BLOCK DESIGN GAMES

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In this paper, we define and begin the study of an extensive family of simple n-person games based in a natural way on block designs, and hitherto for the most part unexplored except for the finite projective games **(13)**. They should serve at least as a proving ground for conjectures about simple games. It is shown that many of these games are not strong and that many do not possess main simple solutions. In other cases, it is shown that they have no equitable main simple solution, that is, one in which the main simple vector has equal components. On the other hand, the even-dimensional finite projective games $PG(2s, p^n)$ with $s > 1$ possess equitable main simple solutions, although they are not strong either. These results are obtained by means of the study of the possible blocking coalitions. Interpretations in terms of graph theory, network flows, and linear programming are discusssed, as well as k-stability, automorphism groups, and some unsolved problems.

1. Preliminaries on block designs. Block designs have long been studied from various points of view and have an extensive literature, an introduction and references to which can be found in Hall **(8)**.

By a *block design*[2] we shall mean a set N of v elements $\{1, 2, \ldots, v\}$, and a family of b distinguished subsets $W_1, W_2, \ldots, W_b$ of N called *blocks*, such that

(a) every W_i contains k elements, $k < v$,

(b) every element x belongs to r blocks.

A block design may be specified by means of its *incidence matrix* $A = ||a_{ij}||$ with v rows and b columns, where $a_{ij} = 1$ if the ith element belongs to the jth block and $a_{ij} = 0$ if not. The numbers v, b, k, r are termed *parameters* of the design. Clearly,

$$vr = bk, \tag{1}$$

since each side represents the total number of ones in the incidence matrix. A block design is termed *symmetric* if $v = b$ or, equivalently, $k = r$. A block design is termed *balanced* if every two elements occur together in λ blocks. The numbers v, b, k, r, λ are termed the *parameters* of the balanced block design and satisfy, in addition to (1), the relation

$$r(k - 1) = \lambda(v - 1). \tag{2}$$

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[2]Also referred to as incomplete block design and tactical configuration in the literature. Cf. (**2**; **3**; **4**; **10**; **11**).

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A symmetric balanced block design is often referred to as a (v, k, λ)-system. Perhaps the most familiar balanced block designs are the finite geometries, projective and euclidean, where the points are taken as the elements and the lines as the blocks; for these, we have $\lambda = 1$ since two points determine a line. Other balanced block designs which have long been studied are the *Steiner triple systems* (cf. **8**; **10**; **11**) for which either $v = 6t + 1$, $b = t(6t + 1)$, $r = 3t$, $k = 3$, $\lambda = 1$, or $v = 6t + 3$, $b = (2t + 1)(3t + 1)$, $r = 3t + 1$, $k = 3$, $\lambda = 1$. The Steiner triple systems with $v = 1, 3, 7$ we shall here term *trivial*. The case $v = 7$ is the familiar seven-point projective plane.

A block design is termed *partially balanced* if:

(A) There exist non-negative integers $\lambda_1, \lambda_2, \ldots, \lambda_h$ and positive integers $n_1, n_2. \ldots, n_h$ such that to every element x corresponds n_j other elements, called jth *associates* of x, with the property that any jth associate of x occurs together with x in λ_j blocks, and

(B) if x and y are ith associates then the number of elements which are jth associates of x and kth associates of y is $p_{jk}{}^i$.

The numbers $v, b, k, r, \lambda_1, \ldots, \lambda_h, n_1, \ldots, n_h, p_{jk}{}^i$ are termed the *parameters* of the partially balanced block design. It is understood that the numbers $\lambda_i, n_i, p_{jk}{}^i$ are independent of the choice of element. We shall suppose that $h > 1$. If $h = 1$, so that all λ_i may be replaced by λ and all n_i by $v - 1$, then the block design is balanced.

A partially balanced block design with two associate classes is termed *group divisible* if the elements can be divided into m groups each with n elements so that pairs of elements in the same group occur together in λ_1 blocks and pairs of elements in different groups occur together in λ_2 blocks, $\lambda_1 \neq \lambda_2$. It is clear that $n_1 = n - 1$ and $n_2 = n(m - 1)$. A group divisible design (cf. **2**; **5**) is termed *singular* if $r = \lambda_1$, *semi-regular* if $r > \lambda_1$ and $rk = v\lambda_2$, *regular* if $r > \lambda_1$ and $rk > v\lambda_2$.

Let s_{ij} be the number of elements common to the ith and jth blocks of a design; the matrix $S = ||s_{ij}|| = A^T A$. It is known that in a symmetric balanced block design all $s_{ij} = \lambda$. It is also known (cf. **5**) that: for a regular symmetric group divisible design,

$$\begin{aligned} &\lambda_2(r - \lambda_1)/(r^2 - v\lambda_2) \leqslant s_{ij} \leqslant \lambda_1 \quad \text{if} \quad \lambda_1 > \lambda_2, \\ &\lambda_1 \leqslant s_{ij} \leqslant \lambda_2(r - \lambda_1)/(r^2 - v\lambda_2) \quad \text{if} \quad \lambda_1 < \lambda_2; \end{aligned} \tag{3}$$

for a symmetric regular group divisible design with $r^2 - v\lambda_2$ and $\lambda_1 - \lambda_2$ relatively prime, all

$$s_{ij} = \lambda_1 \quad \text{or} \quad \lambda_2; \tag{4}$$

for a symmetric semi-regular group divisible design

$$\lambda_1 \leqslant s_{ij} \leqslant \frac{2\lambda_2 r^2}{r + v\lambda_2 - \lambda_1} - \lambda_1. \tag{5}$$

If the block design D^* has parameters v^*, b^*, k^*, r^* and incidence matrix

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A, then the *dual* block design D has parameters $v = b^*$, $b = v^*$, $k = r^*$, $r = k^*$ and incidence matrix A^T, the transpose of A (cf. **3**; **15**).

Restriction. We shall henceforth confine ourselves to block designs, designated by D, of which the incidence matrices have no two column vectors equal, and correspondingly to block designs, designated by D^*, of which the incidence matrices have no two row vectors equal. Thus no two blocks of a design D are to be equal sets.[3]

If D^* is a partially balanced design with all $\lambda^*_i > 0$, then in the dual design D every pair of distinct blocks has a non-empty intersection. If D^* is a balanced design with $\lambda^* > 0$, then in the dual design D the intersection of every pair of distinct blocks has λ^* elements.

2. Preliminaries on simple games. Let N be a finite set $\{1, 2, \ldots, v\}$ of v *players*. Let $\mathfrak{N}$ be the class of all subsets of N, each of which is called a *coalition*. If $\mathfrak{S} \subset \mathfrak{N}$, let $\mathfrak{S}^+$ be the class of all supersets of elements of $\mathfrak{S}$, and let $\mathfrak{S}^*$ be the class of all complements of elements of $\mathfrak{S}$. By a *simple game* is meant an ordered pair $G = (N, \mathfrak{W})$ where $\mathfrak{W}$ is a subclass of $\mathfrak{N}$ satisfying

$$(\alpha) \qquad \mathfrak{W} = \mathfrak{W}^+$$

$$(\beta) \qquad \mathfrak{W} \cap \mathfrak{W}^* = \phi.$$

Elements of $\mathfrak{W}$ are termed *winning coalitions*, elements of $\mathfrak{L} = \mathfrak{N} - \mathfrak{W}$ are termed *losing coalitions*, and elements of $\mathfrak{B} = \mathfrak{L} \cap \mathfrak{L}^*$ are termed *blocking* coalitions. The simple game[4] G is termed *strong* if and only if $\mathfrak{B} = \phi$. A simple game may be defined by specifying the class $\mathfrak{W}^m \subset \mathfrak{W}$ of *minimal winning coalitions*, by virtue of condition (α). Let $W_1, W_2, \ldots, W_b$ be the minimal winning coalitions.

A *dummy* is a player i such that $f(S \cup \{i\}) = f(S)$ for all $S \in \mathfrak{N}$ where f is the characteristic function of the game. We shall confine ourselves here to *strictly essential* games, that is, having no dummies. We use the $0 - 1$ normalization.

A vector $(a_1, a_2, \ldots, a_v)$ of non-negative real components such that

$$(6) \qquad \sum_{i \in S} a_i = 1 \qquad \text{for} \qquad S \in \mathfrak{W}^m,$$

$$(7) \qquad \sum_{i \in S} a_i > 1 \qquad \text{for} \qquad S \in (\mathfrak{W} \cup \mathfrak{B}) - \mathfrak{W}^m$$

is termed a *main simple vector* (cf. **12**; **14**; **7**). If there exists a main simple vector, then the finite set of imputations $X = \{x^{(S)} | S \in \mathfrak{W}^m\}$ where

$$x_i^{(S)} = \begin{cases} a_i & \text{if} \quad i \in S \\ 0 & \text{if} \quad i \notin S \end{cases}$$

[3]It is possible for a design with distinct blocks to have a dual with two or more blocks equal; see for example, design S12 in (**2**). If, in a balanced design, $\lambda < r$, then two row vectors of the incidence matrix cannot be equal. If, in a partially balanced design, all $\lambda_i < r$, the same conclusion holds.

[4]In (**12**), the terminology is such that all simple games are those which are termed strong here and in (**14**). We shall use the terminology of (**14**) throughout.

is termed a *main simple solution* of the simple game G. A player i is *indifferent* relative to the main simple solution X if $a_i = 0$. This can occur without i necessarily being a dummy. *We shall suppose that there are no indifferent players throughout.* A main simple solution will be termed *equitable* if all the components a_i of its main simple vector are equal.

A necessary condition for a set $X \subset N$ to be a blocking coalition is that the row vectors R_i, $i \in X$, in the incidence matrix, with columns corresponding to minimal winning coalitions and rows corresponding to players, shall have a boolean sum equal to the unit vector $\mathfrak{U}_b$, all b components of which are ones; that is,

$$\sum_{i \epsilon X} R_i = \mathfrak{U}_b$$

where the summation is boolean.

3. Block design games. Any block design D, subject to the restriction at the end of § 1, in which, furthermore, every pair of blocks has a non-empty intersection, may be used to define a simple game, called a *block design game*, in which the players correspond to the elements of the design and the minimal winning coalitions correspond to the blocks of the design. Particular examples are the finite projective games studied in **(13)**, symmetric balanced block designs, group divisible designs with all $s_{ij} > 0$, the duals of finite euclidean planes, the duals of Steiner triple systems, the duals of balanced block designs with $\lambda > 0$, and the duals of partially balanced block designs with all $\lambda_i > 0$.

The following lemmas will be useful.

LEMMA 1. *If, in any simple game, there exists a blocking coalition B (properly) contained in some minimal winning coalition W, then there exists no main simple solution.*

Proof. If there were a main simple vector we would have

$$\sum_{i \epsilon W} a_i = 1 \qquad \text{but} \qquad \sum_{i \epsilon B} a_i \ngtr 1.$$

LEMMA 2. *If every blocking coalition B in a block design game is such that the number of players in B is greater than k, then there exists an equitable main simple solution.*

Proof. We can take $a_i = 1/k$.

LEMMA 3. *If, in a block design game, there exists a blocking coalition B of which the number of players is less than or equal to k, then there exists no equitable main simple solution.*

Proof. For we would have

$$\sum_{i \epsilon B} a_i \leqslant 1$$

and therefore not > 1 as required.

4. Some theorems on block design games. We establish some theorems concerning blocking coalitions and main simple solutions of various block design games. Examples are collected in § 8.

THEOREM 1. *A block design game is not strong if one of the following conditions hold:*
(a) $v = 2k$, $b < \frac{1}{2}C(v, k)$,
(b) $v < 2k$, b $< C(v, k)$,
(c) $v > 2k$, *and some* $(v - k + 1)$*-tuple of players constitutes a losing coalition.*

Proof. Under hypothesis (a), at least one k-tuple of players is not in $\mathfrak{W}^m$ and has its complementary k-tuple also not in $\mathfrak{W}^m$, for the number of k-tuples in $\mathfrak{W}^m \cup \mathfrak{W}^{m*}$ is $2b < C(v, k)$. Hence $\mathfrak{B} = \mathfrak{L} \cap \mathfrak{L}^* \neq \phi$.

Under hypothesis (b), there exists a k-tuple not in $\mathfrak{W}^m$ whose complementary $(v - k)$-tuple is not in $\mathfrak{W}$ since $v - k < k$, while any set in $\mathfrak{W}$ has at least k members.

Under hypothesis (c), the complement of the given $(v - k + 1)$-tuple is also in $\mathfrak{L}$ since it has only $k - 1$ members.

More precise information concerning blocking coalitions in various block design games is given in the remaining theorems of this section.

THEOREM 2. *In any simple game, if there exists a player x_1 and a minimal winning coalition W_1 containing x_1 such that every other minimal winning coalition W containing x_1 intersects W_1 in more than one element, then there exists a blocking coalition B which is a (proper) subset of W_1. Hence, under these hypotheses, the game is not strong and there exists no main simple solution.*

Proof. Let $W_1 = \{x_1, x_2, \ldots, x_k\}$, say, and let $B = W_1 - \{x_1\} = \{x_2, \ldots, x_k\}$. Now every minimal winning coalition W different from W_1 must intersect W_1, and furthermore, by hypothesis, must intersect B. Consequently, B is a blocking coalition (properly) contained in W_1. The last sentence of the theorem follows from Lemma 1.

COROLLARY. *The hypotheses and hence the conclusions of Theorem 2 are satisfied if the block design game D is any of the following:*

(a) *a symmetric balanced block design with* $\lambda > 1$;

(b) *the dual of any balanced block design with* $\lambda > 1$; *in particular, the dual of the design formed by the s-spaces in a projective or euclidean m-space* $PG(m, p^n)$ *or* $EG(m, p^n)$ *with* $1 < s < m$;

(c) *the dual of a partially balanced block design with all* $\lambda_i > 1$;

(d) *a symmetric regular group divisible design with* $1 < \lambda_1 < \lambda_2$;

(e) *a symmetric regular group divisible design with* $\lambda_1 > \lambda_2$ *and* $\lambda_2(r - \lambda_1) > r^2 - v\lambda_2$;

(f) *a symmetric semi-regular group divisible design with* $\lambda_1 > 1$;

(g) *a symmetric regular group divisible design with* $r^2 - v\lambda_2$ *and* $\lambda_1 - \lambda_2$ *relatively prime and both* λ_1 *and* λ_2 *greater than one.*

THEOREM 3. *If D^* is a balanced block design with parameters v^*, b^*, k^*, r^*, and $\lambda^* = 1$, then its dual D yields a game which is not strong if either*[5] $k = 3$ *and* $r \geqslant 4$, *or* $k \geqslant 4$ *and* $r \geqslant 3$. *In particular, under these hypotheses, there exists a blocking coalition with* $k + r - 2$ *members.*

Proof. In D we have $bk = vr$ and $k(r-1) = b - 1$, and the intersection of every pair of minimal winning coalitions has just one element. Consider any block $W = \{x_1, \ldots, x_k\}$ of D. There are

$$r + (k-2)(r-1) = k(r-1) - r + 2 = b - (r-1)$$

blocks containing at least one of the elements $x_1, \ldots, x_{k-1}$, leaving $r - 1$ blocks intersecting W in x_k only. In these $r - 1$ blocks there are $(r-1)(k-1)$ elements other than x_k. There exist $(k-1)^{r-1}$ possible $(r-1)$-tuples with one element chosen from each of these $r - 1$ blocks. Excluding W, there are $(r-1)(k-1)$ blocks containing elements of $\{x_1, \ldots, x_{k-1}\}$. But, if $k = 3$ and $r \geqslant 4$, or if $k \geqslant 4$ and $r \geqslant 3$, then

$$(k-1)^{r-1} > (k-1)(r-1).$$

Therefore, in this case, at least one such $(r-1)$-tuple $\{y_1, \ldots, y_{r-1}\}$ exists not forming a block together with any member of $\{x_1, \ldots, x_{k-1}\}$. Thus, $\{x_1, \ldots, x_{k-1}, y_1, \ldots, y_{r-1}\}$ is a blocking coalition, which completes the proof.

COROLLARY 1. *The dual D of a non-trivial Steiner triple system D^* is not strong.*

Proof. Except for the cases $v^* = 1, 3, 7$, which we have termed trivial, the Steiner triple systems have $k^* = 3$ and $r^* \geqslant 4$. Hence, in the dual, $r = 3$ and $k \geqslant 4$.

COROLLARY 2. *If D^* is the system of lines in the finite euclidean space $EG(m, p^n)$ of m dimensions over the Galois field $GF(p^n)$ for $m \geqslant 2$ and $p^n \geqslant 3$, then the dual D is not strong.*

Proof. In D^* we have $v^* = p^{nm}$, $b^* = p^{n(m-1)}(1 + p^n + \ldots + p^{n(m-1)})$, $k^* = p^n$, $r^* = 1 + p^n + \ldots + p^{n(m-1)}$, and $\lambda^* = 1$. Hence in the dual, $r \geqslant 3$ and $k \geqslant 4$.

COROLLARY 3. *If D^* is the system of lines in the finite projective space $PG(m, p^n)$ of m dimensions over the Galois field $GF(p^n)$ with $m \geqslant 3$ and $p^n \geqslant 2$, then the dual D is not strong.*

Proof. In D^*, we have $v^* = 1 + p^n + \ldots + p^{mn}$,

$$b^* = \frac{(1 + p^n + \ldots + p^{mn})(1 + p^n + \ldots + p^{(m-1)n})}{1 + p^n}, \quad k^* = 1 + p^n,$$

[5]It is easily verified that the only remaining cases are the triangle and the seven-point projective plane, which are strong, and the duals of complete n-gons, $n \geqslant 4$, also termed triangular association schemes below, which are not strong.

$r^* = 1 + p^n + \ldots + p^{(m-1)n}$, and $\lambda^* = 1$. Hence, in D, we have $k \geqslant 4$, and $r \geqslant 3$.

5. Some games with no equitable main simple solution.

THEOREM 4. *If D^* is the system of hyperplanes in the finite euclidean m-space $EG(m, p^n)$, $m \geqslant 2$, then in the dual D there exists a blocking coalition with p^n members.*

Proof. In $EG(m, p^n)$, consider any family of p^n parallel hyperplanes or $(m - 1)$-spaces, one through each point of a transversal line. Their union contains all the points of the space. In the dual D, these hyperplanes correspond to p^n elements incident with all the blocks, no two of which elements occur together in any block. Therefore, these elements constitute a blocking coalition with p^n members.

COROLLARY. *The block design game D, dual to the system of hyperplanes of a finite euclidean m-space $EG(m, p^n)$, $m \geqslant 2$, is not strong and has no equitable main simple solution.*

Proof. The last conclusion follows at once from Lemma 3 of § 3.

THEOREM 5. *In the dual D of a non-trivial Steiner triple system D^*, there exists a blocking coalition with k members.*

Proof. In D^*, consider the set of r^* triples containing a given element x, say. Delete any two of these triples, say (x, a, b) and (x, c, d) where a, b, c, and d are, of course, distinct elements. Then there exist triples (a, c, y) and (b, d, z) with $y \neq x$, $z \neq x$ in D^*, since $\lambda^* = 1$ and since the trivial systems have been excluded. Replacing the two deleted triples by the latter two, we have a set of triples whose union contains all elements of D^* and does not contain any set of all triples through any particular element. In the dual D, this corresponds to a set of elements incident with all blocks but not containing all elements of any particular block. This is a blocking coalition with $k = r^*$ elements.

COROLLARY. *The block design game D, dual to a non-trivial Steiner triple system D^*, is not strong and has no equitable main simple solution.*

Proof. The latter conclusion follows at once from Lemma 3 of § 3.

Remark 1. The system of all lines in m-dimensional finite projective space $PG(m, 2)$ over the integers modulo 2, and the system of all lines in m-dimensional finite euclidean space $EG(m, 3)$ over the integers modulo 3, are Steiner triple systems. Of course, not every Steiner triple system is of this type.

Remark 2. The conclusion of Theorem 3 does not hold for the dual of a partially balanced design with some $\lambda_i = 1$ and some $\lambda_j > 1$. For instance, the game of Example 2 is not strong but the game of Example 6 is strong (see § 10, below).

By a *triangular association scheme* (cf. **2**) is meant an n by n matrix in which: (a) the elements on the principal diagonal are left blank; (b) the $n(n-1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \ldots, n(n-1)/2$; (c) the matrix is symmetric. If we take the players to be the numbers $1, 2, \ldots, n(n-1)/2$, and the minimal winning coalitions $W_1, \ldots, W_n$ to be the rows of the triangular association scheme, then it is easily seen that we have a block design game with $v = n(n-1)/2$, $b = n$, $k = n-1$, $r = 2$, and every two distinct minimal winning coalitions have one player in common. We shall term such a game a *triangular game.*

THEOREM 6. *A triangular game with $n > 3$ has a blocking coalition with $\leqslant k-1$ members; hence it is not strong and has no equitable main simple solution.*

Proof. Let $\{x_1\} = W_1 \cap W_2$. Since $W_1 \cup W_2$ has $2n-3$ members, and $v > 2n-3$ for $n > 3$, there exists an $x_2 \notin W_1 \cup W_2$. Suppose, for example, that $\{x_2\} = W_3 \cap W_4$. Then we can choose arbitrarily $x_i \in W_i$ $(i = 5, 6, \ldots, n)$, distinct or not. Obviously, the distinct members of the set $\{x_1, x_2, x_5, x_6, \ldots, x_n\}$ form a blocking coalition with $\leqslant n-2 = k-1$ members. The second assertion of the theorem follows from Lemma 3.

Remark 3. In fact, it is easy to see that there exists a blocking coalition with $[(n+1)/2]$ members, where $[x]$ is the largest integer $\leqslant x$, but this stronger result does not seem to have any interesting game-theoretic implications.

6. Even-dimensional finite projective games. In **(13)**, finite projective games $PG(h, p^n)$ were defined as follows: the players are the points of the finite projective space $PG(h, p^n)$ of dimension $h > 1$ over the Galois field $GF(p^n)$, and the minimal winning coalitions are the $(s+1)$-spaces if $h = 2s+1$, and the s-spaces if $h = 2s$. As noted in **(13)**, the odd-dimensional finite projective games are not strong and have no main simple solution since the s-spaces are blocking coalitions contained in the minimal winning coalitions (cf. Lemma 1, above). In **(13)**, it is also proved that the plane games $PG(2, p^n)$ are not strong except for $p^n = 2$, but that all of them have equitable main simple solutions. We shall now round out this discussion by disposing of the games $PG(2s, p^n)$ with $s \geqslant 2$.

THEOREM 7. *The games $PG(2s, p^n)$, $s \geqslant 2$, are not strong.*

Proof. Consider any $(s+1)$-space P_{s+1} in $PG(2s, p^n)$. Since any s-space intersects P_{s+1} in a space of dimension at least one, a set B will be a blocking coalition if it consists of points of P_{s+1} such that B meets every line of P_{s+1} but contains no s-space of P_{s+1}. We show that such a set B exists. Introduce a homogeneous co-ordinate system $(x_0, x_1, \ldots, x_{s+1})$ into P_{s+1} in the usual way by means of an $(s+1)$-simplex of co-ordinates σ_{s+1}.

Case 1. Suppose either $p^n \neq 2$, or $p^n = 2$ and s is even. Let B be the set of all points x of P_{s+1} such that the number $Z(x)$ of zero co-ordinates of point x satisfies $1 \leqslant Z(x) \leqslant s$.

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We prove first that every line l of P_{s+1} intersects B. Clearly, l meets the s-space $x_0 = 0$ in at least one point x. If $Z(x) \neq s + 1$, it is the desired point of B. If $Z(x) = s + 1$, then let the remaining non-zero co-ordinate of x be x_i, $i \neq 0$. Let y be a point of intersection of l with the s-space $x_i = 0$. If $Z(y) \neq s + 1$, it is the desired point of B. If $Z(y) = s + 1$, then the point $x + y$ is a point of l having $Z(x + y) = s$, and is therefore in B.

We must still prove that B contains no s-space of P_{s+1}. Let an equation of an arbitrary s-space of P_{s+1} be

$$a_0x_0 + a_1x_1 + \ldots + a_{s+1}x_{s+1} = 0. \tag{8}$$

If at least one coefficient, say a_0, is equal to zero, then the point $(1, 0, 0, \ldots, 0)$ is a point of the s-space not in B. If all coefficients of (8) are different from zero, we consider two cases, $p^n = 2$ or $p^n \neq 2$. If $p^n = 2$, and s is even, then $s + 2$ is even, and, since all $a_i = 1$ by hypothesis, we have

$$\sum_{i=0}^{s+1} a_i = 0 \bmod 2;$$

hence $(1, 1, \ldots, 1)$ is a point of the s-space not in B. If $p^n \neq 2$, let $c \neq 0, 1$ and consider the numbers

$$a_0 + a_1 + \ldots + a_s \tag{9}$$

and

$$a_0 + a_1 + \ldots + a_{s-1} + c\, a_s. \tag{10}$$

At least one of these is not zero, because if both were zero then subtraction would yield $(c - 1)a_s = 0$ and hence $a_s = 0$ contrary to the hypothesis that all $a_i \neq 0$. If (9) is not zero, then the point

$$\left(1, 1, \ldots, 1, 1, -\frac{a_0 + \ldots + a_s}{a_{s+1}}\right)$$

satisfies (8) but is not in B. If (10) is not zero, then the point

$$\left(1, 1, \ldots, 1, c, -\frac{a_0 + \ldots + a_{s-1} + c\, a_s}{a_{s+1}}\right)$$

satisfies (8) but is not in B.

Case 2. Suppose $p^n = 2$, and s is odd, $s \geqslant 3$. Let B be the set of all points x of P_{s+1} with $Z(x) \neq 1, s + 1$.

We prove first that every line l of P_{s+1} intersects B. Let x be a point common to l and the s-space $x_0 = 0$. If $x \neq (0, 1, 1, \ldots, 1)$, $(0, 1, 0, \ldots, 0)$, $(0, 0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$, then $x \in B$. If x is one of these points, let y be a point common to l and the s-space $y_i = 0$, where the ith co-ordinate of x is 1, so that $x \neq y$. Suppose, for example, $i = 1$. If $y \neq (1, 0, 1, \ldots, 1)$, $(1, 0, 0, \ldots, 0)$, $(0, 0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$, then $y \in B$. It is easily seen that in any of the remaining cases, $Z(x + y) = 2$, s, or 0. Hence $x + y$ is a point of l belonging to B.

It is still necessary to prove that B contains no s-space of P_{s+1}. Let (8) be again an equation of an arbitrary s-space of P_{s+1}. If at least one coefficient, say a_0, is zero, then the point $(1, 0, \ldots, 0)$ satisfies (8) but is not in B. If all $a_i \neq 0$, then all $a_i = 1$, and, since s is odd, the point $(0, 1, 1, \ldots, 1)$ satisfies (8) but is not in B. This completes the proof.

Geometrically, in Case 1, B consists of all the points of the face-planes of dimensions $s, \ldots, 1$ of the co-ordinate simplex σ_{s+1} excluding the vertices. In Case 2, B consists of all the points of P_{s+1} excepting the vertices of σ_{s+1} and the points of the s-face-planes not lying on face-planes of lower dimension. It is not difficult to count the number of points in B and to see that this number is greater than the number of points in an s-space. But in the next theorem we shall show that this must be true for any blocking coalition in $PG(2s, p^n)$, $s \geqslant 2$; and hence, by Lemma 2, that there exists an equitable main simple solution.

In $PG(2s, p^n)$, let α_i be the number of points in an i-space, and let $\alpha_j{}^i$ be the number of i-spaces containing a given j-space.

LEMMA 4. *If $r < s$, then $\alpha_{r-1}{}^r > 1 + \alpha_s$.*

Proof. By an easy calculation, we get

$$\alpha_{r-1}^{r} = \frac{p^{rn} + \ldots + p^{2sn}}{p^{rn}} = 1 + p^n + p^{2n} + \ldots + p^{(2s-r)n} = \alpha_{2s-r}.$$

But $r < s$ implies $2s - r > s$, or $2s - r \geqslant s + 1$. Hence $\alpha_{2s-r} - \alpha_s \geqslant \alpha_{s+1} - \alpha_s = p^{(s+1)n}$. Since $p^n > 1$, we have $\alpha_{2s-r} - \alpha_s > 1$.

THEOREM 8. *The games $PG(2s, p^n)$, $s \geqslant 2$, have equitable main simple solutions.*

Proof. By Lemma 2 it suffices to show that if B is any blocking coalition, then the number $|B|$ of points in B is greater than α_s. Suppose, contrarywise, that $|B| \leqslant \alpha_s$.

If every line joining two points of B were contained in B, then B would be a t-space. If $t \geqslant s$, B could not be a blocking coalition since it would contain an s-space or minimal winning coalition. If $t < s$, then there would be an s-space in $PG(2s, p^n)$ not meeting B, contrary to the assumption that B is a blocking coalition. Therefore, there exists a line l with at least two points in B and at least one point x not in B.

We now prove inductively that for each $r \leqslant s - 1$ there exists an r-space containing x but not intersecting B. For $r = 0$, the point x suffices. Suppose the assertion is correct for $r < s - 1$. By Lemma 4, there are more than $1 + \alpha_s$ $(r + 1)$-spaces containing the given r-space; since only one of them can contain l, there are more than α_s $(r + 1)$-spaces containing the given r-space but not containing l. One of these $(r + 1)$-spaces does not intersect B, since at most one of them can meet a given point of B; for, if two of them contained the same point of B, then this point would be in the intersection of these two

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$(r+1)$-spaces which is the given r-space, contradicting the induction hypothesis that this r-space does not intersect B. This completes the induction.

In particular, there exists an $(s-1)$-space containing x but not intersecting B. Every s-space containing this $(s-1)$-space meets B in at least one point since B is a blocking coalition. But the s-space determined by l and the given $(s-1)$-space meets B in at least two points. Therefore $|B| \geqslant 1 + \alpha_s$, contrary to the supposition that $|B| \leqslant \alpha_s$. This completes the proof.

7. Affine resolvable games. In this section, we examine certain simple games formed from block designs but not using all the blocks of the design as minimal winning coalitions.

A balanced design is termed *affine resolvable* if the b blocks can be divided into r classes of n blocks each, such that:

(a) every one of the classes of n blocks contains a complete replication of the v elements;

(b) any two blocks of different classes have the same number of elements in common.

Then (cf. **1**) we have $b = nr$, $v = nk$, $b = v + r - 1$, and $s_{ij} = |B_i \cap B_j| = k^2/v$ if B_i and B_j are in different classes. If we arbitrarily select one block from each class as a minimal winning coalition, we obtain a simple game, with $|\mathfrak{W}^m| = r = b/n$, which we term an *affine resolvable game.* Not all these games formable from a given affine resolvable balanced design need have the same number of players. For example, an affine resolvable balanced design with $v = 12$, $b = 22$, $r = 11$, $k = 6$, $\lambda = 5$, $n = 2$ is given (cf. **1**) by the blocks

$$\begin{aligned}
B_1 &= (1, 3, 4, 5, 9, 11) & B_{12} &= (2, 6, 7, 8, 10, 12)\\
B_2 &= (2, 4, 5, 6, 10, 1) & B_{13} &= (3, 7, 8, 9, 11, 12)\\
B_3 &= (3, 5, 6, 7, 11, 2) & B_{14} &= (4, 8, 9, 10, 1, 12)\\
B_4 &= (4, 6, 7, 8, 1, 3) & B_{15} &= (5, 9, 10, 11, 2, 12)\\
B_5 &= (5, 7, 8, 9, 2, 4) & B_{16} &= (6, 10, 11, 1, 3, 12)\\
B_6 &= (6, 8, 9, 10, 3, 5) & B_{17} &= (7, 11, 1, 2, 4, 12)\\
B_7 &= (7, 9, 10, 11, 4, 6) & B_{18} &= (8, 1, 2, 3, 5, 12)\\
B_8 &= (8, 10, 11, 1, 5, 7) & B_{19} &= (9, 2, 3, 4, 6, 12)\\
B_9 &= (9, 11, 1, 2, 6, 8) & B_{20} &= (10, 3, 4, 5, 7, 12)\\
B_{10} &= (10, 1, 2, 3, 7, 9) & B_{21} &= (11, 4, 5, 6, 8, 12)\\
B_{11} &= (11, 2, 3, 4, 8, 10) & B_{22} &= (1, 5, 6, 7, 9, 12)
\end{aligned}$$

where B_i and B_{i+11} $(i = 1, \ldots, 11)$ constitute the ith class. One affine resolvable game with eleven players has $B_i (i = 1, \ldots, 11)$ as minimal winning coalitions. Another affine resolvable game with twelve players has $B_j (j = 12, \ldots, 22)$ as minimal winning coalitions. In both cases $|B_i \cap B_j| = k^2/v = 3$ if $i \neq j$. In the first case $\{1, 3, 4\}$ is a blocking coalition; in the second case $\{12\}$ is a blocking coalition. Many other affine resolvable games can be formed from the same design.

Another example of an affine resolvable balanced design is an $EG(2, p^n)$, where the classes of blocks are the parallel pencils of lines. Selecting one line from each parallel pencil, we have an affine resolvable game with $|B_i \cap B_j| = 1$ if $i \neq j$.

As an immediate consequence of Theorem 2 and Lemma 1, we have the following theorem.

THEOREM 9. *If an affine resolvable balanced design has* $k^2/v > 1$, *then any affine resolvable game obtained from it as above is not strong and has no main simple solution.*

8. Interpretation in terms of linear graphs and network flows. Any simple game G can be represented as an even (or bipartite, or simple) graph, as follows. Let the two vertex sets be $\mathfrak{W}^m = \{W_1, W_2, \ldots, W_b\}$ and $N = \{1, 2, \ldots, v\}$ and let $W_i \in \mathfrak{W}^m$ and $j \in N$ be joined by an arc if and only if j is a member of W_i. Each vertex W_i has degree $|W_i|$, the number of members of W_i. The many-valued mapping $\Gamma: N \to \mathfrak{W}^m$, where $\Gamma(j)$ is the set of all minimal winning coalitions to which j belongs, is such that $\Gamma^{-1}W_i \cap \Gamma^{-1}W_j \neq \phi$ for $i \neq j$, or, in other words, $\Gamma\Gamma^{-1}W_i = \mathfrak{W}^m$ for each W_i. If G is a block design game then the degree of every vertex W_i of $\mathfrak{W}^m$ is k and the degree of every vertex j of N is r. To a blocking coalition of any simple game G in this representation corresponds a subset B of N such that $\Gamma B = \mathfrak{W}^m$ but $B \not\supset \Gamma^{-1} W_i$ for any $W_i \in \mathfrak{W}^m$.

We can convert this graph theoretic representation into a network flow representation as follows. Join all vertices of N to an input vertex I, and all vertices of $\mathfrak{W}^m$ to an output vertex U, as in Fig. 1, illustrating the dual of $EG(2, 2)$ with $v = 6$, $b = 4$, $k = 3$, $r = 2$ (cf. Example 1, § 10). Putting capacities c_{ij} on the arcs as indicated in the figure, a blocking coalition corresponds to a flow x_{ij} yielding maximum output but with the restriction that

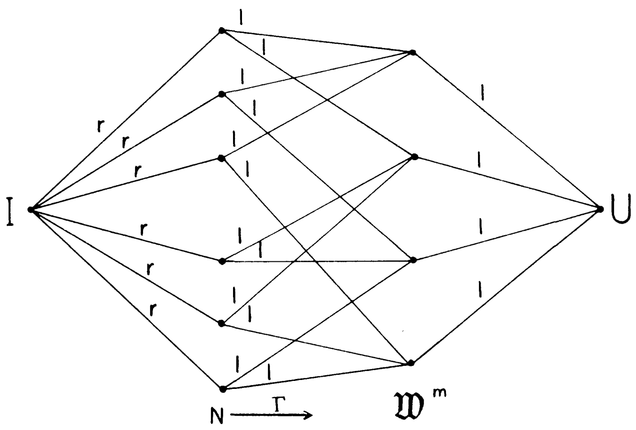

FIG. 1.

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the flow shall not be different from 0 at any entire set of vertices of the form $\Gamma^{-1}W_i$, $W_i \in \mathfrak{W}^m$. This can, in turn, be expressed as a linear programming problem: find x_{ij} such that

$$\sum_{\alpha} x_{\alpha\beta} - \sum_{\gamma} x_{\beta\gamma} = 0$$

for each vertex $\beta \neq I$, U, and such that

(1) $$\sum_{j \in \mathfrak{W}^m} x_{jU} = \max = b$$

subject to the constraints

(2) $$0 \leqslant x_{ij} \leqslant c_{ij},$$

with the additional restriction that

(3) for each $j \in \mathfrak{W}^m$ there exists an $i = i(j) \in \Gamma^{-1}(j)$ such that

$$x_{Ii} = \sum_{j} x_{ij} = 0.$$

If such a flow exists, a blocking coalition is given by the set

$$\left\{ i \in N \,\middle|\, \sum_{j} x_{ij} > 0 \right\}.$$

If such a flow does not exist, the game is strong. By the methods of Goldman and Tucker **(6)**, all extreme feasible vectors of the linear programme given by (1) and (2) can be determined, and then each $W_j \in \mathfrak{W}^m (j = 1, 2, \ldots, b)$ can be examined to see if the additional restriction (3) is satisfied. For if any feasible vector is on a co-ordinate $(n - r)$-plane

$$x_{i_1} = x_{i_2} = \ldots = x_{i_r} = 0$$

then so is some extreme feasible vector.

Thus the results of the preceding sections are readily interpreted in terms of linear graphs or network flows, as desired.

9. Miscellaneous remarks. Unsolved problems. In Luce **(9)**, it is proved that a necessary and sufficient condition for a simple game to be h-unstable is that there exist an $(h + 1)$-element winning coalition and that the intersection of all $(h + 1)$-element winning coalitions be empty.

THEOREM 10. *A block design game is h-stable for* $1 \leqslant h < k - 1$ *and h-unstable for* $k - 1 \leqslant h < v - 1$.

Proof. There exists a winning coalition with $h + 1$ members if $h \geqslant k - 1$. Clearly, for $h = k - 1$, the intersection of all $(h + 1)$-element winning coalitions is empty since $r < b$. As long as there remain two different elements to adjoin to the $(h + 1)$-element sets to obtain $(h + 2)$-element sets, induction shows that the intersection of all $(h + 1)$-element winning coalitions is empty for $h < v - 1$. This completes the proof.

We define an automorphism (collineation or perhaps cowineation) of a block design with incidence matrix $A = ||a_{ij}||$ to be a permutation π_R of the rows of A (elements or players) which carries columns of A (blocks) into columns of A. Let π_C be the permutation of the columns induced by the permutation π_R. *We shall assume that A has neither duplicated columns nor duplicated rows.* Let G_R be the group of all automorphisms π_R and let G_C be the group of all π_C.

LEMMA 5. *If π_R induces π_C then in the dual design the automorphism π_C induces π_R.*

Proof. Let π_R carry the matrix A into the matrix B. Then

$$b_{ij} = a_{\pi_R(i),j} = a_{i,\pi_C(j)}$$

for all i, j. Hence π_R is induced by π_C.

LEMMA 6. *No two different row permutations $\pi_R \neq \pi_R'$ can induce the same column permutation.*

Proof. If so, then

$$a_{i,\pi_C(j)} = a_{i,\pi'_C(j)}$$

for all i, j and hence

$$a_{\pi_R(i),j} = a_{\pi'_R(i),j}$$

for all i, j contrary to hypothesis.

THEOREM 11. *The automorphism groups of dual designs are isomorphic* (*as groups, even though of different degrees*).

Proof. Obviously, the product of two row permutations induces the product of the induced column permutations. Hence, with our restriction of non-duplication of rows and columns of A, the homomorphism is one-to-one, and onto.

The following unsolved problems seem to be difficult:

1. How can one determine the automorphism group of a block design (without examining one by one each of the permutations of the symmetric group on v letters to see if it is an automorphism, although this might be feasible within limits with a computer)? This is solved for the desarguesian finite projective spaces and the associated euclidean spaces. Further, what can be said of the transitivity of this group acting on elements and on blocks?

2. What is the minimum number of members in a blocking coalition of a block design game? This is unsolved even for finite projective planes, except for $PG(2, 3)$.

3. Do the block design games, not covered by **(13)** or the corollaries of Theorem 2, above, possess main simple solutions?

4. To determine all block design games given the parameters v, b, k, r such that $bk = vr$. That is, to determine all $v \times b$ matrices A with $a_{ij} = 0$ or 1 such that the row sums are all equal to r, the column sums are all equal to k, and the elements s_{ij} of $S = A^TA$ are all positive.

5. If $v = b$ and (hence) $k = r$, when do there exist permutation matrices P, Q such that $PAQ = A^T$ where A is the incidence matrix? When do there exist permutation matrices P, Q such that PAQ is a symmetric matrix? When is there a row permutation such that RA is symmetric? What are general criteria for permutation equivalence of matrices with elements equal to 0 or 1? for general matrices?

10. Examples. We collect in this section some concrete examples illustrating some of the preceding theorems. Other examples of block designs yielding simple games can be found in **(2)**.

Example 1. The finite euclidean plane $EG(2, 2)$ has $v^* = 4$, $b^* = 6$, $k^* = 2$, $r^* = 3$, $\lambda^* = 1$ and (Fig. 2) incidence matrix

	a	b	c	d	e	f
1	1	1	1	0	0	0
2	1	0	0	1	1	0
3	0	1	0	1	0	1
4	0	0	1	0	1	1

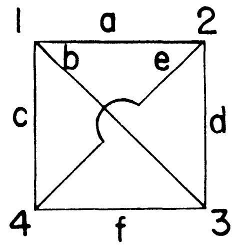

FIG. 2.

In the dual game we have $v = 6$, $b = 4$, $k = 3$, $r = 2$ and the intersection of every pair of blocks has one element. The incidence matrix of the game is the transpose of the above. The sets $\{a, f\}$, $\{b, e\}$, $\{c, d\}$ are blocking coalitions illustrating Theorem 1(a). By Lemma 3, there is no equitable main simple solution. But in this example, it is easy to give a direct proof that no main simple solution exists at all. For the linear system (6), (7) becomes, with obvious changes in notation:

$$\begin{aligned}
&x_a + x_b + x_c &&= 1\\
&x_a + x_d + x_e &&= 1\\
&x_b + x_d + x_f &&= 1\\
&x_c + x_e + x_f &&= 1\\
&x_a + x_f &&> 1\\
&x_b + x_e &&> 1\\
&x_c + x_d &&> 1.
\end{aligned}$$

This implies $x_a = x_f > \frac{1}{2}$, $x_b = x_e > \frac{1}{2}$, $x_c = x_d > \frac{1}{2}$ so that a contradiction would result.

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Example 2. The symmetric partially balanced block design designated as $R1$ in **(2)** has $v^* = b^* = 6$, $k^* = r^* = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$, $n_1 = 1$, $n_2 = 4$ and incidence matrix

	a	b	c	d	e	f
1	1	0	0	1	0	1
2	1	1	0	0	1	0
3	0	1	1	0	0	1
4	1	0	1	1	0	0
5	0	1	0	1	1	0
6	0	0	1	0	1	1

The dual has $v = b = 6$, $k = r = 3$, every pair of blocks has an intersection of one or two elements, and the incidence matrix is the transpose of the above. The set $\{a, b, c\}$ is a blocking coalition. This illustrates Theorem 1.

Example 3. The symmetric regular group divisible partially balanced block design designated as $R2$ in **(2)** has $v^* = b^* = 6$, $r^* = k^* = 4$, $\lambda_1 = 3$, $\lambda_2 = 2$, $n_1 = 2$, $n_2 = 3$ and incidence matrix

	a	b	c	d	e	f
1	1	0	0	1	1	1
2	1	1	1	1	0	0
3	0	1	0	1	1	1
4	1	1	1	0	1	0
5	0	0	1	1	1	1
6	1	1	1	0	0	1

Here $s_{ij} = 2$ or 3 and indeed $\{1, 2\}$ is a blocking coalition contained in the minimal winning coalition a in accordance with Corollary (d) of Theorem 2. The dual has $v = b = 6$, $k = r = 4$, every pair of blocks has an intersection of 2 or 3 elements, and its incidence matrix is the transpose of the above. In accordance with Theorem 2, there is a blocking coalition contained (properly) in a minimal winning coalition, namely, $\{a, b, c\}$, which is a proper subset of blocks 2 or 4, or $\{a, d\}$ which is a proper subset of blocks 1 or 2. In fact the dual is isomorphic to the original design as can be seen by performing the permutations

$$\begin{pmatrix} 123456 \\ 152634 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \text{abcdef} \\ \text{acedfb} \end{pmatrix}$$

on the rows and columns respectively.

Example 4. Let $EG(2, 3)$ be the euclidean plane over the integers modulo 3, with $v^* = 9$, $b^* = 12$, $k^* = 3$, $r^* = 4$, $\lambda^* = 1$. The projective plane $PG(2, 3)$ has the cyclic representation

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
3	4	5	6	7	8	9	10	11	12	0	1	2
9	10	11	12	0	1	2	3	4	5	6	7	8

Taking the line $\{12, 0, 2, 8\}$ as line at infinity, and deleting its points, we derive the incidence matrix of $EG(2, 3)$:

	a	b	c	d	e	f	g	h	i	j	k	l
1	1	1	0	0	0	1	0	0	0	0	0	1
3	1	0	1	1	0	0	0	1	0	0	0	0
4	0	1	0	1	1	0	0	0	1	0	0	0
5	0	0	1	0	1	1	0	0	0	1	0	0
6	0	0	0	1	0	1	1	0	0	0	1	0
7	0	0	0	0	1	0	1	1	0	0	0	1
9	1	0	0	0	0	0	1	0	1	1	0	0
10	0	1	0	0	0	0	0	1	0	1	1	0
11	0	0	1	0	0	0	0	0	1	0	1	1

In the dual, we have $v = 12$, $b = 9$, $k = 4$, $r = 3$, every pair of blocks intersects in one element, and the incidence matrix is the transpose of the above. Note that $EG(2, 3)$ is also the simplest non-trivial Steiner triple system. The process of Theorem 5 applied to $\{a, b, f, l\}$, choosing $f = [1, 5, 6]$, $l = [1, 7, 11]$, yields $e = [4, 5, 7]$ and $k = [6, 10, 11]$ as substitutes, producing the blocking coalition $\{a, b, e, k\}$. But this is not minimal since $\{a, e, k\}$ is also a blocking coalition.

Example 5. One of the Steiner triple systems with $v^* = 13$ (cf. **8**) has the incidence matrix

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	1	1	0	0
4	0	1	0	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
5	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	1
6	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	1	0
7	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	1	1	0	0	0	0	0	0	0	0
8	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0
9	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	1
10	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1
11	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	1	0	0	1	0	1	0	0	0	0	0
12	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	1	0	1	0	0	0	0
13	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	1	1	0

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The dual has $v = 26$, $b = 13$, $k = 6$, $r = 3$, every pair of blocks intersects in one element, the incidence matrix is the transpose of the above, and $\{a, m, s, w, f\}$ is a blocking coalition.

Example 6. The design $T9$ of **(2)** has $v^* = 10$, $b^* = 6$, $k^* = 5$, $r^* = 3$, $\lambda_1 = 1$, $\lambda_2 = 2$, $n_1 = 6$, $n_2 = 3$, and incidence matrix

	a	b	c	d	e	f
1	1	1	0	0	1	0
2	0	0	1	0	1	1
3	1	0	0	1	0	1
4	0	1	1	1	0	0
5	0	1	0	1	0	1
6	0	0	1	1	1	0
7	1	0	1	0	0	1
8	1	1	1	0	0	0
9	1	0	0	1	1	0
10	0	1	0	0	1	1

The dual has $v = 6$, $b = 10$, $k = 3$, $r = 5$, and the incidence matrix is the transpose of the above. This game is strong, and has a main simple vector $a_i = 1/3$ $(i = 1, 2, \ldots, 6)$.

Example 7. The design SR14 of **(2)** is a symmetric semi-regular group divisible design with $v = b = 9$, $k = r = 6$, $\lambda_1 = 3$, $\lambda_2 = 4$, and incidence matrix:

	a	b	c	d	e	f	g	h	i
1	1	1	1	1	1	1	0	0	0
2	1	0	1	1	0	1	1	0	1
3	1	0	1	0	1	1	1	1	0
4	1	1	1	0	0	0	1	1	1
5	1	1	0	1	1	0	1	1	0
6	1	1	0	1	0	1	0	1	1
7	0	0	0	1	1	1	1	1	1
8	0	1	1	0	1	1	0	1	1
9	0	1	1	1	1	0	1	0	1

Here all $s_{ij} \geqslant \lambda_1 = 3$, and indeed $\{1, 2, 3\}$ is a blocking coalition contained in the minimal winning coalition a, as promised by Corollary (f) of Theorem 2.

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