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A Real-analytic Nonpolynomially Convex Isotropic Torus with no Attached Discs

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Abstract. We show by means of an example in $\mathbb{C}^3$ that Gromov's theorem on the presence of attached holomorphic discs for compact Lagrangian manifolds is not true in the subcritical real-analytic case, even in the absence of an obvious obstruction, *i.e.*, polynomial convexity.

A compact set $X \subset \mathbb{C}^n$ is called *polynomially convex* if, for every $z \notin X$, there is a holomorphic polynomial P such that $|P(z)| > \sup_{x \in X} |P(x)|$. It is known that no real compact n-dimensional submanifold $M \subset \mathbb{C}^n$ (without boundary) can be polynomially convex. In the particular case when the inclusion $\iota: M \hookrightarrow \mathbb{C}^n$ is maximally isotropic (or Lagrangian) with respect to $\omega_{\mathrm{st}} = i \sum_1^n dz_j \wedge d\overline{z_j}$, *i.e.*, $\iota^*(\omega_{\mathrm{st}}) = 0$, Gromov [5] proved a stronger statement: there is a holomorphic disc attached to M; *i.e.*, there is a nonconstant holomorphic map from the unit disc $\mathbb{D}$ to $\mathbb{C}^n$ that is continuous up to the boundary and maps $\partial\mathbb{D}$ into M. Gromov's result is not true in the subcritical case (when $\dim M < n$), as there are several examples of polynomially convex isotropic surfaces in $\mathbb{C}^3$. It is natural to ask whether Gromov's result holds in the subcritical case in the absence of polynomial convexity. For $\mathcal{C}^\infty$-smooth manifolds, this is known to be false due to an example in [6] of a nonpolynomially convex two-torus in $\mathbb{C}^3$ that does not have any analytic variety attached to it. Since this torus is the graph of a real-valued function over the standard torus in $\mathbb{C}^2$, it is isotropic in $\mathbb{C}^3$ with respect to ω_{st}. No such examples are known in the real-analytic case.

In this note, we produce an explicit real-analytic nonpolynomially convex two-torus $T \subset \mathbb{C}^3$ that is isotropic with respect to ω_{st}, but has no holomorphic discs attached to it. In view of the example in [6], we note that our example does have a holomorphic annulus attached to it. The isotropicity of T implies that it is both totally real and rationally convex (see [4]). Examples of totally real tori with no attached holomorphic discs have been given by Alexander [1] and Duval–Gayet [3] in $\mathbb{C}^2$, but such examples cannot be rationally convex in view of Duval–Sibony (see [4, Theorem 3.1]) and Gromov's result. In the case of manifolds with boundary, Duval has constructed an example of a nonpolynomially convex Lagrangian surface in $\mathbb{C}^2$ that has no attached discs (see [2] or [4]).

Theorem 1 *There is a real-analytic two-torus in $\mathbb{C}^3$ that is isotropic with respect to ω_{st}, not polynomially convex, but has no holomorphic discs attached to it.*

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Proof Let $p(z,w) := 1 - 4z^2 + 4w^2 - z^2w^2$ and

$$T := \left\{ \left(z, w, \operatorname{Re} p(z,w) \right) \in \mathbb{C}^3 : z, w \in \partial\mathbb{D} \right\}.$$

Being the graph of a real-valued function on the torus $\mathbb{T}^2 := \partial\mathbb{D} \times \partial\mathbb{D}$, T is isotropic with respect to ω_{st}. We claim that T is not polynomially convex, and its polynomial hull (defined below) consists of T and an attached annulus.

Before we proceed, we fix some notation. If $A \subset \overline{\mathbb{D}}^2$ and $f : \overline{\mathbb{D}}^2 \to \mathbb{C}$, then

$$\mathcal{G}_f(A) = \left\{ (z, w, f(z,w)) \subset \mathbb{C}^3 : (z,w) \in A \right\}$$

denotes the graph of $f|_A$. If $\zeta \in \overline{\mathbb{D}}^2$, then $\mathcal{G}_f(\{\zeta\})$ is simplified to $\mathcal{G}_f(\zeta)$. For a compact $X \subset \mathbb{C}^n$, the *polynomial hull* of X is the set

$$\widehat{X} = \left\{ z \in \mathbb{C}^n : |P(z)| \le \sup_{x \in X} |P(x)| \text{ for all polynomials } P \right\}.$$

Now, let $f(z,w) := \operatorname{Re}(p(z,w))$. In our notation, $T = \mathcal{G}_f(\mathbb{T}^2)$. We first consider a related torus $T_1 := \mathcal{G}_{\overline{p}}(\mathbb{T}^2)$. We will show that T_1 has all the required properties except that it is not isotropic with respect to ω_{st}. It will then follow from a simple observation that T is indeed the required example.

We claim that

$$\widehat{T_1} = T_1 \cup \mathcal{G}_p(\mathcal{Z}), \tag{1}$$

where $\mathcal{Z} = \{(z,w) \in \overline{\mathbb{D}}^2 : w^2 = \frac{4z^2-1}{4-z^2}\}$. Since $p|_{\mathcal{Z}} \equiv 0$, $\mathcal{G}_p(\mathcal{Z})$ is isomorphic to $\mathcal{Z}$. Moreover, by a computation due to Rudin (see [8, proof of Theorem B]) $\mathcal{Z}$ is a connected finite Riemann surface of genus 0 with two boundary components in $\mathbb{T}^2$; *i.e*, $\mathcal{G}_p(\mathcal{Z})$ is an annulus attached to T_1.

To prove (1), we use a technique due to Jimbo (see [7]). Following the notation in [7], let

$$\begin{aligned} h(z,w) &= (zw)^{-2}(z^2w^2 - 4w^2 + 4z^2 - 1), \\ L &= (\overline{\mathbb{D}} \times \{0\}) \cup (\{0\} \times \overline{\mathbb{D}}), \\ V &= \left\{ (z,w) \in \overline{\mathbb{D}}^2 \setminus (\mathbb{T}^2 \cup L) : \overline{p(z,w)} = h(z,w) \right\}. \end{aligned}$$

Note that $h(z,w) = \overline{p(z,w)}$ on $\mathbb{T}^2$. Next, we compute

$$\Delta(z,w) = \begin{vmatrix} \dfrac{\partial p}{\partial z}(z,w) & \dfrac{\partial p}{\partial w}(z,w) \\ \dfrac{\partial h}{\partial z}(z,w) & \dfrac{\partial h}{\partial w}(z,w) \end{vmatrix} = \begin{vmatrix} -8z - 2zw^2 & 8w - 2z^2w \\ \dfrac{8}{z^3} + \dfrac{2}{z^3w^2} & -\dfrac{8}{w^3} + \dfrac{2}{z^2w^3} \end{vmatrix},$$

to obtain $\Delta(z,w) = -16(zw)^{-3}(z - iw)(z + iw)p(z,w)$. Setting $q_1 = (z - iw)$, $q_2 = z + iw$, $q_3 = p(z,w)$, and $Q_j := \{(z,w) \in \mathbb{T}^2 : q_j(z,w) = 0\}$, $1 \le j \le 3$, we have that

$$\begin{aligned} Q_1 &= \{(z, iz) \subset \mathbb{T}^2 : z \in \partial\mathbb{D}\}, \\ Q_2 &= \{(z, -iz) \subset \mathbb{T}^2 : z \in \partial\mathbb{D}\}, \\ Q_3 &= \mathcal{Z} \cap \mathbb{T}^2 = \partial\mathcal{Z}. \end{aligned} \tag{2}$$

In [7], Jimbo showed that if $\Delta(z,w) \not\equiv 0$ on $\mathbb{D}^2 \setminus L$ and

$$J := \left\{ 1 \le j \le 3 : \varnothing \ne Q_j \ne \widehat{Q_j},\ \widehat{Q_j} \setminus (\mathbb{T}^2 \cup L) \subset V \right\} \ne \varnothing,$$

then

$$\widehat{\mathcal{G}_{\overline{p}}(\mathbb{T}^2)} = \mathcal{G}_{\overline{p}}(\mathbb{T}^2) \cup \bigcup_{j\in J} \left\{ (z, w, \overline{p(z,w)}) : (z,w) \in \widehat{Q_j} \right\},$$

and p restricts to a constant on each $\widehat{Q_j}$, $j \in J$. In view of (2), $J = \{3\}$, $\widehat{Q_3} = \mathcal{Z}$, and, since $p|_{\mathcal{Z}} = \overline{p}|_{\mathcal{Z}} = 0$, (1) holds; *i.e.*, there is only one annulus attached to T_1. Since $\mathbb{T}^2$ is totally real and rationally convex, and $\overline{p}$ is smooth, $T_1 = \mathcal{G}_{\overline{p}}(\mathbb{T}^2)$ is totally real and rationally convex. Due to a result by Duval and Sibony (see [4]), T_1 is isotropic with respect to some Kähler form on $\mathbb{C}^3$. But, $\iota^*(\omega_{\mathrm{st}}) \ne 0$, where $\iota\colon T_1 \hookrightarrow \mathbb{C}^3$ is the inclusion map.

We now return to $T := \mathcal{G}_f(\mathbb{T}^2)$. Note that the algebraic isomorphism

$$F(z,w,\eta) \mapsto \left(z, w, \frac{1}{2}\big(\eta + p(z,w) \big) \right)$$

maps T_1 onto T and fixes the variety $\mathcal{G}_p(\mathcal{Z})$. Thus, $\widehat{T} = F(\widehat{T_1}) = T \cup \mathcal{G}_p(\mathcal{Z})$. As there are no nontrivial holomorphic discs attached to an annulus, there are none attached to T. ∎

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