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ON SLANT CURVES IN SASAKIAN 3-MANIFOLDS

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A classical theorem by Lancret says that a curve in Euclidean 3-space is of constant slope if and only if its ratio of curvature and torsion is constant. In this paper we study Lancret type problems for curves in Sasakian 3-manifolds.

1. INTRODUCTION

In classical differential geometry of spatial curves, the following result is known (see for example, [**10, 19, 21**]).

THEOREM 1.1. (Bertrand–Lancret–de Saint Venant) *A curve $\gamma(s)$ in Euclidean 3-space $\mathbb{E}^3$ is a curve of constant slope if and only if its ratio of curvature and torsion is constant.*

Here we recall that a curve in $\mathbb{E}^3$ is said to be a *curve of constant slope* (or *cylindrical helix* [**20**]) if the tangent vector field of γ has constant angle with a fixed direction (called the *axis* of the curve). Moreover it is clear that for every curve γ of constant slope, there exists a cylinder on which γ moves in such a way as to cut each ruling at a constant angle. (See [**20**, pp. 72–73].)

Barros [**1**] generalised the above characterisation due to Bertrand–Lancret–de Saint Venant to curves in 3-dimensional space forms. Corresponding results for 3-dimensional Lorentzian space forms are obtained by Ferrández [**11**]. Moreover Ferrández, Giménez and Lucas [**12, 13**] investigated Bertrand–Lancret–de Saint Venant problem for null curves in Minkowski 3-space. (See also [**14, 18**].)

As is well known, the unit 3-sphere S^3 is a typical example of a Sasakian manifold. In 3-dimensional contact metric geometry, Legendre curves play a fundamental role [**2**]. As a generalisation of Legendre curves, in this paper, we introduce the notion of a slant curve.

A curve in a contact 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field. Slant curves appear naturally in differential geometry of Sasakian 3-manifolds. In our recent paper [**9**], it is shown that biharmonic curves in 3-dimensional Sasakian space forms are slant helices.

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In this paper we study Bertrand–Lancret–de Saint Venant type problems for slant curves in Sasakian 3-manifolds.

Our result is

THEOREM. *A curve in a Sasakian 3-manifold is a slant curve if and only if its ratio of "geodesic curvature" and "geodesic torsion ± 1" is constant.*

Moreover, we find the explicit parametric examples of proper slant curves which are not helices in the Heisenberg group $\mathbb{H}_3$ (see Example 4.2).

2. Preliminaries

2.1. Let $\gamma : I \to M = (M^3, g)$ be a Frenet curve parametrised by arc length in a Riemannian 3-manifold M^3 with Frenet frame field (T, N, B). Here T, N, B are the tangent, principal normal and binormal vector fields, respectively. Denote by ∇ the Levi-Civita connection of (M, g). Then the Frenet frame satisfies the following *Frenet-Serret* equations:

$$\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T + \tau B, \quad \nabla_T B = -\tau N, \tag{2.1}$$

where $\kappa = |\nabla_T T|$ and τ are the *geodesic curvature* and *geodesic torsion* of γ, respectively. A Frenet curve is said to be a *helix* if both of κ and τ are constant.

2.2. Next, we recall the fundamental ingredients of 3-dimensional contact metric geometry. Our general reference is [**3**].

Let M be a 3-dimensional manifold. A *contact form* is a one-form η such that $d\eta \wedge \eta \neq 0$ on M. A 3-manifold M together with a contact form η is called a *contact 3-manifold*. The *Reeb vector field* ξ is a unique vector field satisfying

$$\eta(\xi) = 1, \ d\eta(\xi, \cdot) = 0.$$

On a contact 3-manifold (M, η), there exists a structure tensor (φ, ξ, g) such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \varphi Y) = d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.3}$$

The structure (φ, ξ, η, g) is called the *associated contact metric structure* of (M, η). A contact 3-manifold together with its associated contact metric structure is called a *contact metric 3-manifold*. A contact metric 3-manifold M satisfies the following formula ([**22**]).

$$(\nabla_X \varphi)Y = g(X + \mathrm{h}X, Y)\xi - \eta(Y)(X + \mathrm{h}X), \quad X, Y \in \mathfrak{X}(M), \tag{2.4}$$

where $\mathrm{h} = \pounds_\xi \varphi / 2$.

A contact metric 3-manifold $(M, \varphi, \xi, \eta, g)$ is called a *Sasakian manifold* if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.5}$$

for all $X, Y \in \mathfrak{X}(M)$.

A plane section Π_x at a point x of a contact metric 3-manifold is called a *holomorphic plane* if it is invariant under φ_x. The sectional curvature function of holomorphic planes is called the *holomorphic sectional curvature*. Sasakian 3-manifolds of constant holomorphic sectional curvature are called 3-dimensional *Sasakian space forms*. Simply connected and complete 3-dimensional Sasakian space forms are classified as follows:

PROPOSITION 2.1. ([4]) *Simply connected and complete 3-dimensional Sasakian space forms $\mathcal{M}^3(H)$ of constant holomorphic sectional curvature H are isomorphic to one of the following unimodular Lie groups with left invariant Sasakian structures: the special unitary group $\mathrm{SU}(2)$ for $H > -3$, the Heisenberg group $\mathbb{H}_3$ for $H = -3$, or the universal covering group $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the special linear group $\mathrm{SL}(2, \mathbb{R})$ for $H < -3$. The Sasakian space form $\mathcal{M}^3(1)$ is the unit 3-sphere S^3 with the canonical Sasakian structure.*

3. SLANT CURVES

3.1. Let M be a contact metric 3-manifold and $\gamma(s)$ a Frenet curve parametrised by arc length s in M. The *contact angle* $\theta(s)$ is a function defined by $\cos\theta(s) = g(T(s), \xi)$. A curve γ is said to be a *slant curve* if its contact angle is constant. Slant curves of contact angle $\pi/2$ are traditionally called *Legendre curves*. The Reeb flow is a slant curve of contact angle 0.

Now we consider Bertrand–Lancret–de Saint Venant type results for contact geometry. We take an adapted local orthonormal frame field $\{X, \varphi X, \xi\}$ of M such that $\eta(X) = 0$.

Let γ be a non-geodesic Frenet curve in a Sasakian 3-manifold. Differentiating the formula $g(T, \xi) = \cos\theta$ along γ, then it follows that

$$-\theta' \sin\theta = g(\kappa N, \xi) + g(T, -\varphi T) = \kappa\, \eta(N).$$

This equation implies the following result.

PROPOSITION 3.1. *A non-geodesic curve γ in a 3-dimensional Sasakian manifold M is a slant curve if and only if it satisfies $\eta(N) = 0$.*

Hence T, N and ξ of a slant curve $\gamma(s)$ has the form

$$\begin{aligned} T &= \sin\theta\{\cos\beta(s)X + \sin\beta(s)\varphi X\} + \cos\theta\xi, \\ N &= -\sin\beta(s)X + \cos\beta(s)\varphi X, \\ \xi &= \cos\theta T \pm \sin\theta B \end{aligned}$$

for some function $\beta(s)$. Differentiating $0 = g(N, \xi)$ along γ and using the Frenet–Serret equations, we have

$$\kappa \cos\theta + (-1 \pm \tau)\sin\theta = 0. \tag{3.1}$$

This implies that the ratio of $\tau \pm 1$ and κ is a constant. Conversely, if the ratio of $\tau \pm 1$ and $\kappa \neq 0$ is constant, then γ is clearly a slant curve. Thus we obtain the following result.

THEOREM 3.1. *A non-geodesic curve in a Sasakian 3-manifold M is a slant curve if and only if its ratio of $\tau \pm 1$ and κ is constant.*

The equation (3.1) implies the following result (compare with [**2**]).

COROLLARY 3.1. *Let γ be a non-geodesic slant curve. Then $\tau = \pm 1$ if and only if γ is a Legendre helix.*

3.2. A Sasakian 3-manifold M is said to be *regular* if its Reeb vector field ξ generates a one-parameter group K of isometries on M, such that the action of K on M is simply transitive. The Killing vector field ξ induces a regular one-dimensional Riemannian foliation on M. We denote by $\overline{M} := M/\xi$ the orbit space (the space of all leaves) of a regular Sasakian 3-manifold M under the K-action.

The Sasakian structure on M induces a Kähler structure on the orbit space $\overline{M}$. Further the natural projection $\pi : M \to \overline{M}$ is a Riemannian submersion. It is easy to see that M is a Sasakian space form of constant φ-holomorphic sectional curvature H if and only if $\overline{M}$ is a space form of curvature $H + 3$.

Take a curve $\overline{\gamma}$ in the orbit space, then its inverse image $S_{\overline{\gamma}} = \pi^{-1}(\overline{\gamma})$ is a flat surface in M. This flat surface is called the *Hopf cylinder* over γ. The mean curvature of the Hopf cylinder is the half of the geodesic curvature of $\overline{\gamma}$.

In particular, if M is the unit 3-sphere S^3, then π coincides with the Hopf fibreing $S^3(1) \to S^2(4)$. In this case, if $\overline{\gamma}$ is a small circle, then its Hopf cylinder is a non-minimal constant mean curvature torus. If $\overline{\gamma}$ is a great circle, then its Hopf cylinder is the Clifford minimal torus.

Now we consider a slant curve γ with the contact angle θ in a regular Sasakian 3-manifold. Let $\overline{\gamma} = \pi \circ \gamma$ be the projection of γ onto $\overline{M}$. Direct computation shows that the arc length parameter $\overline{s}$ of $\overline{\gamma}$ is

$$\overline{s} = \frac{s}{\sin\theta}. \tag{3.2}$$

The Frenet frame $\{\overline{T}(\overline{s}), \overline{N}(\overline{s})\}$ of $\overline{\gamma}$ is given by

$$\overline{T}(\overline{s}) = \frac{1}{\sin\theta}\pi_* T(s), \quad \overline{N}(\overline{s}) = \pm \pi_* N(s).$$

Thus the signed curvature $\overline{\kappa}$ of $\overline{\gamma}$ is given by

$$\overline{\kappa}(\overline{s}) = \frac{\pm 1}{\sin^2\theta}\kappa(s).$$

We specialise the contact angle of slant curves. Let $\gamma(s)$ be a Legendre curve in a regular contact Riemannian 3-manifold M. Then from (3.2) we see that its projection $\overline{\gamma}(s) = \pi(\gamma(s))$ is a curve with arc length parameter s and that γ is a horizontal lift of $\overline{\gamma}$. Further, the signed curvature $\overline{\kappa}$ is given by $\overline{\kappa}(s) = \pm\kappa(s)$. We note that for the Hopf cylinder $S = \pi^{-1}(\overline{\gamma})$, the Reeb vector field ξ is tangent to S and S contains γ.

4. Examples and Remarks

Let M be a Riemannian 3-manifold and γ a curve in M parametrised by arc length. Then γ is said to be *biharmonic* if

$$\nabla_T^3 T + R(\kappa N, T)T = 0.$$

Caddeo, Montaldo and Piu [**5**] classified biharmonic curves in the unit 3-sphere S^3. Caddeo, Piu and Oniciuc [**7**] classified biharmonic curves in the Heisenberg group. The present authors generalised the results of [**7**] to general 3-dimensional Sasakian space forms [**9**]. Caddeo, Montaldo Oniciuc and Piu generalised the classification of [**9**] to Bianchi–Cartan–Vranceanu spaces [**6**].

Theorem 4.1. ([**9**]) *Every proper biharmonic curve in Sasakian space form with constant holomorphic sectional curvature H is a slant helix satisfying*

$$\kappa^2 + \tau^2 = 1 + (H-1)\sin^2\theta.$$

Thus classification of proper biharmonic curves in a Sasakian 3-space form reduces to solving the equations:

$$\kappa^2 + \tau^2 = 1 + (H-1)\sin^2\theta, \quad \kappa\cos\theta + (-1 \pm \tau)\sin\theta = 0.$$

Remark 1. Let M be one of the following 3-dimensional spaces; Riemannian space form, or Minkowski 3-space. Then the biharmonic equation for non-geodesics in M is given by the following:

(1) M is of constant curvature c, then κ = constant and $\kappa^2 + \tau^2 = c$ ([**5**]),
(2) M is the Minkowski 3-space, then κ = constant and $\kappa^2 - \tau^2 = 0$ ([**8, 15, 16**]).

Example 4.1. ([**7, 9**]) The Heisenberg group $\mathbb{H}_3$ is a Cartesian 3-space $\mathbb{R}^3(x, y, z)$ furnished with the group structure

$$(x', y', z') \cdot (x, y, z) = (x' + x,\ y' + y,\ z' + z + (x'y - y'x)/2).$$

Define the left-invariant metric g by

$$g = \frac{dx^2 + dy^2}{4} + \eta \otimes \eta, \quad \eta = \frac{1}{2}\Big\{dz + \frac{1}{2}(ydx - xdy)\Big\}.$$

We take a left-invariant orthonormal frame field (e_1, e_2, e_3):

$$e_1 = 2\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = 2\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = 2\frac{\partial}{\partial z}.$$

Then the commutation relations are derived as follows:

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = [e_3, e_1] = 0. \tag{4.1}$$

The dual frame field $(\theta^1, \theta^2, \theta^3)$ is given by

$$\theta^1 = \frac{1}{2}dx,\ \theta^2 = \frac{1}{2}dy,\ \theta^3 = \frac{1}{2}dz + \frac{ydx - xdy}{4}.$$

Then the 1-form $\eta = \theta^3$ is a contact form and the vector field $\xi = e_3$ is the Reeb vector field on $\mathbb{H}_3$.

We define a (1,1)-tensor field φ by

$$\varphi e_1 = e_2,\ \varphi e_2 = -e_1,\ \varphi\xi = 0.$$

Then we find

$$d\eta(X, Y) = g(X, \varphi Y), \tag{4.2}$$

and hence, (η, ξ, φ, g) is a contact metric structure. Moreover, we see that it becomes a Sasakian structure. Then every proper biharmonic curve in $\mathbb{H}_3$ is represented as

$$\begin{cases} x(s) = \dfrac{1}{A}\sin\theta\sin(As + a) + b, \\ y(s) = -\dfrac{1}{A}\sin\theta\cos(As + a) + c, \\ z(s) = \left(\cos\theta + \dfrac{\sin^2\theta}{2A}\right)s - \dfrac{b}{2A}\sin\theta\cos(As + a) - \dfrac{c}{2A}\sin\theta\sin(As + a) + d, \end{cases}$$

for a constant contact angle θ, where A, a, b, c, d are constants. These slant helices satisfy $\kappa^2 + \tau^2 = 1 - 4\sin^2\theta$. Note that in [7], the metric on $\mathbb{H}_3$ is chosen as $4g$.

EXAMPLE 4.2. We construct a proper slant curve γ which is not a helix in the above $\mathbb{H}_3$. Let γ be a slant curve in $\mathbb{H}_3$. Then for a constant θ we put

$$\gamma'(s) = T(s) = T_1 e_1 + T_2 e_2 + T_3 e_3$$

and

$$T_1(s) = \sin\theta\cos\beta(s),\ T_2 = \sin\theta\sin\beta(s),\ T_3 = \cos\theta.$$

By using Frenet–Serret equations (2.1) we compute the geodesic curvature κ and the geodesic torsion τ for a slant curve γ in $\mathbb{H}_3$. Then we obtain

$$\begin{aligned} \kappa &= \sin\theta(\beta'(s) - 2\cos\theta), \\ \tau &= \cos\theta(\beta'(s) - 2\cos\theta) + 1, \end{aligned} \tag{4.3}$$

where we assume that $\sin\theta(\beta'(s) - 2\cos\theta) > 0$.

Here, the tangent vector field T of γ is also represented by the following:

$$T = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

Then it follows that

$$\frac{dx}{ds} = 2T_1, \ \frac{dy}{ds} = 2T_2, \ \frac{dz}{ds} = 2T_3 + \left(x\frac{dy}{ds} - y\frac{dx}{ds}\right).$$

In view of (4.3), we take for example $\beta(s) = \ln s$. Then we can find an explicit parametric equations of slant curves γ which are not helices:

$$\begin{cases} x(s) = 2\sin\theta \cdot \dfrac{s}{2}\{\sin(\ln s) + \cos(\ln s)\} + b_1, \\ y(s) = 2\sin\theta \cdot \dfrac{s}{2}\{\sin(\ln s) - \cos(\ln s)\} + c_1, \\ z(s) = 4\left(\dfrac{1}{4}\sin^2\theta\right)s^2 + 2(\cos\theta)s + d_1, \end{cases}$$

where b_1, c_1, d_1 are constants.

EXAMPLE 4.3. (Grassmann geometry) Let M be a Riemannian manifold and $\mathrm{Gr}_\ell(TM)$ its *Grassmann bundle* of all ℓ-planes in TM $(1 \leqslant \ell \leqslant \dim M)$. Take a non-empty subset Σ of $\mathrm{Gr}_\ell(TM)$. An ℓ-dimensional submanifold $\phi : S \to M$ of M is said to be a Σ-*submanifold* of M if $d\phi(TS) \subset \Sigma$. The collection of all Σ-submnaifolds is called the Σ-geometry of M. *Grassmann geometry* is a collected name for such a Σ-geometry. Let us denote by G the identity component of the isometry group of M. Then G naturally acts on $\mathrm{Gr}_\ell(TM)$. If Σ is an G-orbit in $\mathrm{Gr}_\ell(TM)$, the Σ-geometry is called of *orbit type*.

In [**17**], Inoguchi, Kuwabara and Naitoh investigated the Grassmann geometry of orbit type in $\mathbb{H}_3$. In this case, the G-orbit spaces in $\mathrm{Gr}_2(T\mathbb{H}_3)$ are parametrised by the curvature function K and K takes value in the closed interval $[-3, 1]$. The following results were obtained in [**17**]:

PROPOSITION 4.1. *For any $\alpha \in (-3, 1)$, $\mathcal{O}(\alpha)$-surfaces are of constant negative curvature $\alpha - 1$.*

THEOREM 4.2. *For any $\alpha \in (-3, 1)$,*

(1) *$\mathcal{O}(\alpha)$-surfaces are of constant negative curvature $\alpha - 1$.*

(2) *there exist local $\mathcal{O}(\alpha)$-surfaces foliated by circles which are helices of $\mathbb{H}_3$ with the same curvature and torsion* 1.

The helices on $\mathcal{O}(\alpha)$-surfaces are slant helices. In fact, the contact angle θ is computed as

$$\cos\theta = -\sqrt{1-\rho^2}, \quad \rho := \frac{1}{2}\sqrt{1-\alpha}.$$

These helices have geodesic curvature $\kappa = 2\rho/\sqrt{1-\rho^2}$ and geodesic torsion $\tau = 1$, and hence do not satisfy the relation $\kappa^2 + \tau^2 = 1 - 4\sin^2\theta$. Thus these slant helices are non-biharmonic.

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