# UNITARY TRANSFORMATIONS

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**1. Introduction.** We consider the problem of finding a unique canonical form for complex matrices under unitary transformation, the analogue of the Jordan form (**1**, p. 305, §3), and of determining the transforming unitary matrix (**1**, p. 298, l. 2). The term "canonical form" appears in the literature with different meanings. It might mean merely a general pattern as a triangular form (the Jacobi canonical form (**8**, p. 64)). Again it might mean a certain matrix which can be obtained from a given matrix only by following a specific set of instructions (**1**). More generally, and this is the sense in which we take it, it might mean a form that can actually be described, which is independent of the method used to obtain it, and with the property that any two matrices in this form which are unitarily equivalent are identical.

Toeplitz settled the question for normal matrices in 1918. Perhaps the first canonical form for non-normal matrices was given by Röseler (**7**) in 1933. He used Frobenius covariants to obtain various triangular forms for special classes of matrices. Currie (**2**) gave a triangular form for a general matrix, but his work has not yet been published.

In this paper we give a complete solution to our problem as stated above for non-derogatory matrices, and a partial solution for the derogatory case. The solution includes a partial solution to the following allied problem: What conditions on the non-diagonal elements must hold for $T_1$ to be unitarily equivalent to $T_2$ when $T_1$ and $T_2$ are two triangular matrices with the same diagonal elements?

**2. A canonical form.** We begin with some preliminary material.

LEMMA. *If $\phi_1, \ldots, \phi_r$ is a set of normalized orthogonal vectors, then there exists a unitary matrix with $\phi_1, \ldots, \phi_r$ as its first r rows.*

THEOREM 1. *For any matrix A there exist unitary matrices U, V such that $UA = T_1$ is triangular (with 0's above the main diagonal) and $AV = T_2$ is triangular (with 0's below the main diagonal).*

Suppose $A = (a_{ij})$ and $\phi_1 = [y_1, \ldots, y_n]$. The requirement that $\phi_1 A = [*0\ldots 0]$ with 0's in the last $n - 1$ places yields a set of $n - 1$ linear homogenous equations in the $n$ unknowns $y_1, \ldots, y_n$, which always has a non-trivial solution. Thus a non-zero $\phi_1$ may be determined and we may suppose it normalized. If $U_1$ is a unitary matrix with $\phi_1$ as its first row then $U_1 A$ has 0's above the main diagonal in the first row. By induction the proof for $UA$ is complete.

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By working on the other side we may show similarly that $AV = T_2$.

COROLLARY 1 (Schmidt). *If $P$ is non-singular there exists a (non-singular) triangular matrix $T$ with 0's above the main diagonal such that $PT$ is unitary.*

COROLLARY 2. *Any set of matrices which may be simultaneously triangularized by similarity transformation may be simultaneously triangularized by unitary transformation.*

Let $P$ be a matrix which reduces $A$ to Jordan normal form (**3**, chap. 6), $P^{-1}AP = C \equiv C_1 \dotplus \ldots \dotplus C_k$, where the $C_i$ are the non-derogatory blocks in the Jordan form. Let $T$ be a triangular matrix with 0's above the main diagonal such that $PT = U$ is unitary. Then $U^*AU = T^{-1}CT$ is triangular. Moreover if $T$ is partitioned in accordance with $C$ so that the diagonal blocks are $T_1, \ldots, T_k$ then the $i$th diagonal block of $T^{-1}CT$ is $T_i^{-1} C_i T_i$ and hence is similar to $C_i$.

THEOREM 2. *Any matrix may be unitarily transformed to triangular form with diagonal blocks $A_1, \ldots, A_k$ which are respectively similar to the diagonal blocks $C_1, \ldots, C_k$ in the Jordan form.*

This theorem might have been obtained from consideration of linear transformations (**5**). It has been given in terms of matrices since the uniqueness proof is in the latter form.

Suppose $A$ is non-derogatory and $U^*AU = B$ has this form for a unitary $U$, and $C = C_1 \dotplus \ldots \dotplus C_k$ is the Jordan form of $A$. Then $B$ is similar to $C$, say $T^{-1}BT = C$. Partition $B$ and $T$ in accordance with $C$ so that $B = (B_{ij})$, $T = (T_{ij})$; $i,j = 1,2, \ldots, k$.

Consider the elements above the main diagonal in $BT$ and $TC$. Comparison of the elements in the first row and second column gives $B_{11}T_{12} = T_{12}C_2$. As $A$ is non-derogatory, $B_{11}$ and $C_2$ have no characteristic root in common and hence $T_{12} = 0$ (**4**, p. 90). Similarly $T_{13}, \ldots, T_{1k}$ are 0. Following this procedure with the remaining rows shows that $T$ is a triangular block matrix. In particular, then,

$$B_{ii}T_{ii} = T_{ii}C_i \qquad (i = 1,2, \ldots, k).$$

But element-wise comparison of the elements above the main diagonal shows that $T_{ii}$ is triangular for all $i$. Hence $T$ itself is actually a triangular matrix. Since $UT = P$ is a matrix which reduces $A$ to Jordan form, we see that, for a non-derogatory matrix $A$, any unitary matrix which transforms $A$ to the form given in Theorem 2 is obtained from a matrix which reduces $A$ to Jordan form by multiplying it on the right by a triangular matrix.

Let us determine then the degree of uniqueness of a matrix which reduces $A$ to Jordan form. If both $P$ and $Q$ reduce a non-derogatory matrix $A$ to Jordan form $C$, then $A = PCP^{-1} = QCQ^{-1}$ and so $CP^{-1}Q = P^{-1}QC$. Hence we consider the equation $CX = XC$. Partition $X$ according to $C$ so that

$$X = (X_{ij}) \qquad (i,j = 1,2, \ldots, k).$$

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Comparison of the elements off the main diagonal shows, as before, that $X_{ij} = 0$ for $i \neq j$. Comparison of elements on the main diagonal gives $X_{ii}C_i = C_iX_{ii}$ and hence that $X_{ii}$ is triangular with order equal to that of $C_i$. Thus

$$P^{-1}Q = R = R_{11} \dotplus \ldots \dotplus R_{kk}$$

where $R_{ii}$ is triangular with order equal to that of $C_i$, and $Q = PR$.

We now determine the degree of uniqueness of the transforming unitary matrix. Suppose $U$ and $V$ are two unitary matrices which transform a non-derogatory matrix $A$ to the form of Theorem 2. Then there exist triangular matrices $T_1$ and $T_2$ such that $UT_1 = P$, $VT_2 = Q$, where $P$ and $Q$ both reduce $A$ to Jordan form. Hence $Q = PR$, or $VT_2 = UT_1R$. Thus $U^*V = T_1RT_2^{-1}$. Now $U^*V$ is triangular since it is the product of triangular matrices and hence, since it is also unitary, it is diagonal. Thus $V = UD$. That is, the unitary matrix which transforms a non-derogatory matrix to the form of Theorem 2 is unique up to multiplication on the right by a diagonal unitary matrix. The absolute value of every element of a matrix in this form is therefore invariant. Let us agree to go from left to right down the successive diagonals below the main diagonal and pick out each non-zero element as we come to it until we obtain either a total of $n - 1$ non-zero elements or all non-zero elements off the main diagonal, where $n$ is the order of the matrix. These chosen non-zero elements can then be made positive by transforming by a diagonal unitary matrix. We thus obtain a canonical form that is invariant under transformation by a general unitary matrix.

THEOREM 3. *The form of Theorem 2 is unique for a non-derogatory matrix (for a specified ordering of the roots and a convention as to which non-diagonal elements will be made non-negative).*

Consideration of the Jordan normal form of a matrix $A$ shows that it is non-derogatory if and only if $A - \lambda I$ has nullity 1 for every characteristic root $\lambda$; that is, there is precisely one characteristic vector for each characteristic root (**6**, p. 45). Hence a triangular matrix with but one distinct characteristic root is non-derogatory if and only if the elements in the diagonal below the main diagonal are non-zero. These elements can all be made positive on transformation by a diagonal unitary matrix. Hence we could have required that the elements in the diagonal below the main diagonal of each of the diagonal blocks of Theorem 2 be positive.

## REFERENCES


**1.** J. L. Brenner, *The problem of unitary equivalence*, Acta Math., *86* (1951), 297–308.

**2.** J. C. Currie, *Unitary-canonical matrices*, Abstract no. 264, Bull. Amer. Soc., *56* (1950), 321.

**3.** C. C. MacDuffee, *Vectors and matrices*, Carus Mathematical Monograph, no. 7 (Math. Assoc. Amer., 1943).

**4.** ———, *Theory of matrices*, Ergebnisse der Math., vol. 2 (Berlin, 1933).

**5.** B. E. Mitchell, *A canonical form for non-derogatory matrices under unitary transformation*, Abstract no. 51, Bull. Amer. Math. Soc., *58* (1952), 55.

**6.** W. V. Parker, *The matrix equation* $AX = XB$, Duke Math. J., *17* (1950), 43–51.
**7.** H. Röseler, *Normalformen von Matrizen gegenüber unitären Transformationen*, Dissertation (Leipzig, 1933).
**8.** H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices* (Glasgow, 1948).

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