THE $(2, 3)$-GENERATION OF THE SPECIAL LINEAR GROUPS OVER FINITE FIELDS

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Abstract

We complete the classification of the finite special linear groups $\mathrm{SL}_n(q)$ which are $(2, 3)$-generated, that is, which are generated by an involution and an element of order 3. This also gives the classification of the finite simple groups $\mathrm{PSL}_n(q)$ which are $(2, 3)$-generated.

1. Introduction

It is a well-known fact that every finite simple group can be generated by a pair of suitable elements: for alternating groups this is a classical result of Miller [11], for groups of Lie type it is due to Steinberg [17] and for sporadic groups it was proved by Aschbacher and Guralnick [1]. A more difficult problem is to find for a finite nonabelian (quasi)simple group G the minimum prime r, if it exists, such that G is $(2, r)$-generated, that is, such that G can be generated by two elements of respective orders 2 and r. We denote such a minimum prime r by $\varpi(G)$ (setting $\varpi(G) = \infty$ if G is not $(2, r)$-generated for any prime r). Since groups generated by two involutions are dihedral, we must have $\varpi(G) \geq 3$.

Miller himself proved that $\varpi(\mathrm{Alt}(n)) = 3$ if $n = 5$ or $n \geq 9$, while it is easy to verify that $\varpi(\mathrm{Alt}(n)) = 5$ if $n = 6, 7, 8$. The special linear groups were first considered in [21], where Tamburini showed that $\varpi(\mathrm{SL}_n(q)) = 3$ for all $n \geq 25$ and all prime power q. Woldar [29] proved that all simple sporadic groups are $(2, 3)$-generated, except for M_{11}, M_{22}, M_{23} and McL, for which $\varpi(G) = 5$. As proved by Lübeck and Malle [9], all simple exceptional groups are $(2, 3)$-generated with the only exception of the Suzuki groups $\mathrm{Sz}(2^{2m+1})$, for which Suzuki himself [18] proved that $\varpi(\mathrm{Sz}(2^{2m+1})) = 5$.

Hence, we are left to consider the finite simple classical groups. A key result for such groups is due to Liebeck and Shalev, who proved in [8] that, apart from the infinite families $\mathrm{PSp}_4(2^m)$ and $\mathrm{PSp}_4(3^m)$, all finite simple classical groups are $(2, 3)$-generated with a finite number of exceptions. So, the problem of finding the exact value of $\varpi(G)$

reduces to classifying the exceptions to the Liebeck and Shalev theorem. However, their result relies on probabilistic methods and does not provide any estimates on the number or the distribution of such exceptions. We remark that King proved in [6] that $\varpi(G) \neq \infty$ for all finite simple classical groups G, but in general the problem of computing the exact value of $\varpi(G)$ is still wide open (see [14] for a recent survey on this topic).

In this paper we consider the projective special linear groups $\mathrm{PSL}_n(q)$. Many authors, such as Di Martino, Macbeath, Tabakov, Tamburini and Vavilov, already dealt with the problem of the (2, 3)-generation of $\mathrm{SL}_n(q)$. Summarising their results, we have the following list of (2, 3)-generated groups:

(i) $\mathrm{PSL}_2(q)$ if $q \neq 9$ (see [10]);
(ii) $\mathrm{SL}_3(q)$ if $q \neq 4$ (see [15]);
(iii) $\mathrm{SL}_4(q)$ if $q \neq 2$ (see [16]);
(iv) $\mathrm{SL}_n(q)$ if $5 \leq n \leq 11$ (see [4, 5, 15, 19, 20]);
(v) $\mathrm{SL}_n(q)$ if $n \geq 13$ (see [22]);
(vi) $\mathrm{SL}_n(q)$ if $n \geq 5$ and $q \neq 9$ is odd (see [2, 3]).

We observe that the (2, 3)-generation of $\mathrm{SL}_n(q)$ clearly implies the (2, 3)-generation of $\mathrm{PSL}_n(q)$.

Here, using a constructive approach as in many of the above papers and in particular the permutational method illustrated in [22], we solve the last remaining case, that is, we prove the (2, 3)-generation of $\mathrm{SL}_{12}(q)$, obtaining the following classification.

THEOREM 1.1. *The groups* $\mathrm{PSL}_2(q)$ *are* (2, 3)-*generated for any prime power* q, *except when* $q = 9$. *The groups* $\mathrm{SL}_n(q)$ *are* (2, 3)-*generated for any prime power* q *and any integer* $n \geq 3$, *except when* $(n, q) \in \{(3, 4), (4, 2)\}$.

Observe that $\varpi(G) = 5$ if $G \in \{\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6), \mathrm{SL}_3(4), \mathrm{PSL}_3(4), \mathrm{PSL}_4(2) \cong \mathrm{Alt}(8)\}$. Clearly, $\mathrm{SL}_2(q)$ cannot be $(2, r)$-generated when q is odd, as the unique involution is the central one.

Regarding the (2, 3)-generation of the other finite classical groups, we recall that only partial results are available, mainly concerning small or high dimensions; see [12, 13, 15, 16, 23–25].

Finally, we recall that the infinite groups $\mathrm{PSL}_n(\mathbb{Z})$ are (2, 3)-generated if and only if either $n = 2$ or $n \geq 5$, and that the groups $\mathrm{SL}_n(\mathbb{Z})$ are (2, 3)-generated if and only if $n \geq 5$ (see [22, 26–28]).

2. The (2, 3)-generation of $\mathrm{SL}_{12}(q)$

Let $q = p^a$, where p is a prime, and let $\mathbb{F}_q$ be the field of q elements. Let V be a 12-dimensional $\mathbb{F}_q$-space, that we identify with the row vectors of $\mathbb{F}_q^{12}$. Let $C = \{e_1, e_2, \ldots, e_{12}\}$ be the canonical basis of V. For any element $\sigma \in \mathrm{Alt}(C)$, we write $g = \sigma$ to denote the permutation matrix $g \in \mathrm{SL}_{12}(q)$ corresponding to σ with respect to C. This allows us to consider $\mathrm{Alt}(C)$ as a subgroup of $\mathrm{SL}_{12}(q)$.

Now, let

$$y = (e_1, e_2, e_3)(e_4, e_5, e_6)(e_7, e_8, e_9)(e_{10}, e_{11}, e_{12}) \qquad (2.1)$$

and let x be the matrix, written with respect to C, such that:

(a) x swaps e_1 and e_8;
(b) $e_2 x = -e_2$ and $e_5 x = e_5$;
(c) x swaps e_{3i} and e_{3i+1} for all $1 \le i \le 3$;
(d) x acts on $\langle e_{11}, e_{12} \rangle$ as the matrix $\left(\begin{smallmatrix} 1 & 0 \\ t & -1 \end{smallmatrix}\right)$ with $t \in \mathbb{F}_q$.

Clearly, x and y have orders, respectively, 2 and 3, and

$$H = \langle x, y \rangle \qquad (2.2)$$

is a subgroup of $\mathrm{SL}_{12}(q)$.

We first prove the following lemma.

LEMMA 2.1. *If $p \ne 5$, then the group H contains* Alt(C).

PROOF. Let $c = [x, y] = x^{-1} y^{-1} x y$ and define γ according to the following rule:

(a) $\gamma = c^{12}$ if $p = 2$;
(b) $\gamma = c^{12p}$ if $p \equiv 1 \pmod{10}$;
(c) $\gamma = c^{24p}$ if $p \equiv 3 \pmod{10}$;
(d) $\gamma = c^{6p}$ if $p \equiv 7 \pmod{10}$;
(e) $\gamma = c^{18p}$ if $p \equiv 9 \pmod{10}$.

It is easy to see that $e_1 \gamma = -e_3$, $e_3 \gamma = e_5$, $e_5 \gamma = e_4$, $e_4 \gamma = -e_8$ and $e_8 \gamma = e_1$. Furthermore, $e_i \gamma = e_i$ for all $i \in \{2, 6, 7, 9, 10, 11, 12\}$. Also, taking $\delta = \gamma^y$, we define

$$\eta_1 = (\gamma^4 \delta^3 \gamma^2 \delta^2)^2, \quad \eta_2 = (\gamma^4 \delta^3 \gamma^2 \delta^2 \gamma^2 \delta^2)^2, \quad \eta_3 = (\delta \gamma^2 \delta \gamma^2 \delta \gamma^3 \delta^4 \gamma^2)^2.$$

Since

$$\eta_1 = (e_2, e_5)(e_4, e_8), \quad \eta_2 = (e_1, e_6)(e_4, e_9), \quad \eta_3 = (e_1, e_3)(e_2, e_8)(e_4, e_9)(e_5, e_6),$$

we obtain $\langle \eta_1, \eta_2, \eta_3 \rangle = \mathrm{Alt}(\Delta)$, where $\Delta = \{e_1, e_2, e_3, e_4, e_5, e_6, e_8, e_9\} \subset C$. It follows that $\langle \gamma, \delta \rangle$ contains the subgroup $\mathrm{Alt}(\Delta)$ and in particular the element $g = (e_1, e_4, e_9)$. Since $g^x = (e_3, e_{10}, e_8)$, we conclude that H contains the subgroup $\langle \mathrm{Alt}(\Delta), g^x, y \rangle = \mathrm{Alt}(C)$. □

The next key ingredient is the following result, which is a particular case of [22, Lemma 4.1]. As usual, $E_{i,j}$ denotes the elementary matrix having 1 at position (i, j) and 0 elsewhere.

LEMMA 2.2. *Let $t \ne 0, 2$ be such that $\mathbb{F}_q = \mathbb{F}_p(t)$. Then the normal closure N of the involution $w = I_5 - 2E_{5,5} + tE_{5,4}$ under* Alt(5) *is* $\langle \mathrm{SL}_5(q), \mathrm{diag}(-1, 1, 1, 1, 1) \rangle$.

We can now prove the following proposition that, combined with the known results on the $(2, 3)$-generation of $\mathrm{SL}_n(q)$ described in the introduction, immediately gives Theorem 1.1.

PROPOSITION 2.3. *For all primes $p \neq 5$ and all integers $a \geq 1$, the groups $\mathrm{SL}_{12}(p^a)$ are (2, 3)-generated.*

PROOF. Set $q = p^a$. Let $H = \langle x, y \rangle$ be as in (2.2), where the element $t \in \mathbb{F}_q$ in x is chosen in such a way that $t \neq 0, 2$ and $\mathbb{F}_p(t) = \mathbb{F}_q$. As already observed, $H \leq \mathrm{SL}_{12}(q)$. So, we have to prove that $\mathrm{SL}_{12}(q) \leq H$.

First, consider the element $g = (e_1, e_8)(e_9, e_{10}) \in \mathrm{Alt}(C)$. Then $w = gx$ acts on $\langle e_8, \dots, e_{12} \rangle$ as the involution $I_5 - 2E_{5,5} + tE_{5,4}$. By Lemma 2.2, $\mathrm{SL}_5(q)$ is contained in $K = \langle w, \mathrm{Alt}(\{e_8, \dots, e_{12}\}) \rangle$. It follows that $T = \langle K', \mathrm{Alt}(C) \rangle$ is $\mathrm{SL}_{12}(q)$. Since, by Lemma 2.1, $\mathrm{Alt}(C)$ is a subgroup of H, we have $T \leq H$, whence $H = \mathrm{SL}_{12}(q)$. □

For the sake of completeness, using the permutational method we now prove the (2, 3)-generation of $\mathrm{SL}_{12}(5^a)$ for all $a \geq 1$.

Let $\tilde{y} = y$ be as in (2.1) and let $\tilde{x}$ be the matrix, written with respect to C, such that:

(a) $e_1 \tilde{x} = -e_1$, $e_5 \tilde{x} = e_5$ and $e_8 \tilde{x} = e_8$;

(b) $\tilde{x}$ swaps e_{3i} and e_{3i+1} for $i = 2, 3$;

(c) $\tilde{x}$ acts on $\langle e_2, e_3, e_4 \rangle$ as the involution $x_3 = \begin{pmatrix} 3 & 3 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 3 \end{pmatrix}$;

(d) $\tilde{x}$ acts on $\langle e_{11}, e_{12} \rangle$ as the matrix $\begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}$ with $t \in \mathbb{F}_q$.

Again, in this case, $\tilde{x}$ and $\tilde{y}$ have respective orders 2 and 3, and

$$\widetilde{H} = \langle \tilde{x}, \tilde{y} \rangle$$

is a subgroup of $\mathrm{SL}_{12}(q)$.

LEMMA 2.4. *The group $\widetilde{H}$ contains $\mathrm{Alt}(C)$.*

PROOF. Let $\tilde{c} = [\tilde{x}, \tilde{y}]$ and define $\tilde{\gamma} = \tilde{c}^{12}$ and $\tilde{\delta} = \tilde{\gamma}^{y^2}$. We firstly observe that both $\tilde{\gamma}$ and $\tilde{\delta}$ fix the decomposition $V = \langle e_1, \dots, e_8 \rangle \oplus \langle e_9 \rangle \oplus \dots \oplus \langle e_{12} \rangle$. Since $\tilde{\gamma}\tilde{\delta}^2$ and $\tilde{\gamma}\tilde{\delta}\tilde{\gamma}^3\tilde{\delta}^3$ have orders, respectively, 313 and 19 531, we see that $K = \langle \tilde{\gamma}, \tilde{\delta} \rangle$ coincides with the subgroup

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & I_4 \end{array} \right) : A \in \mathrm{SL}_8(5) \right\} \cong \mathrm{SL}_8(5)$$

(use, for instance, [7]). In particular, K contains the elements $g_1 = \mathrm{diag}(1, x_3, I_8)$, $g_2 = (e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ and $g_3 = (e_6, e_7, e_8)$. Now, since $g_3^{\tilde{y}g_1\tilde{x}} = (e_4, e_8, e_{10})$, we see that $\widetilde{H}$ contains the subgroup $\langle g_2, g_3^{\tilde{y}g_1\tilde{x}}, \tilde{y} \rangle = \mathrm{Alt}(C)$. □

COROLLARY 2.5. *For all integers $a \geq 1$, the groups $\mathrm{SL}_{12}(5^a)$ are (2, 3)-generated.*

PROOF. It suffices to repeat the proof of Proposition 2.3 using $\tilde{x}, \tilde{y}, \widetilde{H}$ instead of x, y, H, respectively, and defining $w = g\tilde{x}$, where $g = (e_6, e_7)(e_9, e_{10})$. □

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