Can. J. Math., Vol. XXXIII, No. 2, 1981, pp. 372–380

WALLMAN COMPACTIFICATION AND REPRESENTATION

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Introduction. Let X be any set and A be a uniformly closed algebra of bounded real valued functions on X which contains the constants and separates the points. For a lattice $\mathscr{L}$ of subsets of X (we assume throughout that $\emptyset$ and X belong to $\mathscr{L}$), let $MR(\mathscr{L})$ denote the space of all finite, finitely additive, $\mathscr{L}$-regular measures defined on the field of sets generated by $\mathscr{L}$. Generalizing the notion of an integral representation, in [**5**] Kirk and Crenshaw define a *standard representation* of A^*, the Banach dual of A, in $MR(\mathscr{L})$ to be a linear map I of A^* into $MR(\mathscr{L})$ with the property that if $0 \leqq \phi \in A^*$, then

$$I\phi(W) = \inf \{\phi(f): f \in A, \chi_W \leqq f\}$$

for every W in $\mathscr{L}$. The space $MR(\mathscr{L})$ is said to *represent* A^* if there exists a (unique) standard representation I of A^* onto $MR(\mathscr{L})$ which is a Banach lattice isomorphism. Among other things the following theorem is proved therein: If $\mathscr{L}$ is a normal base for the weak topology generated by A on X and if A consists of precisely those continuous functions on X which have continuous extensions to the Wallman compactification of X relative to $\mathscr{L}$, then $MR(\mathscr{L})$ represents A^*.

The purpose of this paper is to derive some necessary conditions on the lattice $\mathscr{L}$ when $MR(\mathscr{L})$ represents A^*. In particular, we prove that if $MR(\mathscr{L})$ represents A^* then $\mathscr{L}$ is a separating, disjunctive lattice and A contains the algebra of all those functions which have continuous extensions to the Wallman compact space relative to $\mathscr{L}$. Furthermore, the algebra A coincides with the latter one, if and only if $\mathscr{L}$ is normal. Thus if $\mathscr{L}$ is a normal lattice to start with, the converse of Kirk and Crenshaw's theorem holds. This also generalizes an earlier result of [**4**]. Finally we give topological implications of our result.

1. Preliminaries. Throughout the paper, A and $MR(\mathscr{L})$ are as described in the introduction. Let X_A denote the structure space of A, that is, the set of all non-zero real homomorphisms of A topologized by the weak-star topology. Thus X_A is a Hausdorff compactification of X where X carries the relative topology τ_A which is also the weak topology generated by A on X, and A consists of precisely those continuous functions on X which can be continuously extended to X_A.

Received July 27, 1979 and in revised form November 6, 1979.

https://doi.org/10.4153/CJM-1981-031-6 Published online by Cambridge University Press

A lattice $\mathscr{L}$ of subsets of X is called i) *separating* if given distinct x and y in X there exists some L in $\mathscr{L}$ such that $x \in L$ and $y \notin L$, ii) *disjunctive* if for every $L \in \mathscr{L}$ and $x \in X - L$ there exists $M \in \mathscr{L}$ such that $x \in M$ and $L \cap M = \emptyset$, iii) *normal* if whenever L and M are disjoint members of $\mathscr{L}$, there exist U and V in $\mathscr{L}$ such that $L \subset X - U$, $M \subset X - V$ and $U \cup V = X$.

Given a separating, disjunctive lattice $\mathscr{L}$ of subsets of X the *Wallman compact space relative to* $\mathscr{L}$ is the space $\omega(L)$ of all $\mathscr{L}$-ultrafilters having as the base for its closed sets the sets of the form $\bar{L} = \{\mathscr{U} \in \omega(L): L \in \mathscr{U}\}$ where $L \in \mathscr{L}$. Then $\omega(L)$ is a T_1-compactification of X where X carries the topology having $\mathscr{L}$ as a base for its closed sets and $\bar{L}$ is precisely the closure of L in $\omega(\mathscr{L})$. Furthermore, $\omega(\mathscr{L})$ is Hausdorff if and only if $\mathscr{L}$ is normal (see [**1**]).

Given a separating, disjunctive lattice $\mathscr{L}$ of subsets of X, a bounded real valued function f on X is said to be $\mathscr{L}$*-uniformly continuous* if for every $\epsilon > 0$ there exists a finite family $\{L_1, L_2, \ldots L_n\}$ of members of $\mathscr{L}$ such that $\bigcap_{i=1}^{n} L_i = \emptyset$ and the oscillation of f on each $X - L_i$ is less than ϵ. The set $C_u(\mathscr{L})$ of all $\mathscr{L}$-uniformly continuous functions on X is a uniformly closed algebra consisting of all those bounded real valued continuous functions on X which are continuously extendible to $\omega(\mathscr{L})$. (see (**1**), [**3**]. We note that in the proof of Theorem 2 in [**3**], the Hausdorffness of $\omega(\mathscr{L})$ is not needed).

Given a lattice $\mathscr{L}$, the set of all non-zero measures in $MR(\mathscr{L})$ which assume only two values 0 and 1 is denoted by $IR(\mathscr{L})$. In [**6**], Sultan showed that there is a one to one correspondence between the members of $IR(\mathscr{L})$ and those of $\omega(\mathscr{L})$ and the correspondence is given by associating with each μ in $IR(\mathscr{L})$, the $\mathscr{L}$-ultrafilter

$$\mathscr{F}_\mu = \{L \in \mathscr{L}: \mu(L) = 1\}.$$

We give $IR(\mathscr{L})$ the topology of transference from $\omega(\mathscr{L})$ and call it the *Wallman topology* on $IR(\mathscr{L})$. Thus the Wallman topology on $IR(\mathscr{L})$ has a base for its closed sets, the sets of the form

$$\bar{L} = \{\mu \in IR(\mathscr{L}): \mu(L) = 1\}$$

where $L \in \mathscr{L}$.

Finally, as in [**3**], we assume throughout that all lattices $\mathscr{L}$ for which $MR(\mathscr{L})$ represents A^* are lattices of τ_A-closed sets of X.

2. Weak topology on $IR(\mathscr{L})$**.** Let $MR(\mathscr{L})$ represent A^*. The *weak topology* on $MR(\mathscr{L})$ induced by A is the unique topology on $MR(\mathscr{L})$ which makes the standard representation I a topological isomorphism when A^* has the weak-star topology. This topology on $MR(\mathscr{L})$ is characterized by the convergence of nets: A net $\{\mu_\alpha\}$ in $MR(\mathscr{L})$ converges to a $\mu \in MR(\mathscr{L})$ in weak topology if and only if $\int_X f d\mu_\alpha \rightarrow \int_X f d\mu$ for every f in A. As a subset of $MR(\mathscr{L})$, $IR(\mathscr{L})$ inherits the relative topology which we call the *weak topology* on $IR(\mathscr{L})$ induced by A. In the following proposition we prove that $IR(\mathscr{L})$ with this topology is homeomorphic to X_A.

PROPOSITION 2.1. *Let $MR(\mathscr{L})$ represent A^*. Then $IR(\mathscr{L})$ with weak topology induced by A is homeomorphic to X_A.*

Proof. By definition, X_A is a subspace of A^* where A^* has the weak-star topology. Hence to complete the proof it is enough to show that the standard representation I maps X_A onto $IR(\mathscr{L})$. Since I is an integral representation (see [**5**]), the proof of the latter fact is analogous to the proofs of Lemmas 3.1 and 3.2 of [**6**]. We leave the details to the reader.

Thus when $MR(\mathscr{L})$ represents A^*, $IR(\mathscr{L})$ with weak topology induced by A is a Hausdorff compactification of (X, τ_A). For an $x \in X$, let h_x denote the bounded linear functional on A defined by $h_x(f) = f(x)$ for every $f \in A$. Let μ_x be the measure in $MR(\mathscr{L})$ representing h_x. Since $x \rightarrow h_x$ is an embedding of X into X_A, $x \rightarrow \mu_x$ is an embedding of X into $IR(\mathscr{L})$. Later in Section 3, we show that μ_x is precisely the unit mass measure concentrated at x.

Given a Tychonoff space, let $C_b(X)$ denote the space of all bounded real valued continuous functions on X. Recall that the structure space of $C_b(X)$ is homeomorphic to the Stone-Čech compactification βX of X. Hence by Proposition 2.1, it follows that if $\mathscr{L}$ is any lattice of closed subsets of X for which $MR(\mathscr{L})$ represents $C_b(X)^*$, then $IR(\mathscr{L})$ is homeomorphic to βX. In particular, $IR(Z[X])$ is homeomorphic to βX where $Z[X]$ denotes the lattice of zero sets on X. This latter result is due to Varadarajan (Theorem 4 of Part III, [**7**]).

Our next result gives a comparison of the weak topology on $IR(\mathscr{L})$ with the Wallman topology on it.

PROPOSITION 2.2. *Let $MR(\mathscr{L})$ represent A^*. Then the Wallman topology on $IR(\mathscr{L})$ is weaker than the weak topology on $IR(\mathscr{L})$ induced by A.*

Proof. It is enough to prove that the convergence of a net in the weak topology implies its convergence in the Wallman topology on $IR(\mathscr{L})$. Let $\{\mu_\alpha\}$ be a net in $IR(\mathscr{L})$ with $\mu_\alpha \rightarrow \mu \in IR(\mathscr{L})$ weakly. Then $\int_X f d\mu_\alpha \rightarrow \int_X f d\mu$ for every $f \in A$. Since μ is non-negative, for every $W \in \mathscr{L}$,

$$\mu(W) = \inf\left\{\int_X f d\mu : f \in A, \chi_W \leqq f\right\}.$$

Hence given $\epsilon > 0$, there exists $f \in A$ with

$$\chi_W \leqq f \quad \text{and} \quad \int_X f d\mu < \mu(W) + \epsilon.$$

https://doi.org/10.4153/CJM-1981-031-6 Published online by Cambridge University Press

We then have

$$\lim_\alpha \sup \mu_\alpha(W) \leqq \lim_\alpha \int_X f d\mu_\alpha = \int_X f d\mu < \mu(W) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\lim_\alpha \sup \mu_\alpha(W) \leqq \mu(W).$$

In particular if $\mu(W) = 0$, then $\mu_\alpha(W) \to 0$.

Note that a base for the closed sets of Wallman topology is given by $\{\hat{W}: W \in \mathscr{L}\}$ where $\hat{W} = \{\mu \in IR(\mathscr{L}): \mu(W) = 1\}$. Hence if $IR(\mathscr{L}) - \hat{W}$ is any Wallman neighbourhood of μ then $\mu(W) = 0$. But then $\mu_\alpha(W) \to 0$. Since each μ_α is $\{0, 1\}$-valued, if follows that $\mu_\alpha(W) = 0$ for all $\alpha \geqq \alpha_0$ for some α_0. That is $\mu_\alpha \in IR(\mathscr{L}) - \hat{W}$ for all $\alpha \geqq \alpha_0$. This proves that $\mu_\alpha \to \mu$ in the Wallman topology.

Since compact topology is minimal among the Hausdorff topologies, the two topologies on $IR(\mathscr{L})$ coincide if and only if the Wallman topology on $IR(\mathscr{L})$ is Hausdorff. That is, if and only if whenever $\mu_1 \neq \mu_2$, $\mu_1, \mu_2 \in IR(\mathscr{L})$ there exist $W_1, W_2 \in \mathscr{L}$, with $W_1 \cup W_2 = X$ such that $\mu_1(W_1) = 0 = \mu_2(W_2)$.

3. Some necessary conditions on $\mathscr{L}$. We first prove a 'lattice version' of the Urysohn's lemma. Let $\mathscr{L}$ be a separating, disjunctive lattice of subsets of a set X. We call $\mathscr{L}$ a *Urysohn lattice* if for every pair of disjoint sets L and M in $\mathscr{L}$, there exists an $\mathscr{L}$-uniformly continuous function f on X such that $f(L) = 0$ and $f(M) = 1$.

PROPOSITION 3.1. *Let $\mathscr{L}$ be a separating, disjunctive lattice of subsets of a set X. Then $\mathscr{L}$ is normal if and only if L is Urysohn.*

Proof. Let $\mathscr{L}$ be normal. Then the Wallman compact space $\omega(\mathscr{L})$ is a Hausdorff compactification of X where X has the topology with $\mathscr{L}$ as a base for its closed sets. Let $L, M \in \mathscr{L}$ with $L \cap M = \emptyset$. Then $\bar{L} \cap \bar{M} = \emptyset$. By the Urysohn's lemma for normal topological spaces, there exists an $\bar{f} \in C(\omega(\mathscr{L}))$ such that $\bar{f}(\bar{L}) = 0$ and $\bar{f}(\bar{M}) = 1$. Let $f = \bar{f}|X$. Then f is $\mathscr{L}$-uniformly continuous and $f(L) = 0$ and $f(M) = 1$ proving that $\mathscr{L}$ is Urysohn.

Conversely suppose $\mathscr{L}$ is Urysohn. Consider the Wallman compact space $\omega(\mathscr{L})$ relative to $\mathscr{L}$. Recall that $\omega(\mathscr{L})$ is Hausdorff if and only if $\mathscr{L}$ is normal. (See [**1**]). Hence we complete the proof by showing that $\omega(\mathscr{L})$ is Hausdorff.

Let $x, y \in \omega(\mathscr{L})$, $x \neq y$. Since $\bar{\mathscr{L}} = \{\bar{L}: L \in \mathscr{L}\}$ is a base for the closed sets of $\omega(\mathscr{L})$, and $\{x\}$, $\{y\}$ are closed, there is an $L \in \mathscr{L}$ such that $x \in \bar{L}$ and $y \notin \bar{L}$. Again for every $z \in \bar{L}$, there exists $M_z \in \mathscr{L}$ such that $y \in \bar{M}_z$ and $z \notin \bar{M}_z$. Now $\{\omega(\mathscr{L}) - \bar{M}_z : z \in \bar{L}\}$ forms an open cover of the compact set $\bar{L}$. Hence there exists a finite subcover, say $\{\omega(\mathscr{L}) - \bar{M}_{z_i} : i = 1, 2, \ldots, n\}$. Let

$$M = \bigcap_{i=1}^{n} M_{z_i}.$$

Then $M \in \mathscr{L}$ and $\bar{L} \subset \omega(\mathscr{L}) - \bar{M}$. Hence $x \in \bar{L}$, $y \in \bar{M}$ and $\bar{L} \cap \bar{M} = \emptyset$. Then $L \cap M = \emptyset$. By hypothesis, there exists an $\mathscr{L}$-uniformly continuous function f such that $f(L) = 0$ and $f(M) = 1$. But then $\bar{f}(\bar{L}) = 0$ and $\bar{f}(\bar{M}) = 1$ where $\bar{f}$ is the unique continuous extension of f to $\omega(\mathscr{L})$. This guarantees that $\omega(\mathscr{L})$ is Hausdorff.

COROLLARY 3.2. *A topological space X is Tychonoff if and only if X has a Urysohn lattice $\mathscr{L}$ which is a base for its closed sets.*

Proof. A T_1-space X is Tychonoff if and only if X has a normal base for its closed sets [**3**]. If X is Tychonoff then it has a normal base $\mathscr{L}$ which is clearly Urysohn by Proposition 3.1. Conversely suppose X has a Urysohn lattice as a base for its closed sets. Since $\mathscr{L}$ is separating and disjunctive X is a T_1-space. Now the result follows from [**3**].

When X is a normal topological space and $\mathscr{L}$ is the lattice of the closed sets of X, then $MR(\mathscr{L})$ represents $C_b(X)^*$ (see [**2**], p. 262). More generally, let X be any Tychonoff space and $\mathscr{L}$ be a disjunctive, separating lattice of its closed sets which is also a base for its closed sets. If $\mathscr{L}$ is normal then $MR(\mathscr{L})$ represents $C_u(\mathscr{L})^*$. (This follows from Theorem 3.12 of [**5**]). The following corollary contains a converse of these results.

COROLLARY 3.3 *Let $\mathscr{L}$ be a disjunctive, separating lattice of subsets of a set X such that the algebra $C_u(\mathscr{L})$ is point separating. Then $MR(\mathscr{L})$ represents $C_u(\mathscr{L})^*$ if and only if $\mathscr{L}$ is normal.*

Proof. If $\mathscr{L}$ is normal, we have already pointed out that $MR(\mathscr{L})$ represents $C_u(\mathscr{L})^*$. We now prove the converse. It is easy to see that the members of $\mathscr{L}$ are closed sets in the weak topology generated by $C_u(\mathscr{L})$ on X. By Proposition 3.5 of [**5**], if $MR(\mathscr{L})$ represents the dual A^* of an algebra A, then the disjoint sets in $\mathscr{L}$ are separated by a member of A. Thus when $MR(\mathscr{L})$ represents $C_u(\mathscr{L})^*$, $\mathscr{L}$ is Urysohn, and hence normal by Proposition 3.1.

We remark here that if $\mathscr{L}$ is normal then the algebra $C_u(\mathscr{L})$ is point separating. However $C_u(\mathscr{L})$ can be point separating without $\mathscr{L}$ being normal. For example, if X is a non-normal Tychonoff space then the lattice $\mathscr{L}$ of its closed sets is non-normal but $C_u(\mathscr{L}) = C_b(X)$ is point separating.

Next we prove that the point separating property of an algebra A is enough to ensure that $\mathscr{L}$ is separating and disjunctive whenever $MR(\mathscr{L})$ represents A^*. We first prove two lemmas.

For an $x \in X$, let δ_x denote the unit mass measure concentrated at the point x, that is, $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ if $x \notin E$ for every E in the domain of δ_x. Note that for a given lattice $\mathscr{L}$, in general δ_x need not be $\mathscr{L}$-regular.

LEMMA 3.4. *Let $\mathscr{L}$ be a lattice of subsets of a set X. Then $\mathscr{L}$ is disjunctive if and only if the unit mass measure δ_x is $\mathscr{L}$-regular for every $x \in X$.*

Proof. Let $\mathscr{L}$ be disjunctive. Let $x \in X$. Let E be a set in the field of subsets of X generated by $\mathscr{L}$. To prove that δ_x is $\mathscr{L}$-regular it is enough to show that whenever $x \in E$, there exists some $L \in \mathscr{L}$ such that $x \in L \subset E$. Now $E = \bigcup_{i=1}^{n} (L_i - M_i)$ for some n where $L_i, M_i \in L$, $M_i \subset L_i$ and

$$(L_i - M_i) \cap (L_j - M_j) = \emptyset \quad \text{for } i \neq j.$$

Let $x \in E$. Then $x \in L_i - M_i$ for some i, $1 \leqq i \leqq n$. By hypothesis, there exists some $M \in \mathscr{L}$ such that $x \in M$ and $M \cap M_i = \emptyset$. Let $L = M \cap L_i$. Then $x \in L \subset E$.

Conversely suppose δ_x is $\mathscr{L}$-regular for each $x \in X$. Let $L \in \mathscr{L}$ and $x \notin L$. Then $\delta_x(X - \mathrm{L}) = 1$. By the $\mathscr{L}$-regularity of δ_x, there exists $M \in \mathscr{L}$ such that $M \subset X - L$ and $\delta_x(M) = 1$. Then it follows that $x \in M$ and $L \cap M = \emptyset$.

LEMMA 3.5. *Let $\mathscr{L}$ be a disjunctive lattice of subsets of a set X. Then $\mathscr{L}$ is separating if and only if $\delta_x \neq \delta_y$ whenever $x, y \in X$, $x \neq y$.*

Proof. Suppose $\mathscr{L}$ is separating. Then it is easy to see that $\delta_x \neq \delta_y$ whenever $x \neq y$. Conversely let the hypothesis be true. Let $x, y \in X$, $x \neq y$. Then $\delta_x \neq \delta_y$. Hence there exists $L \in \mathscr{L}$ such that $\delta_x(L) \neq \delta_y(L)$. Therefore L contains exactly one of the points x or y. Now together with the disjunctive property of $\mathscr{L}$, it follows that $\mathscr{L}$ is separating.

We now prove

PROPOSITION 3.6. *Let $\mathscr{L}$ be a lattice of τ_A-closed subsets of X such that $MR(\mathscr{L})$ represents A^*. Then $\mathscr{L}$ is separating and disjunctive.*

Proof. For $x \in X$, let $h_x \in X_A$ be the evaluation-at-x homomorphism and let $\mu_x \in IR(\mathscr{L})$ represent h_x (Proposition 2.1). Since μ_x is $\mathscr{L}$-regular, to show that $\mathscr{L}$ is disjunctive we need only show that $\mu_x = \delta_x$. Now for any $L \in \mathscr{L}$

$$\begin{aligned} \mu_x(L) &= \inf \{h_x(f) : f \in A, \chi_L \leqq f\}, \\ &= \inf \{f(x) : f \in A, \chi_L \leqq f\}, \\ &= 1 \text{ if and only if } x \in L. \end{aligned}$$

https://doi.org/10.4153/CJM-1981-031-6 Published online by Cambridge University Press

The last line follows by an application of the Urysohn's lemma to the pair $\bar{L}$ and x in the normal topological space X_A.

For an arbitrary E in the field of subsets of X generated by $\mathscr{L}$ we can show, by a similar argument applied in the proof of Lemma 3.4, that $\mu_x(E) = 1$ if and only if $x \in E$. This proves that $\mu_x = \delta_x$. Since A is point separating, $h_x \neq h_y$ and hence $\mu_x \neq \mu_y$ whenever $x, y \in X$, $x \neq y$. Now the proof of the proposition is complete by direct applications of Lemmas 3.4 and 3.5.

In Corollary 3.3, we proved that if the algebra $C_u(\mathscr{L})$ is point separating, then $\mathscr{L}$ should be normal whenever $MR(\mathscr{L})$ represents $C_u(\mathscr{L})^*$. However $MR(\mathscr{L})$ can represent the dual A^* of a point separating algebra A even if $\mathscr{L}$ is not normal. (See example Appendix (a) of [**5**].) In the following theorem, we prove that $MR(\mathscr{L})$ can not represent the dual of any smaller point separating algebra than $C_u(\mathscr{L})$.

THEOREM 3.7. *Let A be a uniformly closed algebra of bounded real valued functions on a set X which contains the constants and separates the points. Let $\mathscr{L}$ be a lattice of τ_A-closed subsets of X such that $MR(\mathscr{L})$ represents A^*. Then $C_u(\mathscr{L}) \subset A$. Furthermore $C_u(\mathscr{L}) = A$ if and only if $\mathscr{L}$ is normal.*

Proof. Since $MR(\mathscr{L})$ represents A^*, by Proposition 3.6 $\mathscr{L}$ is separating and disjunctive. Hence the Wallman compact space $\omega(L)$ relative to $\mathscr{L}$ is a T_1-compactification of X. That is, $IR(\mathscr{L})$ with Wallman topology is a T_1-compactification of X and each $f \in C_u(\mathscr{L})$ has a unique continuous extension to $IR(\mathscr{L})$ with this topology.

By Proposition 2.2, the Wallman topology on $IR(\mathscr{L})$ is weaker than the weak topology induced by A. By Proposition 2.1, $IR(\mathscr{L})$ with the weak topology induced by A is homeomorphic to X_A.

Let $f \in C_u(\mathscr{L})$ and let $\bar{f}$ be the unique continuous extension of f to $IR(\mathscr{L})$ in the Wallman topology. Then $\bar{f}$ is continuous on $IR(\mathscr{L})$ in the weak topology induced by A as well.

Thus f has a unique continuous extension to X_A. Since A consists of precisely those bounded real valued functions on X which have continuous extensions to X_A, $f \in A$. This proves the first part of the theorem.

Now suppose $\mathscr{L}$ is normal. Then the Wallman topology on $IR(\mathscr{L})$ is Hausdorff and hence coincides with the weak topology induced by A. Therefore by Propositions 2.1 and 2.2 X_A is homeomorphic to the Wallman compactification $\omega(L)$ of X relative to $\mathscr{L}$ and a homeomorphism can be so chosen that it leaves X pointwise fixed. Now it follows that $A = C_u(\mathscr{L})$.

Conversely suppose $C_u(\mathscr{L}) = A$. Then the normality of $\mathscr{L}$ follows from Corollary 3.3.

Remark 1. In [**5**] Kirk and Crenshaw proved that if $X_A = \omega(L)$ for some normal base $\mathscr{L}$ on X with τ_A-topology, then $MR(\mathscr{L})$ represents A^*.

Theorem 3.7 proves that if we start with a normal lattice $\mathscr{L}$ of τ_A-closed sets the converse also holds. That is, if $\mathscr{L}$ is a normal lattice of τ_A-closed sets on X such that $MR(\mathscr{L})$ represents A^*, then $\mathscr{L}$ is a normal base on X and $X_A = \omega(\mathscr{L})$.

Remark 2. Let $Z[A]$ denote the lattice of the zero-sets of members of A. In [**4**] we proved that $MR(Z[A])$ represents A^* if and only if $A = C_u(Z[A])$. Theorem 3.7 and Corollary 3.3 together generalize this result which can be stated as: Let $\mathscr{L}$ be a separating, disjunctive, normal lattice of τ_A-closed subsets of X. Then $MR(\mathscr{L})$ represents A^* if and only if $A = C_u(\mathscr{L})$.

Every Tychonoff space has a normal base which is a separating, disjunctive normal lattice of its closed sets. We show by an example below that a separating, disjunctive, normal lattice of closed sets need not be a base for the closed sets.

Example 3.8. Let N denote the set of all positive integers with discrete topology. Let $\mathscr{L}$ be the lattice of subsets of N generated by the sets of the form

$$A_{m,n} = \{k \in N : k \leqq 2m - 1, \text{ or } k \geqq 2n\},$$
$$B_{m,n} = \{k \in N : 2m \leqq k \leqq 2n - 1\}$$

where $m, n \in N$. Clearly $\mathscr{L}$ consists of finite unions of $B_{m,n}$'s or the unions of a finite number of $B_{m,n}$'s and an $A_{m,n}$. Since $\mathscr{L}$ itself is a field of subsets of N, $\mathscr{L}$ is a separating, disjunctive, normal lattice of closed subsets of N. However $\mathscr{L}$ is not a base for the closed sets of N. In fact, $F = N - \{1\}$ is closed in N and $1 \notin F$. But each member of $\mathscr{L}$ which contains F also contains 1.

For a given separating, disjunctive, normal lattice $\mathscr{L}$ of subsets of X let $\tau_{\mathscr{L}}$ denote the topology on X having $\mathscr{L}$ as a base for its closed sets. Then $\tau_{\mathscr{L}}$ is a completely regular Hausdorff topology on X. Now as an immediate consequence of Theorem 3.7 we have

COROLLARY 3.9. *Let $\mathscr{L}$ be a separating disjunctive normal lattice of τ_A-closed subsets of X. Then $\tau_{\mathscr{L}} = \tau_A$ if and only if $MR(\mathscr{L})$ represents A^*.*

Proof. If $MR(\mathscr{L})$ represents A^*, then as in the proof of Theorem 3.7 $X_A = \omega(L)$. Hence the corresponding relative topologies τ_A and $\tau_{\mathscr{L}}$ on X coincide.

Conversely if $\tau_{\mathscr{L}} = \tau_A$, then $\mathscr{L}$ is a normal base for the τ_A-closed sets and hence $X_A = \omega(\mathscr{L})$. Therefore $MR(\mathscr{L})$ represents A^*.

In particular if X is any Tychonoff space, and $\mathscr{L}$ is a separating, disjunctive, normal lattice of its closed sets then the topology on X having $\mathscr{L}$ as a base for its closed sets coincides with the original topology if and only if $MR(\mathscr{L})$ represents $C_b(X)^*$. In fact, $C_b(X)$ here can be replaced by any "completely regular" algebra A, a subalgebra of $C_b(X)$ which determines the topology of X.

Finally the author thanks Professor Robert F. Wheeler for helpful discussions and the referee for a suggestion regarding the notation.

References

1. R. M. Brooks, *On Wallman compactifications*, Fund. Math. *60* (1967), 157–173.
2. N. Dunford and J. T. Schwartz, *Linear operators, Part I* (Interscience, New York, 1958).
3. O. Frink, *Compactifications and seminormal spaces*, Amer. J. Math. *86* (1964), 602–607.
4. S. Hegde, *A characterization of z-separating algebras*, Proc. Amer. Math. Soc. *73* (1979), 40–44.
5. R. B. Kirk and J. A. Crenshaw, *A generalized topological measure theory*, Trans. Amer. Math. Soc. *207* (1975), 189–217.
6. A. Sultan, *Measure, compactification and representation*, Can. J. Math. *30* (1978), 54–65.
7. V. S. Varadarajan, *Measures on topological spaces*, Amer. Math. Soc. Transl. *48* (1965), 161–228.

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