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DATA NETWORK MODELS OF BURSTINESS

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Abstract

We review characteristics of data traffic which we term *stylized facts*: burstiness, long-range dependence, heavy tails, bursty behavior determined by high-bandwidth users, and dependence determined by users without high transmission rates. We propose an infinite-source Poisson input model which supplies traffic in adjacent time slots. We study properties of the model as slot width decreases and traffic intensity increases. This model has the ability to account for many of the stylized facts.

Keywords: Bursty traffic; M/G/∞ input model; infinite-source Poisson model; network modeling; limit distribution; Lévy process; Gaussian limit

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1. Introduction

Measurements on data networks often show empirical features that are surprising by the standards of classical queueing and telephone network models. Measurements often consist of data giving bit rate or packet rates. This means that a window resolution is selected (e.g. 10 seconds, 1 second, 10 milliseconds, 1 millisecond, etc.) and the number of bits or packets in adjacent windows or slots recorded. Significant examples include Duffy *et al.* (1993), Leland *et al.* (1993), and Willinger *et al.* (1995), (1997).

Despite the fact that collected data is for time slots of modest size, many of the theoretical attempts to create models to explain the empirical observations concentrate on large time scales and cumulative traffic over large time intervals. See, e.g. Heath *et al.* (1998), Kaj and Taqqu (2004), Konstantopoulos and Lin (1998), Levy and Taqqu (2000), Maulik and Resnick (2003), Mikosch *et al.* (2002), and Taqqu *et al.* (1997). For such models, it is difficult to find agreement with many existing data sets (Guerin *et al.* (2003)).

Many network data sets exhibit distinctive properties, which, in analogy with empirical finance, we will term *stylized facts*:

- Heavy tails abound (Leland *et al.* (1994), Willinger *et al.* (1998), Willinger and Paxson (1998), Willinger (1998)) for such things as file sizes (Arlitt and Williamson (1996), Resnick and Rootzén (2000)), transmission rates, and transmission durations (Maulik *et al.* (2002), Resnick (2003)).
- The number of bits or packets per slot exhibits long-range dependence across time slots (see, e.g. Leland *et al.* (1993) or Willinger *et al.* (1995)). There is also a perception of

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self-similarity, as the width of the time slot varies across a range of time scales exceeding a typical round-trip time.

- Network traffic is bursty with rare but influential periods of very high transmission rates punctuating typical periods of modest activity.

Burstiness, a somewhat vague concept, is an important feature of traffic because of the sudden peak loads it introduces into the network. Attempts to understand this phenomenon empirically (Sarvotham *et al.* (2005)) use the α/β-decomposition of users in which the α-users transmit large files at very high rate and the β-users transmit the rest. An alternative description creates a dichotomy between mice and elephants (Ben Azzouna *et al.* (2004)) depending on whether a file is typical or very large. Some stylized facts suggested by the stimulating empirical study of Sarvotham *et al.* (2005) are as follows.

- Large files over fast links contribute to α-traffic. The α-component consitutes a small fraction of the total workload but is entirely responsible for burstiness. Often a single dominant high-rate connection causes a burst.

- Most of the dependence structure across time slots is carried by the β-traffic. The long-range dependence structure of the β-traffic approximates the complete dependence structure.

- The quantity of traffic in a time window is distributionally approximated by the normal distribution when there are high levels of aggregation across users and heavy loading. β-traffic is much more likely to appear Gaussian than is α-traffic.

Owing to measurements being taken for fixed time slots, we begin our attempt to provide models explaining the empirically observed stylized facts by modeling the quantity of data in adjacent time slots of length δ. Then, to obtain approximations and to provide a clarified asymptotic picture of the behavior, we let $\delta \to 0$ and see what limits exist. In particular, we seek a model that explains the origins of burstiness.

2. Model description

The model for data traffic generation is a slight modification of the $M/G/\infty$ input model, also termed the infinite-source Poisson model, as we assume that the transmission rates are random (see Maulik *et al.* (2002)). We assume that a homogeneous Poisson process on $\mathbb{R}$ with points $\{\Gamma_k\}$ activates data transmission *sessions*. The parameter or rate of the Poisson process is $\lambda \equiv \lambda(\delta)$, and each transmission activation time Γ_k has three additional associated quantities, (R_k, L_k, F_k). These three quantities have the following physical interpretations:

- R represents the rate of the transmission.

- L represents the duration of the transmission.

- F represents the size of the transmitted file.

Obviously these three quantities are related by the following relation: $F = RL$.

We assume that the marks $\{(R_k, L_k, F_k), -\infty < k < \infty\}$ are independent and identically distributed and independent of $\{\Gamma_k\}$. The univariate marginal distributions of the triple are

$$G(x) = \mathrm{P}[F_1 \le x], \qquad F_R(x) = \mathrm{P}[R_1 \le x], \qquad F_L(x) = \mathrm{P}[L_1 \le x].$$

We suppose that all three distributions are heavy tailed, i.e.

$$\bar{G}(x) = x^{-\alpha_F} L_F(x), \qquad \bar{F}_R(x) = x^{-\alpha_R} L_R(x), \qquad \bar{F}_L(x) = x^{-\alpha_L} L(x),$$

where L_F, L_R, and L are all slowly varying and we assume that the three tail parameters satisfy

$$1 < \alpha_F, \alpha_R, \alpha_L < 2.$$

There is empirical evidence justifying these assumptions: see Ben Azzouna *et al.* (2004), Campos *et al.* (2005), Guerin *et al.* (2003), Heffernan and Resnick (2005), Leland *et al.* (1994), Maulik *et al.* (2002), Park and Willinger (2000), Resnick (2003), (2004), Riedi and Willinger (2000), Sarvotham *et al.* (2005), and Willinger *et al.* (1995).

Under these assumptions, the counting function of the points $\{(\Gamma_k, R_k, L_k, F_k)\}$ on $\mathbb{R} \times [0, \infty)^3$,

$$N = \sum_k \varepsilon_{(\Gamma_k, R_k, L_k, F_k)}$$

(where $\varepsilon_{\boldsymbol{x}}$ is the probability measure putting all mass at $\boldsymbol{x}$), is a *Poisson random measure* with mean measure

$$\lambda \, \mathrm{d}s \, \mathrm{P}[(R_1, L_1, F_1) \in (\mathrm{d}r, \mathrm{d}l, \mathrm{d}u)] =: \mu^{\#}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}l, \mathrm{d}u).$$

See, e.g. Kallenberg (1983), Neveu (1977), or Resnick (1987, p. 135), (1992, Chapter 4.4).

For a time window of length δ, we will consider weak limits of the process

$$\boldsymbol{A}(\delta) := \{A(k\delta, (k+1)\delta], \ -\infty < k < \infty\}, \tag{2.1}$$

as $\delta \downarrow 0$. Here $A(k\delta, (k+1)\delta]$ represents the total amount of work input to the system in the kth time slot, $(k\delta, (k+1)\delta]$. We will define this precisely for $k = 0$; the definitions for other values of k will be obvious by analogy.

Distinguish four disjoint regions in $\mathbb{R} \times [0, \infty)^3$:

$$\begin{aligned}
\mathcal{R}^{>0,1} &= \{(s, r, l, u) \colon 0 < s \le \delta, \ 0 < s + l \le \delta\}, \\
\mathcal{R}^{>0,2} &= \{(s, r, l, u) \colon 0 < s \le \delta, \ s + l > \delta\}, \\
\mathcal{R}^{<0,1} &= \{(s, r, l, u) \colon s < 0, \ 0 < s + l \le \delta\}, \\
\mathcal{R}^{<0,2} &= \{(s, r, l, u) \colon s < 0, \ s + l > \delta\}.
\end{aligned}$$

The region $\mathcal{R}^{>0,1}$ corresponds to sessions which start and end in $(0, \delta]$ while the region $\mathcal{R}^{>0,2}$ describes sessions starting in $(0, \delta]$ but ending after δ. Region $\mathcal{R}^{<0,1}$ has sessions starting prior to time 0 and ending in $(0, \delta]$ while $\mathcal{R}^{<0,2}$ has sessions initiated prior to 0 and ending after δ. (See Figure 1.)

Corresponding to this decomposition of regions, if we restrict the Poisson random measure to the four regions we obtain four independent Poisson processes,

$$N(\cdot \cap \mathcal{R}^{>0,1}), \qquad N(\cdot \cap \mathcal{R}^{>0,2}), \qquad N(\cdot \cap \mathcal{R}^{<0,1}), \qquad N(\cdot \cap \mathcal{R}^{<0,2}), \tag{2.2}$$

and we use these to express $A(0, \delta) =: A(\delta)$ as the sum of four independent contributions, as follows:

$$A(\delta) = A^{>0,1}(\delta) + A^{>0,2}(\delta) + A^{<0,1}(\delta) + A^{<0,2}(\delta),$$

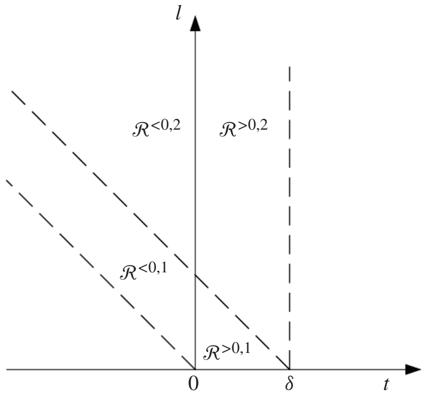

FIGURE 1: Four regions.

where

$$A^{>0,1}(\delta) = \sum_k R_k L_k \mathbf{1}_{\{(\Gamma_k, R_k, L_k, F_k) \in \mathcal{R}^{>0,1}\}},$$

$$A^{>0,2}(\delta) = \sum_k R_k (\delta - \Gamma_k) \mathbf{1}_{\{(\Gamma_k, R_k, L_k, F_k) \in \mathcal{R}^{>0,2}\}},$$

$$A^{<0,1}(\delta) = \sum_k R_k (L_k + \Gamma_k) \mathbf{1}_{\{(\Gamma_k, R_k, L_k, F_k) \in \mathcal{R}^{<0,1}\}},$$

$$A^{<0,2}(\delta) = \sum_k R_k \delta \mathbf{1}_{\{(\Gamma_k, R_k, L_k, F_k) \in \mathcal{R}^{<0,2}\}}.$$

As a further notational device, we will adopt the convention that for a region $\mathcal{R}$ of the (s, r, l, u)-space, $A^{\mathcal{R}}(t_1, t_2]$ will denote the cumulative work input to the system in time interval $(t_1, t_2]$ from points $(\Gamma_k^{\mathcal{R}}, R_k^{\mathcal{R}}, L_k^{\mathcal{R}}, F_k^{\mathcal{R}})$ in region $\mathcal{R}$.

We can represent the restrictions of N to each of the four regions given in (2.2) as empirical measures of a Poisson number of independent, identically distributed points whose joint distributions are respectively the mean measure $\mu^{\#}$ restricted to that region and normalized to be a probability measure (see, e.g. Resnick (1992, p. 341)). For example,

$$N(\cdot \cap \mathcal{R}^{>0,1}) = \sum_{k=1}^{P^{>0,1}(\delta)} \varepsilon_{(\Gamma_k^{>0,1}, R_k^{>0,1}, L_k^{>0,1}, F_k^{>0,1})},$$

where $P^{>0,1}(\delta)$ is Poisson-distributed with parameter

$$\begin{aligned}
\mu^{\#}(\mathcal{R}^{>0,1}) &= \int_{\mathcal{R}^{>0,1}} \lambda \, \mathrm{d}s \, \mathrm{P}[(R_1, L_1, F_1) \in (\mathrm{d}r, \mathrm{d}l, \mathrm{d}u)] \\
&= \int_0^\delta \lambda \, \mathrm{d}s \, \mathrm{P}[L_1 + s < \delta] \\
&= \int_0^\delta \lambda F_L(\delta - s) \, \mathrm{d}s \\
&= \delta \hat{F}_L(\delta)
\end{aligned}$$

(where $\hat{F}_L(x) = \int_0^x F_L(y)\,\mathrm{d}y$) and $\{(\Gamma_k^{>0,1}, R_k^{>0,1}, L_k^{>0,1}, F_k^{>0,1})\}$ are independent and identically distributed with joint distribution

$$\frac{\mu^{\#}(\cdot \cap \mathcal{R}^{>0,1})}{\mu^{\#}(\mathcal{R}^{>0,1})}.$$

In what follows, we sometimes use the convention that $P^A(\delta)$ is Poisson-distributed with parameter equal to the mean measure of the region A.

2.1. Specifying the dependence structure for (R, L, F)

Depending on the dependence structure of the triple (R, L, F), it is possible to have different limit behavior for $\boldsymbol{A}(\delta)$ of (2.1). We distinguish two cases, which we denote by RL and RF:

- In case RL the random variables R and L are independent (cf. Maulik *et al.* (2002)).
- In case RF the random variables R and F are independent.

Standing assumption for this paper: We focus here on the RF model, in which R and F are independent. This choice has statistical justification (see the evidence in Campos *et al.* (2005) and Heffernan and Resnick (2005)) and meets the approval of network engineers, who argue that network rates are assigned without knowledge of the size of the file to be transmitted. The model seems natural since it assumes that file size distributions are unaffected by network state; but even if transmission rates are functions of the network state, we would still have the assumed independence property of the RF model. The RL model is also of interest and undoubtedly leads to different conclusions, but its analysis is not straightforward and certainly not analogous to the RF model. We hope to consider it elsewhere.

Undoubtedly, in practice it is not true that R and F are actually independent, but rather that they satisfy some form of asymptotic independence. However, assuming asymptotic independence rather than full independence would lead to unacceptable complications in the analysis and proofs without changing the conclusions, and thus we feel that choosing full independence of F and R is an appropriate modeling assumption.

2.2. RF model

We assume that the rates of transmission are independent of the file sizes. Durations of transmission are computed using the relation $L = F/R$. From Breiman's theorem (Breiman (1965)), this means that the distribution tail of the random variable L is given by

$$\bar{F}_L(l) \sim \mathrm{E}(1/R)^{\alpha_F} \bar{G}(l), \qquad l \to \infty,$$

provided we assume that

$$\mathrm{E}(1/R)^{\alpha_F+\eta} < \infty$$

for some $\eta > 0$. Here '$\sim$' means the ratio of the two sides converges to 1, and $\bar{G}(l) = 1 - G(l)$.

By using the property that the random variables R and F are heavy tailed, we can easily derive the tail behavior of the random variables $A^{\mathcal{R}}(\delta)$ with

$$\mathcal{R} \in \{\mathcal{R}^{<0,1}, \mathcal{R}^{<0,2}, \mathcal{R}^{>0,1}, \mathcal{R}^{>0,2}\};$$

that is, $\mathcal{R}$ is one of the four regions shown in Figure 1. The tails are given by

$$\mathrm{P}[A^{\mathcal{R}}(\delta) > x] \sim \lambda C^{\mathcal{R}} \frac{\delta^{\alpha_R+1}}{\alpha_F + \alpha_R} \bar{G}(x) \bar{F}_R(x),$$

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where

$$C^{>0,1} = \frac{\alpha_F}{\alpha_R + 1},$$
$$C^{<0,2} = \frac{\alpha_R}{\alpha_F - 1},$$
$$C^{>0,2} = C^{<0,1} = \frac{\alpha_R}{\alpha_R + 1}.$$

Therefore, for finite values of $\delta > 0$, the tails of all the regions are regularly varying with index $-(\alpha_F + \alpha_R)$. In the sequel, we will note that this will be not the case in the limit as $\delta \to 0$.

Since our limiting procedure will shrink the observation window $(0, \delta]$, there is no hope of obtaining a weak limit in (2.1) unless we increase the arrival rate, $\lambda = \lambda(\delta)$, of sessions. Thus, we adopt a heavy-traffic limit theorem philosophy and imagine moving through a family of models indexed by δ, as $\delta \downarrow 0$. A convenient and effective choice of λ is

$$\lambda(\delta) = \frac{1}{\delta \bar{F}_R(\delta^{-1})}. \tag{2.3}$$

Note that, since $1 < \alpha_R < 2$, this choice of λ guarantees that as $\delta \to 0$,

$$\lambda(\delta) = \frac{1}{\delta^{\alpha_R+1} L_R(\delta^{-1})} \to \infty,$$
$$\delta\lambda(\delta) = \frac{1}{\delta^{\alpha_R} L_R(\delta^{-1})} \to \infty.$$

Using assumption (2.3), the behavior of the random variables $A^{(\cdot)}(\delta)$ is as follows:

- $A^{<0,1}(\delta)$ is equal in distribution to $A^{>0,2}(\delta)$.
- $A^{<0,2}(\delta)$ does not converge weakly without scaling, and with centering and scaling converges to a Gaussian random variable.
- $A^{>0,1}(\delta)$ converges in distribution to a compound Poisson random variable.
- $A^{>0,2}(\delta)$, suitably centered, converges weakly to an infinitely divisible random variable with finite variance and a Lévy measure with a regularly varying tail with index $-(\alpha_F + \alpha_R)$, where $\alpha_F + \alpha_R > 2$.

3. Limits for the cumulative input, $A(\delta)$

We now present the details of the limiting arguments yielding distributional approximations for inputs from each of the four regions.

3.1. Region $\mathcal{R}^{>0,2}$

This, recall, is the region contributing input in $(0, \delta]$ from sessions starting in $(0, \delta]$ but terminating after δ.

3.1.1. *Characteristic function.* For $\theta \in \mathbb{R}$, we compute

$$
\begin{aligned}
\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{>0,2}(\delta)}) &= \mathrm{E}\left(\exp\left\{\mathrm{i}\theta \sum_{i=1}^{P^{>0,2}(\delta)} R_i^{>0,2}(\delta-\Gamma_i^{>0,2})\right\}\right)\\
&= \exp\{\mathrm{E}(P^{>0,2}(\delta))[\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta R_1^{>0,2}(\delta-\Gamma_1^{>0,2})})-1]\}\\
&= \exp\left\{\iiint_{0<s<\delta,\, s+l>\delta,\, r>0} (\mathrm{e}^{\mathrm{i}\theta r(\delta-s)}-1)\lambda\,\mathrm{d}s\, F_{L,R}(\mathrm{d}l,\mathrm{d}r)\right\}\\
&= \exp\left\{\int_0^{\delta}\int_{r>0} (\mathrm{e}^{\mathrm{i}\theta r(\delta-s)}-1)\,\mathrm{P}\left[\frac{F}{R}>\delta-s,\, R\in \mathrm{d}r\right]\lambda\,\mathrm{d}s\right\} \qquad (3.1)\\
&= \exp\left\{\int_0^{\delta}\int_0^{\infty} (\mathrm{e}^{i\theta rs}-1)\bar{G}(rs)F_R(\mathrm{d}r)\lambda\,\mathrm{d}s\right\}\\
&= \exp\left\{\lambda\int_0^{\infty}\int_0^{r\delta} (\mathrm{e}^{\mathrm{i}\theta s}-1)\bar{G}(s)r^{-1}F_R(\delta^{-1}\,\mathrm{d}r)\,\mathrm{d}s\right\}\\
&= \exp\left\{\int_0^{\infty}\left(\int_0^{r} (\mathrm{e}^{\mathrm{i}\theta s}-1)\bar{G}(s)\,\mathrm{d}s\right)r^{-1}\frac{F_R(\delta^{-1}\,\mathrm{d}r)}{\bar{F}_R(\delta^{-1})}\right\},
\end{aligned}
$$

where we have used the definition of λ in (2.3) and $F_{L,R}$ is the joint distribution of the random pair (L, R). By interchanging the order of integration, we obtain

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{>0,2}(\delta)}) = \exp\left\{\int_0^{\infty}(\mathrm{e}^{\mathrm{i}\theta s}-1)\bar{G}(s)\left(\int_s^{\infty} r^{-1}\mu_\delta(\mathrm{d}r)\right)\mathrm{d}s\right\},$$

where

$$\mu_\delta(\mathrm{d}r) := \frac{F_R(\delta^{-1}\mathrm{d}r)}{\bar{F}_R(\delta^{-1})}.$$

Writing

$$\nu_\delta^{>0,2}(\mathrm{d}s) = (\nu_\delta^{>0,2})'(s)\,\mathrm{d}s = \bar{G}(s)\left(\int_s^{\infty} r^{-1}\mu_\delta(\mathrm{d}r)\right)\mathrm{d}s,$$

we obtain

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{>0,2}(\delta)}) = \exp\left\{\int_0^{\infty}(\mathrm{e}^{\mathrm{i}\theta s}-1)\nu_\delta^{>0,2}(\mathrm{d}s)\right\}. \qquad (3.2)$$

3.1.2. *Properties of* $\nu_\delta^{>0,2}$.

Proposition 1. *As* $\delta \to 0$,

$$\nu_\delta^{>0,2} \xrightarrow{\mathrm{v}} \nu_0^{>0,2}$$

on $(0, \infty]$*; that is, we have vague convergence to a limit. Furthermore, the limit measure* $\nu_0^{>0,2}$ *is a Lévy measure with density*

$$\frac{\alpha_R}{1+\alpha_R}\bar{G}(x)x^{-\alpha_R-1}.$$

The tail of the Lévy measure is regularly varying with index $-(\alpha_F+\alpha_R)$.

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Proof. Observe that for $s \geq 1$,

$$(\nu_\delta^{>0,2})'(s) = \bar{G}(s) \int_s^\infty r^{-1} \mu_\delta(\mathrm{d}r) \leq \bar{G}(s)\mu_\delta(s, \infty] s^{-1},$$

and that by Potter's bounds (see, e.g. Resnick (1987, p. 23)), for some small η, all $s \geq 1$, and all sufficiently small δ, we have the upper bound

$$(\nu_\delta^{>0,2})'(s) \leq c\bar{G}(s) s^{-(\alpha_R - \eta) - 1},$$

which is integrable with respect to Lebesgue measure on any neighborhood of ∞. Hence, by dominated convergence, for $x > 0$,

$$\begin{aligned} \nu_\delta^{>0,2}(x, \infty] &= \int_x^\infty (\nu_\delta^{>0,2})'(s)\,\mathrm{d}s \\ &\to \int_x^\infty \bar{G}(s)\bigg(\int_s^\infty r^{-1}\alpha_R r^{-\alpha_R - 1}\,\mathrm{d}r\bigg)\,\mathrm{d}s \\ &= \nu_0^{>0,2}(x, \infty] \\ &= \frac{\alpha_R}{1 + \alpha_R} \int_x^\infty \bar{G}(s) s^{-\alpha_R - 1}\,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Regular variation of $\nu_0^{>0,2}(x, \infty]$ follows from the regular variation of the integrand in (3.3) and Karamata's theorem (see, e.g. Resnick (1987, p. 17)). In fact, as $x \to \infty$,

$$\nu_0^{>0,2}(x, \infty] \sim \frac{\alpha_R}{(\alpha_R + 1)(\alpha_F + \alpha_R)} x^{-\alpha_R} \bar{G}(x).$$

To check that $\nu_0^{>0,2}$ is a Lévy measure, note that

$$\int_0^1 s^2 \bar{G}(s) s^{-\alpha_R - 1}\,\mathrm{d}s \leq \int_0^1 s^2 s^{-\alpha_R - 1}\,\mathrm{d}s < \infty$$

since $1 < \alpha_R < 2$.

3.1.3. *Weak limit for* $A^{>0,2}(\delta)$. Now we use (3.2) and write

$$\begin{aligned} &\mathrm{E}\bigg(\exp\bigg\{\mathrm{i}\theta\bigg(A^{>0,2}(\delta) - \int_0^1 s\nu_\delta^{>0,2}(\mathrm{d}s)\bigg)\bigg\}\bigg) \\ &\quad = \exp\bigg\{\int_1^\infty (\mathrm{e}^{\mathrm{i}\theta s} - 1)\nu_\delta^{>0,2}(\mathrm{d}s) + \int_0^1 (\mathrm{e}^{\mathrm{i}\theta s} - 1 - \mathrm{i}\theta s)\nu_\delta^{>0,2}(\mathrm{d}s)\bigg\}. \end{aligned} \tag{3.4}$$

The two integrals on the right-hand side of (3.4) each converge as $\delta \to 0$.

Let '$\xrightarrow{\mathrm{w}}$' denote weak convergence.

Proposition 2. *As* $\delta \to 0$*,*

$$\int_1^\infty (\mathrm{e}^{\mathrm{i}\theta s} - 1)\nu_\delta^{>0,2}(\mathrm{d}s) \to \int_1^\infty (\mathrm{e}^{\mathrm{i}\theta s} - 1)\nu_0^{>0,2}(\mathrm{d}s), \tag{3.5}$$

$$\int_0^1 (\mathrm{e}^{\mathrm{i}\theta s} - 1 - \mathrm{i}\theta s)\nu_\delta^{>0,2}(\mathrm{d}s) \to \int_0^1 (\mathrm{e}^{\mathrm{i}\theta s} - 1 - \mathrm{i}\theta s)\nu_0^{>0,2}(\mathrm{d}s). \tag{3.6}$$

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Therefore, as $\delta \to 0$,

$$A^{>0,2}(\delta) - \int_0^1 s\nu_\delta^{>0,2}(\mathrm{d}s) \xrightarrow{\mathrm{w}} X^{>0,2},$$

where the limit random variable is infinitely divisible with Lévy measure $\nu_0^{>0,2}$ *and characteristic function given by the right-hand side of (3.4) with* $\nu_\delta^{>0,2}$ *replaced by* $\nu_0^{>0,2}$.

Proof. The convergence in (3.5) follows from standard weak convergence, since the integrand is bounded and continuous and

$$\frac{\nu_\delta^{>0,2}(\cdot)}{\nu_\delta^{>0,2}(1,\infty]} \xrightarrow{\mathrm{w}} \frac{\nu_0^{>0,2}(\cdot)}{\nu_0^{>0,2}(1,\infty]}$$

as probability measures on $(1, \infty]$.

To prove (3.6), first observe that

$$|\mathrm{e}^{\mathrm{i}\theta s} - 1 - \mathrm{i}\theta s|(\nu_\delta^{>0,2})'(s) \le \frac{\theta^2 s^2}{2}\bar{G}(s)s^{-1}\mu_\delta(s,\infty] \le cs\frac{\bar{F}_R(\delta^{-1}s)}{\bar{F}_R(\delta^{-1})} = c\frac{V(\delta^{-1}s)}{V(\delta^{-1})},$$

where $V(s) = s\bar{F}_R(s)$ is regularly varying with index $-\alpha_R + 1$. Now, as $\delta \to 0$,

$$|\mathrm{e}^{\mathrm{i}\theta s} - 1 - \mathrm{i}\theta s|(\nu_\delta^{>0,2})'(s) \to |\mathrm{e}^{\mathrm{i}\theta s} - 1 - \mathrm{i}\theta s|(\nu_0^{>0,2})'(s)$$

and

$$\frac{V(\delta^{-1}s)}{V(\delta^{-1})} \to s^{-\alpha_R+1}.$$

Furthermore, by Karamata's theorem,

$$\int_0^1 \frac{V(\delta^{-1}s)}{V(\delta^{-1})}\,\mathrm{d}s \to \int_0^1 s^{-\alpha_R+1}\,\mathrm{d}s = \frac{1}{2-\alpha_R}.$$

The desired result then follows from Pratt's lemma (Pratt (1960), Resnick (1998, p. 164)), since this may be applied to both the real and imaginary parts of

$$(\mathrm{e}^{\mathrm{i}\theta s} - 1 - \mathrm{i}\theta s)(\nu_\delta^{>0,2})'(s)$$

to obtain convergence to the limit after integrating over [0, 1].

3.2. Region $\mathcal{R}^{>0,1}$

The traffic contribution corresponding to this region is

$$A^{>0,1}(\delta) = \sum_{i=1}^{P^{>0,1}(\delta)} F_i^{>0,1}. \tag{3.7}$$

Now,

$$\begin{aligned}\mathrm{E}(P^{>0,1}(\delta)) &= \iiint\limits_{\substack{0<s<\delta,\, r>0,\, u>0,\\ 0<s+u/r<\delta}} \lambda\,\mathrm{d}s\,G(\mathrm{d}u)F_R(\mathrm{d}r)\\ &= \lambda\int_0^\delta \mathrm{P}[F/R \le s]\,\mathrm{d}s = \lambda\int_0^\delta \mathrm{P}[R/F \ge s^{-1}]\,\mathrm{d}s\\ &= \lambda\int_{\delta^{-1}}^\infty \mathrm{P}[R/F \ge s]\frac{\mathrm{d}s}{s^2}.\end{aligned}$$

Apply Breiman's theorem (Breiman (1965)) after assuming that $\mathrm{E}(F^{-(\alpha_R+\eta)}) < \infty$. We then see that for some $\eta > 0$, as $\delta \to 0$, the above is asymptotic to

$$\lambda \int_{\delta^{-1}}^{\infty} \mathrm{P}[R > s] \frac{\mathrm{d}s}{s^2}\, \mathrm{E}(F^{-\alpha_R}),$$

and from Karamata's theorem this is asymptotic to

$$\frac{\lambda\delta}{1+\alpha_R}\, \mathrm{P}[R > \delta^{-1}]\, \mathrm{E}(F^{-\alpha_R}) \sim \frac{\mathrm{E}(F^{-\alpha_R})}{1+\alpha_R}.$$

Thus, as $\delta \to 0$,

$$\mathrm{E}(P^{>0,1}(\delta)) \to \frac{\mathrm{E}(F^{-\alpha_R})}{1+\alpha_R}.$$

This means that

$$P^{>0,1}(\delta) \xrightarrow{\mathrm{w}} P^{>0,1}(0),$$

where $P^{>0,1}(0)$ is Poisson-distributed with parameter $\mathrm{E}(F^{-\alpha_R})/(1+\alpha_R)$.

Now we observe that the distribution of $F_1^{>0,1}$ converges as $\delta \to 0$. For $x > 0$, we have

$$\begin{aligned}
\mathrm{E}(P^{>0,1}(\delta))\, \mathrm{P}[F_1^{>0,1} \le x] &= \iiint\limits_{\substack{0<s<\delta,\, r>0,\\ s+u/r<\delta,\, u\le x}} \lambda\, \mathrm{d}s\, G(\mathrm{d}u) F_R(\mathrm{d}r) \\
&= \int_0^{\delta} \mathrm{P}[FR^{-1} \le s,\ F \le x] \lambda\, \mathrm{d}s \\
&= \lambda \int_0^{\delta} \mathrm{P}[RF^{-1} \ge s^{-1},\ F \le x]\, \mathrm{d}s \\
&= \lambda \int_{\delta^{-1}}^{\infty} \mathrm{P}[RF^{-1}\, \mathbf{1}_{\{F\le x\}} \ge s] \frac{\mathrm{d}s}{s^2} \\
&\sim \frac{\lambda\delta}{1+\alpha_R}\, \mathrm{P}[R > \delta^{-1}]\, \mathrm{E}(F^{-\alpha_R}\, \mathbf{1}_{\{F\le x\}}).
\end{aligned}$$

We conclude that, as $\delta \to 0$,

$$\mathrm{P}[F_1^{>0,1} \le x] \xrightarrow{\mathrm{w}} \frac{\mathrm{E}(F^{-\alpha_R}\, \mathbf{1}_{\{F\le x\}})}{\mathrm{E}(F^{-\alpha_R})}.$$

This leads to the following result.

Proposition 3. *Assume that*

$$\mathrm{E}(F)^{-(\alpha_R+\eta)} < \infty.$$

Then $A^{>0,1}(\delta)$ in (3.7) is a compound Poisson-distributed random variable which, as $\delta \to 0$, converges weakly to a limiting compound Poisson-distributed random variable

$$X^{>0,1} = \sum_{i=1}^{P^{>0,1}(0)} R_i^{>0,1}(0),$$

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where $P^{>0,1}(0)$ *is a Poisson-distributed random variable with parameter* $\mathrm{E}(F^{-\alpha_R})/(1+\alpha_R)$, *independent of the independent, identically distributed sequence* $\{R_i^{>0,1}(0),\ i\geq 1\}$ *which has common distribution function*

$$\frac{\mathrm{E}(F^{-\alpha_R}\mathbf{1}_{\{F\leq x\}})}{\mathrm{E}(F^{-\alpha_R})}.$$

The tail probabilities of this distribution and, hence, of the limiting compound Poisson-distributed random variable, are regularly varying with index $-(\alpha_R+\alpha_F)$ *and, in fact, as* $x\to\infty$,

$$\frac{\mathrm{E}(F^{-\alpha_R}\mathbf{1}_{\{F>x\}})}{\mathrm{E}(F^{-\alpha_R})}\sim\frac{\alpha_F}{\alpha_R+\alpha_F}x^{-\alpha_R}\bar{G}(x).$$

3.3. Region $\mathcal{R}^{<0,2}$

In region $\mathcal{R}^{<0,2}$ we have contributions to traffic in $(0,\delta)$ from sessions starting prior to 0 and ending after δ.

3.3.1. *Characteristic function of* $A^{<0,2}(\delta)$. Since

$$A^{<0,2}(\delta)=\sum_{i=1}^{P^{<0,2}(\delta)}R_i^{<0,2}\delta, \tag{3.8}$$

the characteristic function of $A^{<0,2}(\delta)$ is computed as follows. For $\theta\in\mathbb{R}$,

$$\begin{aligned}\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{<0,2}(\delta)})&=\exp\{\mathrm{E}(P^{<0,2}(\delta))[\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta R_1^{<0,2}\delta})-1]\}\\&=\exp\biggl\{\iiint\limits_{\substack{s<0,\,r>0\\ l>|s|+\delta}}(\mathrm{e}^{\mathrm{i}\theta r\delta}-1)\lambda\,\mathrm{d}s\,F_{L,R}(\mathrm{d}l,\mathrm{d}r)\biggr\}\\&=\exp\biggl\{\lambda\int_0^\infty\int_\delta^\infty(\mathrm{e}^{\mathrm{i}\theta r\delta}-1)\bar{G}(rs)\,\mathrm{d}s\,F_R(\mathrm{d}r)\biggr\},\end{aligned}$$

and, reversing the order of integration and setting $\bar{G}_0(x)=\int_x^\infty\bar{G}(u)\mathrm{d}u/\mathrm{E}(F)$, we find that

$$\begin{aligned}\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{<0,2}(\delta)})&=\exp\biggl\{\lambda\int_0^\infty(\mathrm{e}^{\mathrm{i}\theta r\delta}-1)r^{-1}\bar{G}_0(r\delta)\,\mathrm{E}(F)F_R(\mathrm{d}r)\biggr\}\\&=\exp\biggl\{\lambda\delta\int_0^\infty(\mathrm{e}^{\mathrm{i}\theta r}-1)r^{-1}\bar{G}_0(r)F_R(\delta^{-1}\,\mathrm{d}r)\,\mathrm{E}(F)\biggr\}\\&=\exp\biggl\{\int_0^\infty(\mathrm{e}^{\mathrm{i}\theta r}-1)r^{-1}\bar{G}_0(r)\frac{F_R(\delta^{-1}\,\mathrm{d}r)}{\bar{F}(\delta^{-1})}\,\mathrm{E}(F)\biggr\}.\end{aligned}$$

As before, let

$$\mu_\delta(\mathrm{d}r)=\frac{F_R(\delta^{-1}\,\mathrm{d}r)}{\bar{F}_R(\delta^{-1})}$$

and define

$$\nu_\delta^{<0,2}(\mathrm{d}r)=\mathrm{E}(F)r^{-1}\bar{G}_0(r)\mu_\delta(\mathrm{d}r).$$

We conclude that

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{<0,2}(\delta)})=\exp\biggl\{\int_0^\infty(\mathrm{e}^{\mathrm{i}\theta r}-1)\nu_\delta^{<0,2}(\mathrm{d}r)\biggr\}. \tag{3.9}$$

3.3.2. *Properties of* $\nu_\delta^{<0,2}$. The following properties of the measure $\nu_\delta^{<0,2}$ are self-evident.

1. As $\delta \to 0$, the measures $\nu_\delta^{<0,2}$ converge vaguely on $(0,\infty]$: $\nu_\delta^{<0,2} \xrightarrow{\text{v}} \nu_0^{<0,2}$. Here, for $x > 0$,
$$\nu_0^{<0,2}(x,\infty] = \mathrm{E}(F)\int_x^\infty r^{-1}\bar{G}_0(r)\alpha_R r^{-\alpha_R-1}\,\mathrm{d}r.$$
2. The tail of the measure $\nu_0^{<0,2}(x,\infty]$ is regularly varying with index $-(\alpha_R+\alpha_F)$ and, in fact,
$$\begin{aligned}\frac{\nu_0^{<0,2}(x,\infty]}{x^{-\alpha_R-1}\bar{G}_0(x)} &= \mathrm{E}(F)\int_1^\infty r^{-1}\frac{\bar{G}_0(xr)}{\bar{G}_0(x)}\alpha_R r^{-\alpha_R-1}\,\mathrm{d}r\\ &\to \mathrm{E}(F)\int_1^\infty r^{-1}r^{-(\alpha_F-1)}\alpha_R r^{-\alpha_R-1}\,\mathrm{d}r\\ &= \mathrm{E}(F)\frac{\alpha_R}{\alpha_R+\alpha_F}.\end{aligned}$$
3. The measure $\nu_0^{<0,2}$ is *not* a Lévy measure, since
$$\begin{aligned}\int_0^1 r^2\nu_0^{<0,2}(\mathrm{d}r) &= \alpha_R\,\mathrm{E}(F)\int_0^1 r^{-\alpha_R}\bar{G}_0(r)\,\mathrm{d}r\\ &\geq \alpha_R\,\mathrm{E}(F)\bar{G}_0(1)\int_0^1 r^{-\alpha_R}\,\mathrm{d}r\\ &= \infty.\end{aligned}$$
Since $\nu_0^{<0,2}$ is not a Lévy measure, we do not expect to obtain an infinitely divisible weak limit without a Gaussian component for $A^{<0,2}(\delta)$.

3.3.3. *Gaussian limit.* Observe that the quantity
$$m(\delta) := \mathrm{E}(F)\int_0^1 \bar{G}_0(r)\mu_\delta(\mathrm{d}r) \tag{3.10}$$
is finite, since
$$m(\delta) \leq \mathrm{E}(F)\int_0^1 1\,\mu_\delta(\mathrm{d}r) = \mathrm{E}(F)\frac{F_R(\delta^{-1})}{\bar{F}_R(\delta^{-1})} < \infty.$$
Also define
$$a(\delta) := \left(\mathrm{E}(F)\int_0^1 r\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\right)^{1/2}. \tag{3.11}$$
Note that as $\delta \to 0$ we have $a(\delta) \to \infty$, since for any $\eta > 0$,
$$\begin{aligned}\liminf_{\delta\to 0} a^2(\delta) &\geq \liminf_{\delta\to 0}\mathrm{E}(F)\int_\eta^1 r\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\\ &= \mathrm{E}(F)\int_\eta^1 r\bar{G}_0(r)\alpha_R r^{-\alpha_R-1}\,\mathrm{d}r\\ &\geq \frac{\mathrm{E}(F)}{1-\alpha_R}\,\bar{G}_0(1)r^{-\alpha_R+1}\Big|_\eta^1\\ &\to \infty \quad \text{as } \eta \downarrow 0.\end{aligned}$$

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We now use (3.9) and write

$$\begin{aligned}
&\mathrm{E}\left(\exp\left\{\mathrm{i}\theta\frac{A^{<0,2}(\delta)-m(\delta)}{a(\delta)}\right\}\right)\\
&\quad=\exp\left\{\int_0^\infty (\mathrm{e}^{\mathrm{i}a^{-1}(\delta)\theta r}-1)\nu_\delta^{<0,2}(\mathrm{d}r)-\mathrm{i}\theta\frac{\mathrm{E}(F)}{a(\delta)}\int_0^1 \bar{G}_0(r)\mu_\delta(\mathrm{d}r)\right\}\\
&\quad=\exp\left\{\int_0^1\left(\mathrm{e}^{\mathrm{i}a^{-1}(\delta)\theta r}-1-\mathrm{i}\,\frac{\theta}{a(\delta)}r\right)\mathrm{E}(F)r^{-1}\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\right.\\
&\qquad\quad\left.+\int_1^\infty(\mathrm{e}^{\mathrm{i}a^{-1}(\delta)\theta r}-1)\nu_\delta^{<0,2}(\mathrm{d}r)\right\}\\
&\quad=:\exp\{A+B\}.
\end{aligned}$$

Consider B. We have

$$|B|\le \mathrm{E}(F)\int_1^\infty|\theta|\frac{r}{a(\delta)}r^{-1}\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\le O\left(\frac{1}{a(\delta)}\right)\to 0\quad\text{as } \delta\to 0,$$

since $a(\delta)\to\infty$. For A we have $A\to-\theta^2/2$, since

$$\begin{aligned}
&\left|\int_0^1\left(\mathrm{e}^{\mathrm{i}\theta a^{-1}(\delta)r}-1-\mathrm{i}\,\frac{\theta}{a(\delta)}\,r\right)\mathrm{E}(F)r^{-1}\bar{G}_0(r)\mu_\delta(\mathrm{d}r)+\frac{\theta^2}{2}\right|\\
&\quad=\left|\int_0^1\left(\mathrm{e}^{\mathrm{i}\theta a^{-1}(\delta)r}-1-\mathrm{i}\,\frac{\theta}{a(\delta)}\,r-\frac{1}{2}\left(\frac{\mathrm{i}\theta r}{a(\delta)}\right)^2\right)\mathrm{E}(F)r^{-1}\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\right|\\
&\quad\le\frac{1}{a^3(\delta)}\int_0^1\frac{1}{3!}|\theta|^3r^3\,\mathrm{E}(F)r^{-1}\bar{G}_0(r)\mu_\delta(\mathrm{d}r)
\end{aligned}$$

and

$$\frac{1}{a^3(\delta)}\int_0^1 r^2\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\le\frac{1}{a(\delta)^3}\int_0^1 r\bar{G}_0(r)\mu_\delta(\mathrm{d}r)=\frac{1}{a(\delta)}\to 0.$$

In summary, we present the following proposition.

Proposition 4. *With $m(\delta)$ as defined by (3.10) and $a(\delta)$ as given by (3.11), as $\delta\to 0$ we have*

$$\frac{A^{<0,2}(\delta)-m(\delta)}{a(\delta)}\xrightarrow{\mathrm{w}}X^{<0,2},$$

a N(0, 1) *random variable.*

Remark 1. The centering may be changed from $m(\delta)$ to

$$m^{\#}(\delta):=\mathrm{E}\left(\sum_{i=1}^{P^{<0,2}(\delta)}R_i^{<0,2}\delta\right)=\mathrm{E}(P^{<0,2}(\delta))\,\mathrm{E}(R^{<0,2}\delta),$$

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(recall the notation introduced just before the start of Section 2.1), since

$$\begin{aligned}
\mathrm{E}(P^{<0,2}(\delta))\,\mathrm{E}(R^{<0,2}\delta) &= \iiint\limits_{\substack{s<0,\,r>0\\ l>\delta+|s|}} r\delta\lambda\,\mathrm{d}s\,F_L(\mathrm{d}l,\mathrm{d}r)\\
&= \int_\delta^\infty\int_0^\infty r\delta\bar{G}(rs)F_R(\mathrm{d}r)\lambda\,\mathrm{d}s\\
&= \lambda\delta\int_0^\infty \bar{G}_0(r\delta)F_R(\mathrm{d}r)\,\mathrm{E}(F)\\
&= \int_0^\infty \bar{G}_0(r)\frac{F_R(\delta^{-1}\mathrm{d}r)}{\bar{F}_R(\delta^{-1})}\;\mathrm{E}(F)\\
&= m(\delta)+\mathrm{E}(F)\int_1^\infty \bar{G}_0(r)\mu_\delta(\mathrm{d}r)\\
&= m(\delta)+o(a(\delta)).
\end{aligned}$$

Similarly, the scaling may be changed from $a(\delta)$ to

$$a^{\#}(\delta) := \sqrt{\operatorname{var}\Bigg(\sum_{i=1}^{P^{<0,2}(\delta)} R_i^{<0,2}\delta\Bigg)} = \sqrt{\mathrm{E}(P^{<0,2}(\delta))\,\mathrm{E}((R^{<0,2}\delta)^2)}.$$

This follows from

$$\begin{aligned}
\mathrm{E}(P^{<0,2}(\delta))\,\mathrm{E}((R^{<0,2}\delta)^2) &= \iiint\limits_{\substack{s<0,\,r>0\\ l>|s|+\delta}} \delta^2r^2\lambda\,\mathrm{d}s\,F_{L,R}(\mathrm{d}l,\mathrm{d}r)\\
&= \int_{r>0}\int_\delta^\infty \delta^2r^2\lambda\,\mathrm{d}s\,\bar{G}(rs)F_R(\mathrm{d}r)\\
&= \lambda\int_0^\infty \delta^2r^2\int_\delta^\infty \bar{G}(rs)\,\mathrm{d}s\,F_R(\mathrm{d}r)\\
&= \lambda\delta\int_0^\infty \delta r\int_{r\delta}^\infty \bar{G}(s)\,\mathrm{d}s\,F_R(\mathrm{d}r)\\
&= \lambda\delta\int_0^\infty r\bar{G}_0(r)F_R(\delta^{-1}\,\mathrm{d}r)\,\mathrm{E}(F)\\
&= \int_0^\infty r\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\,\mathrm{E}(F).
\end{aligned}$$

Note that as $\delta\to 0$,

$$\int_1^\infty r\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\to\int_1^\infty r\bar{G}_0(r)\alpha_R r^{-\alpha_R-1}\,\mathrm{d}r<\infty$$

since $1<\alpha_R<2$. Therefore,

$$a^2(\delta)\sim\int_0^\infty r\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\,\mathrm{E}(F),$$

as claimed.

3.3.4. *Further properties of the scaling function $a(\delta)$.* The scaling function plays a significant role in understanding dependence across time slots. Here are two properties we need in the next section.

Proposition 5. (a) *For any $t > 0$,*

$$\lim_{\delta\to 0} \frac{\int_0^1 r\bar{G}_0(tr)\mu_\delta(\mathrm{d}r)}{\int_0^1 r\mu_\delta(\mathrm{d}r)} = 1. \tag{3.12}$$

(b) *The growth rate of $a(\delta)$ is given by*

$$\begin{aligned} a^2(\delta) &= \mathrm{E}(F)\int_0^1 r\bar{G}_0(r)\mu_\delta(\mathrm{d}r) \\ &\sim \mathrm{E}(F)\int_0^1 r\mu_\delta(\mathrm{d}r) \\ &\sim \mathrm{E}(F)\,\mathrm{E}(R)\frac{(\delta^{-1})^{(\alpha_R-1)}}{L_R(\delta^{-1})} \\ &= \frac{\mathrm{E}(F)\,\mathrm{E}(R)}{\delta^{-1}\bar{F}_R(\delta^{-1})} \\ &\to \infty. \end{aligned}$$

Proof. (a) Since $\bar{G}_0 \leq 1$, we see that 1 is an upper bound of the ratio in (3.12). To obtain a lower bound, observe that $\int_0^1 r\mu_\delta(\mathrm{d}r) \to \infty$ as $\delta \to 0$, since for any $\eta > 0$,

$$\begin{aligned} \int_0^1 r\mu_\delta(\mathrm{d}r) &\geq \int_\eta^1 r\mu_\delta(\mathrm{d}r) \\ &\to \int_\eta^1 r\alpha_R r^{-\alpha_R-1}\,\mathrm{d}r \\ &= \frac{\alpha_R}{\alpha_R-1}\,[\eta^{-(\alpha_R-1)}-1] \\ &\to \infty \quad \text{as } \eta \downarrow 0, \end{aligned}$$

since $1 < \alpha_R < 2$. Therefore,

$$\begin{aligned} \frac{\int_0^1 r\bar{G}_0(tr)\mu_\delta(\mathrm{d}r)}{\int_0^1 r\mu_\delta(\mathrm{d}r)} &\geq \frac{\int_0^\eta r\bar{G}_0(tr)\mu_\delta(\mathrm{d}r)}{\int_0^1 r\mu_\delta(\mathrm{d}r)} \\ &\geq \frac{\bar{G}_0(t\eta)\int_0^\eta r\mu_\delta(\mathrm{d}r)}{\int_0^1 r\mu_\delta(\mathrm{d}r)} \\ &= \bar{G}_0(t\eta)\frac{\int_0^1 r\mu_\delta(\mathrm{d}r)}{\int_0^1 r\mu_\delta(\mathrm{d}r)} + o(1) \\ &\to \bar{G}_0(t\eta) \quad \text{as } \delta \to 0 \\ &\to 1 \quad \text{as } \eta \to 0. \end{aligned}$$

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(b) To find the growth rate of $a(\delta)$, observe that

$$\begin{aligned}\int_0^1 r\mu_\delta(\mathrm{d}r) &= \int_0^1 \bigg(\int_0^r \mathrm{d}v\bigg)\mu_\delta(\mathrm{d}r)\\ &= \int_0^1 \int_v^1 \mu_\delta(\mathrm{d}r)\,\mathrm{d}v\\ &= \int_0^1 \mu_\delta(v,\infty]\,\mathrm{d}v - \mu_\delta(1,\infty]\\ &= \frac{\int_0^{\delta^{-1}} \bar{F}_R(v)\,\mathrm{d}v}{\delta^{-1}\bar{F}_R(\delta^{-1})} - 1.\end{aligned}$$

We conclude that

$$\delta^{-1}\bar{F}_R(\delta^{-1})\bigg[1+\int_0^1 r\mu_\delta(\mathrm{d}r)\bigg] \to \mathrm{E}(R).$$

This coupled with $\int_0^1 r\mu_\delta(\mathrm{d}r) \to \infty$ proves the result.

3.4. Region $\mathcal{R}^{<0,1}$

In this section we prove that

$$A^{<0,1}(\delta) \stackrel{\mathrm{D}}{=} A^{>0,2}(\delta).$$

The reasoning behind this is as follows. Recall that

$$A^{<0,1}(\delta) = \sum_k R_k(L_k + \Gamma_k)\,\mathbf{1}_{\{(\Gamma_k, R_k, L_k, F_k)\in\mathcal{R}^{<0,1}\}}\,.$$

It is well known that in the M/G/∞ model the departure process has the same distribution as the arrival process, namely the Poisson distribution with rate λ. The process $A^{<0,1}(\delta)$ accumulates the contribution from 0 to δ of those sessions ending in $(0,\delta)$ and starting before time 0. However, we may re-index Poisson points by swapping the termination and start times. The region $A^{<0,1}(\delta)$ will then correspond to the sessions starting in $(0,\delta]$ and terminating outside it, which is exactly the contribution of the region $\mathcal{R}^{>0,2}$, namely $A^{>0,2}(\delta)$. A more formal proof is given in the following proposition.

Proposition 6. *We have*

$$A^{<0,1}(\delta) \stackrel{\mathrm{D}}{=} A^{>0,2}(\delta)$$

and, therefore, as $\delta \to 0$,

$$A^{<0,1}(\delta) - \int_0^1 s\nu_\delta^{<0,1}(\mathrm{d}s) \xrightarrow{\mathrm{W}} X^{<0,1},$$

where $\nu_\delta^{<0,1} = \nu_\delta^{>0,2}$ *and* $X^{<0,1} \stackrel{\mathrm{D}}{=} X^{>0,2}$*, with the quantities indexed by '*$>0,2$*' defined as in Proposition 3.*

Proof. We compute the characteristic function as follows. For $\theta \in \mathbb{R}$,

$$\begin{aligned}
&\mathrm{E}\left(\exp\left\{\mathrm{i}\theta \sum_{i=1}^{P^{<0,1}(\delta)} R_i^{<0,1}(\Gamma_i^{<0,1} + L_i^{<0,1})\right\}\right) \\
&\quad = \exp\{\mathrm{E}(P^{<0,1}(\delta))[\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta R_1^{<0,1}(\Gamma_1^{<0,1}+L_1^{<0,1})}) - 1]\} \\
&\quad = \exp\left\{\iiint_{\substack{s<0,\, r>0 \\ |s|<l\le|s|+\delta}} (\mathrm{e}^{\mathrm{i}\theta r(s+l)} - 1)\lambda\, \mathrm{d}s\, F_{L,R}(\mathrm{d}l, \mathrm{d}r)\right\},
\end{aligned}$$

and by changing variable according to $s' = \delta - (l + s)$ we obtain

$$\begin{aligned}
&\mathrm{E}\left(\exp\left\{\mathrm{i}\theta \sum_{i=1}^{P^{<0,1}(\delta)} R_i^{<0,1}(\Gamma_i^{<0,1} + L_i^{<0,1})\right\}\right) \\
&\quad = \exp\left\{\iiint_{\substack{r>0,\, s'+l>\delta \\ 0<s'<\delta}} (\mathrm{e}^{\mathrm{i}\theta(\delta-s')} - 1)\lambda\, \mathrm{d}s'\, F_{L,R}(\mathrm{d}l, \mathrm{d}r)\right\} \\
&\quad = \mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{>0,2}(\delta)}).
\end{aligned}$$

In the last step we used (3.1).

3.5. Discussion and summary

We here summarize the contributions of the four regions to cumulative traffic in $(0, \delta)$.

1. For region $\mathcal{R}^{>0,2}$, we have

$$X^{>0,2}(\delta) := A^{>0,2}(\delta) - \int_0^1 s\nu_\delta^{>0,2}(\mathrm{d}s) \xrightarrow{\mathrm{w}} X^{>0,2}, \quad \text{as } \delta \to 0,$$

   a spectrally positive, infinitely divisible random variable with Lévy measure $\nu_0^{>0,2}$ whose tail probabilities are regularly varying with index $-(\alpha_F + \alpha_R)$. Observe that $\alpha_F + \alpha_R > 2$, whence $\mathrm{E}(X^{>0,2})^2 < \infty$.

2. For region $\mathcal{R}^{>0,1}$, we have

$$A^{>0,1}(\delta) \xrightarrow{\mathrm{w}} X^{>0,1},$$

   a compound Poisson-distributed random variable with tail probabilities which are regularly varying with index $-(\alpha_F + \alpha_R)$. Note that, for some $c > 0$,

$$\mathrm{P}[X^{>0,1} > x] \sim c\,\mathrm{P}[X^{>0,2} > x] \quad \text{as } x \to \infty.$$

3. For region $\mathcal{R}^{<0,2}$, we have

$$X^{<0,2}(\delta) := \frac{A^{<0,2}(\delta) - m(\delta)}{a(\delta)} \xrightarrow{\mathrm{w}} X^{<0,2} \sim \mathrm{N}(0, 1).$$

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4. For region $\mathcal{R}^{<0,1}$, we have

$$A^{<0,1}(\delta) \stackrel{\mathrm{D}}{=} A^{>0,2}(\delta),$$

whence

$$X^{<0,1}(\delta) := A^{<0,1}(\delta) - \int_0^1 s\nu_\delta^{>0,2}(\mathrm{d}s) \xrightarrow{\mathrm{w}} X^{<0,1} \stackrel{\mathrm{D}}{=} X^{>0,2}.$$

We may thus write

$$A(\delta) = X^{>0,2}(\delta) + \int_0^1 s\nu_\delta^{>0,2}(\mathrm{d}s) + A^{>0,1}(\delta) + a(\delta)X^{<0,2}(\delta) + m(\delta) + X^{<0,1}(\delta) + \int_0^1 s\nu_\delta^{>0,2}(\mathrm{d}s).$$

We conclude that

$$A(\delta) - m(\delta) - 2\int_0^1 s\nu_\delta^{>0,2}(\mathrm{d}s) = X^{>0,2}(\delta) + A^{>0,1}(\delta) + a(\delta)X^{<0,2}(\delta), \tag{3.13}$$

where the summands on the right-hand side are independent and

$$\begin{aligned} X^{<0,1}(\delta) \stackrel{\mathrm{D}}{=} X^{>0,2}(\delta) &\xrightarrow{\mathrm{w}} X^{>0,2} && \text{(infinitely divisible)},\\ A^{>0,1}(\delta) &\xrightarrow{\mathrm{w}} X^{>0,1} && \text{(compound Poisson)},\\ X^{<0,2}(\delta) &\xrightarrow{\mathrm{w}} X^{<0,2} && \text{(normal)}. \end{aligned}$$

Also,

$$\frac{A(\delta) - m(\delta) - 2\int_0^1 s\nu_\delta^{>0,2}(\mathrm{d}s)}{a(\delta)} \xrightarrow{\mathrm{w}} X^{<0,2} \sim \mathrm{N}(0,1). \tag{3.14}$$

Inspection of the decomposition in (3.13) and (3.14) reveals that the centered cumulative traffic input in the time slot $(0, \delta]$ has an asymptotically normal component on spatial scale $a(\delta)$ plus a component which asymptotically mixes an infinitely divisible and a compound Poisson distribution. For traffic at fine time scales with a high degree of aggregation, resulting in a large number of sessions, the Gaussian component will thus obscure the more 'spikey', high-rate transmissions represented in our model by the infinitely divisible and compound Poisson-distributed components.

This helps to explain why measurements with very high traffic aggregation reveal a Gaussian distribution. The normal component is due to the sessions that start before the time slot and end after the time slot. Thus, they will also be responsible for the dependence structure of the process. As we will see in the next section, the contribution of the infinitely divisible and compound Poisson-distributed components to the dependence across time slots is of lower order than is the contribution of the Gaussian components.

4. Dependence structure across time slots

We now analyze the weak limits of the stochastic process

$$\boldsymbol{A}(\delta) := \{A(k\delta, (k+1)\delta], \ -\infty < k < \infty\}$$

defined in (2.1). We will see that the family of $\mathbb{R}^\infty$-valued random elements indexed by δ converges to a limiting Gaussian sequence,

$$\boldsymbol{X}_\infty = (X_\infty(k),\ -\infty < k < \infty\},$$

with $\operatorname{corr}(X_\infty(0), X_\infty(k)) = 1$. The price paid for letting $\delta \to 0$ is thus the introduction of a limit sequence with degenerate dependence structure. The consequence of sampling at too high a frequency (using economic terminology) is perfect correlation. However, we will see that for fixed $\delta > 0$ we have long-range dependence across time slots.

We begin by considering convergence of finite-dimensional distributions.

4.1. Convergence of finite-dimensional distributions

In this section we prove the following result.

Proposition 7. *For any nonnegative integer k, as $\delta \to 0$ we have*

$$\frac{1}{a(\delta)}\begin{pmatrix} A(0,\delta] - b(\delta) \\ A(\delta, 2\delta] - b(\delta) \\ \vdots \\ A(k\delta, (k+1)\delta] - b(\delta) \end{pmatrix} \xrightarrow{\mathrm{w}} \begin{pmatrix} X_\infty(0) \\ X_\infty(1) \\ \vdots \\ X_\infty(k) \end{pmatrix}$$

in $\mathbb{R}^{k+1}$, where

$$b(\delta) = 2\int_0^1 v\bar{G}(v)\int_v^\infty r^{-1}\mu_\delta(\mathrm{d}r)\,\mathrm{d}v - \int_0^1 \mathrm{E}(F)\bar{G}_0(r,\infty]\mu_\delta(\mathrm{d}r) \tag{4.1}$$

and $X_\infty(i) \sim \mathrm{N}(0,1)$, $0 \le i \le k$, with $\operatorname{corr}(X_\infty(i), X_\infty(j)) = 1$.

Proof. Along with the regions $\mathcal{R}^{<0,1}$, $\mathcal{R}^{<0,2}$, $\mathcal{R}^{>0,1}$, and $\mathcal{R}^{>0,2}$, used to analyze the convergence in distribution of $A(0,\delta]$, we need the analogously defined regions $\mathcal{R}^{<k\delta,1}$, $\mathcal{R}^{<k\delta,2}$, $\mathcal{R}^{>k\delta,1}$, and $\mathcal{R}^{>k\delta,2}$, where, e.g.

$$\mathcal{R}^{<k\delta,2} = \{(s,r,l,u)\colon s < k\delta,\ s+l > (k+1)\delta\},$$
$$\mathcal{R}^{>k\delta,2} = \{(s,r,l,u)\colon k\delta < s < (k+1)\delta,\ s+l > (k+1)\delta\}.$$

(See Figure 2.)

Additionally, for analyzing dependence between $A(0,\delta]$ and $A(k\delta,(k+1)\delta]$ we will need the regions $\mathcal{R}_{11}$, $\mathcal{R}_{12}$, $\mathcal{R}_{21}$, and $\mathcal{R}_{22}$, which contain points $(\Gamma_k, R_k, L_k, F_k)$ contributing to both $A(0,\delta]$ and $A(k\delta,(k+1)\delta]$. (See Figure 3.) In particular, points in $\mathcal{R}_{22} = \mathcal{R}^{<0,2} \cap \mathcal{R}^{<k\delta,2}$ make a contribution

$$A^{22} = \sum_{\{k\colon (\Gamma_k, R_k, L_k, F_k)\in\mathcal{R}_{22}\}} R_k\delta$$

to both $A(0,\delta]$ and $A(k\delta,(k+1)\delta]$.

4.1.1. *Behavior of A^{22}.* By analogy with (3.8), A^{22} is a Poissonized sum of independent, identically distributed random variables, and we compute its characteristic function in a similar manner, to obtain

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{22}}) = \exp\biggl\{\iiint\limits_{\substack{s<0,\,r>0\\ l>(k+1)\delta+|s|}} (\mathrm{e}^{\mathrm{i}\theta r\delta} - 1)\lambda\,\mathrm{d}s\,F_{L,R}(\mathrm{d}l,\mathrm{d}r)\biggr\}.$$

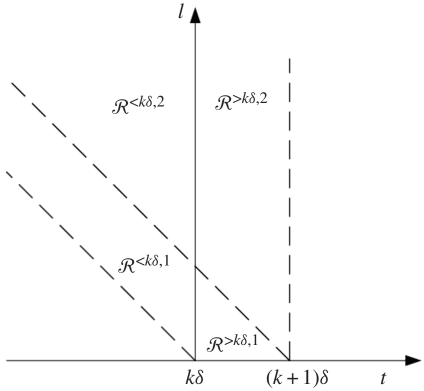

FIGURE 2: Four regions for analyzing contributions in the kth time slot.

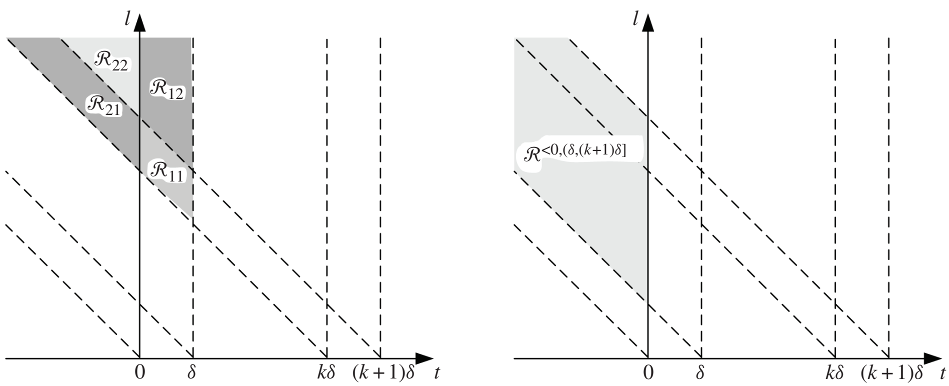

FIGURE 3: Regions for dependence analysis.

By repeating the calculation which led to (3.9), we find that this is equal to

$$\exp\left\{\int_0^\infty (\mathrm{e}^{\mathrm{i}\theta r}-1)\,\mathrm{E}(F)\bar{G}_0((k+1)r)r^{-1}\mu_\delta(\mathrm{d}r)\right\}. \tag{4.2}$$

With

$$a_k^2(\delta) := \int_0^1 \mathrm{E}(F)\bar{G}_0((k+1)r)r\mu_\delta(\mathrm{d}r),$$
$$m_k(\delta) := \int_0^1 \mathrm{E}(F)\bar{G}_0((k+1)r)\mu_\delta(\mathrm{d}r),$$

we find that

$$\frac{A^{22}-m_k(\delta)}{a_k(\delta)} \xrightarrow{\mathrm{w}} X^{22} \sim \mathrm{N}(0,1)$$

in $\mathbb{R}$, as $\delta \to 0$.

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Note that, from Proposition 5, we have

$$a_k(\delta) \sim a(\delta) \sim \left(\mathrm{E}(F) \int_0^1 r\mu_\delta(\mathrm{d}r) \right)^{1/2} \quad \text{as } \delta \to 0.$$

4.1.2. *Contributions from other regions.* Let

$$\mathcal{R}^{<0,(\delta,(k+1)\delta]} = \{(s, r, l, u) \colon s < 0,\ \delta < |s| + l \le (k+1)\delta\}$$

(see Figure 3) and write

$$\begin{aligned} A(0, \delta] &= A^{>0,1}(0, \delta] + A^{>0,2}(0, \delta] + A^{<0,1}(0, \delta] + A^{<0,2}(0, \delta] \\ &= A^{>0,1}(0, \delta] + A^{>0,2}(0, \delta] + (A^{<0,1}(0, \delta] + A^{<0,(\delta,(k+1)\delta]}(0, \delta]) + A^{22}(0, \delta]. \end{aligned}$$

Now, by following the calculation in Section 3.3.1, we find that

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{<0,(\delta,(k+1)\delta]}(0,\delta]}) = \exp\left\{ \int_0^\infty (\mathrm{e}^{\mathrm{i}\theta r} - 1)\, \mathrm{E}(F) r^{-1} G_0(r, (k+1)r]\mu_\delta(\mathrm{d}r) \right\},$$

and

$$\nu_\delta^{<0,(\delta,(k+1)\delta]}(\mathrm{d}r) := \mathrm{E}(F) r^{-1} G_0(r, (k+1)r]\mu_\delta(\mathrm{d}r)$$

converges to a Lévy measure with density

$$\mathrm{E}(F) r^{-1} G_0(r, (k+1)r]\alpha_R r^{-\alpha_R - 1}\mathrm{d}r. \tag{4.3}$$

This means that

$$A^{<0,(\delta,(k+1)\delta]}(0, \delta] - \int_0^1 \mathrm{E}(F) G_0(r, (k+1)r]\mu_\delta(\mathrm{d}r)$$

converges to an infinitely divisible random variable with Lévy measure whose density is given by (4.3). Hence,

$$A^{<0,(\delta,(k+1)\delta]}(0, \delta] - \int_0^1 \mathrm{E}(F) G_0(r, (k+1)r]\mu_\delta(\mathrm{d}r)$$

is $o_p(a(\delta))$. We conclude that

$$\begin{aligned} &A(0, \delta] - 2\int_0^1 v\bar{G}(v) \int_v^\infty r^{-1}\mu_\delta(\mathrm{d}r)\,\mathrm{d}v \\ &- \int_0^1 \mathrm{E}(F) G_0(r, (k+1)r]\mu_\delta(\mathrm{d}r) - \int_0^1 \mathrm{E}(F)\bar{G}_0((k+1)r)\mu_\delta(\mathrm{d}r) \\ &\quad = A(0, \delta] - 2\int_0^1 v\bar{G}(v) \int_v^\infty r^{-1}\mu_\delta(\mathrm{d}r)\,\mathrm{d}v - \int_0^1 \mathrm{E}(F)\bar{G}_0(r, \infty]\mu_\delta(\mathrm{d}r) \\ &\quad = A^{22}(0, \delta] - m_k(\delta) + o_p(a(\delta)). \end{aligned}$$

Likewise, we consider $A(i\delta, (i+1)\delta]$ for $1 \le i \le k$. We set (see Figure 4)

$$\begin{aligned} \mathcal{R}^{<0,((i+1)\delta,(k+1)\delta]} &= \{(s, r, l, u) \colon s < 0,\ (i+1)\delta < s + l < (k+1)\delta\}, \\ \mathcal{R}^{(0,i\delta],((i+1)\delta,\infty]} &= \{(s, r, l, u) \colon 0 < s \le i\delta,\ s + l > (i+1)\delta\}, \end{aligned}$$

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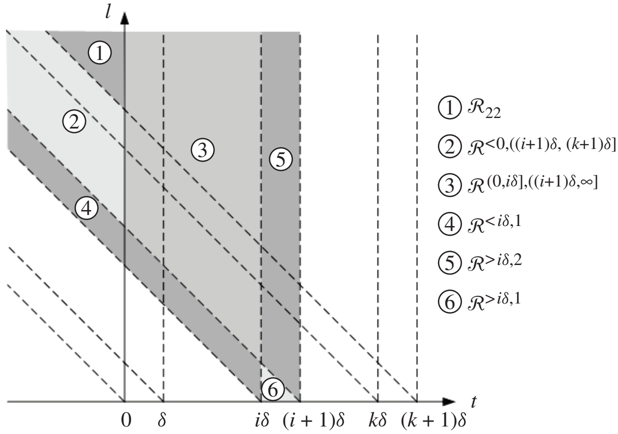

FIGURE 4: Regions for dependence analysis.

and write

$$\begin{aligned}A(i\delta,(i+1)\delta] &= A^{>i\delta,1}(i\delta,(i+1)\delta] + A^{>i\delta,2}(i\delta,(i+1)\delta] + A^{<i\delta,1}(i\delta,(i+1)\delta]\\ &\quad + (A^{<0,((i+1)\delta,(k+1)\delta]}(i\delta,(i+1)\delta] + A^{22}(i\delta,(i+1)\delta]\\ &\quad + A^{(0,i\delta],((i+1)\delta,\infty]}(i\delta,(i+1)\delta]).\end{aligned}$$

Now,

$$\begin{aligned}&\mathrm{E}(\exp\{\mathrm{i}\theta A^{<0,((i+1)\delta,(k+1)\delta]}(i\delta,(i+1)\delta]\})\\ &\quad = \exp\biggl\{\int_0^\infty (\mathrm{e}^{\mathrm{i}\theta r}-1)\,\mathrm{E}(F)r^{-1}G_0((i+1)r,(k+1)r]\mu_\delta(\mathrm{d}r)\biggr\},\\ &\mathrm{E}(\exp\{\mathrm{i}\theta A^{(0,i\delta],((i+1)\delta,\infty]}(i\delta,(i+1)\delta]\})\\ &\quad = \exp\biggl\{\int_0^\infty (\mathrm{e}^{\mathrm{i}\theta r}-1)\,\mathrm{E}(F)r^{-1}G_0(r,(i+1)r]\mu_\delta(\mathrm{d}r)\biggr\}.\end{aligned}$$

Therefore, keeping in mind that

$$A^{22}(k\delta,(k+1)\delta] = A^{22}(i\delta,(i+1)\delta] = A^{22}(0,\delta],$$

we have

$$\begin{aligned}&A(i\delta,(i+1)\delta] - 2\int_0^1 v\bar{G}(v)\int_v^\infty r^{-1}\mu_\delta(\mathrm{d}r)\,\mathrm{d}v - \int_0^1 \mathrm{E}(F)\bar{G}_0(r)\mu_\delta(\mathrm{d}r)\\ &\quad = A(i\delta,(i+1)\delta] - 2\int_0^1 v\bar{G}(v)\int_v^\infty r^{-1}\mu_\delta(\mathrm{d}r)\,\mathrm{d}v - \int_0^1 \mathrm{E}(F)G_0(r,(i+1)r]\mu_\delta(\mathrm{d}r)\\ &\qquad - \int_0^1 \mathrm{E}(F)G_0((i+1)r,(k+1)r]\mu_\delta(\mathrm{d}r) - m_k(\delta)\\ &\quad = A^{22}(0,\delta] - m_k(\delta) + o_p(a(\delta)).\end{aligned}$$

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We thus have

$$\begin{pmatrix} A(0,\delta]-b(\delta) \\ A(\delta,2\delta]-b(\delta) \\ \vdots \\ A(k\delta,(k+1)\delta]-b(\delta) \end{pmatrix} = \begin{pmatrix} A^{22}(0,\delta]-m_k(\delta) \\ A^{22}(0,\delta]-m_k(\delta) \\ \vdots \\ A^{22}(0,\delta]-m_k(\delta) \end{pmatrix} + \begin{pmatrix} o_p(a(\delta)) \\ o_p(a(\delta)) \\ \vdots \\ o_p(a(\delta)) \end{pmatrix}$$

and the conclusion of Proposition 7 follows.

4.2. Correlation structure

Despite the fact that in the limit as $\delta \to 0$ we get a degenerate dependence structure, for a *fixed* $\delta > 0$, decay of correlations over time slots spaced by k exhibits long-range dependence as $k \to \infty$. In this section we will prove the following result.

Proposition 8. *For any fixed $\delta > 0$, as $k \to \infty$ we have*

$$\operatorname{cov}(A(0,\delta], A(k\delta,(k+1)\delta]) \sim \text{const.}\, \bar{G}_0(k) \sim \text{const.}\, k^{-(\alpha_F-1)} L_F(k)$$

and, thus, the stationary sequence $\{A(k\delta,(k+1)\delta],\ -\infty < k < \infty\}$ exhibits long-range dependence.

Referring to Figure 3, we see that we can write

$$\begin{aligned} A(0,\delta] &= A^{11}(0,\delta] + A^{12}(0,\delta] + A^{22}(0,\delta] + A^{21}(0,\delta] + I_1 \\ &= \sum_{\{k\colon (\Gamma_k,L_k,R_k,F_k)\in\mathcal{R}_{11}\}} R_k(\delta-\Gamma_k) + \sum_{\{k\colon (\Gamma_k,L_k,R_k,F_k)\in\mathcal{R}_{12}\}} R_k(\delta-\Gamma_k) \\ &\quad + \sum_{\{k\colon (\Gamma_k,L_k,R_k,F_k)\in\mathcal{R}_{22}\}} R_k\delta + \sum_{\{k\colon (\Gamma_k,L_k,R_k,F_k)\in\mathcal{R}_{21}\}} R_k\delta + I_1 \end{aligned}$$

and

$$\begin{aligned} A(k\delta,(k+1)\delta] &= A^{11}(k\delta,(k+1)\delta] + A^{12}(k\delta,(k+1)\delta] + A^{22}(k\delta,(k+1)\delta] \\ &\quad + A^{21}(k\delta,(k+1)\delta] + I_2 \\ &= \sum_{\{k\colon (\Gamma_k,L_k,R_k,F_k)\in\mathcal{R}_{11}\}} R_k(\Gamma_k+L_k-\delta k) + \sum_{\{k\colon (\Gamma_k,L_k,R_k,F_k)\in\mathcal{R}_{12}\}} R_k\delta \\ &\quad + \sum_{\{k\colon (\Gamma_k,L_k,R_k,F_k)\in\mathcal{R}_{22}\}} R_k\delta + \sum_{\{k\colon (\Gamma_k,L_k,R_k,F_k)\in\mathcal{R}_{21}\}} R_k(\Gamma_k+L_k-k\delta) + I_2, \end{aligned}$$

where I_1 and I_2 are independent of the other summands and of each other and do not affect covariance calculations. We thus have

$$\begin{aligned} &\operatorname{cov}(A(0,\delta], A(k\delta,(k+1)\delta]) \\ &\quad = \operatorname{cov}(A^{11}(0,\delta], A^{11}(k\delta,(k+1)\delta]) + \operatorname{cov}(A^{12}(0,\delta], A^{12}(k\delta,(k+1)\delta]) \\ &\qquad + \operatorname{cov}(A^{22}(0,\delta], A^{22}(k\delta,(k+1)\delta]) + \operatorname{cov}(A^{21}(0,\delta], A^{21}(k\delta,(k+1)\delta]). \qquad (4.4) \end{aligned}$$

The dominant term comes from the region $\mathcal{R}_{22}$, as we now show.

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4.2.1. *Contribution to the covariance from $\mathcal{R}_{22}$.* Since

$$A^{22}(0, \delta] = A^{22}(k\delta, (k+1)\delta],$$

we have

$$\operatorname{cov}(A^{22}(0, \delta], A^{22}(k\delta, (k+1)\delta]) = \operatorname{var}(A^{22}(0, \delta]).$$

If P is a Poisson-distributed random variable independent of the sequence $\{\xi_n,\ n \geq 1\}$, then

$$\operatorname{var}\left(\sum_{i=1}^{P} \xi_i\right) = \mathrm{E}(P)\,\mathrm{E}(\xi_1^2).$$

Therefore, we have

$$\begin{aligned}
\operatorname{var}(A^{22}(0, \delta]) &= \iiint\limits_{\substack{s<0,\, r>0\\ l>(k+1)\delta+|s|}} r^2\delta^2\lambda\,\mathrm{d}s\,F_{L,R}(\mathrm{d}l, \mathrm{d}r)\\
&= \int_{(k+1)\delta}^{\infty} \lambda\,\mathrm{d}s \int_0^{\infty} r^2\delta^2\bar{G}(rs)F_R(\mathrm{d}r)\\
&= \lambda\delta^2 \int_0^{\infty} r\bar{G}_0((k+1)r\delta)F_R(\mathrm{d}r)\,\mathrm{E}(F)\\
&= \lambda\delta \int_0^{\infty} r\bar{G}_0((k+1)r)F_R(\delta^{-1}\,\mathrm{d}r)\,\mathrm{E}(F)\\
&= \int_0^{\infty} r\bar{G}_0((k+1)r)\mu_\delta(\mathrm{d}r)\\
&=: a_k^{\#}(\delta)^2.
\end{aligned}$$

We now discuss the growth rate of $a_k^{\#}(\delta)^2$ as a function of k, keeping $\delta > 0$ fixed.

Proposition 9. *For any fixed $\delta > 0$, as $k \to \infty$ we have*

$$a_k^{\#}(\delta)^2 = \int_0^{\infty} r\bar{G}_0((k+1)r)\mu_\delta(\mathrm{d}r) \sim \bar{G}_0(k)\int_0^{\infty} r^{2-\alpha_F}\mu_\delta(\mathrm{d}r),$$

where the integral on the right-hand side is finite.

Proof. It is no loss of generality to suppose, for convenience, that $\delta = 1$ and to neglect $F_R(1)$ in the denominator of μ_δ. We must then show that

$$\int_0^{\infty} r\,\frac{\bar{G}_0(kr)}{\bar{G}_0(k)}F_R(\mathrm{d}r) \to \int_0^{\infty} r^{2-\alpha_F}F_R(\mathrm{d}r). \tag{4.5}$$

By Fubini's theorem, the left-hand side is equal to

$$\int_0^{\infty}\int_0^{s} rF_R(\mathrm{d}r)\frac{\bar{G}(ks)k}{\mathrm{E}(F)\bar{G}_0(k)}\,\mathrm{d}s.$$

Since

$$\bar{G}_0(k) \sim \frac{k\bar{G}(k)}{\mathrm{E}(F)(\alpha_F - 1)},$$

it suffices to show that

$$\int_0^\infty \int_0^s r F_R(\mathrm{d}r) \frac{\bar{G}(ks)}{\bar{G}(k)} \,\mathrm{d}s(\alpha_F - 1) \tag{4.6}$$

converges to the right-hand side of (4.5).

To do so, we break the integral in (4.6) into an integration over [0, 1] and an integration over $(1, \infty)$. For $s > 1$, we have, by Potter's bounds, that for any small $\eta > 0$, all sufficiently large k, and some constant c,

$$\frac{\bar{G}(kr)}{\bar{G}(k)} \le c r^{-\alpha_F + \eta}.$$

On $[1, \infty)$, the integrand is bounded by $\mathrm{E}(R) c r^{-\alpha_F+\eta}$, which is integrable there. Thus, by dominated convergence we may integrate to the limit over $[1, \infty)$. On [0, 1] the integrand in (4.6) is bounded (neglecting constants) by $s\bar{G}(ks)/\bar{G}(k) \to s^{1-\alpha_F}$. Since Karamata's theorem implies that

$$\int_0^1 s \frac{\bar{G}(ks)}{\bar{G}(k)} \,\mathrm{d}s \to \int_0^1 s^{1-\alpha_F} \,\mathrm{d}s = \frac{1}{2 - \alpha_F},$$

the desired result follows from Pratt's lemma (Pratt (1960), Resnick (1998, p. 164)).

4.2.2. *Contribution to the covariance from other terms.* We now show that the contribution to the covariance from the other three terms in (4.4) is $o(\bar{G}_0(k))$. We use the following formula in doing so: if P is a Poisson-distributed random variable independent of the independent, identically distributed sequence $\{(\xi_n, \eta_n),\ n \ge 1\}$, then

$$\operatorname{cov}\left(\sum_{i=1}^{P} \xi_i, \sum_{i=1}^{P} \eta_i\right) = \mathrm{E}(P)\,\mathrm{E}(\xi_1 \eta_1). \tag{4.7}$$

Contribution from $\mathcal{R}_{11}$. Evaluating the expression on the right-hand side of (4.7) for region $\mathcal{R}_{11}$ yields the following formula:

$$\begin{aligned}
&\iiint\limits_{\substack{0<s\le\delta,\, r>0 \\ k\delta<l+s\le(k+1)\delta}} r^2(\delta - s)(s + l - k\delta)\lambda \,\mathrm{d}s\, F_{L,R}(\mathrm{d}l, \mathrm{d}r) \\
&\quad \le \lambda\delta^2 \int_0^\delta \int\limits_{\substack{r>0 \\ k\delta - s < l \le k\delta+\delta-s}} r^2 F_{L,R}(\mathrm{d}l, \mathrm{d}r)\,\mathrm{d}s \\
&\quad = \lambda\delta^2 \int_{(k-1)\delta}^{k\delta} \int_{r>0} r^2 \int_{sr<u\le(\delta+s)r} G(\mathrm{d}u) F_R(\mathrm{d}r)\,\mathrm{d}s \\
&\quad = \lambda\delta^2 \int_{(k-1)\delta}^{k\delta} \int_{r>0} r^2 G(sr, (\delta+s)r] F_R(\mathrm{d}r)\,\mathrm{d}s \\
&\quad \le \lambda\delta^2 \int_{(k-1)\delta}^{k\delta} \int_{r>0} r^2 \bar{G}(sr) F_R(\mathrm{d}r)\,\mathrm{d}s \\
&\quad = \lambda\delta^2 \int_{r>0} r G_0((k-1)r\delta, kr\delta]\,\mathrm{E}(F) F_R(\mathrm{d}r) \\
&\quad = \int_{r>0} r G_0((k-1)r, kr]\,\mathrm{E}(F)\mu_\delta(\mathrm{d}r) \\
&\quad = o(\bar{G}_0(k)),
\end{aligned}$$

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where the first inequality holds because on the region of integration both $(\delta - s)$ and $(s + l - k\delta)$ are bounded by δ, and where the last assertion follows because $a_k^{\#}(\delta)$ is of order $\bar{G}_0(k)$.

Contribution from $\mathcal{R}_{12}$. For region $\mathcal{R}_{12}$, the expression on the right-hand side of (4.7) yields the following formula:

$$\begin{aligned}
\iiint\limits_{\substack{0<s\le\delta,\, r>0\\ s+l>(k+1)\delta}} r^2\delta(\delta - s)\lambda\,\mathrm{d}s\,F_{L,R}(\mathrm{d}l,\mathrm{d}r) &= \lambda\delta\int_0^{\delta}\int\limits_{\substack{r>0\\ l>k\delta+s}} r^2 s\,F_{L,R}(\mathrm{d}l,\mathrm{d}r)\,\mathrm{d}s\\
&\le \lambda\delta^2\int_0^{\delta}\int\limits_{\substack{r>0\\ l>k\delta+s}} r^2 F_{L,R}(\mathrm{d}l,\mathrm{d}r)\,\mathrm{d}s\\
&= \lambda\delta^2\int_{k\delta}^{(k+1)\delta}\int_{r>0}\bar{G}(rs)r^2F_R(\mathrm{d}r)\,\mathrm{d}s\\
&= \int_{r>0} rG_0(kr,(k+1)r]\,\mathrm{E}(F)\mu_\delta(\mathrm{d}r)\\
&= o(\bar{G}_0(k)).
\end{aligned}$$

In a similar way, we can show that the contribution from region $\mathcal{R}_{21}$ is $o(\bar{G}_0(k))$. This completes the proof of Proposition 8.

5. Dependence structure on a different time scale

In the previous section we discussed dependence over successive slots of length δ. The asymptotic normality statement of Proposition 7 leads to a degenerate limit, because $\delta \downarrow 0$ shrinks the distance between $A(0,\delta]$ and $A(k\delta,(k+1)\delta]$. Here we investigate $(A(0,\delta], A(t,t+\delta])$ for $t > \delta$ and find that as $\delta \downarrow 0$ this vector is asymptotically normal with a limiting correlation $\rho(t)$. The function $\rho(t)$ satisfies

$$\rho(t) \sim c\bar{G}_0(t) \to 0, \quad \text{as } t \to \infty,$$

which may be compared with the result of Proposition 8. This provides another interpretation of the long-range dependence in the model.

Proposition 10. *Suppose that* $t > 0$. *As* $\delta \downarrow 0$,

$$a^{-1}(\delta)\begin{pmatrix} A(0,\delta] - b(\delta)\\ A(t,t+\delta] - b(\delta)\end{pmatrix} \xrightarrow{\mathrm{w}} \begin{pmatrix} N_1 + N\\ N_2 + N\end{pmatrix},$$

where $b(\delta)$ *is given by (4.1),* N_1, N_2, *and* N *are independent normal variables with*

$$N_1 \stackrel{\mathrm{D}}{=} N_2 \sim N(0,\sigma^2(t)), \qquad N \sim N(0,\rho(t)),$$

and

$$\sigma^2(t) = \frac{\int_0^\infty rG_0(tr)F_R(\mathrm{d}r)}{\mathrm{E}(R)},$$

$$\rho(t) = \frac{\int_0^\infty r\bar{G}_0(tr)F_R(\mathrm{d}r)}{\mathrm{E}(R)},$$

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whence $\sigma^2(t) + \rho(t) = 1$. *Thus,*

$$\operatorname{cov}(N_1 + N, N_2 + N) = \operatorname{var}(N) = \rho(t).$$

Furthermore, as $t \to \infty$,

$$\rho(t) \sim \frac{\int_0^\infty r^{2-\alpha_F} F_R(\mathrm{d}r)}{\mathrm{E}(R)} \bar{G}_0(t).$$

Proof. As in the proof of Proposition 7, we write

$$\begin{aligned} A(0, \delta] &= A^{<0,(\delta,t+\delta]}(\delta) + A^{<0,(t+\delta,\infty]}(\delta) + o_p(a(\delta)), \\ A(t, t+\delta] &= A^{(0,t],(t+\delta,\infty]}(t, t+\delta] + A^{<0,(t+\delta,\infty]}(t, t+\delta] + o_p(a(\delta)), \end{aligned}$$

and keep in mind that

$$A^{<0,(t+\delta,\infty]}(\delta) = A^{<0,(t+\delta,\infty]}(t, t+\delta]$$

and

$$A^{<0,(\delta,t+\delta]}(\delta) \stackrel{\mathrm{D}}{=} A^{(0,t],(t+\delta,\infty]}(t, t+\delta].$$

The characteristic functions are

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{<0,(\delta,t+\delta]}(\delta)}) = \exp\left\{\int_0^\infty (\mathrm{e}^{\mathrm{i}\theta r} - 1) r^{-1} G_0(r, r(1 + t/\delta)) \mu_\delta(\mathrm{d}r)\, \mathrm{E}(F)\right\}$$

(see (4.2)) and

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta A^{<0,(t+\delta,\infty]}(\delta)}) = \exp\left\{\int_0^\infty (\mathrm{e}^{\mathrm{i}\theta r} - 1) r^{-1} G_0\left(\frac{r}{\delta}(t+\delta)\right) \mu_\delta(\mathrm{d}r)\, \mathrm{E}(F)\right\}.$$

If we let

$$\begin{aligned} m_t(\delta) &= \int_0^1 \mathrm{E}(F) \bar{G}_0\left(\frac{r}{\delta}(t+\delta)\right) \mu_\delta(\mathrm{d}r), \\ a_t^2(\delta) &= \int_0^1 \mathrm{E}(F) \bar{G}_0\left(\frac{r}{\delta}(t+\delta)\right) r \mu_\delta(\mathrm{d}r), \end{aligned}$$

we obtain

$$a_t(\delta)^{-1}(A^{<0,(t+\delta,\infty]}(\delta) - m_t(\delta)) \xrightarrow{\mathrm{W}} \mathrm{N}(0, 1)$$

from the characteristic function and

$$\begin{aligned} \frac{a_t^2(\delta)}{a^2(\delta)} &\sim \frac{\int_0^1 \bar{G}_0((r/\delta)(t+\delta)) r \mu_\delta(\mathrm{d}r)}{\mathrm{E}(R)/\delta^{-1} \bar{F}_R(\delta^{-1})} \\ &= \frac{\delta \int_0^{\delta^{-1}} \bar{G}_0(r(t+\delta)) r F_R(\mathrm{d}r)/\bar{F}_R(\delta^{-1})}{\mathrm{E}(R)/\delta^{-1} \bar{F}_R(\delta^{-1})} \\ &\to \frac{\int_0^\infty \bar{G}_0(rt) r F_R(\mathrm{d}r)}{\mathrm{E}(R)} \\ &= \rho(t) \end{aligned}$$

from Proposition 5 by dominated convergence, since $\mathrm{E}(R) < \infty$.

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Similarly, the appropriate scaling for $A^{<0,(\delta,t+\delta]}(\delta)$ to achieve asymptotic normality is $\tilde{a}_t(\delta)$, where

$$\tilde{a}_t(\delta)^2 := \mathrm{E}(F)\int_0^1 rG_0(r, r(1+t/\delta))\mu_\delta(\mathrm{d}r)$$

and, as $\delta \downarrow 0$,

$$\begin{aligned}\tilde{a}_t(\delta)^2 &\sim \frac{\int_0^\infty \mathrm{E}(F)rG_0(rt)F_R(\mathrm{d}r)}{\delta^{-1}\bar{F}_R(\delta^{-1})}\\ &\sim a^2(\delta)\frac{\int_0^\infty rG(rt)F_R(\mathrm{d}r)}{\mathrm{E}(R)}\\ &= a^2(\delta)\sigma^2(t).\end{aligned}$$

The asymptotic form of $\rho(t)$ as $t \to \infty$ is obtained using arguments similar to those yielding (4.5).

6. Concluding remarks

As summarized in Table 1, our model does a sound job of explaining certain empirically observed facts that we have termed *stylized facts*. For a fixed time slot, there is observable Gaussian behavior for cumulative input as the rate increases and the slot width decreases. This Gaussian behavior is on a spatial scale $a(\delta) \to \infty$, and is responsible for most of the traffic volume. Hence, this component can model what Sarvotham *et al.* (2005) called β-traffic. The spatial scaling obscures heavy-tailed behavior approximated by infinitely divisible random variables with heavy tails. This heavy-tailed component, which disappears in the limit, is what generates, for finite-scale δ, the bursty behavior, and seems to be the right candidate to model the α-traffic component. Sarvotham *et al.* (2005) pointed out that as the aggregation increases the traffic becomes more and more Gaussian, which implies that the Gaussian character is dominant over the bursty character. In addition, for a fixed slot width, dependence across time slots exhibits long-range dependence, and this dependence is carried mostly by the Gaussian part, i.e. the β-component.

As the slot width goes to 0, the centered and scaled sequence of inputs in successive slots converges to a perfectly correlated limiting Gaussian sequence. This is a consequence of the

TABLE 1: How the model incorporates the stylized facts.

Stylized fact	Model
Presence of heavy tails	Built-in
LRD across slots	Lag-k covariance (see Proposition 8) is of the order $c\bar{G}_0(k)$ for δ fixed and $k \to \infty$
Burstiness	Traffic from regions $\mathcal{R}^{<0,1\cup>0,2\cup>0,1}$ is infinitely divisible and has a heavy tail
Cumulative traffic per slot is approximately normal	$(A(0,\delta] - b(\delta))/a(\delta) \overset{\mathrm{D}}{\approx} \mathrm{N}(0,1)$
Dependence carried by β-traffic	Covariance from infinitely divisible pieces is of smaller order than that from the Gaussian piece

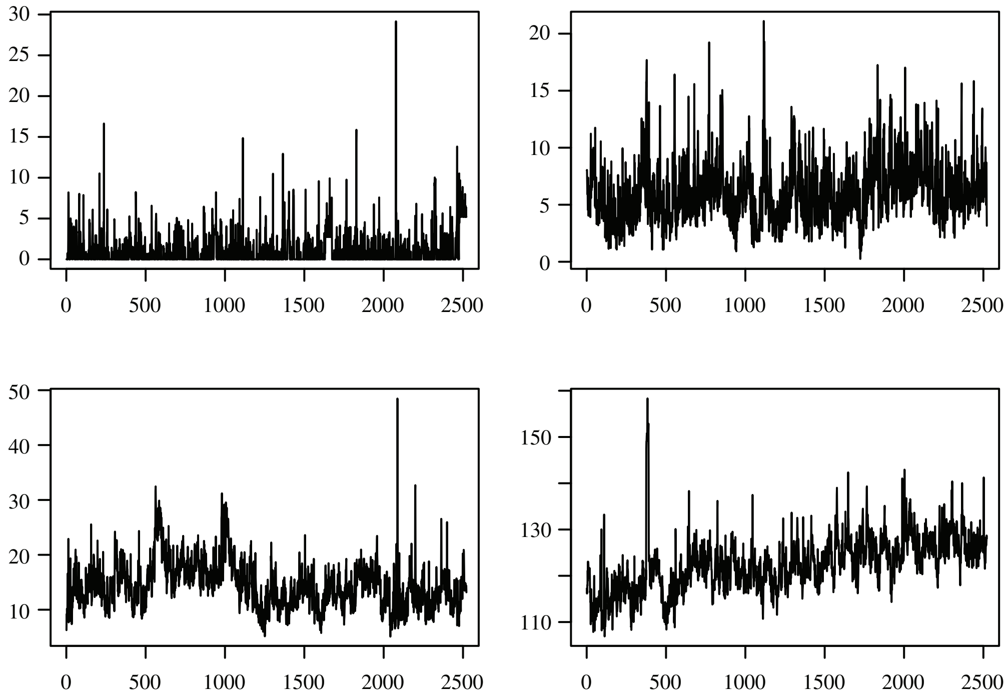

FIGURE 5: Four time series plots of input per slot, corresponding to $\delta = 10$, $\lambda = 0.3372$ (*top-left*), $\delta = 10^{-0.5}$, $\lambda = 4.763$ (*top-right*), $\delta = 10^{-0.75}$, $\lambda = 17.9$ (*bottom-left*), and $\delta = 10^{-1.5}$, $\lambda = 950.3$ (*bottom-right*).

shrinking slot width and is not surprising. The higher the frequency of sampling, the more correlation is to be expected. However, inputs in two slots, $(0, \delta]$ and $(t, t + \delta]$, are jointly asymptotically normal with a correlation $\rho(t)$, which decays slowly according to Proposition 10, giving another interpretation of the long-range dependence in the model.

We intend to assume other dependence structures in our modeling in order to investigate how the conclusions change. In particular, the model LR, where (L_k, R_k) are assumed to be independent, is a subject for future work. A mixture of the models LR and FR might also be worth considering.

The model used here offers a reasonable match to what experimenters actually measure. Another virtue is that it is relatively easy to simulate such a model. For example, in Figure 5 we give four time series plots of length 2449 for the cumulative input per slot, and in Figure 6 we give six normal quantile–quantile plots to graphically assess a normal fit to the simulated data. Finally, in Figure 7 we give an autocorrelation plot over 700 lags, to illustrate the slow rate of decay of the dependence as a function of lag.

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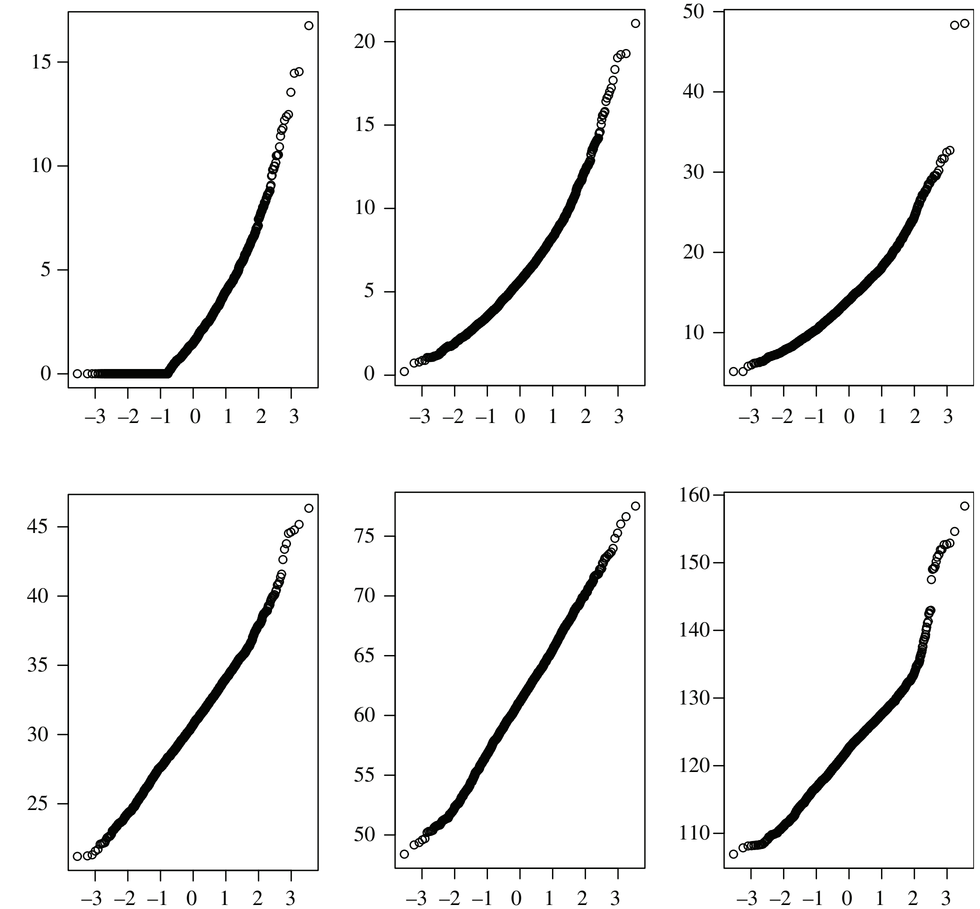

FIGURE 6: Normal quantile–quantile plots corresponding to $\delta = 10^{-0.25}$ (*top-left*), $\delta = 10^{-0.5}$ (*top-middle*), $\delta = 10^{-0.75}$ (*top-right*), $\delta = 10^{-1}$ (*bottom-left*), $\delta = 10^{-1.25}$ (*bottom-middle*), and $\delta = 10^{-1.5}$ (*bottom-right*). Sample quantiles are plotted on the vertical axes and theoretical quantiles on the horizontal axes.

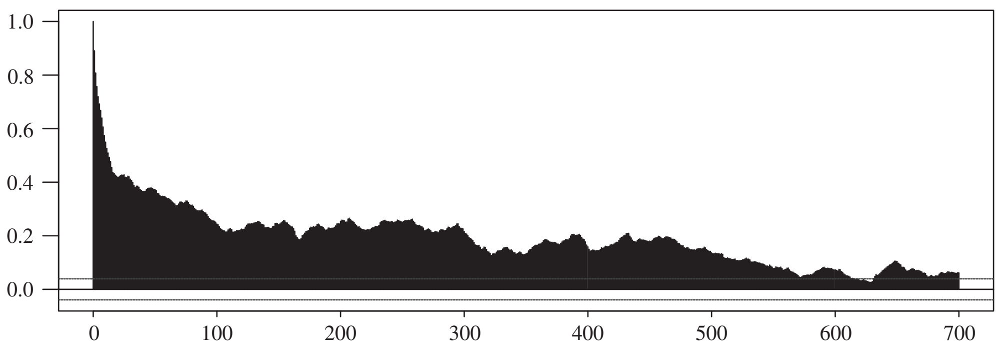

FIGURE 7: Plot of autocorrelation versus the number of lags, for $\delta = 0.032$.

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