Canad. Math. Bull. Vol. **27** (3), 1984

DIVISORIAL PRIME IDEALS IN PRÜFER DOMAINS

BY

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ABSTRACT. Given a Prüfer domain R and a prime ideal P in R, we study some conditions which force P to be a divisorial ideal of R. This paper extends some recent work of Huckaba and Papick.

1. **Introduction.** Let R be an arbitrary Prüfer domain and $P \in \mathrm{Spec}(R)$. In this paper we study some conditions which force P to be divisorial, i.e., $P = P_v$. This work expands upon a recent paper of Huckaba and Papick [5]. In particular we generalize [5, Proposition 3.10] and [5, Proposition 3.11]. Unexplained terminology and unreferenced facts about Prüfer domains may be found in [3].

2. **Some sufficient conditions for P to be divisorial.** Let R be an arbitrary Prüfer domain with quotient field K, and P a nonzero prime ideal of R. It is known that if P is maximal, then P is divisorial if and only if P is invertible [5, Corollary 3.4]. Hence, we shall concentrate on nonzero, non-maximal prime ideals of R.

Let P be a nonzero, non-maximal prime ideal of R. We know that P^{-1} is a subring of K [5, Theorem 3.8] and in particular $P^{-1} = (P :_K P)$ [5, Proposition 2.3], as well as $P^{-1} = R_P \cap (\bigcap_\alpha R_{M_\alpha})$, where $\{M_\alpha\}$ is the set of maximal ideals of R not containing P [5, Theorem 3.2]. Hence we have the following inclusion of rings:

$$R \subseteq P^{-1} \subseteq S \equiv K \cap \left(\bigcap_\alpha R_{M_\alpha}\right).$$

We shall prove that if $P^{-1} \subsetneq S$, then P is divisorial. However, first let us consider a somewhat novel result which is at the opposite extreme of our Prüfer setting.

PROPOSITION 2.0. *Let R be an arbitrary integral domain with quotient field K and $(0) \neq P \in \mathrm{Spec}(R)$. If P^{-1} is not a subring of K, then P is divisorial.*

Proof. Since P^{-1} is not a subring of K, then $(P :_K P) \subsetneq P^{-1}$. Let $J = (R : P^{-1})$. Recall that $J = P_v$ [5, Lemma 2.1]. To complete the proof we will show that

Received by the editors April 5, 1983

AMS(MOS) subject classification: 13A17, 13FO5

Key words and phrases: divisorial, ideal transform, Prüfer domain.

* This work was supported in part by a Nato Senior Fellowship and by a grant from the University of Missouri-Columbia.

https://doi.org/10.4153/CMB-1984-049-9 Published online by Cambridge University Press

$J = P$. It suffices to prove that $J \subseteq P$. Let $r \in J$. Since $rP^{-1}P \subseteq P$ and $PP^{-1} \nsubseteq P$, it follows that $r \in P$. Hence, $J = P$.

We are now prepared to state our main result.

THEOREM 2.1. *Let R be a Prüfer domain with quotient field K, and P a nonzero, non-maximal prime ideal of R. If $P^{-1} \subsetneq S = K \cap (\bigcap R_{M_\alpha})$, where $\{M_\alpha\}$ is the set of maximal ideals of R not containing P, then P is divisorial.*

Before we establish Theorem 2.1, a lemma is needed.

LEMMA 2.2. *Same notation as the theorem. Then $P^{-1} \neq S$ if and only if there exists a finitely generated ideal I of R such that $I \subseteq P$ and $I \nsubseteq M_\alpha$ for each α.*

Proof. Recall that $P^{-1} = R_P \cap S$, and use [4, Corollary 2].

Proof of Theorem 2.1. Since R is a Prüfer domain, it suffices to show that P is an intersection of finitely generated ideals of R. Let I be a finitely generated ideal of R such that $I \subseteq P$ and $I \nsubseteq M_\alpha$ for each α. For $a \in R \backslash P$, we claim that $P \subseteq (I, a)$. It is enough to check this assertion locally. For $M \in \{M_\alpha\}$, we obviously have $R_M = (I, a)R_M = PR_M$. If $M \notin \{M_\alpha\}$, then $PR_M \subseteq aR_M = (I, a)R_M$ in the valuation ring R_M. Finally, we wish to show that $P = \bigcap \{(I, r) : r \in R \backslash P\}$. Since P is non-maximal, it will suffice to show for M maximal with $P \subseteq M$, and $r \in M \backslash P$ that $r \notin (I, r^2)$. This follows since $r \notin (r^2)R_M = (I, r^2)R_M$.

COROLLARY 2.3. *Same notation as the theorem. If $P \nsubseteq \cup M_\alpha$, then P is divisorial.*

Proof. Let $a \in P \backslash \cup M_\alpha$ and set $I = (a)$. The desired conclusion follows from Lemma 2.2 and Theorem 2.1.

COROLLARY 2.4. *Same notation as the theorem. If P is the radical of an invertible ideal I, then P is divisorial.*

Proof. Apply Lemma 2.2 and Theorem 2.1.

COROLLARY 2.5 [5, Proposition 3.10]. *Same notation as the theorem. If P is contained in all but a finite number of maximal ideals, then P is divisorial.*

Proof. Use Corollary 2.3 and Theorem 2.1 to obtain the result.

Before stating our final corollary, we need some terminology. A domain R has property (#) if $\bigcap_{M \in V_1} R_M \neq \bigcap_{M \in V_2} R_M$ for any two distinct subsets V_1 and V_2 of $\operatorname{Max}(R)$; $\operatorname{Max}(R)$ being the set of maximal ideals of R.

COROLLARY 2.6. *Let R be a Prüfer domain having each overring satisfy property (#). If P is a nonzero, non-maximal prime ideal of R, then P is divisorial.*

Proof. This follows immediately from [4, Theorem 3], Lemma 2.2, and Theorem 2.1.

COROLLARY 2.7. *Same notation as the theorem. If $P = PR_P$, then P is divisorial.*

Proof. The fact that $P = PR_P$, implies that P is comparable with all ideals of R, and in particular, P is contained in each maximal ideal of R. Hence P is divisorial by Corollary 2.5.

REMARK 2.8. There exists a nonzero, non-maximal prime ideal P of the ring of entire functions R(R is a Bézout domain) such that P is not divisorial. In fact, $P^{-1} = R$ [5, Example 3.12].

3. **The ideal transform of *P*.** In this final section we study an interesting special case arising from the previous section. More specifically, let R be a Prüfer domain and P a nonzero, non-maximal prime ideal of R. Recall the ideal transform of P, $T(P) = \bigcup_{n=1}^{\infty} (R :_K P^n)$, and note that $T(P) = R_{P_0} \cap (\bigcap_\alpha R_{M_\alpha})$, where $P_0 = \bigcap_{n=1}^{\infty} P^n$ and $\{M_\alpha\}$ is the set of maximal ideals of R not containing P [3, Exercise 11, p. 331]. Hence, since $P^{-1} = R_P \cap (\bigcap_\alpha R_{M_\alpha})$ [5, Theorem 3.2], we have the following tower of rings:

$$R \subseteq P^{-1} \subseteq T(P) \subseteq S.$$

Note that if $P^{-1} \neq T(P)$, it is immediate from Theorem 2.1 that P is divisorial. It is our intent to study when $P^{-1} \neq T(P)$, and as one consequence of our efforts we will give a different proof of the fact that P is divisorial in this setting.

LEMMA 3.0. *Let R be a Prüfer domain and P a nonzero, non-maximal prime ideal of R. Then, P is a prime ideal of P^{-1}.* (Recall that P is an ideal of P^{-1}, since $P^{-1} = (P :_K P)$ [5, Proposition 2.3].)

Proof. Since $P \in \operatorname{Spec}(R)$, we know that $PR(x) \in \operatorname{Spec}(R(x))$, where $R(x) = R[x]_U$, $U = \{f \in R[x] : c(f) = R\}$ [1, Theorem 4]. Also, $R(x)$ is a Bézout domain, as R is a Prüfer domain [1, Theorem 4 and p. 558]. Hence the overring $P^{-1}(x)$ is a quotient ring of $R(x)$. Notice that $P(P^{-1}(x)) \neq P^{-1}(x)$ [3, Proposition 33.1(4)]. Hence, $PR(x)(P^{-1}(x)) = P(P^{-1}(x))$ is a prime ideal of $P^{-1}(x)$. Whence, there exists a $Q \in \operatorname{Spec}(P^{-1})$ such that $P(P^{-1}(x)) = Q(P^{-1}(x))$ [1, Theorem 4]. Therefore $P = Q$ [3, Proposition 33.1(4)], and so P is a prime ideal of P^{-1}.

We are now ready to analyze when $P^{-1} \subsetneq T(P)$.

THEOREM 3.1. *Let R be a Prüfer domain and P a nonzero, non-maximal prime ideal of R. If $P^{-1} \subsetneq T(P)$, then*

(a) *$P^{-1} \subsetneq T(P)$ is a minimal extension, i.e., there are no rings properly between P^{-1} and $T(P)$.*

(b) *P is an invertible maximal ideal of P^{-1}.*

(c) *P is a divisorial ideal of R.*

(d) $T(P)=\bigcap_\alpha R_{Q_\alpha}\equiv S'$ *where* $\{Q_\alpha\}$ *is the set of prime ideals of* R *not containing* P.

(e) P^{-n} *is never a ring for* $n>1$.

Proof. (a). Let us suppose A is a ring satisfying $P^{-1}\subseteq A\subsetneq T(P)$. Since $T(P)$ and A are intersections of localizations of R at certain prime ideals of R (R is a Prüfer domain), there exists a prime ideal Q in R such that $A\subseteq R_Q$ and $T(P)\nsubseteq R_Q$. We claim $P\subseteq Q$, for if $P\nsubseteq Q$ there exists $Q'\in \operatorname{Spec}(T(P))$ such that $T(P)_{Q'}=R_Q$ [6, Exercise 16(c), p. 149]. This contradiction establishes our claim. Hence $A\subseteq R_Q\subseteq R_P$, and so $A\subseteq R_P\cap(\bigcap_\alpha R_{M_\alpha})=P^{-1}$ [5, Theorem 3.2]. Therefore $A=P^{-1}$, and the proof is complete.

(b) Assume P is not a maximal ideal of P^{-1}. (Recall by Lemma 3.0 that P is a prime ideal of P^{-1}.) Since $P^{-1}\subsetneq T(P)$ is a minimal extension, we know that $P_P^{-1}=T(P)_{P^{-1}\setminus P}$ [2, Théorème 2.2]. However $P_P^{-1}=R_P$, since $R\subseteq P^{-1}\subseteq R_P$, and so $T(P)\subseteq R_P\cap(\bigcap_\alpha R_{M_\alpha})=P^{-1}$, a contradiction. Hence P is a maximal ideal of P^{-1}.

To show that P is invertible in P^{-1} we will assume to the contrary. Thus the inverse of P with respect to P^{-1} equals P^{-1}, i.e., $(P^{-1}:P)=P^{-1}$ [5, Corollary 3.4]. However, $(P^{-1}:P)=(R:P^2)\equiv P^{-2}$. Thus, $P^{-1}=P^{-2}$. So, since $P^{-n}=(R:P^n)=((R:P^{n-1}):P)$, we can conclude by induction that $P^{-n}=P^{-1}$ for each positive integer n. Therefore $P^{-1}=T(P)$, the desired contradiction.

(c) As P is a non-maximal prime ideal of R, we see by (b) that $P^{-1}\neq R$, and thus $P_v\neq R$. Therefore, $P=P_v$, as P_v is an ideal of P^{-1} [5, Lemma 2.1].

(d) Since $T(P)\subseteq\bigcap_\alpha R_{Q_\alpha}\equiv S'$ [6, Exercise 16(d), p. 149], it suffices to show $S'\subseteq T(P)$. Assume otherwise. As in part (a), there exists a prime ideal $Q\in\operatorname{Spec}(R)$ such that $T(P)\subseteq R_Q$ and $S'\nsubseteq R_Q$. Hence $P\subseteq Q$, and so $T(P)\subseteq R_Q\subseteq R_P$. Whence, $T(P)\subseteq R_P\cap(\bigcap_\alpha R_{M_\alpha})=P^{-1}$, a contradiction. Therefore, $T(P)=S'$.

(e) Suppose P^{-n} is a ring for some $n>1$. Then $P^{-n}=R_P\cap(\bigcap_\alpha R_{M_\alpha})=P^{-1}$ [5, Theorem 3.2], and by induction $P^{-1}=T(P)$. This contradiction completes the proof.

Remark 3.2. (a) Let R be an arbitrary integral domain with quotient field K, and $P\in\operatorname{Spec}(R)$. Note that if P^{-1} is a subring of K, then $P^{-1}=(P:_KP)$ [5, Proposition 2.3]. Hence P is an ideal of P^{-1}, but P need not be a prime ideal of P^{-1} [5, Example 2.5]. However, if R is a Prüfer domain, then Lemma 3.0 shows that $P\in\operatorname{Spec}(P^{-1})$.

(b) The converse of Theorem 3.1 (b) is valid, i.e.; under the assumptions of Theorem 3.1, if P is an invertible maximal ideal of P^{-1}, then $P^{-1}\subsetneq T(P)$. To see this notice that $P^{-1}\subsetneq(P^{-1}:_KP)=P^{-2}\subseteq T(P)$.

(c) The converse of Theorem 3.1(c) is not generally true. Let R be a valuation domain, and P a nonzero, non-maximal prime ideal of R such that $P=P^2$. Then $P^{-1}=T(P)$, yet $P=P_v$ (Corollary 2.5).

(d) The converse of Theorem 3.1(d) is not generally true. Let R be a valuation domain and P a nonzero, non-maximal prime ideal of R such that P is unbranched, i.e., $P = \bigcup_{Q \in \mathrm{Spec}(R)}^{Q \subsetneq P} Q$. Observe that $P^{-1} = R_P$ [5, Corollary 3.6] and $S' = \bigcap_{\substack{Q \subsetneq P \\ Q \in \mathrm{Spec}(R)}} R_Q = R_P$. Therefore,

$$P^{-1} = R_P \subseteq T(P) \subseteq S' = R_P,$$

and so $T(P) = S'$, yet $P^{-1} = T(P)$.

(e) The converse of Theorem 3.1(e) is obviously true.

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