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HOPF BIFURCATION ANALYSIS FOR A RATIO-DEPENDENT PREDATOR–PREY SYSTEM INVOLVING TWO DELAYS

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Abstract

The aim of this paper is to give a detailed analysis of Hopf bifurcation of a ratio-dependent predator–prey system involving two discrete delays. A delay parameter is chosen as the bifurcation parameter for the analysis. Stability of the bifurcating periodic solutions is determined by using the centre manifold theorem and the normal form theory introduced by Hassard et al. Some of the bifurcation properties including the direction, stability and period are given. Finally, our theoretical results are supported by some numerical simulations.

2010 *Mathematics subject classification*: primary 34K18; secondary 34K13, 34K20, 37G15, 92D25.

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1. Introduction

Differential equations which include parameters are common for modelling in dynamical systems (see [2, 4, 6, 13–15, 20]). Furthermore, studying solutions and their behaviour that depends on a parameter is crucial to understand biological systems. In order to reflect dynamical behaviour of models that depend on the past history of the system, it is often necessary to incorporate time delays into these models [4, 6, 15, 20]. The response of a biological system to a particular input is often not immediate but is delayed. For example, in the pharmacokinetics model, there may be a delay before the drug enters the blood stream [2].

Exploring the dynamical behaviours of models involving time delays has attracted very much interest in mathematical biology, medicine, ecology, economics and so on. In past decades, many theoreticians and experimentalists have paid great attention to differential equations with delay which has significant biological and physical

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meaning. More specifically, they concentrated on the stability of solutions and Hopf bifurcation occurrence (see [1, 5, 7–12, 17–19, 21–27] and the references therein). In delay differential equations, periodic solutions can arise through the *local* Hopf bifurcation. Several methods for analysing the nature of Hopf bifurcations have been described in the literature. Integral averaging has been used by Chow and Mallet-Paret, the Fredholm alternative has been used by Iooss and Joseph, the implicit function theorem by Hale and Lunel, multi-scale expansion by Nayfeh et al. and the centre manifold projection by Hassard et al., Stépán and Kalmár-Nagy (see [4] and the references therein). The centre manifold theory is one of the rigorous mathematical tools to study bifurcations of delay differential equations [13, 14].

In this paper, we consider the following ratio-dependent Michaelis–Menton type (see [3, 16]) delayed predator–prey system

$$\begin{aligned}\frac{dN(t)}{dt} &= r_1 N(t) - \varepsilon P(t)N(t)\\ \frac{dP(t)}{dt} &= P(t)\Big(r_2 - \theta\frac{P(t-\tau_2)}{N(t-\tau_1)}\Big),\end{aligned} \tag{1.1}$$

where r_1, r_2, ε, θ are positive constants and τ_1 and τ_2 are the delay terms which are also positive. Here, $N(t)$ and $P(t)$ represent population densities of prey and predators at time t, respectively, and $N(t-\tau_1)$ and $P(t-\tau_2)$ represent the juveniles of prey and predators who were born at time $t-\tau_1$ and time $t-\tau_2$, respectively, and survive at time t. In this particular model, the prey population has a propensity for unbounded exponential growth, r_1N, which is limited by predation: the effect of the predators upon the prey population is measured by the functional response term, εPN. The predator population is logistic with time delay and its carrying capacity is proportional to the mature prey population; in other words, the predator selects its prey to be mature. Also, the predator population is limited by the number of *mature* prey per predator rather than by the number of prey (see [16, 27] for further discussion). We have incorporated two delay terms in the model which is more suitable for the real world. For example, some predator species need some time, say τ_2, for the ability of predation, that is, predators must be mature enough to capture prey. Predators capture only the adult prey with a certain maturation time, say τ_1, that is, prey must be mature enough to be captured.

The model (1.1) for $\tau_1 = \tau_2 = 0$ was considered by Zhou et al. [27]. They first determined stability conditions for a positive equilibrium point of the system. Later, they added a new term, which is called the Allee effect, into the model and analysed the impact of it on the dynamics of this predator–prey system. Following this work, Çelik analysed Hopf bifurcations of equation (1.1) for two cases, namely, (i) $\tau_1 \neq 0$, $\tau_2 = 0$ in [8] and (ii) $\tau_1 = 0$, $\tau_2 \neq 0$ in [7].

Our aim in this paper is to give a detailed Hopf bifurcation analysis of equation (1.1) for the first case $\tau_1 = \tau_2 = \tau$. The second case, in which $\tau_1 \neq \tau_2$ is to be studied later since the corresponding characteristic equation of the linearized system is very complex to find its roots. In this study, we chose τ as bifurcation parameter. We

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investigate the linear stability and the existence of Hopf bifurcation by analysing the associated characteristic equation, and determine the required conditions on parameters. In other words, we use the Hopf bifurcation theorem [13, 14] to investigate the effect of delay on solutions of equation (1.1) and to show that when τ passes through a certain critical value, the positive equilibrium loses its stability and a Hopf bifurcation occurs. Furthermore, when τ takes a sequence of critical values, the system (1.1) undergoes a Hopf bifurcation near positive equilibrium at these critical values of τ. We determine the direction of the bifurcation, and the stability and the period of the bifurcating periodic solutions by using the centre manifold theorem and the normal form theory introduced by Hassard et al. [14]. Finally, we give some numerical simulations in order to support our theoretical results.

This paper is organized as follows. In Section 2, stability analysis of a constant equilibrium point and existence of Hopf bifurcation is investigated. In Section 3, the bifurcation properties including direction, stability and period of the periodic solutions are studied. Finally, in Section 4, we consider a predator–prey model involving two delays and simulate it using MATLAB to show the effect of the delay term and support our theoretical results.

2. Stability analysis and Hopf bifurcation

In this section, we consider equation (1.1) when $\tau_1 = \tau_2 = \tau$, and investigate stability of equilibrium points and conditions on the parameters to show existence of Hopf bifurcation. From now on, we will consider the following model

$$\begin{aligned} \frac{dN(t)}{dt} &= r_1 N(t) - \varepsilon P(t)N(t) \\ \frac{dP(t)}{dt} &= P(t)\Big(r_2 - \theta\frac{P(t-\tau)}{N(t-\tau)}\Big). \end{aligned} \tag{2.1}$$

When there is no time delay, that is $\tau_1 = \tau_2 = 0$, the positive equilibrium point of the system (1.1) is asymptotically stable (see [27] for details). The system (2.1) has a unique positive equilibrium point, namely, $E^* = (N^*, P^*)$, where $N^* = r_1\theta/r_2\varepsilon$ and $P^* = r_1/\varepsilon$. From a biological point of view, we only consider the positive equilibrium points. Note that this equilibrium point is also the equilibrium point of equation (1.1). From the equilibria conditions, we know that

$$r_1 - \varepsilon P^* = 0 \quad \text{and} \quad r_2 - \theta\frac{P^*}{N^*} = 0.$$

If one shifts the equilibrium point $E^* = (N^*, P^*)$ to $(0, 0)$ by using linear transformations, namely, $x(t) = N(t) - N^*$ and $y(t) = P(t) - P^*$, the equations become

$$\frac{dx(t)}{dt} = -\varepsilon N^* y(t) - \varepsilon x(t)y(t)$$

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$$\frac{dy(t)}{dt} = \theta\frac{(P^*)^2}{(N^*)^2}x(t-\tau) - \theta\frac{P^*}{N^*}y(t-\tau) - \frac{\theta}{N^*}y(t)y(t-\tau) + \theta\frac{P^*}{(N^*)^2}x(t-\tau)y(t-\tau) + \theta\frac{P^*}{(N^*)^2}x(t-\tau)y(t) - \theta\frac{(P^*)^2}{(N^*)^3}x^2(t-\tau) + H.O.T., \tag{2.2}$$

where $H.O.T.$ denotes the higher-order terms, so that the system (2.2) is linearly equal to

$$\begin{aligned}\frac{dx(t)}{dt} &= -\varepsilon N^* y(t)\\ \frac{dy(t)}{dt} &= \theta\frac{P^{*2}}{N^{*2}}x(t-\tau) - \theta\frac{P^*}{N^*}y(t-\tau).\end{aligned} \tag{2.3}$$

Characteristic equation associated with equation (2.3) is

$$\lambda^2 + r_2\lambda e^{-\lambda\tau} + be^{-\lambda\tau} = 0, \tag{2.4}$$

where $r_2 = \theta(P^*/N^*)$ and $b = r_2P^*\varepsilon$. If $\tau = 0$, that is, when there is no time delay, equation (2.4) will be

$$\lambda^2 + r_2\lambda + b = 0. \tag{2.5}$$

The eigenvalues associated with equation (2.5) are $\lambda_{1,2} = (-r_2 \pm \sqrt{r_2^2 - 4b})/2$. Since r_2 and b are positive values, one has the following lemma, which was given in [27] and [7] before.

Lemma 2.1. *The roots $\lambda_{1,2} = (-r_2 \pm \sqrt{r_2^2 - 4b})/2$ of equation (2.5) have always negative real parts, that is, the equilibrium point of the system (2.1) with $\tau = 0$ is asymptotically stable.*

Now, let us take $\tau \neq 0$. We shall investigate the roots of the transcendental equation (2.4) since the stability of the equilibrium point $(0, 0)$ of the linear system (2.3) depends on the locations of the roots of the characteristic equation (2.4).

Lemma 2.2. *The transcendental equation (2.4) has one purely imaginary root.*

Proof. Let $\lambda = iw$ be a root of the characteristic equation (2.4) with $w > 0$. Substituting this into (2.4) and separating real and imaginary parts yields the following equations

$$-w^2 + r_2 w \sin w\tau + b\cos w\tau = 0, \tag{2.6}$$

$$r_2 w \cos w\tau - b \sin w\tau = 0. \tag{2.7}$$

By taking square of (2.6) and (2.7) and then adding them up, one obtains

$$w^4 - r_2^2 w^2 - b^2 = 0. \tag{2.8}$$

From (2.8), it follows that

$$w_\pm^2 = \frac{r_2^2 \pm \sqrt{r_2^4 + 4b^2}}{2}.$$

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Since w^2 must be positive, one concludes

$$w_+^2 = \frac{r_2^2 + \sqrt{r_2^4 + 4b^2}}{2}$$

so that

$$w_+^{(1)} = \sqrt{\frac{r_2^2 + \sqrt{r_2^4 + 4b^2}}{2}}.$$

Now, substituting $w_+^{(1)}$ into both (2.6) and (2.7) one calculates a sequence of the critical values of τ defined by

$$\tau_k = \frac{1}{w_+^{(1)}}\left\{\cos^{-1}\left(\frac{b}{(w_+^{(1)})^2}\right)\right\} + \frac{2k\pi}{w_+^{(1)}} \quad (k = 0, 1, 2, 3, \ldots).$$

This completes the proof. □

Note. One may see easily that the purely imaginary root $iw_+^{(1)}$ is simple.

Now, suppose that $\lambda_k(\tau) = \alpha_k(\tau) + iw_k(\tau)$, where w_0 denotes the root of equation (2.4) near $\tau = \tau_k$ satisfying $\alpha_k(\tau_k) = 0$ and $w_k(\tau_k) = w_0 := w_+^{(1)}$ for $k = 0, 1, 2, 3, \ldots$. Then, we have the following transversality condition.

Lemma 2.3. *The following transversality conditions are satisfied*

$$\frac{dRe\lambda_k(\tau_k)}{d\tau} > 0 \quad (k = 0, 1, 2, 3, \ldots),$$

that is, the system (2.1) undergoes a Hopf bifurcation at the positive equilibrium point (N^*, P^*) *when* $\tau = \tau_k$ $(k = 0, 1, 2, 3, \ldots)$.

Proof. Differentiating equation (2.4) with respect to τ,

$$\frac{d\lambda}{d\tau} = \frac{r_2\lambda^2 e^{-\lambda\tau} + b\lambda e^{-\lambda\tau}}{2\lambda + r_2 e^{-\lambda\tau} - r_2\lambda e^{-\lambda\tau} - b\tau e^{-\lambda\tau}}.$$

Now, one can obtain

$$\left(Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right) = Re\left[\frac{2}{w_0^2} + \frac{r_2}{iw_0(r_2 iw_0 + b)} - \frac{\tau}{iw_0}\right]$$

$$\left(Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right)_{\lambda = iw_0} = \frac{\sqrt{r_2^4 + 4b^2}}{w_0^4} > 0.$$

□

From Lemma 2.3, $d(Re\lambda)/d\tau > 0$ is satisfied at τ_0, when the root crosses the imaginary axis from left to right as τ increases. From Cooke and Grossman's article [11], if there is one imaginary root, then one can say that an unstable zero solution never becomes stable. If it is stable for $\tau = 0$, then it becomes unstable at the smallest value of τ, for which an imaginary root exists.

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Lemma 2.4. *The following statements hold.*

1. *The equilibrium point (N^*, P^*) is asymptotically stable for $\tau = 0$.*
2. *The equilibrium point (N^*, P^*) is asymptotically stable for $\tau < \tau_0$ and unstable $\tau > \tau_0$, where $\tau_0 = (1/w_0)\{\cos^{-1}(b/w_0^2)\}$. Furthermore, the system (2.1) undergoes a Hopf bifurcation at (N^*, P^*) when $\tau = \tau_0$.*

Proof. From Lemma 2.1, statement (1) is clear. For statement (2), we have shown that when $\tau = \tau_0$, equation (2.1) has a pair of purely imaginary roots $\pm iw_0$, which are simple. So, only we need to show that all remaining roots have negative real parts and equation (2.4) has at least one root with a strictly positive real part, when $\tau > \tau_0$.

We know that τ_0 is the first value of τ, at which the equation (2.4) has a pair of purely imaginary roots. Moreover, when $\tau = 0$

$$\lambda^2 + r_2\lambda + b = 0$$

has roots $\lambda_{1,2} = (-r_2 \pm \sqrt{r_2^2 - 4b})/2$. Obviously, both λ_1 and λ_2 have negative real parts when $b > 0$. By Rouche's theorem, as τ varies, the sum of the multiplicities of the roots of equation (2.4) in the open right half-plane can change only if a root appears on or crosses the imaginary axis. Since τ_0 is the minimal positive value of τ such that equation (2.4) has a pair of purely imaginary roots, we can see that all of its roots with $\tau \in [0, \tau_0)$ have strictly negative real parts. Suppose equation (2.4) with $\tau = \tau_0$ has a root with positive real part, say $\lambda(\tau) = \alpha(\tau) + iw(\tau)$, where $\alpha(\tau_0) > 0$. Since $\alpha(\tau)$ is continuous for $\tau \in \delta(\tau_0)$, a neighbourhood of τ_0, we have $\alpha(\tau) > 0$ for $\tau < \tau_0$ and close to τ_0. It follows that equation (2.4) has a root with positive real part for $\tau < \tau_0$, $\tau \in \delta(\tau_0)$, which contradicts the above discussion. □

3. Direction and stability of the Hopf bifurcation

In this section, we determine some of the properties of the Hopf bifurcation, namely, its direction, stability and period by applying the normal form theory and the centre manifold theory presented in Hassard et al. [14]. Following the procedure in [9], we compute the coordinates of centre manifold C_0 at $\mu = 0$.

Let $q(\theta)$ and $q^*(s)$ be the eigenvectors of adjoint operators A and A^* (defined in the Appendix), respectively. Define

$$z(t) = \langle q^*, x_t\rangle, \quad w(t,\theta) = x_t - 2Re\{z(t)q(\theta)\},$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1, 0)$. On the centre manifold, we have

$$w(t,\theta) = w(z(t), \bar{z}(t), \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots$$

where z and $\bar{z}$ are local coordinates for the centre manifold C_0 in the direction of q and q^*, respectively. For $x_t \in C_0$, since $\mu = 0$,

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{x}_t\rangle = \langle q^*, Ax_t + Rx_t\rangle \\ &= i\omega_0\langle q^*, x_t\rangle + \overline{q^*}(0)f_0(z,\bar{z}) \equiv i\omega_0 z(t) + g(z,\bar{z}),\end{aligned}$$

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where

$$g(z,\overline{z}) = \overline{q^*}(0) f_0(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots .$$

Here $f_0(z,\overline{z})$ denotes $f(z,\overline{z})$ at $\mu = 0$.

Here, the first four coefficients will be used for determining properties of the bifurcation and are of the form

$$g_{20} = 2\overline{D}\tau_k \begin{bmatrix} -\varepsilon\alpha\overline{\alpha^*} - \dfrac{\theta}{N^*}\alpha^2 e^{-i\omega_0} + \theta\dfrac{P^*}{(N^*)^2}\alpha e^{-2i\omega_0} \\ +\theta\dfrac{P^*}{(N^*)^2}\alpha e^{-i\omega_0} - \theta\dfrac{(P^*)^2}{(N^*)^3} e^{-2i\omega_0} \end{bmatrix},$$

$$g_{11} = \overline{D}\tau_k \begin{bmatrix} -\varepsilon\overline{\alpha}\overline{\alpha^*} - \varepsilon\alpha\overline{\alpha^*} - \dfrac{\theta}{N^*}\alpha\overline{\alpha} e^{i\omega_0} - \dfrac{\theta}{N^*}\alpha\overline{\alpha} e^{-i\omega_0} \\ +\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha} + \theta\dfrac{P^*}{(N^*)^2}\alpha \\ +\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha} e^{-i\omega_0} + \theta\dfrac{P^*}{(N^*)^2}\alpha e^{i\omega_0} - 2\theta\dfrac{(P^*)^2}{(N^*)^3} \end{bmatrix},$$

$$g_{02} = 2\overline{D}\tau_k \begin{bmatrix} -\varepsilon\overline{\alpha}\overline{\alpha^*} - \dfrac{\theta}{N^*}\overline{\alpha}^2 e^{i\omega_0} + \theta\dfrac{P^*}{(N^*)^2}\overline{\alpha} e^{2i\omega_0} \\ +\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha} e^{i\omega_0} - \theta\dfrac{(P^*)^2}{(N^*)^3} e^{2i\omega_0} \end{bmatrix}$$

and

$$g_{21} = 2\overline{D}\tau_k \begin{bmatrix} -\varepsilon\overline{\alpha}\overline{\alpha^*}\dfrac{w_{20}^{(1)}(0)}{2} - \varepsilon\overline{\alpha^*}\dfrac{w_{20}^{(2)}(0)}{2} - \varepsilon\overline{\alpha}\overline{\alpha^*}\dfrac{w_{20}^{(1)}(0)}{2} - \varepsilon\alpha\overline{\alpha^*} w_{11}^{(1)}(0) \\ -\varepsilon\overline{\alpha^*} w_{11}^{(2)}(0) - \dfrac{\theta}{N^*}\overline{\alpha}\dfrac{w_{20}^{(2)}(0)}{2} e^{i\omega_0} - \dfrac{\theta}{N^*}\overline{\alpha}\dfrac{w_{20}^{(2)}(-1)}{2} \\ -\dfrac{\theta}{N^*}\alpha w_{11}^{(2)}(0) e^{-i\omega_0} - \dfrac{\theta}{N^*}\alpha w_{11}^{(2)}(-1) \\ +\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha}\dfrac{w_{20}^{(1)}(-1)}{2} e^{i\omega_0} + \theta\dfrac{P^*}{(N^*)^2}\dfrac{w_{20}^{(2)}(-1)}{2} e^{i\omega_0} \\ +\theta\dfrac{P^*}{(N^*)^2}\alpha w_{11}^{(1)}(-1) e^{-i\omega_0} + \theta\dfrac{P^*}{(N^*)^2} w_{11}^{(2)}(-1) e^{-i\omega_0} \\ +\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha}\dfrac{w_{20}^{(2)}(-1)}{2} + \theta\dfrac{P^*}{(N^*)^2}\dfrac{w_{20}^{(2)}(0)}{2} e^{i\omega_0} \\ +\theta\dfrac{P^*}{(N^*)^2}\alpha w_{11}^{(1)}(-1) + \theta\dfrac{P^*}{(N^*)^2} w_{11}^{(2)}(0) e^{-i\omega_0} \\ -\theta\dfrac{(P^*)^2}{(N^*)^3} 2 w_{11}^{(1)}(-1) e^{-i\omega_0} - \theta\dfrac{(P^*)^2}{(N^*)^3} w_{20}^{(1)}(-1) e^{i\omega_0} \end{bmatrix},$$

where the terms $w_{11}^{(i)}(\theta)$ and $w_{20}^{(i)}(\theta)$ (for $i = 1, 2$ and $\theta = -1, 0$) and $\overline{D}$ are calculated in the Appendix. Now, using these coefficients we can evaluate the following values

$$c_1(0) = \frac{i}{2\omega\tau_k}\Big(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\Big) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{Re\{c_1(0)\}}{Re\{\lambda'(\tau_k)\}}, \tag{3.1}$$
$$\beta_2 = 2Re\{c_1(0)\}$$

and

$$T_2 = -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tau_k)\}}{\omega\tau_k}.$$

Finally, using the quantities above, some properties of Hopf bifurcation can be determined, which are given by the following theorem.

THEOREM 3.1. *The quantity μ_2 determines the direction of Hopf bifurcation: if $\mu_2 > 0$, then Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $\tau > \tau_0$; and if $\mu_2 < 0$, then Hopf bifurcation is subcritical and the bifurcating periodic solutions exist for $\tau < \tau_0$. Here β_2 determines the stability of the bifurcating periodic solutions: bifurcating periodic solutions are stable if $\beta_2 < 0$; unstable if $\beta_2 > 0$. In addition T_2 determines the period of the bifurcating solution: the period increases if $T_2 > 0$ and decreases if $T_2 < 0$.*

In the following section, we will give a numerical example to verify the theoretical results.

4. Numerical simulations

In this section, we simulate a predator–prey model by using MATLAB to support our theoretical results. As an example, we consider equation (2.1) by setting the parameters as $r_1 = 0.45$, $r_2 = 0.1$, $\varepsilon = 0.03$, $\theta = 0.05$, that is,

$$\frac{dN(t)}{dt} = 0.45N(t) - 0.03P(t)N(t)$$
$$\frac{dP(t)}{dt} = P(t)\Big(0.1 - 0.05\frac{P(t-\tau)}{N(t-\tau)}\Big). \tag{4.1}$$

The system (4.1) has only one positive equilibrium point, namely, $E^* = (N^*, P^*) = (7.5, 15)$. When there is no time delay, that is, $\tau = 0$, the equilibrium point $E^* = (7.5, 15)$ is asymptotically stable. From the results in Section 2, we find

$$w_0 = \sqrt{\frac{r_2^2 + \sqrt{r_2^4 + 4b^2}}{2}} = 0.2242, \quad \tau_0 = \frac{1}{w_0}\Big\{\cos^{-1}\Big(\frac{b}{w_0^2}\Big)\Big\} = 1.7631.$$

So, by Lemma (2.4), the equilibrium point E^* is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > 1.7631$. Hopf bifurcation occurs when $\tau = \tau_0 = 1.7631$.

Next, we determine the direction of the Hopf bifurcation and the stability and the period of the periodic solution of the system (4.1). Using the formulae obtained

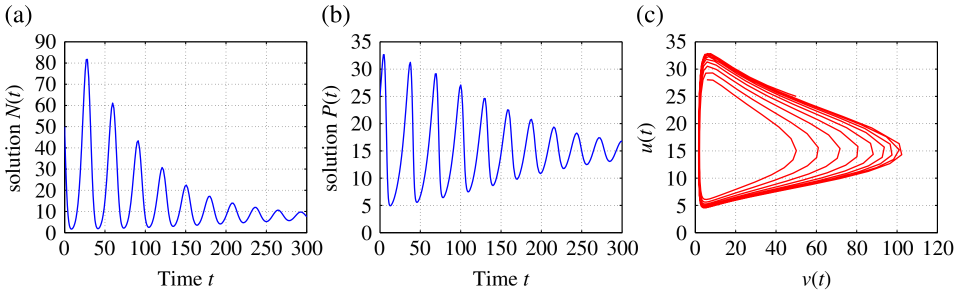

FIGURE 1. The trajectories of prey and predator densities versus time with the initial conditions $N_0 = 50$, $P_0 = 25$ when $\tau = 1.7 < \tau_0$ in (a) and (b). The phase portrait of prey density versus predator density for the same parameters when $\tau = 1.7 < \tau_0$ in (c).

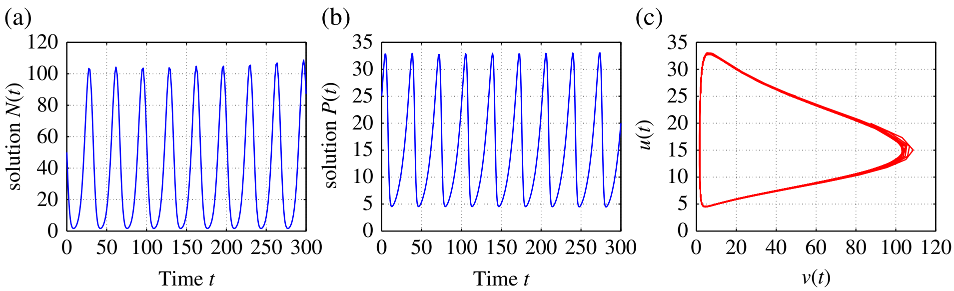

FIGURE 2. The trajectories of prey and predator densities versus time with the initial conditions $N_0 = 50$, $P_0 = 25$ when $\tau = \tau_0 = 1.7631$ in (a) and (b). The phase portrait of prey density versus predator density for the same parameters when $\tau = \tau_0 = 1.7631$ in (c).

for μ_2, β_2 and T_2 in Section 3, we compute these values as

$$\mu_2 = 0.1259 > 0, \quad \beta_2 = -0.0070 < 0, \quad T_2 = 0.0091 > 0.$$

These values show that since $\mu_2 > 0$, Hopf bifurcation is supercritical and the bifurcating periodic solution exists, when τ crosses τ_0 from left to the right. Finally, since $\beta_2 < 0$, the bifurcating periodic solution is stable with the increasing period. For simulations, initial conditions are taken as $(N_0, P_0) = (50, 25)$ and the MATLAB DDE (Delay Differential Equations) solver is used to simulate the system (4.1).

Figure 1 clearly shows that the equilibrium point E^* is asymptotically stable when $\tau \in [0, 1.7631)$. Here, we take $\tau = 1.7 < \tau_0$ for simulations in Figure 1. From Figure 2, one can see that when $\tau = \tau_0$, a bifurcating periodic solution occurs. Figure 3 shows that for the values of $\tau > \tau_0$ (we take, for example, $\tau = 1.77 > \tau_0$ for simulations in Figure 2) the equilibrium point is unstable.

5. Conclusions and remarks

Former studies show that time delay affects dynamics of population models and plays an important role in stability analysis [1, 5, 7–12, 17–19, 21–27]. In this paper,

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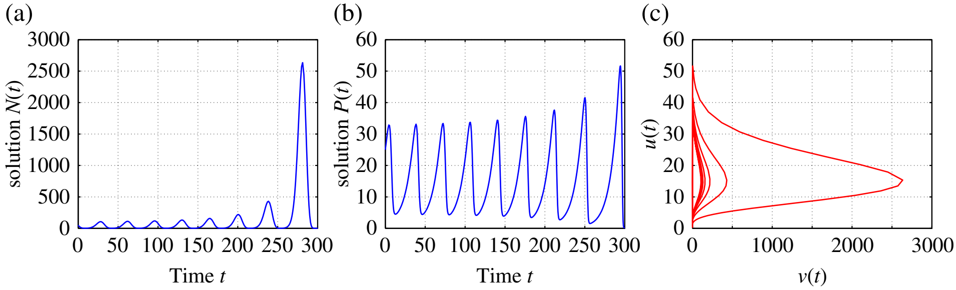

FIGURE 3. The trajectories of prey and predator densities versus time with the initial conditions $N_0 = 50$, $P_0 = 25$ when $\tau = 1.77 > \tau_0$ in (a) and (b). The phase portrait of prey density versus predator density for the same parameters when $\tau = 1.77 > \tau_0$ in (c).

we give a detailed Hopf bifurcation analysis of a ratio-dependent predator–prey system involving two discrete time delays. The system is defined by two differential equations in equation (1.1). We have included two delay terms in the model, which is more suitable than the model involving a single delay for the real world problems. For example, some predator species need some time, say τ_2, for the ability of predation, that is, predators must be mature enough to capture. Also, predators capture only the adult prey with a certain maturation time, say τ_1, that is, prey must be mature enough to be captured. In other words, the predator selects its prey from within mature prey. For the model, choosing the delay parameter τ as bifurcation parameter, we investigated necessary conditions on parameters at which the Hopf bifurcation occurs. Using the normal form theory and the centre manifold theorem given by Hassard and Kazarinoff [14], the formulae that determine direction, period and stability of the periodic solution are obtained. Finally, we supported our theoretical results via some numerical simulations. We showed that when the bifurcation parameter τ passes through a sequence of critical bifurcation values, the stability of the positive equilibrium point of equation (1.1) changes from stable to unstable, so that the Hopf bifurcation occurs at this critical value. The sign of μ_2 given by (3.1) determines whether it is a supercritical or a subcritical bifurcation.

Appendix

In Section 2, we have shown that the system (2.1) undergoes a Hopf bifurcation at the equilibrium point when $\tau = \tau_0$. In Section 3, we have summarized the bifurcation properties. The appendix involves the details of these calculations. In order to compute the properties of the Hopf bifurcation, we use the method on the normal form theory and the centre manifold theory presented by Hassard and Kazarinoff [14].

We first begin with normalizing the delay via $t \to \tau t$ and transforming equation (2.1) by using $x_1(t) = N(t) - N^*$, $x_2(t) = P(t) - P^*$, $\tau = \tau_k + \mu$. This leads to the following functional differential equation system in $C = C([-1, 0], \mathbb{R}^2)$

$$x'(t) = L_\mu(x_t) + f(\mu, x_t), \tag{A.1}$$

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where $x(t) = (x_1(t), x_2(t))^{\mathrm{T}} \in \mathbb{R}^2$, and $L_\mu : C \longrightarrow \mathbb{R}^2$, $f : \mathbb{R} \times C \to \mathbb{R}^2$ are given, respectively, by

$$L_\mu(\phi) = (\tau_k + \mu)\begin{bmatrix} 0 & -a_{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} + (\tau_k + \mu)\begin{bmatrix} 0 & 0 \\ a_{21} & -a_{22} \end{bmatrix}\begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \end{bmatrix},$$

where $a_{12} = \varepsilon N^*$, $a_{21} = \theta((P^*)^2/(N^*)^2)$, $a_{22} = \theta(P^*/N^*)$ and

$$f(\mu, \phi) = (\tau_k + \mu)\begin{bmatrix} f_{11} \\ f_{12} \end{bmatrix}, \tag{A.2}$$

where

$$f_{11} = -\varepsilon\phi_1(0)\phi_2(0),$$

and

$$f_{12} = -\frac{\theta}{N^*}\phi_2(0)\phi_2(-1) + \theta\frac{P^*}{(N^*)^2}\phi_1(-1)\phi_2(-1) + \theta\frac{P^*}{(N^*)^2}\phi_1(-1)\phi_2(0) - \theta\frac{(P^*)^2}{(N^*)^3}\phi_1^2(-1),$$

where $\phi = (\phi_1, \phi_2) \in C$.

By the Riesz representation theorem, there exists a matrix-valued function $\eta(\theta, \mu) : [-1, 0] \longrightarrow \mathbb{R}^4$ whose components have bounded variation and for $\theta \in [-1, 0]$

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, 0)\phi(\theta). \tag{A.3}$$

We can choose

$$\eta(\theta, \mu) = (\tau_k + \mu)\begin{bmatrix} 0 & -a_{12} \\ 0 & 0 \end{bmatrix}\delta(\theta) + (\tau_k + \mu)\begin{bmatrix} 0 & 0 \\ a_{21} & -a_{22} \end{bmatrix}\delta(\theta + 1),$$

where δ is the Dirac delta function. For $\phi \in C^1([-1, 0]$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta} & \text{if } \theta \in [-1, 0) \\[2ex] \displaystyle\int_{-1}^{0} d\eta(\mu, s)\,\phi(s) & \text{if } \theta = 0 \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0 & \text{if } \theta \in [-1, 0) \\ f(\mu, \phi) & \text{if } \theta = 0. \end{cases}$$

Then system (A.1) is equivalent to

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \tag{A.4}$$

where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1, 0)$. For $\psi \in C^1([-1, 0], (\mathbb{R}^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds} & \text{if } s \in (0, 1] \\[2ex] \displaystyle\int_{-1}^{0} d\eta^{\mathrm{T}}(\mu, t)\psi(-t) & \text{if } s = 0 \end{cases}$$

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and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)\, d\eta(\theta)\phi(\xi)\, d\xi, \tag{A.5}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then, $A(0)$ and $A^*(0)$ are adjoint operators. Suppose that $q(\theta)$ and $q^*(s)$ are eigenvectors of A and A^*, corresponding to $\lambda = i\omega_0$ and $\overline{\lambda} = -i\omega_0$, respectively. Let us take $q(\theta) = (1, \alpha)^{\mathrm{T}} e^{i\omega_0\theta}$. Since $A(0)q(\theta) = i\omega_0 q(\theta)$, we can easily find α from the definitions of $A(0), L_\mu\phi$ and $\eta(\theta, \mu)$. Thus, $q(\theta) = (1, \alpha)^{\mathrm{T}} e^{i\omega_0\theta}$ where $\alpha = (-iw_0)/(a_{12}\tau_k)$. Similarly, let $q^*(s) = D(\alpha^*, 1)e^{i\omega_0 s}$. We need to find the value of D and α^*. From the definition of A^*, we can find $\alpha^* = (-\tau_0 a_{21} e^{iw_0})/(iw_0)$. To calculate D, the relation $\langle q^*(s), q(\theta)\rangle = 1$ is used. From (A.5), one obtains

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \overline{D}(\overline{\alpha^*}, 1)(1, \alpha)^{\mathrm{T}} - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{D}(\overline{\alpha^*}, 1)e^{-i\omega_0(\xi-\theta)}\, d\eta(\theta)(1, \alpha)^{\mathrm{T}} e^{i\omega_0\xi}\, d\xi \\ &= \overline{D}\Big\{\overline{\alpha^*} + \alpha - \int_{-1}^{0} (\overline{\alpha^*}, 1)\theta e^{i\omega_0\theta}\, d\eta(\theta)(1, \alpha)^{\mathrm{T}}\Big\}.\end{aligned}$$

Thus, we can choose $\overline{D}$ as

$$\overline{D} = \frac{1}{\overline{\alpha^*} + \alpha + \tau_k e^{-i\omega_0}(a_{21} - a_{22}\alpha)}$$

such that $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \overline{q}(\theta)\rangle = 0$.

Using similar notation as in Hassard et al. [14], we first compute the coordinates to describe the centre manifold C_0 at $\mu = 0$. Let x_t be the solution of equation (A.4) when $\mu = 0$. Define

$$z(t) = \langle q^*, x_t\rangle, \quad w(t, \theta) = x_t - 2Re\{z(t)q(\theta)\}. \tag{A.6}$$

On the centre manifold, we have

$$w(t, \theta) = w(z(t), \overline{z}(t), \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots,$$

where z and $\overline{z}$ are local coordinates for the centre manifold C_0 in the direction of q and q^*, respectively. For $x_t \in C_0$, we have

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{x}_t\rangle = \langle q^*, Ax_t + Rx_t\rangle \\ &= i\omega_0\langle q^*, x_t\rangle + \overline{q^*}(0)f_0(z, \overline{z}) \equiv i\omega_0 z(t) + g(z, \overline{z})\end{aligned}$$

since $\mu = 0$, where

$$g(z, \overline{z}) = \overline{q^*}(0)f_0(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{A.7}$$

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To evaluate the coefficients of $g(z,\bar{z})$, we need to rewrite equation (A.2). From (A.6), we have $x_t(x_{1t}(\theta), x_{2t}(\theta)) = w(t,\theta) + zq(\theta) + \overline{zq(\theta)}$. Since $q(\theta) = (1,\alpha)^{\mathrm{T}} e^{i\omega_0\theta}$, we obtain

$$x_{1t}(0) = z + \bar{z} + w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\bar{z} + w_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3),$$
$$x_{2t}(0) = z\alpha + \overline{z\alpha} + w_{20}^{(2)}(0)\frac{z^2}{2} + w_{11}^{(2)}(0)z\bar{z} + w_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3),$$
$$x_{1t}(-1) = ze^{-i\omega_0} + \bar{z}e^{i\omega_0} + w_{20}^{(1)}(-1)\frac{z^2}{2} + w_{11}^{(1)}(-1)z\bar{z} + w_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3),$$
$$x_{2t}(-1) = z\alpha e^{-i\omega_0} + \overline{z\alpha}e^{i\omega_0} + w_{20}^{(2)}(-1)\frac{z^2}{2} + w_{11}^{(2)}(-1)z\bar{z} + w_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3).$$

From (A.3), we have

$$g(z,\bar{z}) = \overline{q^*}(0)f_0(z,\bar{z}) = \overline{D}\tau_k(\overline{\alpha^*}, 1)\begin{bmatrix} f_{11}^0 \\ f_{12}^0 \end{bmatrix},$$

where

$$f_{11}^0 = -\varepsilon x_{1t}(0)x_{2t}(0)$$

and

$$\begin{aligned} f_{12}^0 = &-\frac{\theta}{N^*}x_{2t}(0)x_{2t}(-1) + \theta\frac{P^*}{(N^*)^2}x_{1t}(-1)x_{2t}(-1) \\ &+ \theta\frac{P^*}{(N^*)^2}x_{1t}(-1)x_{2t}(0) - \theta\frac{(P^*)^2}{(N^*)^3}x_{1t}^2(-1). \end{aligned}$$

Thus,

$$\begin{aligned} g(z,\bar{z}) = {} & \overline{D}\tau_k z^2 \left[\begin{gathered} -\varepsilon\alpha\overline{\alpha^*} - \frac{\theta}{N^*}\alpha^2 e^{-i\omega_0} + \theta\frac{P^*}{(N^*)^2}\alpha e^{-2i\omega_0} \\ + \theta\frac{P^*}{(N^*)^2}\alpha e^{-i\omega_0} - \theta\frac{(P^*)^2}{(N^*)^3}e^{-2i\omega_0} \end{gathered} \right] \\ & + \overline{D}\tau_k z\bar{z} \left[\begin{gathered} -\varepsilon\bar{\alpha}\overline{\alpha^*} - \varepsilon\alpha\overline{\alpha^*} - \frac{\theta}{N^*}\alpha\bar{\alpha}e^{i\omega_0} - \frac{\theta}{N^*}\alpha\bar{\alpha}e^{-i\omega_0} \\ + \theta\frac{P^*}{(N^*)^2}\bar{\alpha} + \theta\frac{P^*}{(N^*)^2}\alpha + \theta\frac{P^*}{(N^*)^2}\bar{\alpha}e^{-i\omega_0} \\ + \theta\frac{P^*}{(N^*)^2}\alpha e^{i\omega_0} - 2\theta\frac{(P^*)^2}{(N^*)^3} \end{gathered} \right] \\ & + \overline{D}\tau_k \bar{z}^2 \left[\begin{gathered} -\varepsilon\bar{\alpha}\overline{\alpha^*} - \frac{\theta}{N^*}\bar{\alpha}^2 e^{i\omega_0} + \theta\frac{P^*}{(N^*)^2}\bar{\alpha}e^{2i\omega_0} \\ + \theta\frac{P^*}{(N^*)^2}\bar{\alpha}e^{i\omega_0} - \theta\frac{(P^*)^2}{(N^*)^3}e^{2i\omega_0} \end{gathered} \right] \end{aligned}$$

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$$
+\overline{D}\tau_k z^2\overline{z}\begin{bmatrix}
-\varepsilon\overline{\alpha}\overline{\alpha^*}\dfrac{w_{20}^{(1)}(0)}{2}-\varepsilon\overline{\alpha^*}\dfrac{w_{20}^{(2)}(0)}{2}-\varepsilon\overline{\alpha}\overline{\alpha^*}\dfrac{w_{20}^{(1)}(0)}{2}\\
-\varepsilon\alpha\overline{\alpha^*}w_{11}^{(1)}(0)-\varepsilon\overline{\alpha^*}w_{11}^{(2)}(0)-\dfrac{\theta}{N^*}\overline{\alpha}\dfrac{w_{20}^{(2)}(0)}{2}e^{i\omega_0}\\
-\dfrac{\theta}{N^*}\overline{\alpha}\dfrac{w_{20}^{(2)}(-1)}{2}-\dfrac{\theta}{N^*}\alpha w_{11}^{(2)}(0)e^{-i\omega_0}-\dfrac{\theta}{N^*}\alpha w_{11}^{(2)}(-1)\\
+\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha}\dfrac{w_{20}^{(1)}(-1)}{2}e^{i\omega_0}+\theta\dfrac{P^*}{(N^*)^2}\dfrac{w_{20}^{(2)}(-1)}{2}e^{i\omega_0}\\
+\theta\dfrac{P^*}{(N^*)^2}\alpha w_{11}^{(1)}(-1)e^{-i\omega_0}+\theta\dfrac{P^*}{(N^*)^2}w_{11}^{(2)}(-1)e^{-i\omega_0}\\
+\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha}\dfrac{w_{20}^{(2)}(-1)}{2}+\theta\dfrac{P^*}{(N^*)^2}\dfrac{w_{20}^{(2)}(0)}{2}e^{i\omega_0}\\
+\theta\dfrac{P^*}{(N^*)^2}\alpha w_{11}^{(1)}(-1)+\theta\dfrac{P^*}{(N^*)^2}w_{11}^{(2)}(0)e^{-i\omega_0}\\
-\theta\dfrac{(P^*)^2}{(N^*)^3}2w_{11}^{(1)}(-1)e^{-i\omega_0}-\theta\dfrac{(P^*)^2}{(N^*)^3}w_{20}^{(1)}(-1)e^{i\omega_0}
\end{bmatrix}
$$

$$
+H.O.T.
$$

If we compare the coefficients with (A.7), we get

$$
g_{20}=2\overline{D}\tau_k\Big[-\varepsilon\alpha\overline{\alpha^*}-\frac{\theta}{N^*}\alpha^2e^{-i\omega_0}+\theta\frac{P^*}{(N^*)^2}\alpha e^{-2i\omega_0}+\theta\frac{P^*}{(N^*)^2}\alpha e^{-i\omega_0}-\theta\frac{(P^*)^2}{(N^*)^3}e^{-2i\omega_0}\Big],
$$

$$
g_{11}=\overline{D}\tau_k\begin{bmatrix}
-\varepsilon\overline{\alpha}\overline{\alpha^*}-\varepsilon\alpha\overline{\alpha^*}-\dfrac{\theta}{N^*}\alpha\overline{\alpha}e^{i\omega_0}-\dfrac{\theta}{N^*}\alpha\overline{\alpha}e^{-i\omega_0}+\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha}+\theta\dfrac{P^*}{(N^*)^2}\alpha\\
+\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha}e^{-i\omega_0}+\theta\dfrac{P^*}{(N^*)^2}\alpha e^{i\omega_0}-2\theta\dfrac{(P^*)^2}{(N^*)^3}
\end{bmatrix},
$$

$$
g_{02}=2\overline{D}\tau_k\Big[-\varepsilon\overline{\alpha}\overline{\alpha^*}-\frac{\theta}{N^*}\overline{\alpha}^2e^{i\omega_0}+\theta\frac{P^*}{(N^*)^2}\overline{\alpha}e^{2i\omega_0}+\theta\frac{P^*}{(N^*)^2}\overline{\alpha}e^{i\omega_0}-\theta\frac{(P^*)^2}{(N^*)^3}e^{2i\omega_0}\Big],
$$

$$
g_{21}=2\overline{D}\tau_k\begin{bmatrix}
-\varepsilon\overline{\alpha}\overline{\alpha^*}\dfrac{w_{20}^{(1)}(0)}{2}-\varepsilon\overline{\alpha^*}\dfrac{w_{20}^{(2)}(0)}{2}-\varepsilon\overline{\alpha}\overline{\alpha^*}\dfrac{w_{20}^{(1)}(0)}{2}-\varepsilon\alpha\overline{\alpha^*}w_{11}^{(1)}(0)\\
-\varepsilon\overline{\alpha^*}w_{11}^{(2)}(0)-\dfrac{\theta}{N^*}\overline{\alpha}\dfrac{w_{20}^{(2)}(0)}{2}e^{i\omega_0}-\dfrac{\theta}{N^*}\overline{\alpha}\dfrac{w_{20}^{(2)}(-1)}{2}-\dfrac{\theta}{N^*}\alpha w_{11}^{(2)}(0)e^{-i\omega_0}\\
-\dfrac{\theta}{N^*}\alpha w_{11}^{(2)}(-1)+\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha}\dfrac{w_{20}^{(1)}(-1)}{2}e^{i\omega_0}+\theta\dfrac{P^*}{(N^*)^2}\dfrac{w_{20}^{(2)}(-1)}{2}e^{i\omega_0}\\
+\theta\dfrac{P^*}{(N^*)^2}\alpha w_{11}^{(1)}(-1)e^{-i\omega_0}+\theta\dfrac{P^*}{(N^*)^2}w_{11}^{(2)}(-1)e^{-i\omega_0}\\
+\theta\dfrac{P^*}{(N^*)^2}\overline{\alpha}\dfrac{w_{20}^{(2)}(-1)}{2}+\theta\dfrac{P^*}{(N^*)^2}\dfrac{w_{20}^{(2)}(0)}{2}e^{i\omega_0}+\theta\dfrac{P^*}{(N^*)^2}\alpha w_{11}^{(1)}(-1)\\
+\theta\dfrac{P^*}{(N^*)^2}w_{11}^{(2)}(0)e^{-i\omega_0}-\theta\dfrac{(P^*)^2}{(N^*)^3}2w_{11}^{(1)}(-1)e^{-i\omega_0}-\theta\dfrac{(P^*)^2}{(N^*)^3}w_{20}^{(1)}(-1)e^{i\omega_0}
\end{bmatrix}.
$$

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In order to determine g_{21}, we need to compute $w_{11}(\theta)$ and $w_{20}(\theta)$. From (A.6), we can write

$$\begin{aligned}\dot{w}(t,\theta) &= \dot{x}_t - 2Re\{\dot{z(t)}q(\theta)\}\\ &= \begin{cases} Aw - 2Re\{\overline{q^*}(0) f_0 q(\theta)\} & \text{if } \theta \in [-1,0)\\ Aw - 2Re\{\overline{q^*}(0) f_0 q(\theta)\} + f_0 & \text{if } \theta = 0\end{cases}\\ &:\equiv Aw + H(z,\bar{z},\theta),\end{aligned}$$

where

$$H(z,\bar{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{A.8}$$

On the other hand,

$$\dot{w} = w_z\dot{z} + w_{\bar{z}}\dot{\bar{z}}$$

on the centre manifold. Thus, comparing the coefficients one obtains that

$$(A - 2i\omega_0)w_{20}(\theta) = -H_{20}(\theta), \quad Aw_{11}(\theta) = -H_{11}(\theta). \tag{A.9}$$

For $\theta \in [-1,0)$,

$$H(z,\bar{z},\theta) = -2Re\{\dot{z}(t)q(\theta)\}.$$

Comparing the coefficients of (A.9) with those of (A.8), we obtain the following

$$\begin{aligned} H_{20}(\theta) &= -(q(\theta)g_{20} + \bar{q}(\theta)\bar{g}_{02}),\\ H_{11}(\theta) &= -(q(\theta)g_{11} + \bar{q}(\theta)\bar{g}_{11}),\\ H_{02}(\theta) &= -(q(\theta)g_{02} + \bar{q}(\theta)\bar{g}_{20}).\end{aligned}$$

From (A.9) and the definition of A, we get

$$w'_{20}(\theta) - 2i\omega_0 w_{20}(\theta) = q(\theta)g_{20} + \bar{q}(\theta)\bar{g}_{02}.$$

Then, since $q(\theta) = q(0)e^{i\omega_0\theta}$, we have

$$w_{20}(\theta) = \frac{i}{\omega_0}g_{20}q(0)e^{i\omega_0\theta} + \frac{i}{3\omega_0}\bar{g}_{02}\bar{q}(0)e^{-i\omega_0\theta} + E_1e^{2i\omega_0\theta},$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^{\mathrm{T}} \in \mathbb{R}^2$ is a constant vector. Similarly,

$$w_{11}(\theta) = \frac{-i}{\omega_0}g_{11}q(0)e^{i\omega_0\theta} + \frac{i}{\omega_0}\bar{g}_{11}\bar{q}(0)e^{-i\omega_0\theta} + E_2$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^{\mathrm{T}} \in \mathbb{R}^2$ is a constant vector. Let us find the values of E_1 and E_2. If we take $\theta = 0$ at (A.9), then

$$\int_{-1}^{0} d\eta(\theta)\, w_{20}(\theta) = 2i\omega_0 w_{20}(0) - H_{20}(0), \tag{A.10}$$

$$\int_{-1}^{0} d\eta(\theta)\, w_{11}(\theta) = -H_{11}(0). \tag{A.11}$$

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Also, for $\theta = 0$,

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau_k \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \tag{A.12}$$

where $n_1 = -\varepsilon\alpha$ and

$$n_2 = -\frac{\theta}{N^*}\alpha^2 e^{-i\omega_0} + \theta\frac{P^*}{(N^*)^2}\alpha e^{-2i\omega_0} + \theta\frac{P^*}{(N^*)^2}\alpha e^{-i\omega_0} - \theta\frac{(P^*)^2}{(N^*)^3}e^{-2i\omega_0},$$

in addition

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + \tau_k \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \tag{A.13}$$

where $s_1 = -2\varepsilon Re(\alpha)$ and

$$\begin{aligned} s_2 = &-\frac{\theta}{N^*}\alpha\overline{\alpha}e^{i\omega_0} - \frac{\theta}{N^*}\alpha\overline{\alpha}e^{-i\omega_0} + \theta\frac{P^*}{(N^*)^2}2Re(\alpha) + \theta\frac{P^*}{(N^*)^2}\alpha e^{i\omega_0} \\ &+ \theta\frac{P^*}{(N^*)^2}\overline{\alpha}e^{-i\omega_0} - 2\theta\frac{(P^*)^2}{(N^*)^3}. \end{aligned}$$

On the other hand, since $A(0)q(0) = iw_0q(0)$ and $A(0)\overline{q}(0) = iw_0\overline{q}(0)$, we can write

$$\left[iw_0 I - \int_{-1}^{0} e^{iw_0\theta}\, d\eta(\theta)\right]q(0) = 0, \tag{A.14}$$

$$\left[-iw_0 I - \int_{-1}^{0} e^{-iw_0\theta}\, d\eta(\theta)\right]\overline{q}(0) = 0. \tag{A.15}$$

Substituting (A.12) into (A.10) and using (A.14) we obtain

$$\left[2iw_0 I - \int_{-1}^{0} e^{2iw_0\theta}\, d\eta(\theta)\right]E_1 = 2\tau_k \begin{bmatrix} n_1 \\ n_2 \end{bmatrix},$$

which is equal to

$$\begin{bmatrix} 2iw_0 & \tau_k a_{12} \\ -\tau_k a_{21}e^{-2iw_0} & 2iw_0 + a_{22}e^{-2iw_0} \end{bmatrix} E_1 = 2\tau_k \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}.$$

Now, if one solves this system for E_1 one obtains

$$E_1^{(1)} = \frac{2\tau_k}{A_1} \begin{vmatrix} n_1 & \tau_k a_{12} \\ n_2 & 2iw_0 + a_{22}e^{-2iw_0} \end{vmatrix},$$

$$E_1^{(2)} = \frac{2\tau_k}{A_1} \begin{vmatrix} 2iw_0 & n_1 \\ -\tau_k a_{21}e^{-2iw_0} & n_2 \end{vmatrix}_{2\times 2},$$

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where

$$A_1 = \begin{vmatrix} 2iw_0 & \tau_k a_{12} \\ -\tau_k a_{21} e^{-2iw_0} & 2iw_0 + a_{22} e^{-2iw_0} \end{vmatrix}.$$

Similarly, substituting (A.13) into (A.11) and utilizing (A.15) we can easily get

$$\begin{bmatrix} 0 & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} E_2 = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix},$$

so that

$$E_2^{(1)} = \frac{1}{A_2} \begin{vmatrix} s_1 & -a_{12} \\ s_2 & -a_{22} \end{vmatrix},$$

$$E_2^{(2)} = \frac{1}{A_2} \begin{vmatrix} 0 & s_1 \\ a_{21} & s_2 \end{vmatrix},$$

where

$$A_2 = \begin{vmatrix} 0 & -a_{12} \\ a_{21} & -a_{22} \end{vmatrix}.$$

Now, we can substitute E_1 and E_2 into $w_{11}(\theta)$ and $w_{20}(\theta)$ and find the coefficients of $g(z, \bar{z})$ which determine the stability, direction and the period of Hopf bifurcation as one can see in Section 3.

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