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EIGENVALUE HOMOGENISATION PROBLEM WITH INDEFINITE WEIGHTS

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Abstract

In this work we study the homogenisation problem for nonlinear elliptic equations involving p-Laplacian-type operators with sign-changing weights. We study the asymptotic behaviour of variational eigenvalues which consist of a double sequence of eigenvalues. We show that the kth positive eigenvalue goes to infinity when the average of the weights is nonpositive, and converges to the kth variational eigenvalue of the limit problem when the average is positive for any $k \geq 1$.

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1. Introduction

We consider the following nonlinear eigenvalue problem with indefinite weight,

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x,\nabla u)) = \lambda\rho_\varepsilon(x)|u|^{p-2}u & \text{in } \Omega\subset\mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is the eigenvalue, ρ_ε is a bounded weight function with nontrivial positive and negative parts and the operator $\operatorname{div}(a_\varepsilon(x,\nabla u))$ is quasilinear $(p-1)$-homogeneous in the second variable with some precise hypotheses that are stated below (see assumptions (H0)–(H8) in Section 2). Throughout the paper, p will denote a constant that satisfies $1 < p < \infty$. The most relevant example of such an operator is

$$a_\varepsilon(x,\xi) = |A_\varepsilon(x)\xi\cdot\xi|^{(p-2)/2}A_\varepsilon(x)\xi,$$

with $A_\varepsilon(x) \in \mathbb{R}^{N\times N}$ a bounded symmetric matrix and positive definite uniformly in $\varepsilon > 0$. The domain Ω is assumed to be bounded but no regularity hypotheses are imposed on $\partial\Omega$.

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For this eigenvalue problem (1.1), as discussed in [8], it is known that there exist two sequences of eigenvalues $\{\lambda^+_{k,\varepsilon}\}_{k\geq 1}$, $\{\lambda^-_{k,\varepsilon}\}_{k\geq 1}$, such that

$$\lambda^+_{k,\varepsilon} \to \infty, \quad \lambda^-_{k,\varepsilon} \to -\infty \quad \text{as } k \to \infty.$$

The asymptotic behaviour as $\varepsilon \downarrow 0$ of these nonlinear eigenvalues in the unweighted case (that is, $\rho_\varepsilon \equiv 1$) was studied in [1, 3, 4]. In particular, for the problem

$$-\mathrm{div}(a_\varepsilon(x, \nabla u)) = \lambda |u|^{p-2} u \quad \text{in } \Omega,$$

it was proved in [3] that the kth variational eigenvalue converges to the kth variational eigenvalue of the limit problem

$$-\mathrm{div}(a(x, \nabla u)) = \lambda |u|^{p-2} u \quad \text{in } \Omega,$$

where $a(x, \xi)$ is the so-called G-limit of the operators $a_\varepsilon(x, \xi)$. Of course, the convergence is understood up to a subsequence. See Section 2 for the definition and some elementary properties of G-convergence. The G-convergence of monotone operators has a long history and there are many results in the literature establishing the usefulness of this concept in the limit behaviour of boundary value problems, especially in *homogenisation theory* (see, for example, [2, 4, 9] and references therein).

The purpose in this paper is to extend the results of [3] to the indefinite-weighted case. The main result of this work is the following theorem.

THEOREM 1.1. *Assume that $a_\varepsilon(x, \xi)$ satisfies (H0)–(H8) as defined in Section 2. Moreover, assume that $a_\varepsilon(x, \xi)$ G-converges to $a(x, \xi)$. Let $\rho_\varepsilon \in L^\infty(\Omega)$ be such that $\rho_\varepsilon \rightharpoonup \rho$ weak* in $L^\infty(\Omega)$. Then:*

(1) *if $\rho^+ = 0$, $\lambda^+_{k,\varepsilon} \to \infty$ as $\varepsilon \downarrow 0$;*
(2) *if $\rho^+ \neq 0$, $\lambda^+_{k,\varepsilon} \to \lambda^+_k$ as $\varepsilon \downarrow 0$, where $\{\lambda^+_k\}_{k\geq 1}$ are the positive eigenvalues associated to the operator $a(x, \xi)$ with weight ρ.*

Our approach follows closely the one in [3]. The main difference is the fact that we cannot work with a uniform normalisation condition as in the unweighted case. The normalisation condition varies with ε and that has to be accommodated.

REMARK 1.2. An analogous statement holds for the negative eigenvalues with the obvious modifications.

For second-order linear elliptic operators, the eigenvalue convergence for the problem of periodic homogenisation with sign-changing weights was studied recently in [10]. Our results here are closely related, although several differences arise. In our setting we are not able to use asymptotic expansions, nor orthogonality of eigenfunctions, so our proofs are different, based mainly on the variational arguments developed in [3]. The main drawback of our approach is that we were unable to obtain one of the results of [10] dealing with the convergence of the rescaled sequences of eigenvalues and the corresponding limit problem. On the other hand, our hypotheses on a_ε go beyond periodic homogenisation and we have relaxed the regularity hypotheses on Ω, since in [10] the domains are of class $C^{2,\alpha}$. Also, as in [5], different boundary conditions can be handled in this way.

A natural question is the study of a quantitative version of Theorem 1.1, that is, to give some precise rate of convergence or divergence of the eigenvalues. Recently, in [7], we obtained the rate of convergence of eigenvalues of problem (1.1) in the case where the operator $a_\varepsilon(x,\xi)$ is independent of ε and the weight function is positive and given in terms of a periodic function ρ, as $\rho_\varepsilon(x) = \rho(x/\varepsilon)$. See also the bibliography in [7] for references about the linear problem and [5] for the analysis of different boundary conditions. Moreover, [13] analyses the Fučik eigenvalue problem for the p-Laplacian and [12] studies the case of the fractional Laplacian. We remark that in the one-dimensional case very simple proofs of most of these results can be obtained by means of a Lyapunov-type inequality (see [6]).

In order to perform such analyses, we need to make some further assumptions on the weights ρ_ε and on the operators $a_\varepsilon(x,\xi)$. First, we assume that the weights ρ_ε are given in terms of a periodic function ρ in the form $\rho_\varepsilon(x) = \rho(x/\varepsilon)$. In this case,

$$\rho_\varepsilon \rightharpoonup \bar\rho := \int_Y \rho(y)\,dy,$$

where $Y = [0,1]^N$ and the function ρ is assumed to be Y-periodic. Under these assumptions, we obtain a precise rate of divergence for the eigenvalues.

THEOREM 1.3. *Assume that $a_\varepsilon(x,\xi)$ satisfies (H0)–(H8) defined is Section 2. Let $\rho_\varepsilon(x)$ be given as $\rho_\varepsilon(x) = \rho(x/\varepsilon)$, where ρ is Y-periodic, $Y = [0,1]^N$ and $\rho \in W^{1,n}(\Omega)$. Then:*

(1) *if $\bar\rho = 0$, then $\lambda^\pm_{k,\varepsilon} = O(\varepsilon^{-1})$ as $\varepsilon \downarrow 0$;*
(2) *if $\bar\rho > 0$, then $\varepsilon\lambda^-_{1,\varepsilon}$ is bounded away from zero and $\varepsilon^p\lambda^-_{k,\varepsilon}$ is bounded away from infinity as $\varepsilon \downarrow 0$;*
(3) *if $\bar\rho < 0$, then $\varepsilon\lambda^+_{1,\varepsilon}$ is bounded away from zero and $\varepsilon^p\lambda^+_{k,\varepsilon}$ is bounded away from infinity as $\varepsilon \downarrow 0$.*

REMARK 1.4. In [10], where only linear eigenvalue problems and periodic homogenisation were considered, it was proved that $c^-_k\varepsilon^{-2} \le \lambda^-_{k,\varepsilon} \le C^-_k\varepsilon^{-2}$ when $\bar\rho > 0$, by using a factorisation technique to construct the asymptotic eigenfunctions. We cannot use such arguments here, due to the nonlinear character of the problem, and we get only the upper bound with a worse lower bound.

Finally, in the case where the operators $a_\varepsilon(x,\xi)$ are independent of ε, we can obtain a rate of convergence of the eigenvalues.

THEOREM 1.5. *Let $a_\varepsilon(x,\nabla u) = a(x,\nabla u)$ be fixed, satisfying hypotheses (H0)–(H8) defined in Section 2. Let $\rho_\varepsilon(x) := \rho(x/\varepsilon)$, where $\rho \in L^\infty(\mathbb{R}^N)$ is a Y-periodic function, $Y = [0,1]^N$. Let $\bar\rho = \int_Y \rho\,dy$. If $\bar\rho > 0$,*

$$|\lambda^+_{k,\varepsilon} - \lambda_k| \le C\varepsilon,$$

where C is given explicitly and depends only on k, p, N and $\|\rho\|_\infty$. An analogous result holds when $\bar\rho < 0$.

After this introduction, the paper is organised as follows. In Section 2 we recall some preliminary results needed in the rest of the paper. In Section 3 we prove the main result of the paper, namely Theorem 1.1. In Section 4 we prove our results on the divergence of eigenvalues (Theorem 1.3). Finally, in Section 5, we prove the rate of convergence of the eigenvalues (Theorem 1.5).

2. Preliminary results

2.1. *G*-convergence of monotone operators. The operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is given by

$$\mathcal{A}u := -\mathrm{div}(a(x, \nabla u)).$$

We assume that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies, for every $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$, the following conditions.

(H0) *Measurability:* $a(\cdot, \cdot)$ is a Carathéodory function, that is, $a(x, \cdot)$ is continuous for almost every $x \in \Omega$, and $a(\cdot, \xi)$ is measurable for every $\xi \in \mathbb{R}^N$.

(H1) *Monotonicity:* $0 \le (a(x, \xi_1) - a(x, \xi_2))(\xi_1 - \xi_2)$.

(H2) *Coercivity:* $\alpha|\xi|^p \le a(x, \xi) \cdot \xi$.

(H3) *Continuity:* $|a(x, \xi)| \le \beta|\xi|^{p-1}$.

(H4) $(p - 1)$*-homogeneity:* $a(x, t\xi) = t^{p-1}a(x, \xi)$ for every $t > 0$.

(H5) *Oddness:* $a(x, -\xi) = -a(x, \xi)$.

Let us introduce $\Psi(x, \xi_1, \xi_2) = a(x, \xi_1) \cdot \xi_1 + a(x, \xi_2) \cdot \xi_2$ for all $\xi_1, \xi_2 \in \mathbb{R}^N$ and all $x \in \Omega$, and let $\delta = \min\{p/2, p - 1\}$.

(H6) *Equi-continuity:*

$$|a(x, \xi_1) - a(x, \xi_2)| \le c\Psi(x, \xi_1, \xi_2)^{(p-1-\delta)/p}((a(x, \xi_1) - a(x, \xi_2)) \cdot (\xi_1 - \xi_2))^{\delta/p}.$$

(H7) *Cyclical monotonicity:* $\sum_{i=1}^{k} a(x, \xi_i) \cdot (\xi_{i+1} - \xi_i) \le 0$ for all $k \ge 1$ and $\xi_1, \ldots, \xi_{k+1}$, with $\xi_1 = \xi_{k+1}$.

(H8) *Strict monotonicity:* for $\gamma = \max(2, p)$,

$$\alpha|\xi_1 - \xi_2|^\gamma \Psi(x, \xi_1, \xi_2)^{1-(\gamma/p)} \le (a(x, \xi_1) - a(x, \xi_2)) \cdot (\xi_1 - \xi_2).$$

Under these conditions, $\mathcal{A}$ is a monotone operator. We have the following results.

Proposition 2.1 [1, Lemma 3.3]. *Given $a(x, \xi)$ satisfying (H0)–(H8), there exists a unique Carathéodory function Φ which is even, p-homogeneous, strictly convex and differentiable in the variable ξ, satisfying*

$$\alpha|\xi|^p \le \Phi(x, \xi) \le \beta|\xi|^p \tag{2.1}$$

for all $\xi \in \mathbb{R}^N$, almost every $x \in \Omega$, such that

$$\nabla_\xi \Phi(x, \xi) = p\, a(x, \xi),$$

and normalised such that $\Phi(x, 0) = 0$.

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Let us recall the definitions of G- and Mosco-convergence.

DEFINITION 2.2. The family of operators $\mathcal{A}_\varepsilon u := -\mathrm{div}(a_\varepsilon(x, \nabla u))$ G-converges to $\mathcal{A}u := -\mathrm{div}(a(x, \nabla u))$ if, for every $f \in W^{-1,p'}(\Omega)$ and for every f_ε strongly convergent to f in $W^{-1,p'}(\Omega)$, the solutions u^ε of the problem

$$\begin{cases} -\mathrm{div}(a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)) = f_\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases}$$

satisfy the following conditions:

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega),$$
$$a_\varepsilon(x, \nabla u^\varepsilon) \rightharpoonup a(x, \nabla u) \quad \text{weakly in } (L^p(\Omega))^n,$$

where u is the solution to the equation

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) = f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

DEFINITION 2.3. Let X be a reflexive Banach space and $F_j : X \to [0, +\infty]$ be a sequence of functionals on X. Then F_j *Mosco-converge* to F if and only if the following conditions hold.

(1) *Lower bound inequality*: for every sequence $\{u_j\}_{j\geq 1}$ such that $u_j \rightharpoonup u$ weakly in X as $j \to \infty$,

$$F(u) \leq \liminf_{j\to\infty} F_j(u_j).$$

(2) *Upper bound inequality*: for every $u \in X$, there exists a sequence $\{u_j\}_{j\geq 1}$ such that $u_j \to u$ strongly in X as $j \to \infty$ such that

$$F(u) \geq \limsup_{j\to\infty} F_j(u_j).$$

In the general case, the following results were proved in [1, 4].

THEOREM 2.4 [4, Theorem 4.1]. *Assume that $a_\varepsilon(x, \xi)$ satisfies (H1)–(H3). Then, up to a subsequence, $\mathcal{A}_\varepsilon$ G-converges to a maximal monotone operator $\mathcal{A}$ whose coefficient $a(x, \xi)$ also satisfies (H1)–(H3).*

THEOREM 2.5 [1, Theorem 2.3]. *If $\mathcal{A}_\varepsilon u := -\mathrm{div}(a_\varepsilon(x, \nabla u))$ G-converges to $\mathcal{A}u := -\mathrm{div}(a(x, \nabla u))$ and $a_\varepsilon(x, \xi)$ satisfies (H0)–(H8), then $a(x, \xi)$ also satisfies (H0)–(H8).*

LEMMA 2.6 [1, Lemma 4.2]. *Given $a_\varepsilon(x, \xi)$, $a(x, \xi)$ satisfying (H0)–(H8), and $\Phi_\varepsilon(x, \xi)$, $\Phi(x, \xi)$ as in Proposition 2.1, define $F_\varepsilon, F : L^p(\Omega) \to (-\infty, +\infty]$ by*

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Phi_\varepsilon(x, \nabla u)\, dx, & u \in W_0^{1,p}(\Omega), \\ +\infty, & \textit{otherwise}, \end{cases}$$

$$F(u) = \begin{cases} \displaystyle\int_\Omega \Phi(x, \nabla u)\, dx, & u \in W_0^{1,p}(\Omega), \\ +\infty, & \textit{otherwise}. \end{cases}$$

If $\mathcal{A}_\varepsilon$ G-converges to $\mathcal{A}$, then F_ε Mosco-converges to F in the strong topology of $L^p(\Omega)$.

2.2. Oscillatory integrals. The proof of the main theorem makes use of results on convergence of oscillatory integrals. In the case of periodic oscillations, the result needed here was proved in [7].

Theorem 2.7 [7, Lemma 3.3]. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and denote by Q the unit cube in $\mathbb{R}^N$. Let $g \in L^\infty(\mathbb{R}^N)$ be a Q-periodic function such that $\bar{g} = 0$. Then the inequality*

$$\left| \int_\Omega g(x/\varepsilon) u \, dx \right| \leq \|g\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla u\|_{L^1(\Omega)}$$

holds for every $u \in W_0^{1,1}(\Omega)$, where c_1 is the optimal constant in Poincaré's inequality in $L^1(Q)$.

For $v \in W_0^{1,p}(\Omega)$, applying the previous result to $u = |v|^p$ gives the next corollary.

Corollary 2.8 [7, Theorem 3.4]. *Suppose that $v \in W_0^{1,p}(\Omega)$. Under the assumptions of Theorem 2.7,*

$$\left| \int_\Omega g(x/\varepsilon) |v|^p \, dx \right| \leq \|g\|_{L^\infty(\mathbb{R}^N)} p c_1 \varepsilon \|v\|_{L^p(\Omega)}^{p-1} \|\nabla v\|_{L^p(\Omega)}.$$

In the general case, when no periodicity is assumed, one cannot have a rate of convergence. Nevertheless, the following result holds.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\rho_\varepsilon, \rho \in L^\infty(\Omega)$ be such that $\rho_\varepsilon \rightharpoonup \rho$ *-weakly in $L^\infty(\Omega)$. Let $K \subset L^1(\Omega)$ be a compact set. Then*

$$\lim_{\varepsilon \to 0} \sup_{v \in K} \int_\Omega (\rho_\varepsilon - \rho) v \, dx = 0.$$

Proof. Given $r > 0$, there exists $\{v_i\}_{i=1}^J \subset K$ such that $K \subset \bigcup_{i=1}^J B_r(v_i)$. By the hypotheses,

$$\lim_{\varepsilon \to 0} \max_{1 \leq i \leq J} \int_\Omega (\rho_\varepsilon - \rho) v_i \, dx = 0.$$

Let now $v_\varepsilon \in K$ be such that

$$\sup_{v \in K} \int_\Omega (\rho_\varepsilon - \rho) v \, dx \leq \int_\Omega (\rho_\varepsilon - \rho) v_\varepsilon \, dx + \varepsilon.$$

Then there exists $i_\varepsilon \in \{1, \ldots, J\}$ such that $v_\varepsilon \in B_r(v_{i_\varepsilon})$. Now

$$\begin{aligned} \int_\Omega (\rho_\varepsilon - \rho) v_\varepsilon \, dx &= \int_\Omega (\rho_\varepsilon - \rho) v_{i_\varepsilon} \, dx + \int_\Omega (\rho_\varepsilon - \rho)(v_\varepsilon - v_{i_\varepsilon}) \, dx \\ &\leq \max_{1 \leq i \leq J} \int_\Omega (\rho_\varepsilon - \rho) v_i \, dx + Mr, \end{aligned}$$

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where M is a bound on $\|\rho_\varepsilon\|_\infty + \|\rho\|_\infty$. Therefore,

$$\limsup_{\varepsilon\to 0}\sup_{v\in K}\int_\Omega(\rho_\varepsilon-\rho)v\,dx \le Mr.$$

Since $r > 0$ is arbitrary, the result follows. □

In our application of Theorem 2.9, the compact set will be a bounded set in $W_0^{1,1}(\Omega)$. So, we have the corollary.

Corollary 2.10. *Under the same hypotheses as in the previous theorem,*

$$\lim_{\varepsilon\to 0}\sup_{\substack{v\in W_0^{1,1}(\Omega),\\ \|\nabla v\|_1\le 1}}\int_\Omega(\rho_\varepsilon-\rho)v\,dx = 0.$$

Finally, for bounded sets in $W_0^{1,p}(\Omega)$ the following analogue of Corollary 2.8 holds.

Corollary 2.11. *Under the same assumptions as in Theorem 2.9,*

$$\left|\int_\Omega(\rho_\varepsilon-\rho)|v|^p\,dx\right| \le o(1)\|v\|_p^{p-1}\|\nabla v\|_p.$$

2.3. Eigenvalues of quasilinear operators. We refer the interested reader to the survey [8] for details; only the facts that will be used below are stated here.

In this subsection, we state some results for the eigenvalue problem (1.1) for fixed $\varepsilon > 0$. That is, we analyse the problem

$$\begin{cases}-\operatorname{div}(a(x,\nabla u)) = \lambda\rho(x)|u|^{p-2}u & \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{cases} \tag{2.2}$$

where Ω is a bounded open set in $\mathbb{R}^N$. We assume that $a(x,\xi)$ satisfies (H0)–(H8) of the previous subsection. As a consequence, there exists a potential function $\Phi(x,\xi)$ given by Proposition 2.1.

By using the Ljusternik–Schnirelmann theory, if $\rho^+ \ne 0$, one can construct a sequence of (variational) eigenvalues of (2.2) as

$$\lambda_k^+ = \inf_{C\in C_k}\sup_{v\in C}\frac{\int_\Omega\Phi(x,\nabla v)\,dx}{\int_\Omega\rho|v|^p\,dx}, \tag{2.3}$$

where

$$C_k = \{C\subset M^+ : C \text{ is compact and symmetric}, \gamma(C)\ge k\},$$
$$M^+ = \left\{u\in W_0^{1,p}(\Omega) : \int_\Omega\rho|u|^p\,dx > 0\right\}$$

and $\gamma : \Sigma \to \mathbb{N}\cup\{\infty\}$ is the Krasnoselskii genus (see [11]) defined by

$$\gamma(A) = \min\{k\in\mathbb{N} : \text{there exists } f\in C(A,\mathbb{R}^k\backslash\{0\}),\ f(x) = -f(-x)\}.$$

When $\rho^- \neq 0$, one can construct a sequence of negative eigenvalues in a completely analogous way changing M^+ by M^- given by

$$M^- = \Big\{u \in W_0^{1,p}(\Omega) : \int_\Omega \rho|u|^p\,dx < 0\Big\}.$$

It is customary to reformulate (2.3) as

$$\frac{1}{\lambda_k^+} = \sup_{C\in\mathcal{C}_k} \inf_{v\in C} \frac{\int_\Omega \rho|v|^p\,dx}{\int_\Omega \Phi(x,\nabla v)\,dx}.$$

From the homogeneity condition (H4), we will also use the following equivalent characterisation for the eigenvalues:

$$\frac{1}{\lambda_k^+} = \sup_{C\in\mathcal{D}_k} \inf_{u\in C} \int_\Omega \rho|u|^p\,dx,$$

where

$$\mathcal{D}_k = \{C \subset S : \text{compact and symmetric with } \gamma(C) \geq k \text{ and } 0 \notin C\},$$
$$S = M^+ \cap \Big\{u \in W_0^{1,p}(\Omega) : \int_\Omega \Phi(x,\nabla u)\,dx = 1\Big\}.$$

The following useful Sturm-type theorem will be needed later.

Theorem 2.12. *Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ and let $\{\lambda_{k,i}^+\}_{k\geq 1}$ be the eigenvalues given by (2.3) in Ω_i, $i = 1,2$, respectively. Then*

$$\lambda_{k,2}^+ \leq \lambda_{k,1}^+$$

for any $k \geq 1$. Moreover, let $\rho_1(x) \leq \rho_2(x)$ for almost every $x \in \Omega$ and $\Phi_1(x,\xi) \geq \Phi_2(x,\xi)$ for almost every $x \in \Omega$ for every $\xi \in \mathbb{R}^N$. Then, if $\{\lambda_{k,i}^+\}_{k\geq 1}$ are the eigenvalues given by (2.3) with weight ρ_i and potential Φ_i, $i = 1,2$, respectively,

$$\lambda_{k,2}^+ \leq \lambda_{k,1}^+.$$

The proof follows easily by comparing the Rayleigh quotient and using the inclusion of Sobolev spaces $W_0^{1,p}(\Omega_1) \subset W_0^{1,p}(\Omega_2)$.

3. Proof of the main result

Proof of Theorem 1.1. Assume first that $\rho^+ = 0$ and $\rho_\varepsilon^+ \neq 0$ for every $\varepsilon > 0$. If $u \in W_0^{1,p}(\Omega)$, then

$$\int_\Omega \Phi_\varepsilon(x,\nabla u)\,dx \geq \alpha\|\nabla u\|_p^p.$$

On the other hand, by Corollary 2.11,

$$\int_\Omega \rho_\varepsilon|u|^p\,dx = \int_\Omega \rho|u|^p + o(1)\|u\|_p^{p-1}\|\nabla u\|_p \leq o(1)\|u\|_p^{p-1}\|\nabla u\|_p.$$

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Hence, we get the bound

$$\frac{\int_\Omega \Phi_\varepsilon(x,\nabla u)\,dx}{\int_\Omega \rho_\varepsilon |u|^p\,dx} \geq \frac{\alpha}{o(1)}\Big(\frac{\|\nabla u\|_p^p}{\|u\|_p^p}\Big)^{(p-1)/p}.$$

Now, taking the infimum in the former inequality,

$$\lambda_{1,\varepsilon}^+ \geq \frac{\alpha}{o(1)}\mu_1^{(p-1)/p},$$

where μ_1 is the first eigenvalue of the p-Laplacian with Dirichlet boundary conditions. From this, the first part of the theorem follows.

Now assume that $\rho^+ \neq 0$. Then $\rho_\varepsilon^+ \neq 0$ for every $\varepsilon > 0$ sufficiently small. Fix $\delta > 0$ and let $C \in C_k$ be such that

$$\sup_{v\in C} \int_\Omega \Phi(x,\nabla v)\,dx \leq \lambda_k^+ + \delta,$$

with

$$\int_\Omega \rho |v|^p\,dx = 1 \quad \text{for any } v \in C.$$

The last condition can be imposed without loss of generality by the homogeneity of Φ.

Since C is a compact set, we can choose $r > 0$ and $\{u_i\}_{i=1}^J \subset C$, $J = J(r)$, such that

$$C \subset \bigcup_{i=i}^J B(u_i, r),$$

$$\Big|\int_\Omega |u-u_i|^p\,dx\Big| < \frac{\delta}{\|\rho\|_\infty} \quad \text{if } u \in B(u_i, r).$$

Since $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho$ in $L^\infty(\Omega)$, there exists some ε_0 such that $C \subset M_\varepsilon^+$ for $0 < \varepsilon < \varepsilon_0$. By Lemma 2.6, the functionals $\{F_\varepsilon\}_{\varepsilon>0}$ Mosco-converge to F and this implies that, for any $1 \leq i \leq J$, there exists a sequence $u_{\varepsilon_j,i} \rightharpoonup u_i$ such that

$$\int_\Omega \Phi(x,\nabla u_i)\,dx = \lim_{j\to\infty} \int_\Omega \Phi_{\varepsilon_j}(x,\nabla u_{\varepsilon_j,i})\,dx.$$

Moreover, we can assume that $u_{\varepsilon_j,i} \to u_i$ in $L^p(\Omega)$ and thus

$$1-\delta \leq \int_\Omega \rho_{\varepsilon_j}|u_{\varepsilon_j,i}|^p\,dx \leq 1+\delta.$$

Following [3], let C_{ε_j} be the convex closure of $\{\pm u_{\varepsilon_j,i}\}_{i=i}^J$, a compact convex set (since it has dimension lower than or equal to J). Observe that, since the functions $u_{\varepsilon_j,i}$ are weakly convergent and hence bounded in $W_0^{1,p}(\Omega)$, the sets C_{ε_j} are bounded in $W_0^{1,p}(\Omega)$ uniformly in ε_j.

Define the projection $P_{\varepsilon_j} : C \to C_{\varepsilon_j}$ and observe that, for any $v \in C$, since $v \in B(u_i, r)$ for some i,

$$\|P_{\varepsilon_j}(v) - v\|_p \leq \|u_{\varepsilon_j,i} - v\|_p \leq \|u_{\varepsilon_j,i} - u_i\|_p + \|u_i - v\|_p \leq \delta + r.$$

On the other hand, since $\int_\Omega \rho |v|^p\,dx = 1$,

$$\int_\Omega |v|^p\,dx \geq \frac{1}{\|\rho\|_\infty}\int_\Omega \rho|v|^p\,dx = \frac{1}{\|\rho\|_\infty}.$$

Therefore,

$$\|P_{\varepsilon_j}(v)\|_p \geq \|\rho\|_\infty^{-1/p} - (\delta + r) \geq \theta > 0$$

and $G_{\varepsilon_j} := P_{\varepsilon_j}(C) \subset C_{\varepsilon_j} \backslash B_\theta(0)$ has genus greater than or equal to k for ε small enough. Again, the sets G_{ε_j} are uniformly bounded in $W_0^{1,p}(\Omega)$. Now

$$\begin{aligned}
\lambda^+_{k,\varepsilon_j} &\leq \sup_{v\in G_{\varepsilon_j}} \frac{\int_\Omega \Phi_{\varepsilon_j}(x,\nabla v)\,dx}{\int_\Omega \rho_{\varepsilon_j}|v|^p\,dx}\\
&\leq \sup_{v\in G_{\varepsilon_j}}\Big(\int_\Omega \Phi_{\varepsilon_j}(x,\nabla v)\,dx\Big)\sup_{v\in G_{\varepsilon_j}}\Big(\int_\Omega [\rho_{\varepsilon_j}-\rho]|v|^p + \rho|v|^p\,dx\Big)^{-1}\\
&\leq (1 + O(r) + O(\delta) + O(\varepsilon_j)) \max_{1\leq i\leq J}\int_\Omega \Phi_{\varepsilon_j}(x,\nabla u_{\varepsilon_j,i})\,dx\\
&\leq (1+o(1))\max_{1\leq i\leq J}\int_\Omega \Phi(x,\nabla u_i)\,dx\\
&\leq (1+o(1))(\lambda_k+\delta).
\end{aligned}$$

Therefore, we have obtained the inequality

$$\lambda^+_{k,\varepsilon_j} \leq (1+o(1))(\lambda^+_k + O(\delta)).$$

Observe that, in particular, the sequence $\{\lambda^+_{k,\varepsilon_j}\}_{j\in\mathbb{N}}$ is bounded for each $k\in\mathbb{N}$ and

$$\limsup_{j\to\infty} \lambda^+_{k,\varepsilon_j} \leq \lambda^+_k.$$

In order to prove the reverse inequality, $\lambda^+_k \leq \liminf_{j\to\infty}\lambda^+_{k,\varepsilon_j}$, we start with a family of compact symmetric sets $C_{\varepsilon_j} \subset \{u \in W_0^{1,p}(\Omega) : \|\nabla u\|_p = 1\}$, $\gamma(C_{\varepsilon_j}) \geq k$ such that

$$\sup_{v\in C_{\varepsilon_j}} \frac{\int_\Omega \Phi_{\varepsilon_j}(x,\nabla v)\,dx}{\int_\Omega \rho_{\varepsilon_j}|v|^p\,dx} \leq \lambda^+_{k,\varepsilon_j} + \varepsilon_j.$$

We can extract a sequence that we still denote by $\{C_{\varepsilon_j}\}_{j\in\mathbb{N}}$ and a compact symmetric set C such that $C_{\varepsilon_j} \to C$ in the Hausdorff distance induced by the $L^p(\Omega)$-norm. We have $C \in W_0^{1,p}(\Omega)$ and $\gamma(C) \geq k$ (see Step 2 in [3, Theorem 3.3]).

By the sequential characterisation of Hausdorff convergence, for any $u \in C$, there exists $u_{\varepsilon_j} \in C_{\varepsilon_j}$ such that $u_{\varepsilon_j} \rightharpoonup u$-weakly in $W_0^{1,p}(\Omega)$. From the Mosco-convergence of the functionals, $\Phi_\varepsilon \to \Phi$, and the weak convergence $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho$ in $L^\infty(\Omega)$,

$$\frac{\int_\Omega \Phi(x,\nabla u)\,dx}{\int_\Omega \rho|u|^p\,dx} \le \liminf_{j\to\infty} \frac{\int_\Omega \Phi_{\varepsilon_j}(x,\nabla u_{\varepsilon_j})\,dx}{\int_\Omega \rho_{\varepsilon_j}|u_{\varepsilon_j}|^p\,dx}$$
$$\le \liminf_{j\to\infty} \sup_{v\in C_{\varepsilon_j}} \frac{\int_\Omega \Phi_{\varepsilon_j}(x,\nabla v)\,dx}{\int_\Omega \rho_{\varepsilon_j}|v|^p\,dx}$$
$$\le \liminf_{j\to\infty} \lambda^+_{k,\varepsilon_j} + \varepsilon_j.$$

Taking the supremum in u,

$$\lambda_k^+ \le \sup_{v\in C} \frac{\int_\Omega \Phi(x,\nabla v)\,dx}{\int_\Omega \rho|v|^p\,dx} \le \liminf_{j\to\infty} \lambda^+_{k,\varepsilon_j}.$$

The proof is finished. □

4. Proofs of the divergence results

Proof of Theorem 1.3. We divide the proof into several parts.

Step 1. The case $\bar\rho = 0$: first eigenvalue, lower bound.

It is enough to consider only the first positive eigenvalue $\lambda^+_{1,\varepsilon}$; the result for $\lambda^-_{1,\varepsilon}$ follows by considering the weight $-\rho$. We can bound $\lambda^+_{1,\varepsilon}$ as follows:

$$\frac{1}{\lambda^+_{1,\varepsilon}} = \sup_{v\in W_0^{1,p}} \left(\frac{\int_\Omega \rho(x/\varepsilon)|v|^p\,dx}{\int_\Omega \Phi(x,\nabla v)\,dx}\right)$$
$$\le \frac{\varepsilon c_1 p}{\alpha}\|\rho\|_{L^\infty(\mathbb{R}^N)} \sup_{v\in W_0^{1,p}} \left(\frac{\|u\|^{p-1}_{L^p(\Omega)}\|\nabla u\|_{L^p(\Omega)}}{\int_\Omega |\nabla v|^p\,dx}\right)$$
$$\le \varepsilon C(p,\rho,c_1,\alpha) \sup_{v\in W_0^{1,p}} \left(\frac{\int_\Omega |v|^p\,dx}{\int_\Omega |\nabla v|^p\,dx}\right)^{p-1}$$
$$\le \varepsilon C(p,\rho,c_1,\alpha,|\Omega|),$$

where we used Theorem 2.8 in the first inequality. The constant $C(p,\rho,c_1,\alpha,|\Omega|)$ is obtained from Theorem 2.8 and the isoperimetric inequality, that is, the first eigenvalue in Ω is greater than the first eigenvalue of a ball $B_{|\Omega|}$ with the same measure as Ω,

$$\sup_{v\in W_0^{1,p}} \frac{\int_\Omega |v|^p\,dx}{\int_\Omega |\nabla v|^p\,dx} \le \sup_{v\in W_0^{1,p}} \frac{\int_{B_{|\Omega|}} |v|^p\,dx}{\int_{B_{|\Omega|}} |\nabla v|^p\,dx}.$$

Therefore, the first positive eigenvalue goes to $+\infty$ at least as ε^{-1}.

Step 2. The case $\bar\rho = 0$: first eigenvalue, upper bound.

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The upper bound follows by taking as test function $v_\varepsilon = u(1+\varepsilon\rho(x/\varepsilon))^{1/p}$, with a fixed positive function $u \in C_c^\infty(\Omega)$. Observe that $v_\varepsilon \in W_0^{1,p}(\Omega)$ if $\rho \in W^{1,N}(\Omega)$. Then

$$\begin{aligned}\frac{1}{\lambda^+_{1,\varepsilon}} &= \sup_{v\in W_0^{1,p}} \left(\frac{\int_\Omega \rho(x/\varepsilon)|v_\varepsilon|^p\,dx}{\int_\Omega \Phi(x,\nabla v)\,dx}\right)\\ &\geq \frac{\int_\Omega u^p\rho(x/\varepsilon)+\varepsilon\rho(x/\varepsilon)^2u^p\,dx}{\beta\int_\Omega|\nabla(u(1+\varepsilon\rho(x/\varepsilon))^{1/p})|^p\,dx}\\ &\geq C(u,\|\rho\|_\infty,p,\beta)\int_\Omega u^p\rho(x/\varepsilon)\,dx,\end{aligned}$$

where the constant C is strictly positive for ε small enough. Hence, by using Corollary 2.8, we have proved that $\lambda^+_{1,\varepsilon} = O(\varepsilon^{-1})$.

Step 3. The case $\bar\rho = 0$: higher eigenvalues, lower bound.

This is immediate from Step 1, since $\lambda^+_{k,\varepsilon} \geq \lambda^+_{1,\varepsilon} \geq C\varepsilon^{-1}$.

Step 4. The case $\bar\rho = 0$: higher eigenvalues, upper bound.

Fix $k \in \mathbb{N}$ and $\varepsilon_0 > 0$ sufficiently small so that there exists $\{Q_i\}_{i=1}^k$, where $Q_i \subset \Omega$ is a cube of side length ε_0 and $Q_i \cap Q_j = \emptyset$ (we consider open cubes). By the scaling properties of the eigenvalues, $\mu_{1,\varepsilon}(Q_i) = \varepsilon_0^{-p}\mu_{1,\varepsilon/\varepsilon_0}(Q_0)$, where $\mu_{1,\varepsilon}(U)$ is the first eigenvalue of the p-Laplacian in $U \subset \mathbb{R}^N$ with Dirichlet boundary conditions and weight ρ_ε and Q_0 is the unit cube in $\mathbb{R}^N$.

Let u_i^ε be the first eigenfunction corresponding to $\mu_{1,\varepsilon}(Q_i)$ and extended by 0 to Ω, and define the set

$$C_k^\varepsilon = \operatorname{span}\{u_i^\varepsilon : 1 \leq i \leq k\} \cap B,$$

where B is the unit ball in $W_0^{1,p}(\Omega)$. Since the functions have disjoint support, C_k^ε is a k-dimensional set and so $\gamma(C_k^\varepsilon) = k$. For an arbitrary element $v = \sum b_i u_i^\varepsilon \in C_k^\varepsilon$,

$$\begin{aligned}\frac{\int_\Omega a_\varepsilon(x,\nabla v)\cdot\nabla v\,dx}{\int_\Omega \rho(x/\varepsilon)|v|^p\,dx} &= \frac{\sum_{i=1}^k\int_{Q_i} a_\varepsilon(x,b_i\nabla u_i^\varepsilon)\cdot b_i\nabla u_i^\varepsilon\,dx}{\sum_{i=1}^k\int_{Q_i}\rho(x/\varepsilon)|b_iu_i^\varepsilon|^p\,dx}\\ &\leq \beta\frac{\sum_{i=1}^k|b_1|^p\int_{Q_i}|\nabla u_i^\varepsilon|^p\,dx}{\sum_{i=1}^k|b_i|^p\int_{Q_i}\rho(x/\varepsilon)|u_i^\varepsilon|^p\,dx}\\ &= \beta\varepsilon_0^{-p}\mu_{1,\varepsilon/\varepsilon_0}(Q_0).\end{aligned}$$

Since v was arbitrary,

$$\lambda^+_{k,\varepsilon} \leq \sup_{v\in C_k^\varepsilon}\frac{\int_\Omega a_\varepsilon(x,\nabla v)\cdot\nabla v\,dx}{\int_\Omega\rho(x/\varepsilon)|v|^p\,dx} \leq \beta\varepsilon_0^{-p}\mu_{1,\varepsilon/\varepsilon_0}(Q_0).$$

Finally, since from Step 2 we have $\mu_{1,\varepsilon}(Q_0) \leq C\varepsilon^{-1}$, we obtain the desired result.

Step 5. The case $\bar\rho < 0$: lower bound.

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Set $\sigma = \rho + c$, where we add a positive constant to ρ so that $\bar{\sigma} = 0$. Since

$$\int_\Omega \rho(x)|u|^p\,dx \le \int_\Omega \sigma(x)|u|^p\,dx$$

for any $u \in W_0^{1,p}(\Omega)$, we have the Sturmian-type comparison $\lambda_k^+(\rho) \ge \lambda_k^+(\sigma)$, by comparing the Rayleigh quotients. From this and the previous part of the proof,

$$C\varepsilon^{-1} \le \lambda_k^+(\sigma) \le \lambda_k^+(\rho).$$

Step 6. The case $\bar{\rho} < 0$: upper bound.

Let $Q \subset [0,1]^N$ be a cube such that $\int_Q \rho^+(x)\,dx > 0$. Let $\{\mu_k^+\}$ be the positive eigenvalues of

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu\rho|u|^{p-2}u & \text{in } Q,\\ u = 0 & \text{on } \partial Q. \end{cases}$$

By using Theorem 2.12, the variational characterisation of eigenvalues together with inequality (2.1) and the scaling of eigenvalues,

$$\lambda_{k,\varepsilon}(\Omega) \le \lambda_{k,\varepsilon}(Q_\varepsilon) \le \beta\varepsilon^{-p}\mu_k(Q),$$

and the upper bound is proved.

Step 7. The case $\bar{\rho} > 0$.

This one follows from the previous one, by changing $\rho \to -\rho$.

The proof is finished. □

5. Convergence of eigenvalues

Proof of Theorem 1.5. Consider the case where the operators are independent of ε and the weights are periodic. Recall the characterisation of the eigenvalues:

$$\frac{1}{\lambda_k} = \sup_{C\in C_k}\inf_{u\in C}\int_\Omega \rho(x)|u|^p\,dx,$$

where u is normalised with $\int_\Omega a(x,\nabla u)\nabla u\,dx = 1$.

For fixed k and given $\varepsilon > 0$, we can choose $C_\varepsilon \in C_k$ such that

$$\frac{1}{\lambda_{k,\varepsilon}} \le \inf_{u\in C_\varepsilon}\int_\Omega \rho(x/\varepsilon)|u|^p\,dx + \varepsilon.$$

Since C_ε is compact, we can choose $\{u_i\}_{i=1}^J \subset C_\varepsilon$ and $r = r(\bar{\rho}, p, \varepsilon, C_\varepsilon) > 0)$ such that $C_\varepsilon \subset \bigcup_{i=i}^J B(u_i, r)$ and

$$\left|\int_\Omega \bar{\rho}(|u|^p - |u_i|^p)\,dx\right| < \varepsilon \quad \text{if } u \in B(u_i, r),$$

and we choose u_{i_0} such that

$$\int_\Omega \bar{\rho}|u_{i_0}|^p\,dx = \min_{1\le i\le J}\int_\Omega \bar{\rho}|u_i|^p\,dx.$$

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Therefore,

$$\begin{aligned}\frac{1}{\lambda_{k,\varepsilon}} &\le \inf_{u\in C_\varepsilon}\int_\Omega \rho(x/\varepsilon)|u|^p\,dx+\varepsilon \le \int_\Omega \rho(x/\varepsilon)|u_{i_0}|^p\,dx+\varepsilon\\ &= \int_\Omega(\rho(x/\varepsilon)-\bar\rho)|u_{i_0}|^p\,dx+\int_\Omega\bar\rho|u_{i_0}|^p\,dx+\varepsilon\\ &\le \int_\Omega\bar\rho|u_{i_0}|^p\,dx+O(\varepsilon)\\ &\le \inf_{u\in C_\varepsilon}\int_\Omega\bar\rho|u|^p\,dx+O(\varepsilon)\\ &\le \sup_{C\in C_k}\inf_{u\in C}\int_\Omega\bar\rho|u|^p\,dx+O(\varepsilon)=\frac{1}{\lambda_k}+O(\varepsilon),\end{aligned}$$

where the term $O(\varepsilon)$ is given by Theorem 2.8.

The same arguments can be used interchanging the roles of $\lambda_{k,\varepsilon}$ and λ_k. Hence,

$$\left|\frac{1}{\lambda_{k,\varepsilon}}-\frac{1}{\lambda_k}\right|\le C\varepsilon.$$

Finally, using the asymptotic behaviour of eigenvalues, $\lambda_k \approx Ck^{p/N}$, we obtain the desired bound and the proof is finished. □

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