# TIME EVOLUTION OF THE SOLUTION IN MODEL Z

A.P. ANUFRIEV
*Geophysical Inst. Bulg. Acad. Sci.*
*1113 Sofia, Bulgaria*

I. CUPAL AND P. HEJDA
*Geophysical Inst. Czechosl. Acad. Sci.*
*141 31 Prague 4, Czechoslovakia*

ABSTRACT. Time evolution of the solution of model Z is considered simply as an aid to obtain the steady state solution. Balance equations of the energy of the azimuthal field $E_B$ showes that excluding several beginning time steps the solution exposes the time behaviour with a physical sense.

## 1. Introduction

S.I. Braginsky (1975) introduced an idea of the Model Z for planetary dynamos as a possible solution of hydromagnetic nearly symmetric dynamo problem where $z$-component of poloidal magnetic field dominate in the main volume of the liquid and conductive core. The solution of model Z was found in many cases taking into account both viscous and electromagnetic core-mantle coupling, however, minimum attention was paid to time evolution of the solution which was used rather as an aid to obtain the steady state solution. An important question is which characteristics of the time behaviour of the soulution reflect the physical behaviour of the system and which follow from the limitations of the numerical method. We anticipate that the balance equation of energy provide a good independent test of the solution. The inbalance (caused by a numerical process) then indicates in which time steps the behaviour of the solution has no physical sense.

## 2. Governing equations

The equations governing the model Z in dimensionless form were derived by Braginsky (1975). Relative to spherical coordinates $r, \theta, \phi$ or cylindrical coordinates $s, \phi, z$, we introduce the stream functions of



*F. Krause et al. (eds.), The Cosmic Dynamo,* 447–452.

meridional magnetic field and meridional velocity respectively $\mathbf{B}_p = s^{-1}\nabla\psi \times \mathbf{1}_\phi, \mathbf{v}_p = s^{-1}\nabla\chi \times \mathbf{1}_\phi$. Taking into account only viscous core mantle coupling and that $\nabla^2 B - s^{-2}B = s\nabla \cdot (s^{-2}\nabla sB)$, $\nabla^2 s^{-1}\psi - s^{-2}\psi = \nabla \cdot (s^2\nabla s^{-2}\psi)$ the equations can be re-written in the form (see e.g. Cupal & Hejda 1989):

$$\frac{\partial\psi}{\partial t} = \nabla \cdot (-\psi\mathbf{v}_p + s^2\nabla s^{-2}\psi) + s\alpha B, \tag{2.1}$$

$$\frac{\partial B}{\partial t} = s\nabla \cdot (-s^{-1}B\mathbf{v}_p + s^{-2}\nabla sB + \zeta\mathbf{B}_p), \tag{2.2}$$

$$v_s = s^{-1}\nabla \cdot (sB\mathbf{B}_p), \tag{2.3}$$

$$\zeta = f + s^{-2}B^2 + \omega, \tag{2.4}$$

where $\zeta(s,z) = v(s,z)/s$. $f(s,z)$ is the Archimedean wind, $s^{-2}B^2(s,z)$ is the magnetic wind and $\omega(s) = v(s,z_1)/s$ is the geostrophic shear determined by the equation

$$\omega = \frac{2\sqrt{z_1}}{\varepsilon s^3}\frac{d}{ds}\left(s^3\int_0^{z_1} JB_s\,dz\right), \tag{2.5}$$

where $z_1 = \sqrt{1-s^2}$. The viscous coupling parametr $\varepsilon$ is the same as $\varepsilon_1$ used by Cupal & Hejda (1989). Following Braginsky(1978) the Archimedean wind

$$f = -3f_o s^2(1-r^2), \tag{2.6}$$

is prescribed and a slightly generalized form of $\alpha$-effect is used (Cupal & Hejda, 1992)

$$\alpha = \begin{cases} 0 & s \le 1-\delta, \\ 20\,\alpha_o z[1-(\frac{z}{z_1})^6]\sin[(2\frac{\pi}{\delta}(1-s-\frac{1}{2}\delta)] & 1-\delta \le s \le 1, \end{cases} \tag{2.7}$$

which makes possible to change the thickness, $\delta$, of $\alpha$-layer and to investigate the influence of the changed $\delta$ on the solution.

Appropriate functions, which measure the amplitude of the magnetic field in the volume of the core, are toroidal field energy, $E_B(t) = \frac{1}{2}\int_V B^2 dV$, and the squared poloidal magnetic flux, $E_\psi(t) = \frac{1}{2}\int_V \psi^2 dV$, integrated throughout the volume, $V$, of the core. Multiplying the equation (2.2) by $B$ and integrating it over volume $V$, we obtain the following balance equation for energy:

$$\frac{\partial E_B}{\partial t} = Q_A - Q_J - Q_\nu, \tag{2.8}$$

where only the Archimedean forces, $Q_A$, do work and energy is lost by the Joule's dissipation, $Q_J$, and by the viscous dissipation in the Ekman layer, $Q_\nu$. The terms on r.h.s. of equation (2.8) are

$$Q_A = -\int_V s f v_s \, dV, \qquad Q_J = \int_V s^{-2}[\nabla(sB)]^2 \, dV,$$

$$Q_\nu = 2\pi\varepsilon \int_0^1 (s\omega)^2 / \sqrt{z_1} \, s ds.$$

We call the inbalance the difference between l.h.s. and r.h.s. of eq. (2.8).

## 3.Calculated models

Cupal & Hejda (1992) calculated several cases of model Z. Several solutions tended to oscillate before they reached their steady state. The oscillatory solution for $\varepsilon = 0.01, f_0 = 500, \alpha_0 = 25$ and $\delta = 0.3$ is particularly useful in demonstrating the time behaviour of the solution using the equation (2.8).

Figure 3.1 shows the behaviour of $\partial E_B/\partial t$ (solid line) and the r.h.s. of equation (2.8) (circles). The dashed line represents the inbalance to which a constant 500 is added to aid comparison. The start of the graph is characterized by a large inbalance and there is not sensible to discuss about time behaviour of the solution. After time t=0.02 the inbalance begins to decrease and the time behaviour illustrated by the solution makes physical sense. However even later e.g. after t=0.11 the inbalance remains small, but non-zero as a consequence of the space discretization.

The space discretization and the length of time step can influence the solution, but artificial time behaviour of the numerical origin can easily be distinguished. An experiment was been done with automatical time step change depending on the magnitude of the inbalance to save the time of calculations. Unfortunately, the numerical process had a certain inertia and we received the system like a pendulum with non-decaying oscillations driven by numerical process. The inbalance was

https://doi.org/10.1017/S0074180900174601 Published online by Cambridge University Press

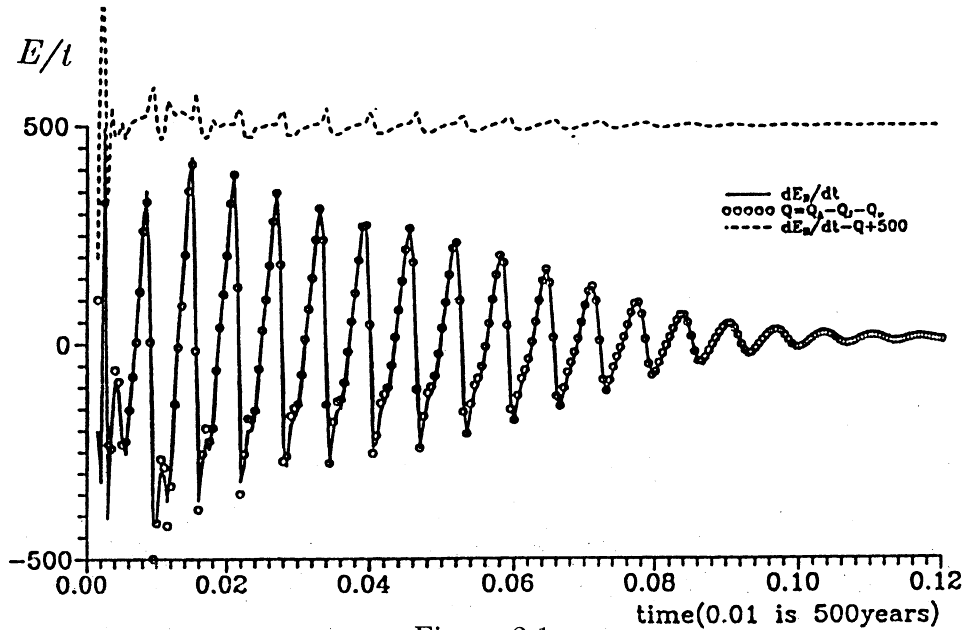


Figure 3.1

periodically too large and in those moments the pendulum received impulses from the numerical method. The balance equation (2.8) isolated this problem and so we knew that our numerical results are unreliable. Generally, the coarse grid was used in the space discretization. The names "coarse" and "fine" grid were introduced by Braginsky & Roberts (1987) and they relate to $32 \times 32$ or $64 \times 64$ space grids in $\theta$ and $r$ respectively. The fine space grid also have been used at the relative steady part of the solution about the time $t = 0.11$. The inbalance was observed essentially smaller (roughly 8times) than in the case of the coarse grid. The inbalance for longer time practically does not vary and the beginning of this time interval may be a time when the searching the steady solution for given space discretization can be stopped. The inbalance also decreased roughly 0.3 times when 5 times shorter time step was used and kept constant during calculation. It was observed mainly in those critical points where balance equation changes its sign quickly from positive to negative values.

It may be useful to see what happen during one period of the oscillation when the inbalance begins to be small. The Figure 3.2b) is an analog of Fig. 3.1 for one period of oscillations when coarse grid is used, however, the inbalance is now plotted without an additive constant. The Fig. 3.2.a) additionaly represents the behaviour of the variables

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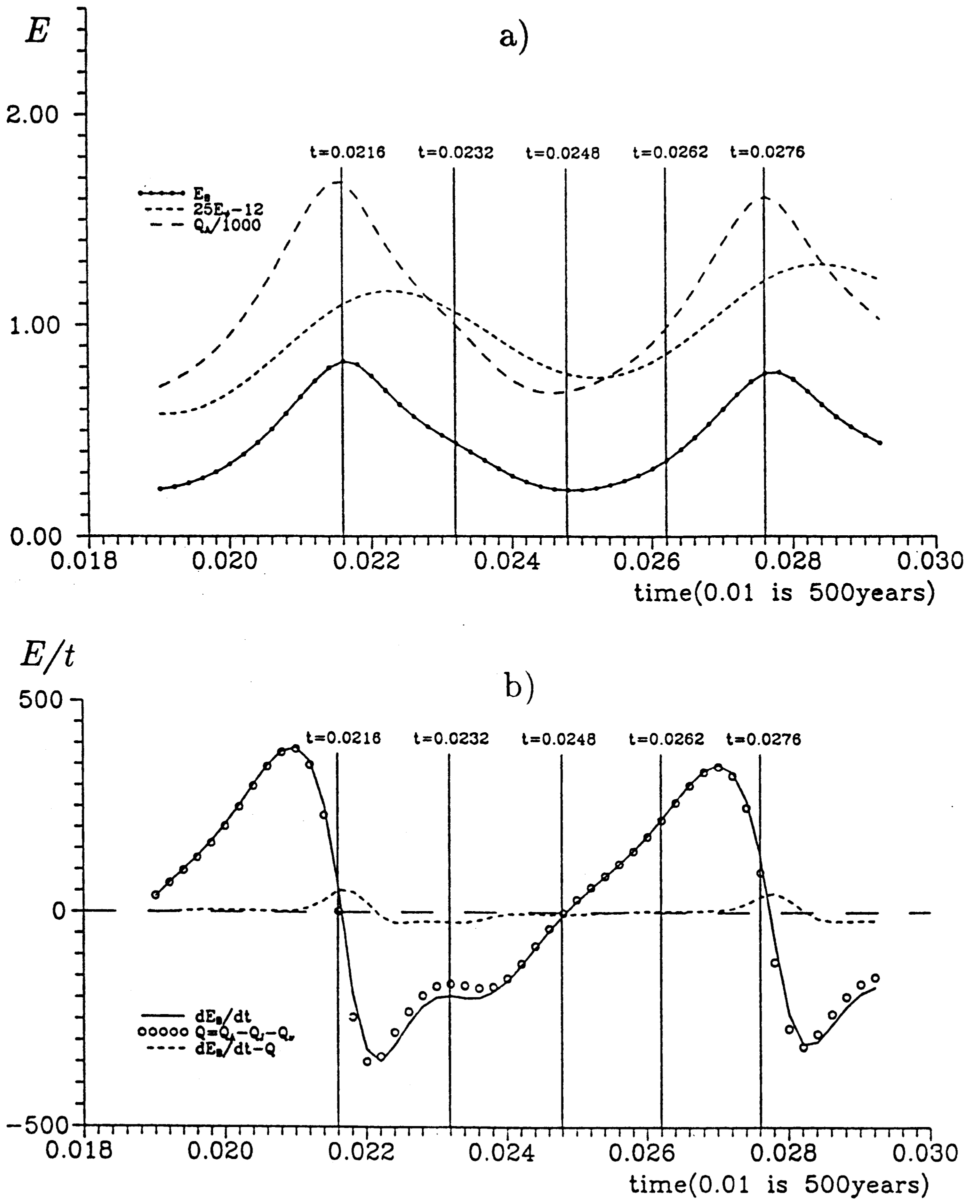


Figure 3.2

$E_B$ (solid line), $E_\psi$ (short dashed line) and $Q_A$ (long dashed line) where the variables are scaled to aid comparison. The inbalance shows some substantial non-zero values at the times $t = 0.0216$ and $t = 0.0276$ of maximum $E_B$. However, this inbalance generally decreases when the fine space grid is used and also it somewhat decreases when the time step is shortened.

## 4. Conclusion

The balance equation helps to decide which time intervals the time dependent behaviour of the solution in model Z makes physical sense. After several initial time steps, where the inbalance caused by the numerical process is large, the later time steps reflect the true time behaviour of the solution. The oscillating solutions are characterized by two time scales. The short is the period of oscillations and the long is the decay time of oscillation. The small time scale seems to be a diffusion time for toroidal field B through the layer at CMB while the long time scale is the diffusion time of poloidal field from the generating region into main volume of the core (see Anufriev et. al.,1992).

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