisfy the algebra of generators of $SU(n_{\mathrm{f}}) \times SU(n_{\mathrm{f}})$, were it not that the symmetry is supposed to be broken spontaneously. It should also be possible to introduce the QCD theta parameter (cf. problem (iv)).

From the chiral-symmetry point of view there are now much better lattice fermion methods. Ginsparg and Wilson made a renormalization-group 'block-spin' transformation for fermions from the continuum to the lattice, paying special attention to chiral symmetry [124]. More recently such transformations were studied in search of 'perfect actions' [125]. The continuum action is chirally symmetric for zero mass parameters but this symmetry is hidden in the resulting lattice action, because the blocking transformation to the lattice breaks chiral symmetry to avoid fermion doubling. Writing the massless fermion action as $S_{\mathrm{F}} = -\bar{\psi} D \psi$, chiral symmetry in the continuum can be expressed as $\gamma_5 D + D\gamma_5 = 0$. On the lattice there is a remnant of this: the blocked D satisfies the Ginsparg–Wilson relation

$$\gamma_5 D + D\gamma_5 = aD\, 2R\gamma_5\, D, \tag{8.92}$$

where we used matrix notation also for the space–time indices; R is a matrix commuting with γ_5 that enters in the renormalization-group blocking transformation. It is *local*, which means that R_{xy} falls off exponentially fast as $|x - y| \to \infty$ (on the lattice scale, in physical units it resembles a delta function). So D_{xy} practically anticommutes with γ_5 for physical separations, provided that it is itself local, as it should be (this is a basic requirement for universality). Taking (8.92) as a starting point, one can take $R_{xy} = \frac{1}{2}\,\delta_{xy}$. Dirac matrices D_{xy} satisfying

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(8.92) are complicated, because for given x all y contribute, albeit with exponentially falling magnitude as $|x-y|$ increases. An explicit solution [126], arrived at via the 'overlap' approach to chiral gauge theories (see the next section), has the form

$$aD = 1 - A(A^\dagger A)^{-1/2}, \quad A = 1 - aD_\mathrm{W}, \quad R = \tfrac{1}{2}, \tag{8.93}$$

where D_W is Wilson's lattice Dirac operator with zero bare mass ($r = 1$, $M = 4/a$). Adding mass terms the resulting lattice QCD action has very nice properties with respect to broken chiral symmetry and topology, which can be studied again by deriving Ward–Takahashi identities [130].

Moreover, the resulting action has (for $m = 0$) an exact chiral symmetry [131] under

$$\delta\psi = i\omega\gamma_5\big(1 - \tfrac{1}{2}aD\big)\,\psi, \quad \delta\bar{\psi} = i\bar{\psi}\big(1 - \tfrac{1}{2}aD\big)\,\omega, \tag{8.94}$$

with infinitesimal ω_{fg}. (Note that such a finite chiral transformation is non-local as it involves arbitrarily high powers of D.) The chiral anomaly in this formulation comes from a non-invariance of the fermion measure [131], similar to continuum derivations [132]. Domain-wall fermions [128, 129] are closely related. At the time of writing the research into these directions is very active; for a review, see [135]. Applications to the topological susceptibility can be found in [136, 137].

8.5 Spontaneous breaking of chiral symmetry

We now turn to the question of *spontaneous* chiral-symmetry breaking. One would like to compute the expectation value of the order field $\bar{\psi}_f P_\mathrm{L}\psi_g$ at vanishing quark masses and verify that $SU(n_\mathrm{f}) \times SU(n_\mathrm{f})$ symmetry is broken spontaneously to $SU(n_\mathrm{f})$. As for the $O(n)$ model (cf. (3.157)), this could be done by introducing explicit symmetry-breaking quark masses and studying the infinite-volume limit.

However, with Wilson fermions we cannot simply use $\bar{\psi}_f P_\mathrm{L}\psi_g$ as an order field because the cancellation of the chiral-symmetry breaking by the M and r terms is a subtle issue. Even for free fermions $\langle\bar{\psi}_f P_\mathrm{L}\psi_g\rangle \neq 0$ at $m_f = M_f - 4r/a = 0$: it diverges in the continuum limit (cf. problem (ii)). The chiral-symmetry breaking causes $\bar{\psi}_f P_\mathrm{L}\psi_g$ to mix with the unit operator, with a coefficient $c(g^2, m)\,\delta_{fg} = [c_0(g^2)a^{-3} + c_1(g^2)ma^{-2} + c_2(g^2)m^2a^{-1}]\,\delta_{fg}$ that diverges in the limit $a \to 0$ (for simplicity we assume here all quark masses to be equal). The identification of $c(g^2, m)$, and a computation of the subtracted expectation value $\langle\bar{\psi}_f P_\mathrm{L}\psi_g\rangle - c(g^2, m)\delta_{fg}$ in the limit of zero quark mass

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is a hazardous endeavor, because several powers of a^{-1} have to cancel out, and, moreover, because the gauge coupling g^2 also depends on a.

On the other hand, we have seen (section 7.5) that the relation (8.36), i.e. $m_{fg}^2 \approx B(m_f + m_g)$, $B = 2v/f_\pi^2$, is borne out by the numerical results when $f \neq g$. So, using this relation, we could *define* $\langle\bar\psi\psi\rangle$ by $\sum_f \langle\bar\psi_f\psi_f\rangle = -n_{\rm f} B f_\pi^2$, with $m_u = m_d \equiv m_{ud} \to 0$, or $\langle\bar\psi_u\psi_u\rangle = -f_\pi^2 m_\pi^2/2m_{ud}$. Using renormalized quark masses instead of the bare m_{ud} would give a renormalized B and a correspondingly renormalized $\langle\bar\psi_f\psi_f\rangle$. Note that $m_{ud}\langle\bar\psi_u\psi_u\rangle$ should be renormalization-group invariant. Using e.g. $m_{ud} = 3.4$ MeV (the result of [97]), the value of $\langle\bar\psi_u\psi_u\rangle$ is about $(290\,\mathrm{MeV})^3$ in the MS-bar scheme on the scale $\mu = 2$ GeV.

We may appeal to continuity at any fixed gauge coupling $0 < g < \infty$ by sending the symmetry-breaking parameters M and r to zero and studying spontaneous breaking of chiral symmetry there. Actually, at $r = M = 0$ the staggered-fermion form (6.66) of the action is more appropriate and it shows that the Dirac labels are to be interpreted as flavor indices. At $M = r = 0$ the symmetry of the action enlarges to $U(4n_{\rm f}) \times U(4n_{\rm f})$. Combining the Dirac ($\alpha$) and flavor ($f$) indices into one label $A = (\alpha, f)$ the transformation is

$$\chi_{Ax} \to \left(\mathcal{V}^{\rm L}_{AB}\frac{1-\epsilon_x}{2} + \mathcal{V}^{\rm R}_{AB}\frac{1+\epsilon_x}{2}\right)\chi_{Bx},$$

$$\bar\chi_{Ax} \to \bar\chi_{Bx}\left(\mathcal{V}^{\rm R\dagger}_{BA}\frac{1-\epsilon_x}{2} + \mathcal{V}^{\rm L\dagger}_{BA}\frac{1+\epsilon_x}{2}\right) \tag{8.95}$$

with $\epsilon_x = (-1)^{x_1+\cdots x_4}$. Moreover, in the scaling region at weak coupling the *staggered-fermion flavors* also emerge, implying a further multiplication of the number of flavors by four. With such a large number of flavors (i.e. $16n_{\rm f}$) and only three colors, asymptotic freedom is lost as soon as $n_{\rm f} > 1$ (recall (7.54)) and we can expect continuity in $M, r \to 0$ only if we consider a sufficiently large number of colors $n_{\rm c}$. Assuming this to be the case, we can get analytic insight at strong coupling [119, 120, 83, 84, 82, 121].

At strong gauge coupling and for a large number of colors the exact continuous symmetry breaks spontaneously as $U(4n_{\rm f}) \times U(4n_{\rm f}) \to U(4n_{\rm f})$, resulting in $16n_{\rm f}^2$ NG bosons. The baryons acquire a mass $\propto n_{\rm c}$ from the spontaneous symmetry breaking. Suppose now $n_{\rm f} = 3$. Turning on the symmetry-breaking parameters M and r, it is possible to keep the pions, kaons and eta massless by choosing $M = M_{\rm c}(g, r)$ and in the process all other NG bosons become massive. We need to

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keep $M - M_{\rm c}(g,r)$ infinitesimally positive to let the order field $\bar\psi_f P_{\rm L}\psi_g$ acquire its expectation value in the direction $-v\delta_{fg}$, with real positive v. Otherwise we might induce complex v, which corresponds to non-zero $\langle\bar\psi_f i\gamma_5\psi_g\rangle$ and spontaneous breaking of parity [119, 122]. The situation is similar in the model using continuous time (the Hamiltonian method), in which the symmetry breaking at strong coupling is actually $U(4n_{\rm f}) \to U(2n_{\rm f})\times U(2n_{\rm f})$. At non-zero r the $U(1)$ problem is also qualitatively resolved by giving the flavor-singlet boson a (small) non-zero mass [119, 122, 123].

So in this way, connecting with $M = r = 0$, we can understand spontaneous breaking of chiral symmetry in multicolor QCD with Wilson fermions. However, it is conceptually simpler to study the corresponding order field for staggered fermions.

The staggered-fermion action (6.67) has for $m = 0$ a chiral $U(1)\times U(1)$ symmetry, which is (8.95) with phase factors $\mathcal{V}^{\rm L,R}$ (since there is no spin–flavor index A to act on). The axial $U(1)$ transformation contained in this $U(1)\times U(1)$, i.e. $\mathcal{V}^{\rm L} = \mathcal{V}^{\rm R*} = \exp(i\omega^{\rm A})$, is in the staggered-fermion interpretation [74] a flavor-*non-singlet* transformation, of the form $\exp(i\omega^{\rm A}\xi_5)$ with ${\rm Tr}\,\xi_5 \neq 0$. In the scaling region, where the symmetry enlarges to $SU(4)\times SU(4)\times U(1)_{\rm V}$, this ξ_5 is a linear combination of the generators of $SU(4)$. So it is natural to study spontaneous breaking of this $U(1)$ remnant of $SU(4)\times SU(4)$ chiral symmetry. A suitable order field for this symmetry is the coefficient of the quark mass m in the action, i.e. $\bar\chi_x\chi_x$, which together with $\epsilon_x\bar\chi_x\chi_x$ forms a doublet under the chiral $U(1)$. In the scaling region $\bar\chi_x\chi_x \to \sum_{f=1}^4 \bar\psi_f(x)\psi_f(x)$ and $\epsilon_x\bar\chi_x\chi_x \to \sum_{fg}\bar\psi_f(x)\xi_{5fg}\gamma_5\psi_g(x)$.

A definition of $\Sigma \equiv -\langle\bar\chi\chi\rangle$ in which the quark mass is introduced as a symmetry breaker, which is to be taken to zero after taking the infinite-volume limit, as in (3.157) for the $O(n)$ model, is hard to implement in practice. This can be circumvented by using a method based on the eigenvalues of the Dirac operator, which we shall denote by $D(U)$, where U is a given gauge-field configuration. In the continuum D is anti-Hermitian, $D(U) = -D(U)^\dagger$, and therefore its eigenvalues are purely imaginary. On the lattice the staggered-fermion Dirac matrix

$$D(U)_{xa,yb} = \sum_\mu \eta_{\mu x}[(U_{xy})_{ab}\delta_{x+\hat\mu,y} - (U_{yx})_{ab}\delta_{y+\hat\mu,x}] \tag{8.96}$$

has the same property (unlike the Wilson–Dirac operator $D(U) = \not{D}(U) + M - W(U)$ which is the sum of an anti-Hermitian and a Hermitian matrix). Let u_r denote the complete orthonormal set of eigenvectors with

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eigenvalues $i\lambda_r$,

$$D\, u_r = i\lambda_r\, u_r, \quad u_r^\dagger u_s = \delta_{rs}, \quad \sum_r u_r u_r^\dagger = 1\!\!1. \tag{8.97}$$

The matrix $\epsilon_{xy} = \epsilon_x \delta_{xy}$ anticommutes with D, $D\epsilon = -\epsilon D$, so

$$D\, \epsilon u_r = -i\lambda_r\, \epsilon u_r, \tag{8.98}$$

and for every eigenvalue λ_r there is also an eigenvalue $-\lambda_r$. The expectation value of the order field at finite quark mass, $\Sigma \equiv -\langle \bar{\chi}_x \chi_x \rangle$, can be written as

$$\begin{aligned}\Sigma &= \frac{1}{V} \sum_x \langle \chi_x \bar{\chi}_x \rangle = \frac{1}{V} \langle \mathrm{Tr}\,[(D+m)^{-1}] \rangle_U \\ &= \frac{1}{V} \left\langle \sum_r \frac{1}{i\lambda_r + m} \mathrm{Tr}\,(u_r u_r^\dagger) \right\rangle_U = \frac{1}{V} \left\langle \sum_r \frac{1}{i\lambda_r + m} \right\rangle_U \\ &= \frac{1}{V} \left\langle \sum_r \frac{m}{\lambda_r^2 + m^2} \right\rangle_U .\end{aligned} \tag{8.99}$$

In terms of the spectral density $\rho(\lambda)$,

$$\rho(\lambda) = \frac{1}{V\,\Delta\lambda} \langle n(\lambda + \Delta\lambda, \lambda) \rangle_U, \quad \Delta\lambda \to 0, \tag{8.100}$$

where $n(\lambda + \Delta\lambda, \lambda) = \sum_r \theta((\lambda + \Delta\lambda - \lambda_r)\theta(\lambda_r - \lambda)$ is the number of eigenvalues of $D(U)$ in the interval $(\lambda, \lambda + \Delta\lambda)$, this can be written as [138]

$$\Sigma = \lim_{m\to 0} \lim_{V\to\infty} \int d\lambda\, \rho(\lambda)\, \frac{m}{\lambda^2 + m^2} \tag{8.101}$$

$$= \pi \rho(0). \tag{8.102}$$

Here we used the identity $\lim_{\epsilon\to 0} \epsilon/(x^2 + \epsilon^2) = \pi\delta(x)$. Note that $\rho(\lambda)$ depends on the gauge coupling, the dynamical quark mass and the volume V; it furthermore satisfies $\rho(\lambda) = \rho(-\lambda)$ because of (8.98).

The spectral density can be computed numerically by counting the number of eigenvalues in small bins and figure 8.3 shows an example for the gauge group $SU(2)$ in the quenched approximation. The quantity $\rho(\lambda)/V$ in the plot is our density $\rho(\lambda)$ in lattice units, i.e. $a^3\rho(\lambda)$. The value $\rho(0)$ may be determined by extrapolating $\lambda \to 0$, it is nearly equal to the value in the first bin. The resulting $a^3\Sigma$ drops rapidly from the value 0.1247(22) to 0.00863(48) as β is increased from 2.0 to 2.4 and the lattice spacing decreases accordingly. Actually, the value $\beta = 2.0$ is near

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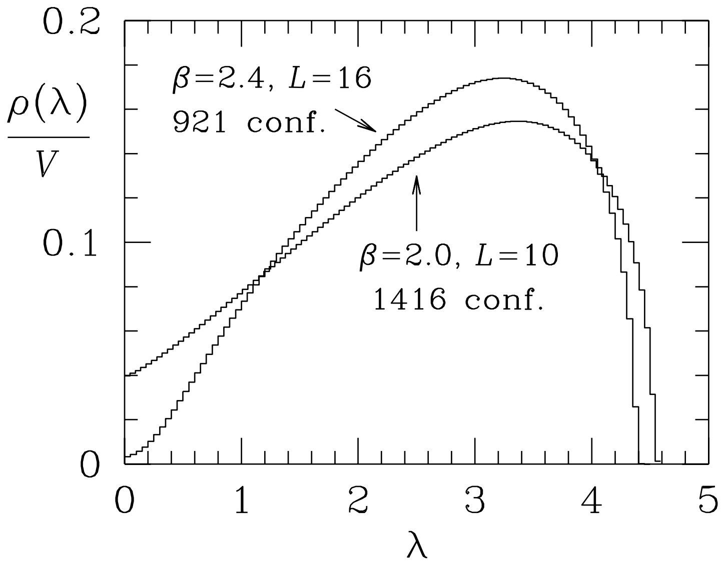

Fig. 8.3. The quenched spectral density in lattice units of the $SU(2)$ staggered-fermion matrix for two values of $\beta = 4/g^2$ and lattice volumes $V = L^4$. The number of gauge-field configurations used is also indicated. From [142].

the edge of the scaling region on the strong-coupling side, while $\beta = 2.4$ is more properly in the scaling region[3] (see e.g. figure 8 in [69]).

The volume dependence of Σ obtained this way is expected to be small. This can be made more precise by using scaling arguments based on a remarkable connection with random-matrix theory (for a review see [140]). From (8.100) and (8.102) we see that, in the neighborhood of the origin, $\rho(\lambda)$ behaves like $1/[V\Sigma\, d(\lambda)]$, with $d(\lambda)$ the average distance between two eigenvalues. This observation leads one to define the *microscopic spectral density* [141]

$$\rho_{\rm s}(\zeta) = \frac{1}{\Sigma}\,\rho\left(\frac{\zeta}{\Sigma V}\right), \tag{8.103}$$

in which the region around the origin is blown up by the factor ΣV. The function $\rho_{\rm s}(\zeta)$ is predicted to be a universal function in random-matrix theory depending only on the gauge group and the representation carried by the fermions, provided that it is evaluated for gauge fields with fixed topological charge $Q_{\rm top} = \nu$. For example for $SU(n_{\rm c} > 2)$ and $n_{\rm f}$

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dynamical fermions it is given by

$$\rho_{\mathrm{s}}^{(\nu)}(\zeta) = \frac{\zeta}{2}\left[J_{n_{\mathrm{f}}+\nu}(\zeta)^2 - J_{n_{\mathrm{f}}+\nu-1}(\zeta)J_{n_{\mathrm{f}}+\nu+1}(\zeta)\right], \tag{8.104}$$

where J is the Bessel function. So, by fitting $\rho^{(\nu)}(\lambda)$ according to (8.103) and (8.104) with only one free parameter (Σ), one obtains the *infinite*-volume value of Σ.

Zero modes corresponding to the index theorem should be ignored here. This is not easy with staggered fermions as the would-be zero modes fluctuate away from zero and can be identified only by the expectation value of the 'staggered γ_5' (cf. (8.134) and (8.135)) [107, 113, 114, 116].

The prediction (8.104) works well using staggered fermions and $SU(2)$ [142] or $SU(3)$ [143] quenched ($n_{\mathrm{f}} = 0$) gauge-field configurations at relatively strong gauge coupling and selecting[4] $\nu = 0$. The dependence on the fermion representation and the pattern of chiral-symmetry breaking is studied for various gauge groups in [144]. A (finite-temperature) study with $n_{\mathrm{f}} = 2$ dynamical fermions is given in [145].

A recent study [146] using related finite-size techniques with Neuberger's Dirac operator (8.93) in quenched $SU(3)$ at $\beta = 5.85$ gave the result $a^3\Sigma = 0.0032(4)$. A further non-perturbative computation [147] of the appropriate multiplicative renormalization factor then allows conversion value $\Sigma_{\overline{\mathrm{MS}}}(\mu = 2\,\mathrm{GeV}) \approx (270\,\mathrm{MeV})^3$ in the MS-bar scheme.

8.6 Chiral gauge theory

In QED and QCD the representation of the gauge group carried by all left- and right-handed fields is *real* up to equivalence. For example, in QCD, let Ω be the fundamental representation of $SU(3)$. The left-handed fields are $\psi_{\mathrm{L}} = P_{\mathrm{L}}\psi$ and $(\bar\psi_{\mathrm{R}}C)^{\mathrm{T}} = P_{\mathrm{L}}(\bar\psi C)^{\mathrm{T}}$, with C the charge-conjugation matrix (cf. appendix D), while the right-handed fields are $\psi_{\mathrm{R}} = P_{\mathrm{R}}\psi$ and $(\bar\psi_{\mathrm{L}}C)^{\mathrm{T}} = P_{\mathrm{R}}(\bar\psi C)^{\mathrm{T}}$. The fields transform as

$$\psi_{\mathrm{L}} \to \Omega\psi_{\mathrm{L}}, \quad (\bar\psi_{\mathrm{R}}C)^{\mathrm{T}} \to \Omega^*(\bar\psi_{\mathrm{R}}C)^{\mathrm{T}}, \text{ left;} \tag{8.105}$$

$$\psi_{\mathrm{R}} \to \Omega\psi_{\mathrm{R}}, \quad (\bar\psi_{\mathrm{L}}C)^{\mathrm{T}} \to \Omega^*(\bar\psi_{\mathrm{L}}C)^{\mathrm{T}}, \text{ right.} \tag{8.106}$$

Taking ψ_{L} and $\bar\psi_{\mathrm{R}}$ in pairs, the representation of the gauge group has the form of a direct sum $\Omega \oplus \Omega^*$, which is real up to the equivalence transformation $\Omega \oplus \Omega^* \to \Omega^* \oplus \Omega$.

The fundamental representation of $U(1)$, a phase factor, is evidently complex, but the fundamental representation of $SU(2)$ is real up to

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equivalence: $\Omega^* = \exp(i\omega_k \frac{1}{2}\sigma_k)^* = \sigma_2\Omega\sigma_2$. It is not difficult to see, e.g. by looking at the element $\exp(i\omega_8 \frac{1}{2}\lambda_8)$, that the fundamental representation of $SU(3)$ is complex. The adjoint representation of $SU(n)$ is real for all n.

Chiral gauge theories are models in which the representation of the gauge group is truly *complex* (no reality up to equivalence). The Standard Model, which has the gauge group $U(1) \times SU(2) \times SU(3)$, is a chiral gauge theory, as can be seen by looking at the $U(1)$ charges of the left- and right-handed fields. Since this model is able to describe all known interactions up till now, it is evidently desirable to give it a non-perturbative lattice formulation.† This turns out to be very difficult.

To get a glimpse of the problem, consider a $U(1)$ model with continuum action

$$S_{\rm F} = -\int d^4x\, \bar{\psi}_{\rm L}\gamma^\mu(\partial_\mu - igq_{\rm L}A_\mu)\psi_{\rm L} + {\rm L} \to {\rm R}, \tag{8.107}$$

assuming for the moment no further quantum numbers (no 'flavors'). The fields transform as

$$\psi_{\rm L} \to e^{i\omega q_{\rm L}}\psi_{\rm L}, \quad \bar{\psi}_{\rm R} \to e^{-i\omega q_{\rm R}}\bar{\psi}_{\rm R}, \text{ left}; \tag{8.108}$$

$$\psi_{\rm R} \to e^{i\omega q_{\rm R}}\psi_{\rm R}, \quad \bar{\psi}_{\rm L} \to e^{-i\omega q_{\rm L}}\bar{\psi}_{\rm L}, \text{ right}, \tag{8.109}$$

and we see, e.g. from the pair $\psi_{\rm L}$ and $\bar{\psi}_{\rm R}$, that the model is chiral if the charges $q_{\rm R}$ and $q_{\rm L}$ are not equal. Assuming this to be the case, it follows that $\bar{\psi}\psi = \bar{\psi}_{\rm R}\psi_{\rm L} + \bar{\psi}_{\rm L}\psi_{\rm R}$ is not gauge invariant. Consequently there can be no mass term for the fermions. We also cannot use $P_{\rm L} + P_{\rm R} = 1$ and eliminate γ_5 from the action. So the gauge-field couples also to an axial-vector current (there is a term $\bar{\psi}i\gamma^\mu\gamma_5\psi A_\mu$ in the action), instead of only to vector currents as in QED and QCD. These features are generic for chiral gauge theories: no mass terms and axial-vector currents that are dynamical (rather than being just symmetry currents of global chiral symmetry). With γ_5 prominent in the vertex functions we may expect chiral anomalies to play a role. This has been analyzed in perturbation theory in the continuum, with the conclusion that the above model is unsatisfactory because gauge invariance is spoilt by anomalies due to contributions involving triangle diagrams (cf. figure 8.1). These problems can be avoided by extending the model to contain more than one 'flavor', with charges $q_{{\rm L}f}$ and $q_{{\rm R}f}$, such that the anomalies cancel out between the different flavors, which requires $\sum_f (q_{{\rm L}f}^3 - q_{{\rm R}f}^3) = 0$. The model

† We consider $U(1)$-neutral right-handed neutrino fields $\psi_{\rm R}$ (and $\bar{\psi}_{\rm R}$) as part of the Standard Model.

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and its representation of the gauge group are then called 'anomaly-free'. Such considerations played an important role in the construction of the Standard Model. It was noticed that the anomalies in the lepton sector could cancel out against those in the quark sector [148, 151]. This is how the Standard Model is anomaly-free.

We continue with the choice of integer $q_{\rm L} = q$, $q_{\rm R} = 0$. With just one flavor the continuum model is then anomalous. Let us see what happens if we put the above model naively on the lattice.

A Euclidean naive lattice action is easy to write down:

$$S_{\rm F} = -\sum_{x\mu} \tfrac{1}{2}\left[\bar{\psi}_x \gamma_\mu (U_{\mu x} P_{\rm L} + P_{\rm R})\psi_{x+\hat{\mu}} - \bar{\psi}_{x+\mu}\gamma_\mu (U_{\mu x} P_{\rm L} + P_{\rm R})\psi_x\right], \tag{8.110}$$

with a path integral

$$Z = \int D\bar{\psi} D\psi\, DU\, e^S, \tag{8.111}$$

in which $S = S_{\rm F} + S_{\rm U}$ with $S_{\rm U}$ the usual plaquette action. The lattice action and measure are gauge invariant (for the fermion measure this follows from (8.82) with $V_{{\rm L}x} = \exp(i\omega_x)$, $V_{{\rm R}x} = 1$). In this model the right-handed $\psi_{\rm R}$ and the left-handed $(\bar{\psi}_{\rm R} C)^{\rm T}$ are just free fields, they are not coupled to the gauge fields. However, the species-doubling phenomenon induces 16 fermion flavors in the scaling region. What are the charges of these fermions?

To answer this question consider a fermion line in a diagram with a gauge-field line attached to it. The corresponding mathematical expression is

$$\cdots S(p) V_\mu(p, q; k) S(q) \cdots, \tag{8.112}$$

where $S(p)$ is the massless naive fermion propagator and $V_\mu(p, q; k)$ the bare vertex function for the model ($p = q + k$). Such vertex functions have been determined in problem (i) in section 6.6 for the case of QED, and to get these for the present case we only have to make the substitution $g\gamma_\mu \to g\gamma_\mu P_{\rm L}$ in (6.99), giving

$$V_\mu(p, q; k) = ig\gamma_\mu P_{\rm L} \tfrac{1}{2}\left(e^{iaq_\mu} + e^{iap_\mu}\right). \tag{8.113}$$

To interpret this expression in the scaling region for fermion species A we use (6.26) and (6.31) and substitute $p \to k_A + p$ and $q \to k_A + q$ into

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(8.112) ($k_A = \pi_A/a$),

$$\cdots S(k_A+p)V_\mu(k_A+p, k_A+q; k)S(k_A+q)\cdots \tag{8.114}$$
$$= \cdots S_A^\dagger \left[\frac{-i\gamma_\kappa p_\kappa}{p^2}\, gi\gamma_\mu \tfrac{1}{2}(1-\epsilon_A\gamma_5)\, \frac{-i\gamma_\lambda q_\lambda}{q^2} + O(a) \right] S_A \cdots,$$

where we used (6.30) and (6.28) for the terms not involving γ_5 and

$$\epsilon_A \gamma_\mu \gamma_5 = S_A \gamma_\mu \gamma_5 S_A^\dagger \cos(\pi_{A\mu}). \tag{8.115}$$

Using (6.29) we find $\epsilon_A = +1$ for $\pi_A = \pi_0$, $\epsilon_A = -1$ for $\pi_A = \pi_1, \ldots, \pi_4$, $\epsilon_A = +1$ for $\pi_A = \pi_{12}, \ldots, \pi_{34}$, $\epsilon_A = -1$ for $\pi_A = \pi_{123}, \ldots, \pi_{234}$ and $\epsilon_A = +1$ for $\pi_A = \pi_{1234}$, such that

$$\sum_A \epsilon_A = 1 - 4 + 6 - 4 + 1 = 0. \tag{8.116}$$

From (8.114) we conclude that in the scaling region we have eight continuum fields with $q_{\mathrm{L}}^{\mathrm{cont}} = 1$ ($\epsilon_A = 1$), and eight with $q_{\mathrm{R}}^{\mathrm{cont}} = 1$ ($\epsilon_A = -1$), in addition to the uncharged fields: the lattice has produced flavors (the species doublers) such that the anomalies cancel out.[5] However, since all the $q_{\mathrm{L}}^{\mathrm{cont}}$ and $q_{\mathrm{R}}^{\mathrm{cont}}$ are equal, the model is not a chiral gauge theory! It is just QED with eight equal-mass Dirac fermions (plus eight neutral Dirac fermions).

A natural suggestion for a lattice formulation of the Standard Model is to give the doubler fermions masses of order of the lattice cutoff through Wilson-type Yukawa couplings with the Higgs field [119, 149, 150]. Because the Standard Model is anomaly-free the set of doublers in such a formulation is anomaly-free too: the set of 15 doublers of some fermion contributes to anomalies with the same strength as this fermion (opposite in sign, $\sum_{A=2}^{16} \epsilon_A = -1$). Insofar as anomalies are concerned there is no objection to the decoupling of the doublers. Other objections [151, 152], namely that masses of the order of the cutoff might not be possible because renormalized couplings cannot be arbitrarily strong (triviality is expected to play a role here), do not apply if new phases come into play. This is indeed the case. On turning on the Wilson–Yukawa couplings one runs into a new phase, called the paramagnetic strong-coupling (PMS) phase [153]. Unfortunately, in this phase the doublers bind with the Higgs field to give right-handed fields transforming in the same representation as the left-handed fields, or vice-versa, and the result is a non-chiral (vector) gauge theory in the scaling region [154, 155]. Other models [156] (see also [157]) which can be put into this Wilson–Yukawa framework have been argued to fare

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the same fate [158]. Another approach is to keep the doublers as heavy physical particles in mirror fermion models [159].

How to formulate a lattice chiral gauge model? This problem is difficult because of the peculiar symmetry breaking of chiral anomalies. We want them to be there without interfering with gauge invariance. Nielson and Ninomiya [160] formulated a no-go theorem that has to be overcome first. They used a Hamiltonian description (continuous time and spatial lattice), and, loosely speaking, the theorem states that, under cherished conditions such as translation invariance, locality and Hermiticity, a free-fermion lattice model with a $U(1)$ invariance has always an equal number of left- and right-handed fermions of a given $U(1)$ charge. The $U(1)$ is supposed to be contained in the gauge group and the implication is that the model can be extended only into an interacting gauge theory that is 'vector' and not chiral. A simpler Euclidean formulation is given in [161]. An extension to an effective action formulation is given in [129].

The Euclidean reasoning runs as follows. Suppose that we replace $\sin k_\mu \to F_\mu(k)$ in the naive fermion propagator. This corresponds to the translation-invariant action of the form (ignoring possible neutral fields)

$$S_{\rm F} = -\sum_{xy\mu} \bar\psi_x \gamma_\mu P_{\rm L} \tilde F_\mu(x-y)\psi_y, \quad iF_\mu(k) = \sum_x \exp(-ikx)\tilde F_\mu(x), \tag{8.117}$$

which has a $U(1)$ invariance $\psi \to \exp(i\omega q)\,\psi$, $\bar\psi \to \exp(-i\omega q)\,\bar\psi$. Hermiticity is easy to state in the Hamiltonian formulation: $\hat H^\dagger = \hat H$. In the Euclidean formulation we require the spatial part of the action ($\mu = 1,2,3$) to be Hermitian and extend this to $\mu = 4$ by covariance. Then Hermiticity means that $F_\mu(x)^* = -F_\mu(-x)$, so $F_\mu(k)$ is real. Locality means that $\tilde F_\mu(x)$ approaches zero sufficiently fast as $|x| \to \infty$. This implies that its Fourier transform is not singular and we shall assume $F_\mu(k)$ to be smooth, i.e. it and all its derivatives are continuous. If $F_\mu(k)$ has isolated zeros of first order then the model has a particle interpretation. Near a zero at $k = \bar k$,

$$F_\mu(k) = Z_{\mu\nu}(k_\nu - \bar k_\nu) + O((k-\bar k)^2), \tag{8.118}$$

with coefficients $Z_{\mu\nu}$ forming a matrix Z with $\det Z \neq 0$. We write

$$Z = RP, \tag{8.119}$$

with R an orthogonal matrix and P a symmetric positive matrix. The

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matrix R can be absorbed in a unitary transformation,

$$\gamma_\mu(1-\gamma_5)R_{\mu\nu} = \Lambda^\dagger\gamma_\nu(1-\epsilon\gamma_5)\Lambda, \quad \epsilon = \det R = \pm 1. \tag{8.120}$$

For $\epsilon = 1$, Λ is a rotation $\exp(\frac{1}{2}\varphi_{\mu\nu}[\gamma_\mu, \gamma_\nu])$, for $\epsilon = -1$, Λ is e.g. γ_4 times a rotation (cf. appendix D). So for k near $\bar{k}$ the fermion propagator is equivalent to the continuum expression

$$S(k) \approx \frac{-i\gamma_\mu p_\mu}{p^2}\,\frac{1-\epsilon\gamma_5}{2}, \quad p_\mu \equiv P_{\mu\nu}(k-\bar{k})_\nu, \tag{8.121}$$

which corresponds to a left- ($\epsilon = +1$) or right-handed ($\epsilon = -1$) fermion field.

Now comes input from topology: ϵ is the index of the vector field $F_\mu(k)$ of its zero at $k = \bar{k}$, i.e. the degree of the mapping $F_\mu/|F| = R_{\mu\nu}p_\nu/|p|$ onto S^4. The Poincaré–Hopf theorem states that the global sum of the indices equals the Euler characteristic χ_{E} of the manifold on which the vector field is defined: $\sum\epsilon = \chi_{\mathrm{E}}$. In our case this manifold is the momentum-space torus T^4, for which $\chi_{\mathrm{E}} = 0$. Hence, there must be an even number of zeros and in the continuum limit we have an equal number of left- and right-handed fermion fields with the same charge. The naive $U(1)$ model above is a typical illustration of the theorem.

To avoid these theorems we have to avoid some of their assumptions (including hidden assumptions). Giving up translation invariance (e.g. using a random lattice), Hermiticity (e.g. $S_{\mathrm{F}} = -\sum_{x\mu}\bar{\psi}_x\gamma_\mu\partial_\mu\psi_x$, which gives the complex $F_\mu(k) = (e^{ik_\mu}-1)/i$), or locality (e.g. the discontinuous $F_\mu(k) = 2\sin(k_\mu/2)$ (mod 2π) has only a zero at the origin but corresponds to $F_\mu(x)$ falling only like $|x|^{-1}$) tends to lead to other trouble (for a review, see [162]). The basic reason is that, with an exactly gauge-invariant action and fermion measure, there can be no anomaly, which means that it cancels out in one way or another, generically without the desired particle interpretation.

One line of approach is to give up gauge invariance at finite lattice spacing by working in a fixed gauge and adding counterterms such that gauge invariance is restored in the continuum limit [163, 164]. However, non-perturbative gauge fixing has its own complications, not least the existence of Gribov copies, i.e. configurations differing by a gauge transformation satisfying the same gauge condition. A gauge-fixed $U(1)$ model appears to have passed basic tests [166]. For a review see [168]. One may try to keep the fermions in the continuum, or on a finer lattice than the gauge-field lattice, and invoking restoration of gauge symmetry by the mechanism of [167]. See [168] for a review. Gauge-symmetry restoration was also invoked in models gauging non-invariant models, using Wilson fermions or gauging the staggered flavors [162, 169] but it failed in its simplest realization [170]. Further information can be found in the reviews presented at the Lattice meetings [171].

New developments that constitute a major advancement can be classified under the heading 'overlap' and 'Ginsparg–Wilson' fermions. Theorems of the Nielson–Ninomiya type are avoided by having an infinite number of fermion field components ('overlap'), and changing the definition of γ_5, such that it is as usual for $\bar{\psi}$ but for ψ it involves replacing γ_5 by

$$\hat{\gamma}_5 = \gamma_5(1 - aD), \tag{8.122}$$

where D is a Ginsparg–Wilson Dirac operator, together with an elaborate definition of the fermion measure in the path integral (apparently giving up Hermiticity on the lattice) [172]. The subject is beautiful and erudite and the reader is best introduced by the reviews [173, 174] ('overlap') and [175] ('Ginsparg–Wilson'). One may feel uncomfortable, though, about using formulations with an infinite number of field components; it runs contrary to the basic idea of being able to approach infinity from the finite.

8.7 Outlook

There is of course a lot more to lattice field theory than has been presented here. An introduction to finite temperature can be found in [9]. Simulation algorithms are introduced in [4, 10]; improved actions and electroweak matrix elements are discussed in [14, 15]. See also [16] for advanced material. For an introduction to simplicial gravity[6] see [17]. Non-perturbative lattice formulations of quantum fields out of equilibrium are still in their infancy.[7] For the current status of all this, see the proceedings of the 'Lattice' meetings.

8.8 Problems

(i) *The pion–nucleon σ model*

Consider an effective nucleon field N that is a doublet in terms of Dirac proton (p) and neutron (n) fields

$$N(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix}. \tag{8.123}$$

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The effective action of the pion–nucleon sigma model is given by

$$S_{\text{eff}} = -\int d^4x\, [\bar{N}\gamma^\mu\partial_\mu N + G\bar{N}(\phi P_{\text{R}} + \phi^\dagger P_{\text{L}})N] + S_{O(4)}, \quad (8.124)$$

where $S_{O(4)}$ is the scalar field action of the $O(4)$ model (equations (3.1) and (3.4)) and ϕ is a matrix field constructed out of the scalar fields,

$$\phi = \varphi^0 \mathbb{1} + i\sum_{k=1}^{3} \varphi^k \tau_k. \quad (8.125)$$

The τ_k are the three Pauli matrices, which act on the p and n components of N and G is the pion–nucleon coupling constant.

Show that the action is invariant under $SU(2) \times SU(2)$ transformations

$$N \to VN, \quad \bar{N} \to \bar{N}\bar{V}, \quad \phi \to V_{\text{L}}\phi V_{\text{R}}^\dagger, \quad V_{\text{L,R}} \in SU(2). \quad (8.126)$$

Verify that the transformation on the matrix scalar field ϕ is equivalent to an $SO(4)$ rotation on the φ^α. Hint: check that $\phi^\dagger\phi = \varphi^2\mathbb{1}$, $\det\phi = \varphi^2$; and hence that ϕ may be written as $\phi = \sqrt{\varphi^2}\,U$, $U \in SU(2)$.

This chiral invariance of the sigma-model action is a nice expression of the symmetry properties of the underlying quark–gluon theory. When the symmetry is spontaneously broken, such that the ground-state value of the scalar field is $\phi_{\text{g}} = f\mathbb{1}$, $f = \varphi^0_{\text{g}}$, the action acquires a mass term $Gf\bar{N}N$: the nucleon gets its mass from spontaneous breaking of chiral symmetry, $m_N = Gf$. This relation is in fair agreement with experiment. On introducing the weak interactions into the model one finds that f equals the pion decay constant, $f = f_\pi \approx 93$ MeV, while $G \approx 13$ from pion–nucleon-scattering experiments, so with $m_N = 940$ MeV we have to compare $m_N/f \approx 10$ with 13.

The field φ^0 is often denoted by σ, and φ^k by π^k, the sigma and pion fields. The pions are stable within the strong interactions but the σ is a very unstable particle with mass m_σ in the range 600–1200 MeV. Given $m_\pi = 140$ MeV and $m_\sigma = 900$ MeV, determine the other parameters in the action.

Reanalyze the model in 'polar coordinates' $\phi = \rho U$, $U \in SU(2)$ with ρ a single-component scalar field. Note that ρ plays the role of the matrix field H introduced in (8.19). What is its mass?

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In sections 8.1 the effective field ϕ is a general complex 4×4 matrix, which has eight independent real parameters, whereas the above ϕ has only the four real φ^α, which cannot incorporate chiral $U(1)$ transformations. Verify this, and work out a generalization in which ϕ has the general form. (Include in the action terms that break chiral $U(1)$.)

(ii) *Free fermion* $\langle\bar{\psi}\psi\rangle$

Consider free 'naive' fermions on the lattice (one flavor). Show that

$$\Sigma \equiv -\langle\bar{\psi}\psi\rangle = a^{-3} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{4am}{a^2m^2 + \sum_\mu \sin^2 k_\mu}, \tag{8.127}$$

and that it has the expansion

$$\Sigma = c_1 m a^{-2} + m^3[c_3 \ln(am) + c_3'] + \cdots, \tag{8.128}$$

where the $\cdots$ vanish as $a \to 0$. Hint: use (3.66).

Now consider free Wilson fermions. Show that for this case the expansion takes the form

$$\Sigma = c_0 a^{-3} + c_1 m a^{-2} + c_2 m^2 a^{-1} + m^3[c_3 \ln(am) + c_3'] + \cdots, \tag{8.129}$$

where $m = M - 4r/a$. Find expressions for the coefficients c_k.

(iii) *Research project: the index theorem*

Go through the following formal arguments.

In the continuum, let $D = \gamma_\mu[\partial_\mu - iG_\mu(x)]$ be the Dirac operator in an external gauge field G_μ in a finite volume with periodic (up to gauge transformations) boundary conditions. Consider the divergence equation for the flavor-singlet axial current

$$\partial_\mu \bar{\psi} i\gamma_\mu\gamma_5\psi = 2m\bar{\psi}i\gamma_5\psi + 2iq, \tag{8.130}$$

where we assumed that there is only one flavor. Taking the fermionic average and integrating over (Euclidean) space–time gives

$$0 = -2m\,\mathrm{Tr}\,[\gamma_5(m+D)^{-1}] + 2i\nu, \tag{8.131}$$

where the trace is over space–time and Dirac indices and $\nu = Q_{\rm top}$ is the topological charge.

Verify that iD is a Hermitian operator, $(iD)^\dagger = iD$.

Let f_s be the eigenvectors of D with (purely imaginary) eigenvalues $i\lambda_s$,

$$Df_s = i\lambda_s f_s, \quad D\gamma_5 f_s = -i\lambda_s \gamma_5 f_s, \tag{8.132}$$

and assume the eigenvectors to be orthogonal and complete,

$$f_s^\dagger f_t = \delta_{st}, \quad \sum_s f_s f_s^\dagger = 1. \tag{8.133}$$

Because f_s and $\gamma_5 f_s$ correspond generically to different eigenvalues,

$$f_s^\dagger \gamma_5 f_s = 0, \quad \lambda_s \neq 0. \tag{8.134}$$

For $\lambda_s = 0$, $[D, \gamma_5] f_s = 0$, so in this subspace we can look for simultaneous eigenvectors of D and γ_5. The eigenvalues of γ_5 are ± 1,

$$\gamma_5 f_s = \pm f_s, \quad \lambda_s = 0. \tag{8.135}$$

It follows that

$$\nu = m \sum_s \frac{\mathrm{Tr}\,(\gamma_5 f_s f_s^\dagger)}{m + i\lambda_s} = \sum_{s,\, \lambda_s = 0} f_s^\dagger \gamma_5 f_s = n_+ - n_-, \tag{8.136}$$

with $n_\pm$ the number of zero modes with chirality $\gamma_5 = \pm 1$.

Periodicity modulo gauge transformations is needed in order to allow non-zero topological charge. For the proper mathematical setting in the continuum, see e.g. [12]. Lattice studies using Wilson and staggered fermions are in [107, 108, 114, 115, 90], while [135] gives an introduction to Ginsparg–Wilson fermions. Choose one of these studies and reproduce (and possibly extend) its results.

(iv) *Research project: the theta parameter of QCD*

Consider the QCD action with generalized mass term

$$\int d^4x\, \bar{\psi}' m \psi', \quad m = m_{\mathrm{L}} P_{\mathrm{L}} + m_{\mathrm{L}}^\dagger P_{\mathrm{R}}, \tag{8.137}$$

in which m_{L} is a fairly arbitrary complex matrix. Assume that it can be transformed into a diagonal matrix by the transformation

$$V_{\mathrm{L}}^\dagger m_{\mathrm{L}} V_{\mathrm{R}} = m_{\mathrm{diag}} = \text{diagonal with entries} \geq 0. \tag{8.138}$$

Suppose this transformation is the result of a chiral transformation on the fermion fields (cf. (8.2)),

$$\psi' = V\psi, \quad \bar{\psi}' = \bar{\psi}\bar{V}. \tag{8.139}$$

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In continuum treatments the fermion measure is not invariant under such a transformation, but is produces the chiral anomaly in the form [132]

$$D\bar{\psi}' D\psi' = D\bar{\psi} D\psi \, e^{i\theta \int d^4x \, q(x)}, \quad \theta = \arg(\det m_{\rm L}). \tag{8.140}$$

So in terms of the un-primed fermion fields we have an additional term in the (Euclidean) action proportional to the topological charge,

$$S = -\int d^4x \left[\frac{1}{2g^2} \mathrm{Tr}\,(G_{\mu\nu} G_{\mu\nu}) + \bar{\psi}\gamma_\mu D_\mu \psi + \bar{\psi} m_{\rm diag} \psi - i\theta q \right]. \tag{8.141}$$

The original mass m may be the result of electroweak symmetry breaking. Experiments constrain the value of θ, which violates CP invariance, to be less than 10^{-9} in magnitude.

Our problem is to give a rigorous version of the above reasoning using the lattice regularization. With Wilson's fermion method the following steps get us going.

Consider the fermion determinant $\exp[\mathrm{Tr}\ln(\not{D} - W + M)]$, where M is arbitrary. In the scaling region M is close to the critical value $M_{\rm c}$; if not, then there is no continuum physics. So assume that $M = M_{\rm c} + m$, with $M_{\rm c} \propto r\mathbb{1}$ and m arbitrary as in the above continuum outline. With Wilson's fermion method the fermion measure is invariant under chiral transformations and the anomaly comes from the non-invariant term $\bar{\psi}(W - M_{\rm c})\psi$ in the action. So we have

$$\mathrm{Tr}\,[\ln(\not{D} - W + M_{\rm c} + m)] = \mathrm{Tr}\,\{\ln[\not{D} + \bar{V}(M_{\rm c} - W)V + m_{\rm diag}]\}. \tag{8.142}$$

To evaluate this consider a change δV of V. Then the above expression changes by

$$\begin{aligned} &\mathrm{Tr}\,\{[\delta\bar{V}(M_{\rm c} - W)V + \bar{V}(M_{\rm c} - W)\,\delta V] \\ &\qquad \times [\not{D} + \bar{V}(M_{\rm c} - W)V + m_{\rm diag}]^{-1}\}. \end{aligned} \tag{8.143}$$

Expanding this expression in terms of the gauge field leads to an infinite number of diagrams with external gauge-field lines impinging upon a closed fermion loop. The crucial point is now that the factor $M_{\rm c} - W$ in the numerator above suppresses the region of loop-momentum integration where $m_{\rm diag}$ has any influence. For example in momentum space at lowest order, $M_{\rm c} - W \to$

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$ra^{-1}\sum_\mu[1-\cos(ak_\mu)]$, and we therefore need a loop momentum k of order $a^{-1} \gg m_{\rm diag}$ to give a non-vanishing contribution. See [70] for an explicit computation of the triangle-diagram-like contributions. So we may as well set $m_{\rm diag}=0$ in (8.143). Then (8.143) can be rewritten in the form

$$\begin{aligned} &\mathrm{Tr}\,[\bar{V}^{-1}\,\delta\bar{V}(M_{\rm c}-W)(\not{D}+M_{\rm c}-W)^{-1} \\ &\qquad + \delta V\,V^{-1}(\not{D}+M_{\rm c}-W)^{-1}(M_{\rm c}-W)] \\ &= \mathrm{Tr}\,[(V_{\rm L}\,\delta V_{\rm L}^{-1}-V_{\rm R}\,\delta V_{\rm R}^{-1})\,\gamma_5\,(M_{\rm c}-W)(\not{D}+M_{\rm c}-W)^{-1}], \end{aligned} \tag{8.144}$$

where we used the fact that $\not{D}$, $M_{\rm c}$ and W are all flavor diagonal, and the cyclic property of the trace. Denoting the trace over space–time plus Dirac indices (excluding the flavor indices) by $\mathrm{Tr}_{\rm st}$ we have the result [70, 133, 134]

$$\mathrm{Tr}_{\rm st}\,[\gamma_5\,(M_{\rm c}-W)(\not{D}+M_{\rm c}-W)^{-1}] = Q_{\rm top}, \quad a\to 0. \tag{8.145}$$

Note that this result is *independent* of the r parameter [70], as long as it is non-zero. The coefficient of $Q_{\rm top}$ is given by

$$\mathrm{Tr}_{\rm flavor}\,(V_{\rm L}\,\delta V_{\rm L}^{-1}-V_{\rm R}\,\delta V_{\rm R}^{-1}) = \delta\ln[\det(V_{\rm R}V_{\rm L}^{-1})] = i\,\delta\arg(\det m_{\rm L}). \tag{8.146}$$

So one concludes that, in the continuum limit,

$$\begin{aligned} &\exp\{\mathrm{Tr}\,[\ln(\not{D}+M_{\rm c}-W+m)]\} \\ &\qquad = e^{i\theta Q_{\rm top}}\,\exp\{\mathrm{Tr}\,[\ln(\not{D}+M_{\rm c}-W+m_{\rm diag})]\}, \end{aligned} \tag{8.147}$$

$$\theta = \arg(\det m_{\rm L}), \tag{8.148}$$

which is equivalent to the continuum result.

By taking the continuum *limit* we have happily been able to ignore finite renormalization factors κ ($\kappa = 1+O(g^2)\to 1$, g^2 is the bare gauge coupling).

The problem with the above reasoning, taken from [149], is how to improve it such that it applies in a practical scaling region with g^2 not much less than 1.