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Reflection Subquotients of Unitary Reflection Groups

To H. S. M. Coxeter on the occasion of his ninetieth birthday.

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Abstract. Let G be a finite group generated by (pseudo-) reflections in a complex vector space and let g be any linear transformation which normalises G. In an earlier paper, the authors showed how to associate with any maximal eigenspace of an element of the coset gG, a subquotient of G which acts as a reflection group on the eigenspace. In this work, we address the questions of irreducibility and the coexponents of this subquotient, as well as centralisers in G of certain elements of the coset. A criterion is also given in terms of the invariant degrees of G for an integer to be regular for G. A key tool is the investigation of extensions of invariant vector fields on the eigenspace, which leads to some results and questions concerning the geometry of intersections of invariant hypersurfaces.

1 Introduction and Statement of Results

Let G be a finite reflection group acting on $V = \mathbb{C}^\ell$. We suppose V to be endowed with a positive definite Hermitian form such that $G \subset U(V)$, the corresponding unitary group. Fix an integer d, let ζ be a primitive d-th root of unity and for $x \in \mathrm{GL}(V)$ write $V(x, \zeta)$ for the ζ-eigenspace of x. The following theorem was proved in [LS1], to which this paper is a sequel.

Theorem 1.1 **([LS1, (2.5), (4.2)])** *Assume that $g \in N_{U(V)}(G)$ is such that* $\dim\big(V(g,\zeta)\big) \geq \dim\big(V(gx,\zeta)\big)$ *for all $x \in G$. Write $E = V(g,\zeta)$, $N = \{x \in G \mid xE = E\}$ and $C = \{x \in G \mid xe = e$ for all $e \in E\}$. Let $f_1, \dots, f_\ell$ be a set of basic invariants for G acting on V, such that $g.f_i = \delta_i f_i$, $i = 1, \dots, \ell$. Write $d_i = \deg f_i$.*

Then $\overline{N} = N/C$ acts as a unitary reflection group on E, whose reflecting hyperplanes are the intersections with E of those of G. A set of basic invariants for $\overline{N}$ is given by the restrictions to E of those f_i satisfying $\zeta^{d_i}\delta_i = 1$.

If the pointwise isotropy group C is trivial, or equivalently, if there is no reflecting hyperplane of G which contains E, we say that the triple (G, g, ζ) is *regular*.

If, in (1.1), we take g to be in G, then $\overline{N}$ is a reflection group which is determined up to linear isomorphism by the integer d and the group G. We denote $\overline{N}$ by $G(d)$; if C is trivial, we say that d is regular. It was shown in [LS1, (2.9)] that if e divides d, then

$$G(d) \cong G(e)(d). \tag{1.2}$$

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In this paper we shall prove several results which supplement those of [LS1]. Given a triple (G, g, ζ) as above, we say it is *twisted*, or that we are in the *twisted case* if we allow the possibility that $g \notin G$. Theorems A and B apply to the twisted case. Notation is as above.

Theorem A *If G is irreducible, so is the reflection group $\overline{N} = N/C$ on E.*

When the triple (G, g, ζ) is regular, Theorem A was proved in the untwisted case by Denef and Loeser [DL], using a case by case analysis.

Theorem B *There is an element $c \in C$ such that if $Z_G(x)$ denotes the centraliser in G of x, we have $N = Z_G(gc) \cdot C$.*

Theorem C *Let d be any integer and let $G(d)$ be as above. Then*

(i) The coexponents of $G(d)$ form a subset (in the sense of multisets—i.e., including multiplicities) of those coexponents of G which are congruent to $1 \pmod{d}$.
(ii) The two multisets of (i) coincide if d is regular.
(iii) The converse of (ii) is true.

Part (ii) was treated by Denef and Loeser in [DL], using a case by case argument. We shall deduce a stronger version of (ii) which applies to the twisted case from a more general statement concerning extensions of invariant regular functions. Our proofs of (i) and (iii) involve a little case by case checking. Some corollaries of the main theorems are given below.

Our groups $G(d)$ arise in the context of the "maximal Φ_d-subgroups" of Broué and Malle [BM1] as the automorphism groups $W_G(L)$ of the Levi subgroups L which are centralisers of maximal Φ_d-subgroups. In this context they have been investigated empirically in [BM2], [BMM], [BrM] and [M]. These groups provide the framework for the study of ramification groups of "induced" representations (*cf.* [HL]).

2 Irreducibility

In this section we shall prove Theorem A. Maintaining the notation above, write $\overline{N} = N/C$ and assume that G is irreducible. Then all degrees d_i are greater than 1, so from (1.1) it follows that all basic invariants of $\overline{N}$ have degree greater than 1. Hence the fixed point set $E^{\overline{N}}$ is zero.

Assume that $\overline{N}$ acts reducibly on E. Then there is a decomposition $E = E_1 \oplus E_2$ such that any reflection in $\overline{N}$ fixes either E_1 or E_2 pointwise. Since the reflecting hyperplanes of $\overline{N}$ are the intersections of those of G with E (by [LS1, (2.7)]), the same is true of any reflection in G. If a reflection of G fixes E_1 and E_2 pointwise, it fixes E pointwise. Since G is irreducible, this can not apply to all reflections of G unless $E = \{0\}$ or V, cases in which the result is trivial. Thus we may assume that there are reflections in G which fix E_1 pointwise, but which do not fix E_2 pointwise. Denote this set of reflections by R_1, let R be the set of all reflections of G and write $R_2 = R - R_1$; then R_2 is the set of all reflections of G which fix E_2 pointwise.

Let G_1 be the subgroup of G generated by R_1. We show that G_1 is a normal subgroup of G. For this, it suffices since G is generated by R, to show that for $r \in R$, $rG_1r^{-1} = G_1$. If $r \in R_1$ this is clear; thus we are reduced to the case $r \in R_2$. Let $r_1 \in R_1$, $r \in R_2$. Then

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$s = rr_1r^{-1}$ is a reflection. If s fixes E_2 pointwise, then so does r_1 (since r does). But $r_1 \in R_1$, so that r_1 does not fix E_2 pointwise, whence the same is true of s. Thus $s \in R_1$ and it follows that G_1 is a normal subgroup of G.

Let $V_1 = V^{G_1}$ be the fixed point subspace of G_1. It is non-zero since $E_1 \subset V_1$ and is a proper subspace. Since G_1 is normal, V_1 is stabilized by G, whence G is reducible. This completes the proof of Theorem A. ∎

3 Proof of Theorem B

In the proof of Theorem B, we shall require the following result from [Sp, (6.2)]. Notation is as in (1.1).

Lemma 3.1 **([Sp. (6.2)])** *The maximal spaces among the eigenspaces $\{V(x,\zeta) \mid x \in gG\}$ are all conjugate under the action of G.*

Let $V = E_0 \oplus \cdots \oplus E_t$ be an ordered g-eigenspace decomposition of V, where $E_i = V(g, \zeta_i)$ and $\zeta_0 = \zeta$, the ζ_i being distinct. For $i = 0, 1, \ldots, t$, write C_i for the pointwise stabiliser in G of the subspace $E_0 \oplus \cdots \oplus E_i$ and define C_{-1} to be G. Then $C_i \supseteq C_{i+1}$ for all i, $C_0 = C$ and $C_t = \{1\}$. Say that g is *quasi-regular* if there is an ordering of its eigenvalues such that $\dim E_i = \max_{c \in C_{i-1}} \dim V(gc, \zeta_i)$ for each i.

Lemma 3.2 *Let $E_0 = V(g,\zeta)$ and assume $\dim E_0 = \max_{x\in G} \dim V(gx,\zeta)$. Then there is an element $c \in C_0 = C = G_{E_0}$ such that gc is quasi-regular.*

Proof We shall inductively define subspaces E_i of V and elements $g_i \in gC_0$ $(i = 0, 1, \ldots)$ such that

(a) $E_{i+1} (\neq 0)$ is orthogonal to $E_0 \oplus \cdots \oplus E_i$ (all i).
(b) If $C_i = C_{E_0 \oplus \cdots \oplus E_i}$, then $g_{i+1} = g_i c_i \in g_i C_i$ for all i.
(c) For each i and each k with $0 \le k \le i$, $E_k = V(g_i, \zeta_k)$ and the ζ_k are distinct.
(d) With i, k as in (c), $\dim E_k = \max_{c \in C_{k-1}} \dim V(g_i c, \zeta_k)$.

If t is the smallest integer such that $V = E_0 \oplus \cdots \oplus E_t$, then $g_t = gc_0c_1 \cdots c_{t-1} \in gC_0$ is quasi-regular. This construction will therefore prove the lemma. Notice that property (c) implies that g_i normalises each of the reflection groups C_i.

To start the construction, we have E_0 and $g_0 = g$ given. Suppose we have $E_0, \ldots, E_j$ and $g_0, \ldots, g_j$ satisfying the above conditions. To produce E_{j+1} and g_{j+1} proceed as follows. Let ζ_{j+1} be any eigenvalue of g_j which is distinct from $\zeta_0, \ldots, \zeta_j$; such an eigenvalue exists unless $V = E_0 \oplus \cdots \oplus E_j$, in which case we are finished. Let $c_j \in C_j$ $(= G_{E_0 \oplus \cdots \oplus E_j})$ be such that $\dim V(g_j c_j, \zeta_{j+1}) = \max_{c \in C_j} \dim V(g_j c, \zeta_{j+1})$ and take $g_{j+1} = g_j c_j$, $E_{j+1} = V(g_j c_j, \zeta_{j+1})$.

Then (a) is clear, as is (b). To prove (c), we must show that for $0 \le k \le j+1$, we have $E_k = V(g_{j+1}, \zeta_k)$. Now $C_i \supseteq C_{i+1}$ for each i, whence $g_{j+1} \in g_k C_k$. But by hypothesis, $\dim E_k = \max_{c \in C_{k-1}} \dim V(g_k c, \zeta_k) \ge \max_{c \in C_k} \dim V(g_k c, \zeta_k)$, so that $\dim V(g_{j+1}, \zeta_k) \le \dim E_k$. But $E_k \subseteq V(g_{j+1}, \zeta_k)$ and (c) follows.

For (d) we require that for $0 \le k \le j+1$, $\dim E_k = \max_{c \in C_{k-1}} \dim V(g_{j+1} c, \zeta_k)$. If $k = j+1$, this holds by construction; if $k \le j$, then $g_{j+1} C_{k-1} = g_j C_{k-1}$ by construction and the statement follows from the induction hypothesis. This completes the proof of the lemma. ■

Proposition 3.3 *In the above notation, if $g \in N_{U(V)}(G)$ is quasi-regular then $N = \{g \in G \mid gE_0 = E_0\} = Z_G(g) \cdot C$.*

Proof Assume g is quasi-regular and let $n \in N$. Write $V = E_0 \oplus \cdots \oplus E_t$ as above, with $E_i = V(g, \zeta_i)$. We shall construct a sequence $n_0, n_1, \ldots, n_t$ such that

(a) $n_i \in N$, $n_0 = n$,
(b) $n_{i+1} \in C_i n_i$ (all i) and
(c) $n_i g n_i^{-1} g^{-1} \in C_i$ (all i).

If we have such a sequence, then $n_{i+1} \in C_i C_{i-1} \cdots C_0 n_0 = C_0 n_0$ for all i. But $n_t \in Z_G(g)$ since $C_t = \{1\}$ and so $n_t = c_0 n$ implies that $n = c_0^{-1} n_t \in C_0 Z_G(g)$. Hence $N = C.Z_G(g) = Z_G(g).C$, since C is normal in N.

To construct the sequence, proceed as follows. Suppose we have $n_0, \ldots, n_j$; then $n_j g n_j^{-1} g^{-1}$ fixes $E_0 \oplus \cdots \oplus E_j$ pointwise and since E_{j+1} is the ζ_{j+1}-eigenspace of g, $n_j E_{j+1}$ is an eigenspace for $n_j g n_j^{-1}$ with eigenvalue ζ_{j+1}. Moreover $n_j g n_j^{-1} = n_j g n_j^{-1} g^{-1} \cdot g \in C_j g = g C_j$, so that

$$\dim n_j E_{j+1} = \max_{c \in C_j} \dim V(cg, \zeta_{j+1})$$

by the quasi-regular nature of g. It follows, using (3.1) applied to the pair (C_j, g), that there is an element $c_j \in C_j$ such that $n_j E_{j+1} = c_j E_{j+1}$. Let $n_{j+1} = c_j^{-1} n_j \in C_j n_j$.

Then $n_{j+1} g n_{j+1}^{-1} g^{-1}$ fixes $E_0 \oplus \cdots \oplus E_j$ pointwise since $n_j g n_j^{-1} g^{-1}$ does. Furthermore, since $n_{j+1} E_{j+1} = E_{j+1}$ and g acts as a scalar on E_{j+1}, $n_{j+1} g n_{j+1}^{-1} g^{-1} \in C_{j+1}$. This proves the properties (a), (b) and (c) for $(n_0, \ldots, n_{j+1})$ and completes the proof of the proposition by induction. ■

Theorem B now follows immediately from (3.2) and (3.3).

We remark that the centralisers of elements of G are of significance in the equivariant cohomology of the hyperplane complement corresponding to G, *cf.* [L].

Corollary 3.4 *Suppose G is a reflection group in V and let $g \in N_{U(V)}(G)$. If $E = V(g, \zeta)$ and $\dim E = \max_{x \in G} \dim V(gx, \zeta)$ then there is an element $c \in C = G_E$ such that $Z_G(gc)/\big(Z_G(gc) \cap C\big)$ acts as a reflection group in E.*

This follows immediately from Proposition (3.3) because

$$\overline{N} \cong Z_G(gc).C/C \cong Z_G(gc)/\big(Z_G(gc) \cap C\big).$$

For $c \in C$, write $\gamma(c) = g^{-1} c g \in C$. Say that c and $c' \in C$ are γ-conjugate if there is an element $d \in C$ such that $c' = \gamma(d) c d^{-1}$. Let $\sim$ be the equivalence relation on C generated by conjugacy under $Z_G(g)$ and γ-conjugacy.

Corollary 3.5 *With notation as in (3.4), assume (as we may by (3.2)) that g is quasi-regular. Then*

(i) *If* $h \in G$ *then* $\dim V(g, \zeta) = \dim V(gh, \zeta)$ *if and only if gh is G-conjugate to an element of gC.*
(ii) *The G-conjugacy classes of elements gh such that* $\dim V(gh, \zeta)$ *is maximal are in bijection with the set of equivalence classes* $C/\sim$.

Proof (i) By (1.1), $\dim V(gh, \zeta) = \dim V(g, \zeta)$ if and only if there is an element $x \in G$ such that $V(g, \zeta) = xV(gh, \zeta)$. This is equivalent to requiring that ${}^{x}(gh) \in gC$.

(ii) By (i), each G-conjugacy class of elements of the required type is represented in gC. But if two elements ga and gb of gC satisfy $gb = x(ga)x^{-1}$, then $xE = E$, so that $x \in N = Z_G(g) \cdot C$ (by quasi-regularity). The result follows. ∎

The authors are grateful to Jean Michel for discussions which led to the next two observations.

Remark 3.6 It is clear that, with notation as in (3.4), the coset gC may be characterised as the set of elements $x \in gG$ such that the eigenspace $V(x, \zeta) = E$. For if this set is denoted S, then clearly $gC \subset S$. Conversely, if g_1 and g_2 are any two elements of S then $g_1^{-1}g_2 \in C$, so that $g_2 \in g_1C$. Taking $g_1 = g$, we see that $S \subset gC$, which proves the assertion. It follows easily that

$$N = N_G(gC) = \{x \in G \mid xgC = gCx\} \tag{3.6.1}$$

To see this, observe that for any pair x, y of linear transformations of V such that y is invertible, and element $\lambda \in \mathbb{C}$, we have $yV(x, \lambda) = V(yxy^{-1}, \lambda)$. The above characterisation of gC therefore yields (3.6.1).

In case $g \in G$ (the "untwisted" case), we say that the coset $gH \in N_G(H)/H$ of the parabolic subgroup H of G is *(ζ-)special* for some $\zeta \in \mathbb{C}$ if there is a subspace $E \subset V$ such that gH is the set of elements $x \in G$ such that $E = V(x, \zeta)$ and E is a maximal ζ-eigenspace among the elements of G. We refer to gH as a special parabolic coset. The next result is a generalisation of the statement that any power of a regular element of G is regular.

Proposition 3.7 *Any power of a special parabolic coset contains a special parabolic coset.*

Proof Suppose gC is ζ-special, where ζ is a primitive d-th root of unity. We shall show that for any integer e such that $e \mid d$, some element of the coset $g^{d/e}C$ has a maximal $\zeta_e = \zeta^{d/e}$-eigenspace. As in the proof of (2.9) in [LSp1], we may take $E(d) = V(g, \zeta) \subseteq E(e)$, where for any integer k, $E(k)$ denotes a maximal ζ_k-eigenspace $V(x, \zeta_k)$, where ζ_k is a primitive k-th root of unity. Thus for the respective pointwise stabilisers (parabolic subgroups), we have $C(d) = C \supseteq C(e)$. Moreover by the argument in [LSp1, *loc. cit.*], $G(e)(d) = G(d)$, from which it follows that $N(d) \subseteq N(e)C(d)$, where $N(k)$ denotes the setwise stabiliser of $E(k)$. Hence in particular, $g^{d/e}c \in N(e)$ for some element $c \in C(d)$. Thus $g^{d/e}C \supseteq g^{d/e}cC(e)$, which proves (3.7) since $g^{d/e}cC(e)$ is special. ∎

4 Extension of Invariant Regular Functions

In this section, although we continue to work in the general setting where $g \in N_{U(V)}(G)$, we shall as far as possible use the notation of [LS1]. However we shall require some extensions of the results of [*loc. cit.*, Section 3] to the twisted case. These will be stated in a short digression below without proof, since the proofs are entirely analogous to those in [LS1]. Denote by $\mathcal{A}(G)$ and $\mathcal{A}(\overline{N})$ respectively the sets of reflecting hyperplanes of G on V and $\overline{N}$ on E. By [LS1, (2.7)], $\mathcal{A}(\overline{N})$ consists of the set of intersections with E of those elements of $\mathcal{A}(G)$ which do not contain E.

We use a basis $v_1, \dots, v_a, \dots, v_s, \dots, v_\ell$ of g-eigenvectors in V, so that

(4.1) $$gv_i = \zeta_i v_i \quad \text{for } i = 1, \dots, \ell$$

and we assume notation chosen so that $\langle v_1, \dots, v_a \rangle = E$ and $\langle v_1, \dots, v_s \rangle = \bigcap_{\substack{H \supset E \\ H \in \mathcal{A}(G)}} H$. As in (1.1), we suppose that the basic invariants $\{f_1, \dots, f_\ell\}$ satisfy

(4.2) $$gf_i = \delta_i f_i \quad \text{for } i = 1, \dots, \ell$$

and numbering is chosen so that if $a = a(G, g, \zeta) = |\{i \mid \delta_i \zeta^{d_i} = 1\}|$, then $f_1, \dots, f_a$ are precisely the invariants satisfying this condition. We take as basis of the free S_V^G-module $(S_V \otimes V)^G$ the set

(4.3) $$\phi_i = \sum_{j=1}^{\ell} f_{ij} \otimes v_j, \quad i = 1, \dots, \ell$$

where degree $(f_{ij}) = n_i$ (the i-th coexponent of G). A simple argument shows that we may assume the ϕ_i chosen as eigenfunctions of g, so that

(4.4) $$g\phi_i = \xi_i \phi_i \quad \text{for } i = 1, \dots, \ell.$$

The elements of $S_V \otimes V$ may be interpreted as polynomial vector fields on V in the usual way and we have the map

$$\rho \colon S_V \otimes V \longrightarrow S_E \otimes E$$

given by $\rho = \mathrm{res}_E^V \otimes p_E$, where res_E^V is the restriction map and p_E is the projection onto the eigenspace E; ρ is clearly surjective. Its restriction to $(S_V \otimes V)^G$ will be denoted

$$\rho^0 \colon (S_V \otimes V)^G \longrightarrow (S_E \otimes E)^N.$$

Using the automorphism g, one shows that $\rho^0(\phi_i) = 0$ unless $\xi_i \zeta^{n_i - 1} = 1$. Write $b = b(G, g, \zeta)$ for the number of ϕ_i satisfying $\xi_i \zeta^{n_i - 1} = 1$. The following statements are analogues of [LS1, (3.3) and (3.5)] and are proved in a similar way. There are permutations π and κ of $\{1, \dots, \ell\}$ such that for $i = 1, \dots, s$,

(4.5) $$\zeta_i = \xi_{\pi i} \zeta^{n_{\pi i}} = \delta_{\kappa i}^{-1} \zeta^{1 - d_{\kappa i}}.$$

If we write $\alpha_i = \zeta^{d_i} \delta_i$ and $\beta_i = \xi_i \zeta^{n_i - 1}$, then we have the following analogue of [LS1, (3.6)].

Proposition 4.6 *Suppose (G, g, ζ) is a regular triple. Then there is a numbering of the ϕ_i such that for each i, $\beta_i = \alpha_i^{-1}$. In particular, in the regular case, $a = b$.*

Definition 4.7 We say that the *extension property* holds for (G, g, ζ) if the map ρ^0 defined above is surjective.

Proposition 4.8 *If the extension property holds for (G, g, ζ) then the coexponents of $\overline{N}$ form a subset (in the sense of multisets) of those of G.*

Proof Write $M_V = (S_V \otimes V)^G$. Let $\widetilde{M_V} = M_V/IM_V$, where I is the augmentation ideal of S_V^G. Then $\widetilde{M_V}$ is a graded vector space and the degrees of the elements of a homogeneous basis of $\widetilde{M_V}$ coincide with the coexponents of G. Similarly, we have a graded vector space $\widetilde{M_E}$. If ρ^0 is surjective we obtain a surjective linear map $\widetilde{M_V} \to \widetilde{M_E}$, and the lemma follows. ∎

To study the extension property, we start with the following easy observations. Let V_0 be the subspace of fixed points of G on V. Let V_1 be its orthogonal complement, on which G acts as a reflection group. Write $E_i = E \cap V_i$, $i = 0, 1$. These spaces are normalised by g and N.

Lemma 4.9

(i) With the above notation, (G, g, ζ) has the extension property if and only if $(G|_{V_1}, g|_{V_1}, \zeta)$ does.

(ii) Assume that there is a hyperplane $\overline{H}$ in E which is contained in all reflecting hyperplanes of G. Then the extension property holds for (G, g, ζ).

Proof (i) We have, since $S_V = S_{V_0} \otimes S_{V_1}$ and G acts trivially on S_{V_0},

$$(S_V \otimes V)^G = (S_V^G \otimes V_0) \oplus \big(S_{V_0} \otimes (S_{V_1} \otimes V_1)^G\big)$$

and similarly

$$(S_E \otimes E)^N = (S_E^N \otimes E_0) \oplus \big(S_{E_0} \otimes (S_{E_1} \otimes E_1)^N\big).$$

The map ρ^0 is induced by restriction of functions on V, V_0, V_1 respectively to their intersections with E and projection of these spaces onto their intersections with E. The statement (i) follows by observing that the maps $S_V^G \to S_E^N$, $V_0 \to E_0$ and $S_{V_0} \to S_{V_0}$ are all surjective.

(ii) By (i), we may assume that $V_0 = 0$, so that $\overline{H} \subset V_0$ implies that $\overline{H} = 0$, whence $\dim E$ is either 0 or 1. If $E = 0$ there is nothing to prove. If $\dim E = 1$, write the basis of V^* dual to $v_1, \ldots, v_\ell$ as $X_1, \ldots, X_\ell$. Then $\phi = \sum_i X_i \otimes v_i \in (S_V \otimes V)^G$ and $\rho^0(\phi) = X_1 \otimes v_1$, which is a generator of the S_E^N-module $(S_E \otimes E)^N$. ∎

The main result of this section is

Theorem D *If (G, g, ζ) is a regular triple, then ρ^0 is surjective, i.e., (G, g, ζ) enjoys the extension property.*

Of course given Theorem D, Theorem C(ii) follows immediately from (4.8).

The set of common zeros of $f_{a+1}, \dots, f_\ell$ is the union $X = X_G$ of the translates xE of E ($x \in G$) (see [Sp, Section 6]). These translates are the irreducible components of X. Let $A = S_V/(f_{a+1}, \dots, f_\ell)$. Then $\mathbb{C}[X]$ is the quotient of A by its radical; it is equal to A if and only if A is reduced. The group G acts on both A and $\mathbb{C}[X]$ and we denote the respective rings of G-invariants by A^G and $\mathbb{C}[X]^G$.

Lemma 4.10 *Let $A = S_V/(f_{a+1}, \dots, f_\ell)$. Then*

(i) $A^G \cong S_V^G/(f_{a+1}, \dots, f_\ell) \cong \mathbb{C}[f_1, \dots, f_a]$.
(ii) A is a free graded module over A^G.
(iii) The map $A \to \mathbb{C}[X]$ induces an isomorphism $A^G \cong \mathbb{C}[X]^G$.

Proof The sequence $(f_1, \dots, f_\ell)$ is homogeneous and regular in S_V, *i.e.*, the image of f_i in $S_V/(f_{i+1}, \dots, f_\ell)$ is not a zero divisor (see [B, p. 115]). It is clear that the canonical surjection $S_V \to A$ induces a surjective map $S_V^G \to A^G$, since if $a \in A^G$ is the image of $f \in S_V$, a is also the image of $Av_G(f) = |G|^{-1} \sum_{h \in G} hf \in S_V^G$. To prove (i) it therefore suffices to prove the following fact: if $f \in S_V^G$ and

$$f = g_i f_i + \cdots + g_\ell f_\ell$$

with $g_j \in S_V$ then there is also such a relation with g_j in S_V^G for $i \le j \le \ell$. In fact, if we have such a relation it follows from the definition of a regular sequence that for all $x \in G$ we have $c(x) = xg_i - g_i \in (f_{i+1}, \dots, f_\ell)$. Then c is a 1-cocycle of G with values in $(f_{i+1}, \dots, f_\ell)$; *i.e.*, for any two elements $x, y \in G$, we have $c(xy) = xc(y) + c(x)$. Such a cocycle being a coboundary (take the average of the relation for $c(xy)$ over all $y \in G$), it follows that we may modify g_i by an element of $(f_{i+1}, \dots, f_\ell)$ so as to become G-invariant. Using induction this establishes what we wanted, and proves (i). Then (ii) follows from the fact that S_V is free as S_V^G-module [B]. The map $A^G \to \mathbb{C}[X]^G$ is surjective by the averaging argument above, and its kernel consists of nilpotent elements. Since A^G is an integral domain by (i), the kernel must be zero and (iii) follows. ∎

We shall freely use the identification between $\mathcal{P}(V, V)$, the space of regular functions $\phi\colon V \longrightarrow V$ and $S_V \otimes V$, via $\sum h_i \otimes v_i \mapsto \phi\colon V \longrightarrow V$ defined by $\phi(v) = \sum h_i(v)v_i$. More generally, we write $\mathcal{P}(A, B)$ for the space of regular functions (*i.e.*, morphisms) from a variety A to a variety B. Still more generally, for any sets A, B, let $\mathcal{F}(A, B)$ be the set of all functions from A to B. If H is a group which acts on both A and B, then $\mathcal{F}(A, B)$ has the obvious H-action, given by

$$(h\phi)(a) = h\big(\phi(h^{-1}a)\big)$$

Lemma 4.11 *The restriction map $\tau\colon \mathcal{P}(V, V)^G \longrightarrow \mathcal{P}(X, V)^G$ is surjective. Hence ρ^0 is surjective if and only if $\sigma^0\colon \mathcal{P}(X, V)^G \to \mathcal{P}(E, E)^N$ is surjective.*

Proof The first statement follows by the averaging argument given in the proof of (4.10). The second assertion is clear. ∎

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In view of (4.11), we are reduced to the study of extensions of N-invariant functions on E to G-invariant functions on X. Moreover since every G-orbit on X intersects E, any N-invariant function on E has at most one extension to X which is G-invariant.

In addition to the space of regular functions, we shall speak of the space $\mathcal{R}(A, B)$ of rational functions from an affine variety A to an affine variety B. This is the space of functions f which are defined and regular on a dense open subvariety U_f of A, modulo the equivalence relation which identifies two functions which agree on a dense open subvariety of A. Generalities concerning such functions may be found in [G, Section 20.2, p. 231]. Any element $\omega \in \mathcal{R}(A, B)$ has a domain (of definition), which is the largest open set U such that ω contains a representative function $f\colon U \to B$. When $B = \mathbb{C}$, the space of rational functions from A to B and the domain of definition of such a function are identified in

Proposition 4.12

(i) *(cf. [GD, (8.1.9)], [G, (20.2.3)]) The space of rational functions $\mathcal{R}(A, \mathbb{C})$ is isomorphic to the localisation $\mathbb{C}[A]_\Sigma$, where $\Sigma = \Sigma(A)$ is the set of non zero-divisors in the coordinate ring $\mathbb{C}[A]$.*

(ii) *(cf. [G, (20.2.14)]) Let $\omega \in \mathcal{R}(A, \mathbb{C})$ be a rational function. Define the "ideal of denominators" $D(\omega)$ of ω by*

$$D(\omega) = \{h \in \mathbb{C}[A] \mid h\omega \in \mathcal{P}(A, \mathbb{C})\}.$$

Then the domain of definition of ω is the complement in A of the set of zeros of $D(\omega)$.

If $f \in \Sigma, g \in \mathbb{C}[A]$, the domain of $f^{-1}g$ contains the principal open set $A_f := \{a \in A \mid f(a) \neq 0\}$. Note that for any affine variety A, the set of non zero-divisors in the coordinate ring $\mathbb{C}[A]$ is the set of those regular functions on A which have non-zero restriction to each irreducible component of A. In particular, if A is irreducible, $\mathcal{R}(A, \mathbb{C})$ is the quotient field of $\mathbb{C}[A]$, *i.e.*, the function field of A. Note also, that if H is a group which acts on the varieties A and B, then H acts on $\mathcal{R}(A, B)$ in analogous fashion to its action on $\mathcal{F}(A, B)$.

The space $\mathcal{R}(A, \mathbb{C})$ is clearly a $\mathbb{C}[A]$-module; moreover it is clear that for any two varieties B and C, $\mathcal{R}(A, B \times C) \cong \mathcal{R}(A, B) \times \mathcal{R}(A, C)$ (as sets). Similarly, if B is an affine space $\mathbb{A}^r$ of dimension r, then

$$\mathcal{R}(A, B) \cong \mathcal{R}(A, \mathbb{C}) \otimes \mathbb{C}^r \cong \mathbb{C}[A]_\Sigma \otimes \mathbb{C}^r \tag{4.13}$$

where the right hand side has a natural structure as $\mathbb{C}[A]$-module. Assume for the rest of this discussion that B is an affine space. The space $\mathcal{P}(A, B)$ is a $\mathbb{C}[A]$-submodule of $\mathcal{R}(A, B)$ and clearly

$$\mathcal{R}(A, B) \cong \mathbb{C}[A]_\Sigma \otimes_{\mathbb{C}[A]} \mathcal{P}(A, B) \cong \mathcal{P}(A, B)_\Sigma.$$

Similar remarks to those in the preceding paragraph apply to the space $\mathcal{F}(A, B)$. This is also a $\mathbb{C}[A]$-module, which contains $\mathcal{P}(A, B)$ as a submodule.

Proposition 4.14

(i) *Suppose $v_1, v_2 \in E$ and assume that $v_2 = hv_1$ for some $h \in G$. Then $v_2 = nv_1$, for some $n \in N$.*

(ii) *If $\phi \in \mathcal{F}(E,V)^N$ and $e \in E$ is such that $he = e (h \in G)$, then $h\phi(e) = \phi(e)$.*

(iii) *If $\phi \in \mathcal{F}(E,V)^N$, there is a unique function $\widetilde{\phi} \in \mathcal{F}(X,V)^G$ which extends ϕ.*

Proof (i) If $v_2 = hv_1$, then $f(v_1) = f(v_2)$ for all $f \in S_V^G$; since $\mathrm{res}_G^V \colon S_V^G \longrightarrow S_E^{\overline{N}}$ is surjective (see [LS1, (2.5)]), this implies that v_1 and v_2 lie in the same $\overline{N}$-orbit in E.

(ii) Since $he = e$, by Steinberg's theorem [St1], h is a product of reflections in hyperplanes through E. It therefore suffices to prove the result when h is such a reflection $r_H, H \in \mathcal{A}(G)$. If $H \supseteq E$, then $r_H \in N$ and the result is clear. If not, then there is a non-trivial element $n_H \in N$, which represents a reflection in $\overline{N}$ and which fixes e. Thus $\phi(n_H e) = \phi(e) = n_H\phi(e)$. Hence $\phi(e)$ is contained in the fixed point set of n_H, which is contained in H; thus $r_H\phi(e) = \phi(e)$.

(iii) The definition $\widetilde{\phi}(he) = h\phi(e) (h \in G, e \in E)$ is unambiguous by (ii). ■

Proposition 4.15

(i) *Let $h \in G - N$. Then $E \cap hE$ is contained in a reflecting hyperplane of $\overline{N}$.*

(ii) *Let v be any point of E. The isotropy group G_v acts transitively on the components of X which contain v.*

(iii) *Fix $\overline{H} \in \mathcal{A}(\overline{N})$ (this is a hyperplane in E). If $hE \cap E = \overline{H}$, there is an element $x \in G_{\overline{H}}$ such that $hE = xE$.*

Proof (i) If $e = he' \in E \cap hE$, then there is an element $n \in N$ such that $e = ne' = nh^{-1}e$. Since $h \notin N$, $nh^{-1} \notin N$. Thus nh^{-1} is a product of reflections in hyperplanes which contain e (by Steinberg's theorem), at least one of which does not contain E. So e lies in some hyperplane $H \in \mathcal{A}(G)$ with $H \not\supseteq E$. Thus $E \cap E'$ is contained in the union of all such hyperplanes and being irreducible, must be in one of them.

(ii) If E and hE are two components of X which contain v, then as in (i), we take $n \in N$ such that $nh^{-1} \in G_v$. Then $hE = hn^{-1}E \subset G_vE$, as stated.

(iii) Take $e \in \overline{H}$ such that e is in no reflecting hyperplane of G which does not contain $\overline{H}$. Then $G_e = G_{\overline{H}}$ and the result is immediate from (ii). ■

Fix $\phi \in \mathcal{P}(E,E)^N$. We are interested in showing that the extension $\widetilde{\phi}$ of (4.14)(iii) is a regular function. In the next result, we show first that $\widetilde{\phi} \in \mathcal{R}(X,V)^G$ and our strategy is to show that the domain of definition of $\widetilde{\phi}$ has complement of codimension at least two, *i.e.*, that $\widetilde{\phi}$ is regular except on a subvariety of codimension at least two; in the case where (G,g,ζ) is regular, this will suffice, because in that case we shall show that if $\widetilde{\phi}$ is not regular, it is irregular on a hypersurface of X.

Suppose $\omega \in \mathcal{R}(X,V)^G$ and that U is an open set in X. To show that ω is regular on U (or equivalently that its domain includes U), it suffices (by, *e.g.*, [Mu, Prop. 1, p. 42]) to show that for any point $v \in U$, there is a Zariski open neighbourhood of v in X on which ω is regular.

Proposition 4.16

(i) *For any $\phi \in (S_E \otimes V)^N$, the function $\widetilde{\phi}$ of (4.14)(iii) is in $\mathcal{R}(X,V)^G$.*

(ii) Let E_1 be the set of all points of E which lie in at most one reflecting hyperplane of $\overline{N}$ on E. Let $X_1 = \bigcup_{h\in G} hE_1$. Then $\widetilde{\phi}$ is regular on X_1.

Proof (i) Let E_0 be the set of points of E which lie on no reflecting hyperplane of $\overline{N}$ and let $X_0 = \bigcup_{h\in G} hE_0$. This is an open dense subvariety of X and each of its points has a Zariski neighbourhood contained in just one irreducible component of X. Since the restriction of $\widetilde{\phi}$ to any component is a G-translate of ϕ, which is regular, it follows that $\widetilde{\phi}$ is regular on X_0, which proves (i).

(ii) It suffices, since $\widetilde{\phi}$ is G-invariant to show that $\widetilde{\phi}$ is regular on E_1, which by the remarks above, will follow if we show that for each element $v \in E_1$, there is a Zariski open subset $U_v \ni v$ of X_1 on which $\widetilde{\phi}$ is regular.

If $v \in E_1$ does not lie on any reflecting hyperplane, such a neighbourhood exists by (i). Suppose $v \in E_1$ lies in a unique reflecting hyperplane $\overline{H}$ of $\overline{N}$ on E. We prove by induction on $|G|$ that $\widetilde{\phi}$ is regular at v. Clearly g normalises $G_{\overline{H}}$. Let X_1' be the complement in X_1 of the union of the components of X which do not contain v. Then X_1' is an open neighbourhood of v in X_1 which by (4.15)(ii) coincides with a neighbourbood in $X_1(\overline{H}) = \bigcup_{h\in G_{\overline{H}}} hE_1$. If $G \neq G_{\overline{H}}$, then applying the induction hypothesis to the pair $(g, G_{\overline{H}})$ we deduce that the unique $G_{\overline{H}}$-invariant extension of ϕ to $\bigcup_{h\in G_{\overline{H}}} hE$ is regular at $v \in X_1(\overline{H})$. Of course this extension coincides with the restriction of $\widetilde{\phi}$ to $X_1(\overline{H})$. This proves the statement, except when $G_{\overline{H}} = G$. But in this case (4.9)(ii) shows that $\widetilde{\phi}$ is regular. ■

We remark that (i) also follows from a result about the rank of the matrix $\big(f_{ij}(v)\big)$ for any $v \in V$ (see (4.3) above), which is analogous to that of Steinberg [St2].

We have seen in (4.10) that restriction from X to E defines an isomorphism $\mathbb{C}[X]^G \cong S_E^N$ and in (1.1) that we have a natural isomorphism $\mathbb{C}[f_1,\dots,f_a] \cong S_E^N$, given by restriction from V to X. We shall identify all three algebras without further comment. Thus S_E^N will be thought of as an algebra of functions on E, V and X.

Let $X(\widetilde{\phi}) \subset X$ be the complement of the domain of definition of $\widetilde{\phi}$, regarded as a rational function. This is a G-invariant, closed subvariety of X.

Corollary 4.17 *The irreducible components of $X(\widetilde{\phi})$ have codimension at least 2 in X.*

Proof This is an immediate consequence of (4.16). ■

The ring $\mathbb{C}[X]^G$ is the coordinate ring of the affine variety $G \setminus X$, whose points are the orbits of G on X. Under the identification $\mathbb{C}[X]^G \cong \mathbb{C}[f_1,\dots,f_a]$, $G \setminus X$ is identified with $\mathbb{C}^a$ and the quotient map

$$\pi\colon X \longrightarrow G \setminus X$$

is realised as the map $x \mapsto \big(f_1(x),\dots,f_a(x)\big)$.

Lemma 4.18 *Consider the ideals $D^0(\phi) = \{h \in S_E^N \mid h\phi \in \operatorname{Im}\rho^0\}$ and $D(\widetilde{\phi}) = \{h \in S_E^N \mid h\widetilde{\phi} \in \mathcal{P}(X,V)^G\}$ of $\mathbb{C}[G \setminus X]$. We have $D^0(\phi) = D(\widetilde{\phi}) \neq 0$.*

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Proof If h is any element of $D^0(\phi)$ then $h\phi = \sigma^0(\psi)$ for some $\psi \in \mathcal{P}(X, V)^G$. But then $\psi = \widetilde{(h\phi)}$ and by uniqueness, $\widetilde{h\phi} = h\widetilde{\phi}$, whence $h\widetilde{\phi} \in \mathcal{P}(X, V)^G$. Since $h \in \Sigma(X)$ by (4.10)(ii), we conclude that $h \in D(\widetilde{\phi})$, so that $D^0(\phi) \subset D(\widetilde{\phi})$. Conversely if $k \in D(\widetilde{\phi})$, then $k\widetilde{\phi} \in \mathcal{P}(X, V)^G$, whence by (4.11) $k\widetilde{\phi}$ is the restriction to X of some $\psi \in \mathcal{P}(V, V)^G$. But then $k\phi = \rho^0\psi$, so that $k \in D^0(\phi)$. ∎

Let Z be the closed subvariety of $G \setminus X$ defined by the ideal $D^0(\phi) = D(\widetilde{\phi})$ of its coordinate ring.

Lemma 4.19 *The image $\pi X(\widetilde{\phi})$ contains Z.*

Proof We need to show that if U is the domain of definition of the rational function defined by $\widetilde{\phi}$, then $\pi(U) \cap Z = \varnothing$. Take a point $v \in U$. By (4.12)(ii), v is not in the zero set of the denominator ideal of $\widetilde{\phi}$. Hence there is a function $k \in \mathbb{C}[X]$ such that $k(v) \neq 0$ and $k\widetilde{\phi} \in \mathcal{P}(X, V)$.

We shall show that there is an element $k' \in \mathbb{C}[G \setminus X]$ such that $k'(v) \neq 0$ and $k'\widetilde{\phi} \in \mathcal{P}(X, V)^G$. This will imply that $\pi(v) \notin Z$, from which the lemma follows.

To prove the existence of k', observe first that for any element $y \in G$, $y(k\widetilde{\phi}) = (yk)\widetilde{\phi} \in \mathcal{P}(X, V)$. Let σ_j be the j-th elementary symmetric function in the translates yk (over all $y \in G$). Then for each j, $\sigma_j \in \mathbb{C}[G \setminus X]$ and by the first observation, $\sigma_j\widetilde{\phi} \in \mathcal{P}(X, V)^G$. To see that $\sigma_j(v) \neq 0$ for some j, observe that if $\sigma_j(v) = 0$ for all j, then

$$\prod_{y \in G} \bigl(1 - tk(yv)\bigr) = 1$$

for any $t \in \mathbb{C}$. This would imply that $k(yv) = 0$ for any $y \in G$, contradicting the fact that $k(v) \neq 0$. ∎

Corollary 4.20 *Each component of Z has codimension at least 2 in $G \setminus X$.*

Proof Since G is a finite group, we have $\dim \pi Y = \dim Y$ for any closed irreducible subset Y of X. The result is now immediate from (4.19) and (4.17). ∎

Proposition 4.21 *Suppose that in the notation of (4.3), $a = b$ (in particular this is the case if (G, g, ζ) is a regular triple, i.e., $C = 1$). If $\phi \in (S_E \otimes E)^N$ is not in $\operatorname{Im} \rho^0$, then each component in $G \setminus X$ of Z has codimension at most 1.*

Proof Recall that $\rho^0\phi_i = 0$ unless $i \in \{1, \dots, b\}$. It follows from (4.18) that if $a = b$, each element ψ of $(S_E \otimes E)^N$ is expressible uniquely in the form $\psi = \sum_{i=1}^a k_i\phi_i'$, where $\phi_i' = \rho^0(\phi_i)$ and k_i is in the quotient field K of $S_E^N = \mathbb{C}[f_1, \dots, f_a]$. Hence $h \in D(\psi)$ if and only if $hk_i \in S_E^N$ for each i. If $\phi \notin \operatorname{Im} \rho^0$, then for some i, $k_i \notin S_E^N$, so that if $h \in D^0(\phi)$ then h is in the ideal $\{h \in S_E^N \mid hk_i \in S_E^N\}$, which is principal since S_E^N is a unique factorisation domain. Thus its components, which are subvarieties of those of Z, have codimension 1. ∎

Proof of Theorem D For any $\phi \in (S_E \otimes E)^N$, we know (4.20) that the variety of $D^0(\phi)$ has codimension at least 2. From (4.21) we have that when d is regular, $D^0(\phi)$ has codimension one if ϕ is not in the image of ρ^0. Hence there is no such element ϕ and ρ^0 is surjective. ■

We conclude this section with some results and questions concerning the variety X.

Remark The results of this section have involved the reducible affine variety $X = \bigcup_{g \in G} gE$ and its coordinate ring $\mathbb{C}[X]$. We have seen that $\mathbb{C}[X]$ is the quotient of $A = S_V/(f_{a+1}, \dots, f_\ell)$ by its radical. It is therefore natural to ask when A itself is reduced, or equivalently, when do we have $\mathbb{C}[X] = A$?

The next Proposition answers this.

Proposition 4.22 *The ring A is reduced if and only* (G, g, ζ) *is a regular triple.*

Sketch of Proof By [LS1, 2.3] (for the twisted case, see [*loc. cit.*, Section 4]), (G, g, ζ) is regular if and only if the intersection multiplicity $\mu = i(E, H_{a+1}, \dots, H_\ell; V)$ of E as a component of the intersection of the divisors H_i defined by the f_i $(a + 1 \le i \le l)$ equals 1.

Let P be the prime ideal of E in S_V. Let $\mathcal{O}_P$ be the localization of S_V at P. Then $\mu = \ell(\mathcal{O}_P/I\mathcal{O}_P)$, the length of a local Artinian ring (see [F, 7.1.10]). This shows that $\mu = 1$ if $A = \mathbb{C}[X]$. Conversely, assume that $\mu = 1$. The minimal prime ideals in S_V containing $I = (f_{a+1}, \dots, f_\ell)$ are the transforms xP $(x \in G)$. Denote them by $P = P_1, \dots, P_t$. The radical of I is the intersection $\bigcap_{i=1}^t P_i$ and I is a (minimal) intersection of a set Ω of associated primary ideals. Because S is a polynomial algebra and I is generated by a regular sequence, A is a Cohen-Macaulay ring, and there are no imbedded prime ideals associated to the ideal I. It follows that the associated prime ideals must be our P_j. It also follows that for each j there is a primary ideal $Q_j \in \Omega$ with radical P_j. The interpretation of μ $(= 1)$ as a length shows that $(\bigcap P_i/I)_{P_j} = \{0\}$ for $j = 1$, hence also for all j (use the G-action). It follows that $\bigcap P_j = I$, hence I is reduced and the proposition is proved. ■

From the above, we conclude that if (G, g, ζ) is regular, $\mathbb{C}[X]$ is a free graded module over the algebra of invariants $\mathbb{C}[X]^G \cong \mathbb{C}[f_1, \dots, f_a]$. This leads to the following question.

Question 4.23 *Is* $\mathbb{C}[X]$ *always a free graded module over* $\mathbb{C}[X]^G$ *?*

A positive answer would imply the validity of Theorem D in all cases, *i.e.*, without the regularity assumption. In fact, the induction argument used in the proof of (4.16) would give that the rational map $\widetilde{\phi}$ of [*loc. cit.*] is defined at all non-zero points of E. If $\mathbb{C}[X]$ is free over $\mathbb{C}[X]^G$ it is not hard to show that $\widetilde{\phi}$ is also defined at the point $\{0\}$.

Another version of the question is whether $\mathbb{C}[X]$ is a graded Cohen-Macaulay algebra. Since $\mathbb{C}[X]$ is the quotient of the Cohen-Macaulay ring A $(= S_V/(f_{a+1}, \dots, f_\ell))$ by its radical, the latter statement would follow if it were known that such a quotient is always Cohen-Macaulay. However it has been pointed out to us by Strooker that this is not generally true, as is shown by an example due to Cowsik and Nori [CN].

5 Completion of the Proof of Theorem C

Theorem D of the last section, together with (4.8), imply Theorem C(i) in case d is regular for G. Moreover Theorem C(ii) is immediate from (4.6). Thus to complete the proof of Theorem C, we need to prove part (i) when d is not necessarily regular and part (iii). As indicated above, both of these will require some case by case analysis. For convenience, we paraphrase here the results to be proved. Recall that Theorem C deals only with the untwisted case, *i.e.*, the case where $g \in G$.

5.1

(i) If $a(d) = b(d)$ then d is regular for G.
(ii) The set of coexponents of $G(d)$ is a subset of the (multi-)set of coexponents of G whose degree is congruent to $1 \pmod d$.

Note first that (5.1)(ii) is clear when $a(d) = 1$ (which is the case $\dim E = 1$) by (4.9)(ii) and (4.8). Moreover it is easy to see that it suffices to prove the assertions (5.1) in the case that G is irreducible, which we shall henceforth assume. We shall use the classification of the irreducible groups which is due to Shephard and Todd [ST].

To deal with the imprimitive groups we use

Proposition 5.2 *If G is the group $G(r, m, \ell)$, then $G(d) \cong G(r', m', \ell')$, for some integers r', m', ℓ'.*

For the proof of (5.2), which was proved in the case when $G = S_\ell$ in [BMM], we shall require the following generalisation of [LS1, (2.9)].

Lemma 5.3 *For any reflection group G and integers d, e, we have $G(e)(d) \cong G\big(\operatorname{lcm}(d, e)\big)$.*

Proof Suppose first that H is a finite reflection group each of whose degrees is divisible by the integer e. Write $l = \operatorname{lcm}(d, e)$ and let ζ_l be a primitive l-th root of unity. If $E = V(g, \zeta_l)$ is a maximal ζ_l-eigenspace, then since any degree d_i of H is divisible by d if and only if it is divisible by l, it follows that $E = V(g^{l/d}, \zeta_l^{l/d})$ is a maximal $\zeta_d = \zeta_l^{l/d}$-eigenspace of V (they both have dimension equal to $\#\{i \mid d \text{ divides } d_i\}$). It is therefore clear that $H(d) = H(l)$. Applying this statement with $H = G(e)$, we obtain $G(e)(d) = G(e)(l) = G(l)$, the last equality following from [LS1, (2.9)]. ∎

Proof of Proposition 5.2 We assume that $G = G(r, m, \ell)$. Recall that this group acts in $\mathbb{C}^\ell$, and is the semi-direct product of the symmetric group S_ℓ, acting by permutations of the canonical basis (e_i), and the group of diagonal maps $e_i \mapsto \theta_i e_i$, where $\theta_i^r = 1$ and $(\theta_1 \cdots \theta_\ell)^n = 1$. Here m divides r and $r = mn$. When $r = 1$ this group is the symmetric group S_ℓ; it does not act irreducibly. The degrees of G are

$$r, 2r, \ldots, (\ell - 1)r, \ell n. \tag{5.2.1}$$

If we denote the coordinate function for e_i by X_i, the invariant for the last degree may be chosen to be

$$f_\ell = (X_1 \cdots X_\ell)^n.$$

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Let us begin by observing that (5.2) is true for $G = S_\ell$. For if we write $\ell = qd + r$, with $0 \le r < d$, then we may take $g \in S_\ell$ to be a permutation with q d-cycles and one r-cycle to obtain a maximal ζ-eigenspace $V(g, \zeta)$, where ζ is a primitive d-th root of unity. It is then straightforward to verify that $G(d) \cong G(d, 1, q)$, verifying (5.2).

Write $e = \big(\gcd(d, r)\big)^{-1} d$. By Theorem (1.1) the degrees of $G(d)$ are

$$\begin{cases} er, 2er, \dots, [e^{-1}(\ell - 1)]er & \text{if } d \nmid \ell n, \\ er, 2er, \dots, [e^{-1}(\ell - 1)]er, \ell n & \text{otherwise.} \end{cases}$$

Case 1 Assume $d \mid \ell n$.

Then $d \mid \ell r$ and $e \mid \ell$. Put $G' = G(r, 1, \ell)$. Then $G \subset G'$ and the degrees of G' are

$$r, 2r, \dots, \ell r.$$

Hence the degrees of $G'(d)$ are

$$er, 2er, \dots, [e^{-1}(\ell - 1)]er, \ell n.$$

The number of these degrees is $e^{-1}\ell = [e^{-1}(\ell - 1)] + 1$, which is the same as the number of degrees of $G(d)$. It follows that if $g \in G$ is such that $E = V(g, \zeta)$ has maximal dimension $(= \ell/e)$, the same is true if g is regarded as an element of G'. Hence $G(d)$ is a reflection group in E which is generated by a subset of the reflections in $G'(d)$. Consequently if $G'(d)$ is imprimitive (*i.e.*, respects a direct sum decomposition of E into lines), the same is true of $G(d)$.

Thus we need check (5.2) only for $G = G' \cong G(r, 1, \ell)$. But in this case $G \cong S_{\ell r}(r)$, so that $G(d) \cong S_{\ell r}(r)(d)$, which by (5.3) is isomorphic (as reflection group) to $S_{\ell r}\big(\operatorname{lcm}(r, d)\big)$, a case which has already been dealt with above.

Case 2 Assume $d \nmid \ell n$.

The polynomial f_ℓ vanishes on E. Let H_i be the hyperplane of zeros of X_i. Since E is contained in the union of the H_i, E lies inside one of them, say H_ℓ. Now G has a subgroup $G' = G(r, 1, \ell - 1)$ which may be identified in the obvious way as a reflection group acting in H_ℓ. Viewed thus, its degrees are $r, 2r, \dots, (\ell - 1)r$, whence the degrees of $G'(d)$ coincide with those of $G(d)$ which implies that the two groups have the same cardinality. It also follows that there is an element $g' \in G'$ such that the eigenspace E may be taken as $V(g', \zeta)$. Since $G(d)$ and $G'(d)$ have the same cardinality, all the transformations of E which are induced by its normaliser in G are induced by elements of G', whence $G(d) = G'(d)$. By induction on ℓ, $G'(d)$ is of the form $G(r', m', \ell')$, whence the result for $G(d)$. This completes the proof of (5.2). ■

Proof of (5.1) in the Imprimitive Case We can now prove (5.1) for the case when $G = G(r, m, \ell)$. The degrees are given by (5.2.1) and the coexponents are

$$1, r + 1, 2r + 1, \dots, (\ell - 2)r + 1, (\ell - 1)r + 1 \quad (n \ne 1),$$

$$1, r + 1, 2r + 1, \dots, (\ell - 2)r + 1, (\ell - 1)r + 1 - \ell \quad (n = 1),$$

see [OS, p. 92, Table 2]. Also, the regular degrees for G are the divisors of

$$\ell n \quad (n \neq 1),$$
$$(\ell - 1)r \quad (n = 1, \ell \mid r),$$
$$(\ell - 1)r, \ell n \quad (n = 1, \ell \nmid r),$$

see [Co, (2.11), p. 391]. In view of the results of Section 4, if d is regular, then (5.1)(ii) holds.

Case 1 If $d \mid \ell n$ then d is regular. For if not, then $n = 1$ and $\ell \mid r$; so $d \mid \ell$ and thus $d \mid r$. But then $d \mid (\ell - 1)r$ and so d is regular. Hence (5.1) holds in this case.

Case 2 Suppose $d \nmid \ell n$. The degrees of $G(d)$ are

$$er, 2er, \ldots, [(\ell - 1)/e]er$$

and we may assume $[(\ell - 1)/e] > 1$, since the case $\dim E = 1$ has been taken care of above. The sequence of degrees is therefore an arithmetic progression. By (5.2), we know that $G(d)$ is of the form $G(r', m', \ell')$, whose degrees are

$$r', 2r', \ldots, (\ell' - 1)r', \ell' r'/m'. \tag{5.1.1}$$

The sequence (5.1.1) is an arithmetic progression of the form $k, 2k, 3k, \ldots$ only if at least one of the following conditions holds.

(a) $m' = 1$, in which case $G(d) \cong G\big(er, 1, [(\ell - 1)/e]\big)$.
(b) $\ell' = 2$.

We check the statements (5.1) in these two cases.

In case (a), $m' = 1$, so $G(d) \cong G(r', 1, \ell') = G(er, 1, \ell')$. Since $er > 1$ (it is divisible by d), $G(d)$ has coexponents

$$1, er + 1, \ldots, \big([(\ell - 1)/e] - 1\big)er + 1,$$

which form a subset of those of G because $\{er, 2er, \ldots, \ell' er\} \subset \{r, 2r, \ldots, (\ell - 2)r\}$. Hence (5.1)(ii) holds. As to (5.1)(i), clearly $a(d) = \ell'$; we need to show that if $b(d) = \ell'$, d is regular. If $n \neq 1$ then $1, er + 1, 2er + 1, \ldots, \ell' er + 1$ are among the coexponents of G which are congruent to 1 mod d. Hence if $b(d) = \ell'$ we must have $n = 1$ and $(\ell - 1)r = \ell' er$, so that d divides $(\ell - 1)r$, whence d is regular. This proves (5.1)(i) and completes case (a).

Suppose that we are in case (b). Observe that since $\ell' = 2$, we have $3 > [(\ell - 1)/e] \geq 2$, whence

$$3e > \ell - 1 \geq 2e. \tag{5.1.2}$$

Further, the degrees of $G(d)$ are r' and $\ell' r'/m' = 2r'/m' = 2n'$. Since these are also equal to $er, 2er$, we must have either

$$r' = er \quad \text{and} \quad 2n' = 2er \tag{b1}$$

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or

$$\text{(b2)} \qquad r' = 2er \quad \text{and} \quad 2n' = er.$$

In case (b1), $m' = 1$, which is a case we have treated above (case (a)).

In case (b2), we have $r' = 2er$ and $2n' = er$, so that $r' = 4n'$ and $m' = 4$. Hence $G(d) \cong G(2er, 4, 2)$ and the coexponents of $G(d)$ are given by

$$\text{(5.1.3)} \qquad \begin{matrix} 1, 2er + 1 & \text{if } n' \neq 1 \\ 1, 2er - 1 & \text{if } n' = 1. \end{matrix}$$

If $n' \neq 1$, *i.e.*, $er \neq 2$, then the coexponents $1, 2er + 1$ are among those of G, except possibly if $n = 1$ and $(\ell - 1)r = 2er$. But in this case d is regular for G and (5.1)(ii) holds. Moreover in the non-regular case ($n' \neq 1$), $1 + er$ is a coexponent of G which is not among those of $G(d)$. Hence $a(d) < b(d)$, which confirms (5.1)(i) in this case.

Finally, suppose $n' = 1$, *i.e.*, $er = 2$. Then $d = 2$. Moreover if $r = 1$, we are in the case where $G = S_\ell$ is the symmetric group. We have described $G(d)$ completely in this case in the proof of (5.2) and (5.1) is easily verified as well. Hence we may take $r = 2$, $e = 1$ and $d = 2$. Thus $G(d) \cong G(4, 4, 2)$. Furthermore, from (5.1.2) we see that $\ell = 3$, so that $G = G(2, m, 3)$. The degrees of G are 2, 4 and $3n$; since $a(d) = a(2) = 2$, we have $n = 1$, whence d is regular for G. This completes the proof of (5.1) for $G = G(r, m, \ell)$ and any d. ∎

We next prove (5.1) when G is primitive, which we assume now. We denote by G_i the group with number i in the list of [ST]. We shall use the following result.

Lemma 5.4 *Let z be the order of the centre of G. It suffices to prove (5.1)(ii) for $d = p^a$, where $a \geq 1$ and p is a prime number such that $p^a \nmid z$, $p^{a-1} \mid z$.*

Proof Suppose we know (5.1)(ii) for d of the stated form. Let d be any integer not dividing z and write $d = p^a e$, where p is a prime and $p \nmid e$. Since $G(d) = G(e)(d)$, if $G(e) \neq G$ we are finished by induction on the pair $(|G|, d)$. If $G(e) = G$, then e divides z and $G(d) = G(p^a)$ by (5.3). Similarly, we have $G(p^a) = G(p^{a-1})(p^a)$, so that by induction, we may asume that $p^{a-1} \mid z$. Thus if the result is known for all cases in the statement of (5.4), (5.1)(ii) is known for all G, d. ∎

Proof of (5.1)(ii) for Primitive G As pointed out above, we may assume that d divides at least two degrees of G and that d is not regular. Under these assumptions we have $\ell > 2$. An explicit check, using the results of [Sp, Section 5] and [Co, p. 412] reveals that the only cases with $d = p^a$ as in Lemma (5.4) above which remain after imposing the conditions that $a(d) \geq 2$ and that d be non-regular, are G_{33}, G_{34}, G_{36}, with $d = 4$ in each case. The degrees of $G_{33}(4)$ are 4, 12 and it follows by inspection of the list that $G_{33}(4) \cong G_6$, whose coexponents $\{1, 9\}$ form a subset of those of G_{33}. The degrees of $G_{34}(4)$ are 12, 24. This leaves G_{10}, G_{15}, $G(12, 1, 2)$ and $G(24, 4, 2)$ as possibilities for $G_{34}(4)$. The coexponents are therefore either $\{1, 13\}$ or $\{1, 25\}$. Both possibilities are subsets of the coexponents of G_{34}. Finally, we similarly see that $G_{36}(4)$ is one of G_8, G_{12} or $G(12, 3, 2)$. The possible sets of coexponents are therefore $\{1, 5\}, \{1, 11\}$ or $\{1, 13\}$. In each case we have a subset of the coexponents of G_{36}. This completes the proof of (5.1)(ii). ■

To complete the proof of (5.1), we now give the

Proof of (5.1)(i) for Primitive G We need to show that if d is not regular, then $b(d)$, which is the number of coexponents of G which are congruent to 1 modulo d, is greater than $a(d)$. In view of Lemma (5.3), we have $G(d) = G\big(\mathrm{lcm}(d, z)\big)$, where z is the order of the centre of G. Moreover if $e \mid d$ and $G(e) = G(d)$, then if we know (5.1)(i) for (G, d), we also know it for (G, e) since if a coexponent of $G(d) = G(e)$ is congruent to 1 modulo d, the same is true modulo e. Hence (5.1)(i) need only be verified for the largest integers d such that $G(e) = G(d)$. The results of [Sp, *loc. cit.*] and [Co, pp. 395, 412] now show that only the following cases must be checked:

G_{13} $(d = 8)$, G_{15} $(d = 24)$, G_{24} $(d = 4)$, G_{26} $(d = 12)$, G_{27} $(d = 12)$, G_{29} $(d = 8, 12)$, G_{32} $(d = 18)$, G_{33} $(d = 4, 12)$, G_{34} $(d = 12, 18, 24, 30)$, G_{35} $(d = 5)$, G_{36} $(d = 4, 8, 10, 12)$, G_{37} $(d = 14, 18)$.

Inspection of the list of coexponents of [OS, p. 92, Table 2] reveals that in all these cases we have $a(d) < b(d)$, proving (5.1)(i) and completing the proof of Theorem C. ■

Corollary 5.5 *Let G be a Coxeter group. Then d is regular for G if and only if the number of degrees d_i which are congruent to* $0 \pmod d$ *equals the number of d_i which are congruent to* $2 \pmod d$.

This follows from (5.1)(i), upon remarking that for a Coxeter group the coexponents are the numbers $d_i - 1$.

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