Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 March 2024.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

107.I (Kieren MacMillan)

For n a positive integer, let $Q_n(x)$ be the sum of the nth powers of the first x even squares. For example, $Q_1(x) = \frac{2}{3}\left(2x^3 + 3x^2 + x\right)$. Find the points which all the graphs $y = Q_n(x)$ have in common.

107.J (Tran Quang Hung)

Let ω denote the circumcircle of the acute-angled triangle ABC. Let D be the foot of the altitude from A to BC. The perpendicular bisector of AD meets ω at M and N. Lines MD and ND meet ω again at P and Q respectively. Let R be the midpoint of PQ and let the circumcircle of triangle RBC meet PQ again at K. Prove that the line KD bisects the segment MN.

107.K (Didier Pinchon and George Stoica)

Let a be a non-zero real number and suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f\left(x + f(y)\right) = f(x) + f(y) + ay$ for all $x, y \in \mathbb{R}$. Prove that f is additive, i.e. $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

107.L (Toyesh Prakash Sharma)

Prove the inequality:

$$\int_0^{\pi/2}\int_0^{\pi/2} \frac{x\sqrt{y}\,\sin^2 x\,\sin y}{\sqrt{x}\,\sin x + \sqrt{y}\,\sin y}\,dx\,dy < \frac{\pi^3}{64}.$$

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Solutions and comments on **107.A**, **107.B**, **107.C**, **107.D** (March 2023).

107.A (Prithwijit De)

We say that three numbers such as 75, 84, 93 form a *reversible arithmetic progression* because both 75, 84, 93 and the numbers formed by reversing their digits, 57, 48, 39, are in arithmetic progression.

(a) What property characterises the digits of triples of 2-digit reversible arithmetic progressions?

(b) Does this property also characterise the digits of triples of 3-digit, 4-digit, … reversible arithmetic progressions?

Answer:

(a) Three 2-digit numbers form a reversible arithmetic progression (RAP) if, and only if, the corresponding individual digits in the numbers form arithmetic progressions.

(b) The same property characterises 3-digit, 4-digit, … RAPs.

The solutions received generally went along the following lines.

(a) If the three 2-digit numbers a_1a_0, b_1b_0, c_1c_0 form an arithmetic progression, then

$$10a_1 + a_0 + 10c_1 + c_0 = 2(10b_1 + b_0)$$

so that $$10(a_1 + c_1 - 2b_1) + (a_0 + c_0 - 2b_0) = 0$$

or, writing $d_1 = a_1 + c_1 - 2b_1$ and $d_2 = a_0 + c_0 - 2b_0$,

$$10d_1 + d_0 = 0. \qquad (1)$$

If the numbers form a RAP we also have here, by the same argument,

$$10d_0 + d_1 = 0. \qquad (2)$$

Equations (1) and (2) solve to give $d_1 = d_2 = 0$, so the corresponding individual digits of the three numbers form arithmetic progressions. The converse is a straightforward verification.

(b) If the three $(n + 1)$-digit numbers

$$a_n \dots a_1a_0, \qquad b_n \dots b_1b_0, \qquad c_n \dots c_1c_0$$

form a RAP, then the equations corresponding to (1) and (2) are

$$10^n d_n + 10^{n-1} d_{n-1} + \dots + 10d_1 + d_0 = 0 \qquad (1')$$

and

$$10^n d_0 + 10^{n-1} d_1 + \dots + 10d_{n-1} + d_n = 0, \qquad (2')$$

where $d_k = a_k + c_k - 2b_k$.

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Analysing these equations requires a different approach from (a). We focus first on d_0. From (2')

$$d_0 = -\left[10^{-1}d_1 + 10^{-2}d_2 + \dots + 10^{-n}d_n\right]$$

so, since $|d_k| \leqslant 18$,

$$|d_0| \leqslant 18\left(10^{-1} + 10^{-2} + \dots + 10^{-n}\right) < 18\left(10^{-1} + 10^{-2} + \dots\right) = 2.$$

But from (1'), $10 \mid d_0$ so we must have $d_0 = 0$.

Repeating the argument after removing $d_0 = 0$ from (1') and (2') (or proceeding inductively) we similarly conclude that $d_k = 0$ for all $0 \leqslant k \leqslant n$.

As before, the converse is straightforward.

Peter Johnson showed that the same conclusion holds for numbers written in base b ($b \geqslant 3$), but that there are no non-trivial RAPs in base 2.

Correct solutions were received from: M. V. Channakeshava, S. Dolan, M. G. Elliott, P. F. Johnson, S. Y. Khan, P. M. King, J. A. Mundie, J. Osborne, Z. Retkes, S. Riccarelli, I, D. Sfikas, D. K. Shakyawar, V. R. Shrimali, C. Starr, and the proposer Prithwijit De.

107.B (Neil Curwen)

Points X and Y lie on edges OA and OC, respectively, of a rectangular sheet of card $OABC$. The triangular corner OXY is folded along XY so that O becomes the vertex of a pyramid with pentagonal base $XABCY$. Given that length $OA = a$ and length $OC = b$ (with $a \geqslant b$), determine the positions of X and Y that maximise the volume of the pyramid.

Answer: The volume of the pyramid is maximised by $OX = OY = \sqrt{\frac{2ab}{3}}$ when $b \geqslant \frac{2a}{3}$ and by $OX = t$, $OY = b$ when $b < \frac{2a}{3}$, where t is the (unique) real root of $t^3 + 2b^2t - 2ab^2 = 0$.

This intriguing maximisation problem with its unusual split of cases attracted many detailed and accurate analyses. Although solutions were quite similar, the solution below most closely follows that of the proposer, Neil Curwen.

Writing $OX = x$ and $OY = y$, the base of the pyramid has area $ab - \frac{1}{2}xy$ and maximising the volume requires the folded corner OXY to be perpendicular to the base. The height of the pyramid is then equal to the relevant altitude of triangle OXY, $\dfrac{xy}{\sqrt{x^2 + y^2}}$. So the volume, V, of the pyramid is given by $6V = \dfrac{xy(2ab - xy)}{\sqrt{x^2 + y^2}}$.

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Suppose that $x = p, y = q$ maximises V.

Case 1: $pq \leqslant b^2$

In this case, p and q must be equal otherwise setting $x = y = \sqrt{pq}$ would increase V because, by the inequality of means,

$$\frac{pq(2ab - pq)}{\sqrt{2pq}} > \frac{pq(2ab - pq)}{\sqrt{p^2 + q^2}}.$$

Case 2: $pq > b^2$

In this case $q = b$ (and the pentagonal base degenerates to a quadrilateral) otherwise setting $x = \frac{pq}{b}, y = b$ would increase V since

$$\frac{pq(2ab - pq)}{\sqrt{\left(\frac{pq}{b}\right)^2 + b^2}} > \frac{pq(2ab - pq)}{\sqrt{p^2 + q^2}}$$

because

$$\left(p^2 + q^2\right) - \left(\frac{p^2q^2}{b^2} + b^2\right) = \frac{\left(b^2 - q^2\right)\left(p^2 - b^2\right)}{b^2} > 0$$

since $q < b, p > b$.

Thus for maximum V, in Case 1 we have $x = y = t \leqslant b$ and in Case 2, $x = t, y = b$.

Case 1:

Here, $6\sqrt{2}V = t\left(2ab - t^2\right)$, a cubic with maximum $V = \frac{2\sqrt{3}}{27}(ab)^{3/2}$ when $t = \sqrt{\frac{2ab}{3}}$. This is the required solution provided that $t = \sqrt{\frac{2ab}{3}} \leqslant b$ or $b \geqslant \frac{2a}{3}$.

Case 2:

Here, $b < \frac{2a}{3}$ and $\frac{6V}{b^2} = \frac{t(2a - t)}{\sqrt{t^2 + b^2}}$ which has a maximum stationary value when $V'(t) = 0$. This gives $2(a-t)\left(t^2 + b^2\right) - t^2(2a - t) = 0$ or $f(t) = t^3 + 2b^2t - 2ab^2 = 0$. This cubic has a unique real root which lies in the interval (b, a) because $f(b) = b^2(3b - 2a) < 0 < f(a) = a^3$. The root is given explicitly by Cardano's formula:

$$t = b\left[\left(\sqrt{\lambda^2 + \frac{8}{27}} + \lambda\right)^{1/3} - \left(\sqrt{\lambda^2 + \frac{8}{27}} - \lambda\right)^{1/3}\right],$$

where $\lambda = \frac{a}{b} > \frac{3}{2}$. With this value of t, the maximum volume simplifies to give $V = \frac{1}{6}t^2\sqrt{t^2 + b^2}$.

The proposer noted that the analogous problem starting with a triangular sheet of card and folding up a triangular corner has a similar solution.

Correct solutions were received from: M. V. Channakeshava, S. Dolan, M. G. Elliott, P. F. Johnson, S. Y. Khan, J. A. Mundie, Z. Retkes, S. Riccarelli, C. Starr, L. Wimmer, and the proposer Neil Curwen.

107.C (Toyesh Prakash Sharma)

In this problem you may assume that $\int e^x \tan x \, dx$ is not expressible in terms of elementary functions.

(a) Show that, for integers $n \geqslant 2$, $\int e^x \tan^n x \, dx$ is not expressible in terms of elementary functions.

(b) Show that, for integers $m, n \geqslant 2$ with $m \neq n$, there is a constant a (dependent on m, n) for which $\int e^x \left(\tan^m x + a \tan^n x\right) dx$ is expressible in terms of elementary functions.

The solutions received for this unusual slant on reduction formulae, though few in number, were carefully argued. Stan Dolan's, which follows, deals neatly with the key point that the full reduction formula for $T_n = \int e^x \tan^n x \, dx$ always involves a *non-zero* multiple of the non-elementary integral, T_1.

Let $\mathcal{E}$ denote the set of elementary functions. Then

$$\frac{d}{dx}\left(e^x \tan^n x\right) = e^x\left(\tan^n x + n \tan^{n-1} x \sec^2 x\right)$$
$$= e^x\left(n \tan^{n-1} x + \tan^n x + n \tan^{n+1} x\right)$$

and so $nT_{n-1} + T_n + nT_{n+1} \in \mathcal{E}$.

Consider the recurrence relation $na_{n-1} + a_n + na_{n+1} = 0$, $a_0 = 0$, $a_1 = 1$. Since $T_0 - a_0T_1 = \int e^x dx \in \mathcal{E}$, $T_1 - a_1T_1 = \int 0 \, dx \in \mathcal{E}$ and $n\left(T_{n-1} - a_{n-1}T_1\right) + \left(T_n - a_nT_1\right) + n\left(T_{n+1} - a_{n+1}T_1\right) = 0$, it follows by induction that $T_n - a_nT_1 \in \mathcal{E}$ for all $n \geqslant 0$. For $n \geqslant 1$, let $b_n = (n-1)!\, a_n$. Then $b_1 = 1$, $b_2 = -1$ and $n(n-1)b_{n-1} + b_n + b_{n+1} = 0$ from which, by induction, all the b_n are odd integers. Thus b_n and a_n are non-zero for $n \geqslant 1$.

For (a), $T_n - a_nT_1 \in \mathcal{E}$ with $a_nT_1 \notin \mathcal{E}$ means that $T_n \notin \mathcal{E}$ for $n \geqslant 2$.

For (b), $a_n\left(T_m - a_mT_1\right) - a_m\left(T_n - a_nT_1\right) \in \mathcal{E}$, hence

$$a_nT_m - a_mT_n = a_n\left(T_m - \frac{a_m}{a_n}T_n\right) \in \mathcal{E}$$

so the required constant is $a = -\dfrac{a_m}{a_n}$.

The sequence $\left((-1)^{n-1} b_n\right)$ occurs as A002019 in the OEIS where further properties of it may be found.

Correct solutions were received from: S. Dolan, M. G. Elliott, P. F. Johnson, J. A. Mundie, Z. Retkes, S. Riccarelli, and the proposer Toyesh Prakash Sharma.

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107.D (George Stoica)

Let $0 < a < b < 1$ and $\varepsilon > 0$ be given. Prove the existence of positive integers m and n such that $(1 - b^m)^n < \varepsilon$ and $(1 - a^m)^n > 1 - \varepsilon$.

First, as a number of respondents commented, this problem has previously appeared in *Mathematics Magazine* with a solution in the October 2022 issue, pp. 409-410. My apologies for this oversight: the alternative solution which follows is due to Kee-Wai Lau.

There is nothing to prove if $\varepsilon \geqslant 1$, so assume that $0 < \varepsilon < 1$. For $0 < a < b < 1$, $\lim_{t \to \infty} \left(\frac{\varepsilon}{a^t} - \frac{\ln(1/\varepsilon)}{b^t} \right) = \infty$, so there exist positive integers m, n such that $\frac{\ln(1/\varepsilon)}{b^m} < n < \frac{\varepsilon}{a^m}$. Hence $e^{-nb^m} < \varepsilon$ and $na^m < \varepsilon$.

Since $1 + x \leqslant e^x$ for all real x, it follows that $1 - b^m \leqslant e^{-b^m}$ and $(1 - b^m)^n \leqslant e^{-nb^m} < \varepsilon$. And, using Bernoulli's inequality, $(1 - x)^n \geqslant 1 - nx$ for $0 \leqslant x \leqslant 1$, $n \geqslant 1$, we obtain $(1 - a^m)^n \geqslant 1 - na^m > 1 - \varepsilon$ to complete the solution.

Correct solutions were received from: Y. Aliyev, U. Abel, M. V. Channakeshava, S. Dolan, M. G. Elliott, S. Y. Khan, K-W. Lau, Z. Retkes, I, D. Sfikas, and the proposer George Stoica.

N.J.L

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