Can. J. Math., Vol. XXIII, No. 1, 1971, pp. 176–188

THE DIMENSIONS OF IRREDUCIBLE TENSOR REPRESENTATIONS OF THE ORTHOGONAL AND SYMPLECTIC GROUPS

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1. Introduction. It is well known [**13**] that the irreducible tensor representations (IRs) of the unitary, orthogonal, and symplectic groups in an n-dimensional space may be specified by means of Young tableaux associated with partitions $(\sigma)_s = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ with $\sigma_1 + \sigma_2 + \ldots + \sigma_p = s$. Formulae for the dimensions of the corresponding representations have been established [**1**; **8**; **9**; **13**] in terms of the row lengths of these tableaux. It has been shown [**12**] for the unitary group, $\mathrm{U}(n)$, that the formula may be written as a quotient whose numerator is a polynomial in n containing s factors, and whose denominator is a number independent of n, which likewise may be expressed as a product of s factors. This formula is valid for all n.

In contrast to this, the existing formulae [**1**; **13**] for the dimensions of IRs of the orthogonal group $\mathrm{O}(n)$ and the symplectic group $\mathrm{Sp}(n)$ are not valid for all n in the sense that they apply only to standard IRs of these groups and not to non-standard IRs, which are related to standard IRs through modification rules [**9**; **10**]. Moreover, for the orthogonal group $\mathrm{O}(n)$ the formulae appropriate to the cases n even and n odd are distinct. Our aim in this paper is to present derivations of expressions for the dimensions of the IRs of $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$ specified by $(\sigma)_s$ which are in the form of a quotient whose numerator is a polynomial in n containing s factors and whose denominator is the same as that appropriate to the IR of $\mathrm{U}(n)$ specified by $(\sigma)_s$. In addition, these expressions are valid for all n, including those cases for which the IRs are non-standard. For $\mathrm{O}(n)$, the unique expression is valid for both n even and n odd.

Applications can be easily made to the calculation of the dimensions of IRs of $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$ appropriate to the atomic and nuclear *LS* and *jj* coupling schemes.

2. Partitions and Young tableaux. The irreducible tensor representations of $\mathrm{U}(n)$, $\mathrm{O}(n)$, and $\mathrm{Sp}(n)$ may be specified by the symbols $\{\sigma\}_s$, $[\sigma]_s$, and $\langle\sigma\rangle_s$, respectively, associated with the partition of s into p parts, denoted by $(\sigma)_s = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ with $\sigma_1 + \sigma_2 + \ldots + \sigma_p = s$ and

$$\sigma_1 \geqq \sigma_2 \geqq \ldots \geqq \sigma_p > 0.$$

Received May 25, 1970.

https://doi.org/10.4153/CJM-1971-017-2 Published online by Cambridge University Press

Such a partition is represented diagrammatically by means of a regular Young tableau (or diagram) consisting of p rows of boxes (or nodes). The ith row of the tableau contains σ_i boxes and each row begins from the same vertical line. The tableau also defines a partition of s into q parts, denoted by $(\sigma')_s = (\sigma_1', \sigma_2', \ldots, \sigma_q')$ with $\sigma_1' + \sigma_2' + \ldots + \sigma_q' = s$ and

$$\sigma_1' \geqq \sigma_2' \geqq \ldots \geqq \sigma_q' > 0,$$

where σ_j' is the number of boxes in the jth column of the Young tableau defined by the partition $(\sigma)_s$. The partition $(\sigma')_s$ is said to be conjugate to the partition $(\sigma)_s$, so that the conjugacy operation simply involves interchanging the rows and columns of the appropriate Young tableau.

It is to be noted that $\sigma_1' = p$ and $\sigma_1 = q$, and that in Frobenius notation [**8**; **12**], the partition $(\sigma)_s$ takes the form

$$(\sigma)_s = \begin{pmatrix} \sigma_1 - 1 & \sigma_2 - 2 & \ldots & \sigma_r - r \\ \sigma_1' - 1 & \sigma_2' - 2 & \ldots & \sigma_r' - r \end{pmatrix},$$

where r is the length of the principal diagonal of the Young tableau. Clearly with this notation, the conjugacy operation yields

$$(\sigma)_s' = (\sigma')_s = \begin{pmatrix} \sigma_1' - 1 & \sigma_2' - 2 & \ldots & \sigma_r' - r \\ \sigma_1 - 1 & \sigma_2 - 2 & \ldots & \sigma_r - r \end{pmatrix}.$$

It is convenient to extend the definition of the partition to that of a generalized partition $(\sigma)_s = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ with $\sigma_1 + \sigma_2 + \ldots + \sigma_p = s$ but with the restriction $\sigma_1 \geqq \sigma_2 \geqq \ldots \geqq \sigma_p > 0$ completely removed, so that any partition number σ_i may be positive, zero or negative. Such a generalized partition may still be represented diagrammatically by means of a Young tableau consisting of p rows. Each row begins from the same vertical line and the ith row consists of σ_i boxes extending to the right of this line if $\sigma_i > 0$, of no boxes if $\sigma_i = 0$, and of $|\sigma_i|$ boxes extending to the left of this line if $\sigma_i < 0$. This generalization is such that differences between partitions may always be defined. Thus if $(\tau)_t = (\tau_1, \tau_2, \ldots)$ and $(\sigma)_s = (\sigma_1, \sigma_2, \ldots)$, then $(\tau - \sigma)_{t-s} = (\tau_1 - \sigma_1, \tau_2 - \sigma_2, \ldots)$, where any or all of these partitions may be unconventional in that their parts may be negative and unordered. However, except where otherwise stated, it is to be assumed that all partitions are conventional with ordered positive parts.

3. Conjugacy relationships. The dimensions of the IR of U(n) specified by $\{\sigma\}_s$ is given by

$$D_n\{\sigma\}_s = N_n\{\sigma\}_s / H(\sigma)_s \tag{3.1}$$

[**12**, p. 60] with

$$N_n\{\sigma\}_s = \prod_{(i,j)} (n - i + j) \tag{3.2}$$

https://doi.org/10.4153/CJM-1971-017-2 Published online by Cambridge University Press

and

$$H(\sigma)_s = \prod_{(i,j)} (1 - i - j + \sigma_i + \sigma_j'), \tag{3.3}$$

where the products are taken over all values of (i, j) corresponding to a box in the ith row and jth column of the tableau defined by $(\sigma)_s$.

In (3.2), replacing j by $(\sigma_i - j + 1)$ yields

$$N_n\{\sigma\}_s = \prod_{i=1}^{p} \prod_{j=1}^{\sigma_i} (n - i - j + 1 + \sigma_i), \tag{3.4}$$

whereas replacing i by $(\sigma_j' - i + 1)$ yields

$$N_n\{\sigma\}_s = \prod_{j=1}^{q} \prod_{i=1}^{\sigma_j'} (n + i + j - 1 - \sigma_j'). \tag{3.5}$$

Comparison of (3.4) and (3.5) then indicates that

$$N_n\{\sigma\}_s = (-1)^s N_{-n}\{\sigma'\}_s.$$

In addition, it is clear from (3.3) that

$$H(\sigma)_s = H(\sigma')_s,$$

so that

$$D_n\{\sigma\}_s = (-1)^s D_{-n}\{\sigma'\}_s. \tag{3.6}$$

The reduction of the outer product of two IRs of U(n) into a sum of IRs of U(n) is determined by the formula

$$\{\sigma\}_s \cdot \{\tau\}_t = \sum_{\lambda} m_{\sigma\tau,\lambda}\{\lambda\}_{s+t}$$

together with the well-established procedure [**8**, p. 94] for determining the multiplicities $m_{\sigma\tau,\lambda}$. The conjugacy relationship is such that

$$m_{\sigma\tau,\lambda} = m_{\sigma'\tau',\lambda'} \tag{3.7}$$

[**8**, p. 110].

The IR $\{\sigma\}_s$ of U(n) decomposes into IRs of O(n) in accordance with the formula

$$\{\sigma\}_s = \sum_{\tau} d_{\sigma\tau}[\tau]_t = \sum_{\delta\tau} m_{\delta\tau,\sigma}[\tau]_t \tag{3.8}$$

[**8**, p. 240], where $(\delta)_d$ is a partition into even parts only, so that

$$(\delta)_d = (2d_1, 2d_2, \ldots).$$

The inverse of the formula (3.8) is the identity

$$[\sigma]_s = \sum_{\tau} c_{\sigma\tau}\{\tau\}_t = \sum_{\gamma\tau} (-1)^{c/2} m_{\gamma\tau,\sigma}\{\tau\}_t, \tag{3.9}$$

where $(\gamma)_c$ is a partition, which in Frobenius notation takes the form

$$(\gamma)_c = \begin{pmatrix} c_1+1 & c_2+1 & \dots \\ c_1 & c_2 & \dots \end{pmatrix}.$$

In the same way the IR $\{\sigma\}_s$ of U(n) decomposes into IRs of Sp(n) in accordance with the formula

$$\{\sigma\}_s = \sum_{\tau} b_{\sigma\tau}\langle\tau\rangle_t = \sum_{\beta\tau} m_{\beta\tau,\sigma}\langle\tau\rangle_t \tag{3.10}$$

[**8**, p. 295], where $(\beta)_b$ is a partition in which each distinct part is repeated an even number of times, so that

$$(\beta)_b = (b_2{}^2, b_2{}^2, \dots).$$

The inverse of (3.10) is the identity

$$\langle\sigma\rangle_s = \sum_{\tau} a_{\sigma\tau}\{\tau\}_t = \sum_{\alpha\tau} (-1)^{a/2} m_{\alpha\tau,\sigma}\{\tau\}_t, \tag{3.11}$$

where $(\alpha)_a$ is a partition which in Frobenius notation takes the form

$$(\alpha)_a = \begin{pmatrix} a_1 & a_2 & \dots \\ a_1+1 & a_2+1 & \dots \end{pmatrix}.$$

The existence of the relations (3.9) and (3.11) indicates that the dimensions of the IRs of O(n) and Sp(n) specified by $[\sigma]_s$ and $\langle\sigma\rangle_s$, and denoted by $D_n[\sigma]_s$ and $D_n\langle\sigma\rangle_s$, respectively, may be expressed, by the use of (3.1), as polynomials in n of degree s. Moreover, it must be stressed that since (3.1), (3.9), and (3.11) are valid for all n, the corresponding polynomials $D_n[\sigma]_s$ and $D_n\langle\sigma\rangle_s$ are unique. That is to say these polynomials give the dimensions of the IRs of the orthogonal and the symplectic groups specified by the partition $(\sigma)_s$ for all possible values of n. In particular, the polynomial $D_n[\sigma]_s$ is applicable for both even and odd values of n.

It is well known that the complete set of standard IRs of O$(2k)$, O$(2k+1)$, and Sp$(2k)$ may be specified by partitions consisting of not more than k parts. However, if $n < 2p$, where the partition $(\sigma)_s$ consists of p parts, then the reductions (3.8) and (3.10) give rise to non-standard IRs. There exist well-established modification rules [**9**, p. 282; **10**] which relate each non-standard IR to an equivalent standard IR. However, since the non-standard IRs of O(n) and Sp(n) may be defined by (3.9) and (3.11), it follows that the polynomials $D_n[\sigma]_s$ and $D_n\langle\sigma\rangle_s$ give the dimensions of the IRs of O(n) and Sp(n) specified by the partition $(\sigma)_s$ even if these IRs are non-standard.

In § 4 the unique polynomial formulae for $D_n[\sigma]_s$ and $D_n\langle\sigma\rangle_s$ are derived. Necessarily there is a conjugacy relationship between these formulae which arises from the fact that the identity (3.7) implies that

$$d_{\sigma\tau} = b_{\sigma'\tau'} \qquad \text{and} \qquad c_{\sigma\tau} = a_{\sigma'\tau'}. \tag{3.12}$$

It then follows from (3.9) that

$$D_n[\sigma]_s = \sum_{\tau} c_{\sigma\tau} D_n\{\tau\}_t = \sum_{\tau'} a_{\sigma'\tau'}(-1)^t D_{-n}\{\tau'\}_t,$$

where use has been made of both (3.12) and (3.6). However, $t = s + c$, and c is even, so that

$$D_n[\sigma]_s = (-1)^s \sum_{\tau'} a_{\sigma'\tau'} D_{-n}\{\tau'\}_t.$$

From (3.11),

$$D_{-n}\langle\sigma'\rangle_s = \sum_{\tau'} a_{\sigma'\tau'} D_{-n}\{\tau'\}_t,$$

so that finally

$$D_n[\sigma]_s = (-1)^s D_{-n}\langle\sigma'\rangle_s. \tag{3.13}$$

This important conjugacy relationship is exploited in § 4 in devising procedures for determining $D_n[\sigma]_s$ and $D_n\langle\sigma\rangle_s$.

4. Dimensions of IRs of O(n) **and** Sp(n). Although (3.9) and (3.11) indicate that $D_n[\sigma]_s$ and $D_n\langle\sigma\rangle_s$ may be expressed as polynomials in n of degree s, they do not provide, in general, an easy method of calculating these polynomials. In fact, the only formulae available for these quantities are expressed as quotients whose numerators and denominators are both polynomials in n. Moreover, for O(n), distinct formulae need to be used according as n is even or odd. Thus for O(n) [**1**, p. 250], if $n = 2k$, then

$$D_n[\sigma]_s = D_{2k}[\sigma]_s = \prod_{1\leqq i<j\leqq k} \frac{(\sigma_i - \sigma_j - i + j)}{(-i + j)} \prod_{1\leqq i<j\leqq k} \frac{(\sigma_i + \sigma_j + n - i - j)}{(n - i - j)}, \tag{4.1}$$

and if $n = 2k + 1$, then

$$D_n[\sigma]_s = D_{2k+1}[\sigma]_s = \prod_{1\leqq i<j\leqq k} \frac{(\sigma_i - \sigma_j - i + j)}{(-i + j)} \prod_{1\leqq i\leqq j\leqq k} \frac{(\sigma_i + \sigma_j + n - i - j)}{(n - i - j)}. \tag{4.2}$$

Similarly for Sp(n) [**13**, p. 218], with $n = 2k$,

$$D_n\langle\sigma\rangle_s = D_{2k}\langle\sigma\rangle_s = \prod_{1\leqq i<j\leqq k} \frac{(\sigma_i - \sigma_j - i + j)}{(-i + j)} \prod_{1\leqq i\leqq j\leqq k} \frac{(\sigma_i + \sigma_j + n + 2 - i - j)}{(n + 2 - i - j)}. \tag{4.3}$$

In these expressions the notation of § 2 has been extended slightly in such a way that

$$\begin{aligned} \sigma_i &> 0 \quad \text{if } i \leqq p, \\ \sigma_i &= 0 \quad \text{if } i > p, \end{aligned} \tag{4.4}$$

with $p \leqq k$.

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Multiplying (4.1) by the factor

$$\prod_{i=1}^{k} \frac{2(k-i)(\sigma_i+k-i)(\sigma_i+n-i-k-1)!}{(n-2i)(\sigma_i+k-i)!},$$

which is just unity by virtue of the fact that in (4.1), $n = 2k$, and then rearranging terms yields

$$(4.5) \qquad D_n[\sigma]_s = N_n[\sigma]_s/H[\sigma]_s,$$

where

$$(4.6) \quad N_n[\sigma]_s = \prod_{i=1}^{k} \left[\frac{(n+\sigma_i-i-k-1)!}{(n-2i)!} \prod_{j=i}^{k} (n+\sigma_i+\sigma_j-i-j)\right]$$

and

$$(4.7) \qquad H[\sigma]_s = \prod_{i=1}^{k} \left[\frac{(\sigma_i+k-i)!}{\prod_{j=i+1}^{k} (\sigma_i-\sigma_j-i+j)}\right].$$

Making use of the condition (4.4), then leads to the results

$$(4.8) \quad N_n[\sigma]_s = \prod_{i=1}^{p} \left[\frac{(n+\sigma_i-i-p-1)!}{(n-2i)!} \prod_{j=i}^{p} (n+\sigma_i+\sigma_j-i-j)\right]$$

and

$$(4.9) \qquad H[\sigma_s] = \prod_{i=1}^{p} \left[\frac{(\sigma_i+p-i)!}{\prod_{j=i+1}^{p} (\sigma_i-\sigma_j-i+j)}\right].$$

In this form it is easy to see that

$$(4.10) \qquad H[\sigma]_s = H(\sigma)_s,$$

so that the denominator of (4.5) is just the conventional [**12**, p. 44] hook length factor associated with the partition of s into p parts denoted by $(\sigma)_s$.

Moreover, in the expression (4.8), if $p < i + \sigma_i$, then the factor $(n - 2i)!$ will cancel with the same factor which is contained in $(n + \sigma_i - i - p - 1)!$. In this case, for a particular value of i the contribution to $N_n[\sigma]_s$ is a polynomial in n of degree $(p - i + 1) + (i + \sigma_i - p - 1) = \sigma_i$, where of course σ_i is the length of the ith row of the corresponding tableau. If on the other hand $p \geqq i + \sigma_i$, then, for a particular value of i, the contribution to $N_n[\sigma]_s$ is a polynomial of degree σ_i multiplied by $(p - i - \sigma_i + 1)$ factors of the form

$$(n - 2i - l + \sigma_{i+\sigma_i+l})/(n - 2i - l).$$

Therefore in general

$$(4.11) \quad N_n[\sigma]_s = \prod_{i=1}^{p} \left[\prod_{j=i}^{i+\sigma_i-1} (n+\sigma_i+\sigma_j-i-j) \times \prod_{l=0}^{p-i-\sigma_i} \frac{(n-2i-l+\sigma_{i+\sigma_i+l})}{(n-2i-l)}\right].$$

Replacing j by $i + j - 1$ in this expression then yields in (4.5), using (4.10) and (3.3), the final result

$$(4.12)\quad D_n[\sigma]_s = \prod_{(i,j)} \frac{(n - \sigma_i + \sigma_{i+j-1} - 2i - j + 1)}{(1 + \sigma_i + \sigma_j' - i - j)} \times \prod_{(i,l)} \frac{(n - 2i - l + 1 + \sigma_{i+\sigma_i+l-1})}{(n - 2i - l + 1)},$$

where the first product is taken over all values of (i, j) corresponding to a box in the ith row and jth column of the tableau defined by $(\sigma)_s$, and the second product is taken over all values of (i, l) with $l \geqq 1$ corresponding to a box in the ith row and the lth column of the tableau defined by the generalized partition $(\tau - \sigma)_{t-s}$ with $(\tau)_t = (p, p - 1, p - 2, \ldots, 2, 1)$. Thus formula (4.12) defines the following procedure, (A), for calculating $D_n[\sigma]_s$.

(A) (i) The numbers $n - 1, n - 3, n - 5, \ldots, n - 2p + 1$ are placed in the end boxes of the 1st, 2nd, 3rd, ..., pth rows of the tableau defined by $(\sigma)_s$. An array of s numbers is then constructed by inserting in the remaining boxes of the tableau, numbers which increase by 1 in passing from one box to its left-hand neighbour.

(A) (ii) The series of numbers in any row of this array is then extended outside the tableau defined by $(\sigma)_s$ to the limit of the triangular tableau defined by $(\tau)_t$ with $\tau_1 = p$, if, that is, the corresponding row length defined by $(\tau)_t$ is greater than that defined by $(\sigma)_s$.

(A) (iii) The row lengths $\sigma_1, \sigma_2, \ldots, \sigma_p$ are then added to all of the numbers of this extended array which lie on lines of unit slope passing through the first box of the 1st, 2nd, ..., pth rows, respectively, of the tableau defined by $(\sigma)_s$.

(A) (iv) Each number of the resulting array corresponding to a box of the tableau defined by $(\sigma)_s$ is divided by the associated hook length, whilst each number of the array outside the tableau defined by $(\sigma)_s$ is divided by the number occupying the same position in the array at the end of stage (A) (ii), prior to the addition of the row lengths.

$D_n[\sigma]_s$ is then the product of the numbers of the array obtained after stage (iv) of procedure (A).

For example, if $(\sigma)_s = (4, 3, 1)$, then $p = 3$ and $(\tau)_t = (3, 2, 1)$ so that $(\tau - \sigma)_{t-s} = (-1, -1, 0)$. In this case,

$$(4.13)\quad D_n[4, 3, 1] = \frac{(n + 2 + 4)}{6} \cdot \frac{(n + 1 + 3)}{4} \cdot \frac{(n + 0 + 1)}{3} \cdot \frac{(n - 1)}{1} \cdot \frac{(n - 1 + 3)}{4} \cdot \frac{(n - 2 + 1)}{2} \cdot \frac{(n - 3)}{1} \cdot \frac{(n - 5 + 1)}{1},$$

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so that

(4.14) $D_n[4, 3, 1]$

$$= (n + 6)(n + 4)(n + 2)(n + 1)(n - 1)^2(n - 3)(n - 4)/576.$$

In this particular example it was not necessary to carry out stage (ii) of the procedure (A). On the other hand, if $(\sigma)_s = (3, 2^2, 1)$, then $p = 4$ and $(\tau)_t = (4, 3, 2, 1)$ so that $(\tau - \sigma)_{t-s} = (1, 1, 0, 0)$. In this case,

$$(4.15)\quad D_n[3, 2^2, 1] = \frac{(n+1+3)}{6}\cdot\frac{(n+0+2)}{4}\cdot\frac{(n-1+2)}{1}\cdot\frac{(n-2+1)}{(n-2)}$$
$$\cdot\frac{(n-2+2)}{4}\cdot\frac{(n-3+2)}{2}\cdot\frac{(n-4+1)}{(n-4)}$$
$$\cdot\frac{(n-4+2)}{3}\cdot\frac{(n-5+1)}{1}$$
$$\cdot\frac{(n-7+1)}{1},$$

so that

(4.16) $D_n[3, 2^2, 1] = (n + 4)(n + 2)(n + 1)(n)(n - 1)^2(n - 3)(n - 6)/576.$

In this example it is to be noted that the factors in the denominator dependent upon n cancel to yield a factored polynomial of degree 8 for $D_n[3, 2^2, 1]$. It is not at all obvious from the formula (4.12) that the factors in the denominator dependent upon n will always cancel with identical factors in the numerator to leave a factored polynomial in n of degree s. This is the case however and may be proved by distorting the unextended tableau defined by $(\sigma)_s$ in such a way that each row ends in the same vertical line. With this distortion, cancellations take place between numbers joined by lines of unit slope.

The derivation of the expression (4.12) for $D_n[\sigma]_s$ from (4.1) depended upon the conditions $n = 2k$ and $k \geqq p$. However, since the expression (4.12) is a polynomial in n of degree s, it must be valid for all n as explained in § 3. In fact, it is not difficult to derive (4.12) from the formula (4.2) which is appropriate to IRs of $O(n)$ with $n = 2k + 1$. This confirms that (4.12) is valid for both even and odd n. It is to be stressed, moreover, that (4.12) is valid for all values of n including those for which $n < 2p$. Thus (4.12) provides an excellent means of checking the validity of modification rules appropriate to IRs of $O(n)$ [**9**, p. 282; **10**].

In exactly the same way, it is straightforward for the IRs of $Sp(n)$ to derive from (4.3) the formula

$$(4.17)\qquad D_n\langle\sigma\rangle_s = N_n\langle\sigma\rangle_s/H\langle\sigma\rangle_s,$$

where

(4.18) $N_n\langle\sigma\rangle_s$

$$= \prod_{i=1}^{p} \left[\frac{(n+\sigma_i-i-p+1)!}{(n-2i+1)!} \prod_{j=i+1}^{p} (n+\sigma_i+\sigma_j-i-j+2) \right]$$

and

(4.19) $$H\langle\sigma\rangle_s = H(\sigma)_s.$$

Hence it may be shown that

(4.20) $D_n\langle\sigma\rangle_s$

$$= \prod_{(i,j)} \frac{(n+\sigma_i+\sigma_{i+j}-2i-j+2)}{(1+\sigma_i+\sigma_j{}'-i-j)} \prod_{(i,l)} \frac{(n-2i-l+2+\sigma_{i+\sigma_i+l})}{(n-2i-l+2)},$$

where the first product is taken over all values of (i, j) corresponding to a box in the ith row and jth column of the tableau defined by $(\sigma)_s$, and the second product is taken over all values of (i, l) with $l \geqq 1$ corresponding to a box in the ith row and lth column of the tableau defined by the generalized partition $(\rho - \sigma)_{r-s}$ with $(\rho)_r = (p-1, p-2, p-3, \ldots, 2, 1)$. This formula (4.20) is analogous to (4.12), and defines the following procedure (B) for calculating $D_n\langle\sigma\rangle_s$.

(B) (i) This is the same as (A) (i) with the numbers "$n-1, n-3, n-5, \ldots, n-2p+1$" replaced by the numbers "$n, n-2, n-4, \ldots, n-2p+2$."

(B) (ii) This is the same as (A) (ii) with "$(\tau)_t$ with $\tau_1 = p$" replaced by "$(\rho)_r$ with $\rho_1 = p-1$", and with "$(\tau)_t$" replaced by "$(\rho)_r$".

(B) (iii) This is the same as (A) (iii) with the row lengths "$\sigma_1, \sigma_2, \ldots, \sigma_p$" replaced by "$\sigma_2, \sigma_3, \ldots, \sigma_p$" and with pth replaced by "$(p-1)$st".

(B) (iv) This is the same as (A) (iv) with "(A) (ii)" replaced by "(B) (ii)".

$D_n\langle\sigma\rangle_s$ is then the product of the numbers of the array obtained after stage (iv) of this procedure (B).

Using the same examples as before if $(\sigma)_s = (4, 3, 1)$, then $p = 3$ and $(\rho)_r = (2, 1)$ so that $(\rho - \sigma)_{r-s} = (-2, -2, -1)$. In this case,

(4.21) $$D_n\langle 4, 3, 1\rangle = \frac{(n+3+3)}{6} \cdot \frac{(n+2+1)}{4} \cdot \frac{(n+1)}{3} \cdot \frac{n}{1} \cdot \frac{(n+0+1)}{4} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{1} \cdot \frac{(n-4)}{1},$$

so that

(4.22) $$D_n\langle 4, 3, 1\rangle = (n+6)(n+3)(n+1)^2(n)(n-1)(n-2)(n-4)/576.$$

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Similarly, if $(\sigma)_s = (3, 2^2, 1)$, then $p = 4$ and $(\rho)_r = (3, 2, 1)$ so that $(\rho - \sigma)_{r-s} = (0, 0, -1, -1)$. Then

$$(4.23) \quad D_n\langle 3, 2^2, 1\rangle = \frac{(n+2+2)}{6} \cdot \frac{(n+1+2)}{4} \cdot \frac{(n+0+1)}{1} \cdot \frac{(n-1+2)}{4} \cdot \frac{(n-2+1)}{2} \cdot \frac{(n-3+1)}{3} \cdot \frac{(n-4)}{1} \cdot \frac{(n-6)}{1},$$

so that

$$(4.24) \quad D_n\langle 3, 2^2, 1\rangle = (n+4)(n+3)(n+1)^2(n-1)(n-2)(n-4)(n-6)/576.$$

In these particular examples, stage (ii) of procedure (B) is not necessary and there is no question of the denominator depending upon n. However, for other examples, factors in the denominator depending upon n do occur and, as for $O(n)$, these factors always cancel with identical factors in the numerator. Thus the expression (4.20) for $D_n\langle\sigma\rangle_s$ is a polynomial in n of degree s which is valid for all values of n including those for which $n < 2p$. Thus (4.20) provides an excellent means of checking the validity of modification rules appropriate to IRs of $\mathrm{Sp}(n)$ [**10**].

The identity (3.13), besides illustrating the conjugacy relationship between IRs of $O(n)$ and $\mathrm{Sp}(n)$, also provides new procedures, (C) and (D), for evaluating $D_n[\sigma]_s$ and $D_n\langle\sigma\rangle_s$ when used in conjugation with (4.12) and (4.20). These procedures correspond to the formulae

$$(4.25) \quad D_n[\sigma]_s = \prod_{(i,j)} \frac{(n - \sigma'_{i+j} - \sigma_j' + i + 2j - 2)}{(1 + \sigma_i + \sigma_j' - i - j)} \prod_{(j,l)} \frac{(n + 2j + l - 2 - \sigma'_{j+\sigma_j'+l})}{(n + 2j + l - 2)}$$

and

$$(4.26) \quad D_n\langle\sigma\rangle_s = \prod_{(i,j)} \frac{(n - \sigma'_{i+j-1} - \sigma_j' + i + 2j - 1)}{(1 + \sigma_i + \sigma_j' - i - j)} \prod_{(j,l)} \frac{(n - 2j + l - 1 - \sigma'_{j+\sigma_j'+l-1})}{(n + 2j + l - 1)}$$

in an obvious extension of the notation of (4.12) and (4.20). The formula (4.25) defines the following procedure, (C), for calculating $D_n[\sigma]_s$.

(C) (i) The numbers $n, n+2, n+4, \ldots, n+2q-2$ are placed in the end boxes of the 1st, 2nd, 3rd, ..., qth columns of the tableau defined by $(\sigma)_s$. An array of s numbers is then constructed by inserting in the remaining boxes of the tableau, numbers which decrease by 1 in passing from one box to the one immediately above.

(C) (ii) This is the same as (A) (ii) with "row" replaced by "column", with "$(\tau)_t$ with $\tau_1 = p$" replaced by "$(\lambda)_l$ with $\lambda_1 = q - 1$", with "row" again replaced by "column", and with "$(\tau)_t$" replaced by "$(\lambda)_l$".

(C) (iii) The column lengths $\sigma_2', \sigma_3', \ldots, \sigma_q'$ are then subtracted from all of the numbers of this array which lie on lines of unit slope passing through the first box of the 1st, 2nd, ..., $(q - 1)$st columns, respectively, of the tableau defined by $(\sigma)_s$.

(C) (iv) This is the same as (A) (iv) with "(A) (ii)" replaced by "(C) (ii)" and with "addition of the row lengths" replaced by "subtraction of the column lengths".

$D_n[\sigma]_s$ is then the product of the numbers of the array obtained after stage (iv) of this procedure (C).

For example, as before, if $(\sigma)_s = (4, 3, 1)$, then $q = 4$ and $(\lambda)_l = (3, 2, 1)$ so that $(\lambda - \sigma)_{l-s} = (-1, -1, 0)$. In this case,

$$(4.27)\quad D_n[4,3,1] = D_n[3, 2^2, 1]' = \frac{(n-2-2)}{6}\cdot\frac{(n+1-2)}{4}\cdot\frac{(n+3-1)}{3}$$

$$\cdot\frac{(n+6)}{1}\cdot\frac{(n-1-2)}{4}\cdot\frac{(n+2-1)}{2}\cdot\frac{(n+4)}{1}\cdot\frac{(n+0-1)}{1},$$

in agreement with (4.14). Similarly, if $(\sigma)_s = (3, 2^2, 1)$, then $q = 3$ and $(\lambda)_l = (2, 1)$, so that $(\lambda - \sigma)_{l-s} = (-1, -1, -2, -1)$. Then

$$(4.28)\quad D_n[3, 2^2, 1] = D_n[4, 3, 1]' = \frac{(n-3-3)}{6}\cdot\frac{(n+0-1)}{4}\cdot\frac{(n+4)}{1}$$

$$\cdot\frac{(n-2-1)}{4}\cdot\frac{(n+1)}{2}\cdot\frac{(n-1)}{3}\cdot\frac{(n+2)}{1}\cdot\frac{(n)}{1},$$

in agreement with (4.16). In this particular case it is clear that the array (4.28) is simpler than (4.15) in that it involves no factors outside the tableau defined by $(\sigma)_s$.

Similarly, the formula (4.26) defines the following procedure, (D), for calculating $D_n\langle\sigma\rangle_s$.

(D) (i) This is the same as (C) (i) with the numbers "$n, n + 2, n + 4, \ldots, n + 2q - 2$", replaced by the numbers "$n + 1, n + 3, n + 5, \ldots, n + 2q - 1$".

(D) (ii) This is the same as (C) (ii) with "$(\lambda)_l$ with $\lambda_1 = q - 1$" replaced by "$(\mu)_m$ with $\mu_1 = q$", and with "$(\lambda)_l$" replaced by "$(\mu)_m$".

(D) (iii) This is the same as (C) (iii) with the column lengths "$\sigma_2', \sigma_3', \ldots, \sigma_q'$" replaced by "$\sigma_1', \sigma_2', \ldots, \sigma_q'$" and with "$(q - 1)$st" replaced by "$q$th".

(D) (iv) This is the same as (C) (iv) with (C) (ii) replaced by (D) (ii).

$D_n\langle\sigma\rangle_s$ is then the product of the numbers of the array obtained after stage (iv) of this procedure (D).

https://doi.org/10.4153/CJM-1971-017-2 Published online by Cambridge University Press

For example, if $(\sigma)_s = (4, 3, 1)$, then $q = 4$ and $(\mu)_m = (4, 3, 2, 1)$ so that $(\mu - \sigma)_{m-s} = (0, 0, 1, 1)$. Then

$$(4.29) \quad D_n\langle 4,3,1\rangle = D_n\langle 3, 2^2, 1\rangle' = \frac{(n-1-3)}{6}\cdot\frac{(n+2-2)}{4}\cdot\frac{(n+4-2)}{3} \cdot\frac{(n+7-1)}{1}\cdot\frac{(n+0-2)}{4}\cdot\frac{(n+3-2)}{2}\cdot\frac{(n+5-1)}{1}\cdot\frac{(n+1-2)}{1} \cdot\frac{(n+4-1)}{(n+4)}\cdot\frac{(n+2-1)}{(n+2)},$$

in agreement, after cancellations, with (4.22). Similarly, if $(\sigma)_s = (3, 2^2, 1)$, then $q = 3$ and $(\mu)_m = (3, 2, 1)$ so that $(\mu - \sigma)_{m-s} = (0, 0, -1, -1)$. Then

$$(4.30) \quad D_n\langle 3, 2^2, 1\rangle = D_n\langle 4, 3, 1\rangle' = \frac{(n-2-4)}{6}\cdot\frac{(n+1-3)}{4}\cdot\frac{(n+5-1)}{1} \cdot\frac{(n-1-3)}{4}\cdot\frac{(n+2-1)}{2}\cdot\frac{(n+0-1)}{3}\cdot\frac{(n+3)}{1}\cdot\frac{(n+1)}{1},$$

in agreement with (4.24).

Comparison of (4.14) and (4.24) indicates that

$$D_n[4, 3, 1] = D_{-n}\langle 3, 2^2, 1\rangle,$$

and comparison of (4.16) and (4.22) indicates that

$$D_n[3, 2^2, 1] = D_{-n}\langle 4, 3, 1\rangle,$$

in agreement with (3.13) since in these cases $s = 8$ and $(4, 3, 1)' = (3, 2^2, 1)$.

In addition to this connection between conjugate IRs of $O(n)$ and $Sp(n)$, it follows from (4.8) and (4.18) that if n is even, then

$$(4.31) \qquad D_n[\sigma]_s = D_{n-2}\langle\sigma\rangle_s \prod_{i=1}^{p} \frac{(n-2i+2\sigma_i)}{(n-2i)}.$$

Similarly, it can be shown from (4.25) and (4.26) that

$$(4.32) \qquad D_n[\sigma]_s = D_{n-2}\langle\sigma\rangle_s \prod_{j=1}^{q} \frac{(n+2j-2)}{(n+2j-2-2\sigma_j')}.$$

The identities (4.31) and (4.32) then provide a very elegant way of verifying that since

$$(4.33) \quad D_n[\sigma]_s = D_n[p, \sigma_2', \ldots, \sigma_q']' = D_n[n-p, \sigma_2', \ldots, \sigma_q']',$$

[**13**, p. 156], then

$$(4.34) \quad D_n\langle\sigma\rangle_s = D_n\langle p, \sigma_2', \ldots, \sigma_q'\rangle' = -D_n\langle n-p+2, \sigma_2', \ldots, \sigma_q'\rangle'$$

for the tableau having p rows defined by $(\sigma)_s$. These equalities, of course, correspond to modification rules appropriate to $O(n)$ and $Sp(n)$.

Following the work of Racah [**11**], the IRs of $O(n)$ and $Sp(n)$ have come to play an important role in atomic and nuclear spectroscopy. Calculations of the dimensions of these IRs have been carried out using formulae involving row lengths [**2–7**; **14**]. However, it is more convenient to make use of formulae involving column lengths. Such formulae can be obtained either from (4.8) and (4.18) by the use of (3.13) or from (4.25) and (4.26).

Acknowledgement. I wish to express my gratitude to Professor H. A. Jahn for a careful reading of the manuscript and for his enthusiastic encouragement.

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