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ANALYSIS ON SEMIDIRECT PRODUCTS AND HARMONIC MAPS

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Abstract. We study the analysis of a probability density K on a Lie group G, where G is a semidirect product of a compact group M with a nilpotent group N. To approximate analysis on G with analysis on N, it is natural to consider certain maps ("realizations") of G onto N. In this paper, we prove the existence of a realization of G in N which is K-harmonic (modulo the commutator subgroup of N). By utilizing this result and extending some ideas of Alexopoulos, we can prove the boundedness in L^p spaces of some new Riesz transforms associated with K, and obtain new regularity estimates for the convolution powers of K.

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1. Introduction. Consider a Lie group G which is a semidirect product of a connected compact Lie group M acting on a connected, simply connected nilpotent Lie group N. We will identify M and N with closed subgroups of G, so that

$$G = NM, \quad N \cap M = \{e\}, \tag{1}$$

and N is a normal subgroup of G.

Since M is compact, it is natural to expect that analysis "at infinity" on G is approximated by analysis on the nilpotent group N. This idea of approximating a given group by a nilpotent group with simpler structure has been extensively developed, even in a more general setting where G is replaced by any Lie group of polynomial growth: see, for example, [**1–3**, **6**] and references therein.

To compare analysis on G and N, one usually chooses a map $\Psi: G \to N$ which "realizes" G in N. In view of (1), it is natural to define Ψ by $\Psi(xm) = x$, for $m \in M$, $x \in N$. Note that Ψ is not a homomorphism, except when G is a direct product of M and N.

More generally, let us say that a connected compact subgroup M' of G is a *compact factor* of G if $G = NM'$ and $N \cap M' = \{e\}$. For each such M' we define a "realization"

$$\Psi_{M'}: G \to N, \quad \Psi_{M'}(xm') = x$$

for $m' \in M'$, $x \in N$, and in general we could have $\Psi_{M'} \neq \Psi_M$. Our use of the term "realization" is inspired by Kotani and Sunada [**10**], who studied a different setting of realizations of lattice graphs in Euclidean spaces.

The motivation of this paper is that one can get better analytic results if one chooses M' so that $\Psi_{M'}$ is a harmonic or "almost-harmonic" map. We will develop this idea for analysis of a probability density $K: G \to \mathbb{R}$ on G. Define $\widetilde{K}: G \to \mathbb{R}$ by $\widetilde{K}(g) = K(g^{-1})$. A function $f: G \to \mathbb{R}$ is said to be harmonic with respect to K if $f = f * \widetilde{K}$, or equivalently if $Hf = 0$, where $H = H_{(K)}$ is the discrete Laplacian defined by

$$(Hf)(h) = f(h) - (f * \widetilde{K})(h) = \int_G dg\, K(g)[f(h) - f(hg)], \quad h \in G.$$

Here dg denotes a fixed Haar measure on G and the convolution of functions f_1, f_2 is defined by $(f_1 * f_2)(h) = \int_G dg\, f_1(g) f_2(g^{-1}h)$, $h \in G$.

More generally, a map $F: G \to V$ into a vector space $V \cong \mathbb{R}^d$ is said to be harmonic if its components $F_i = x^i \circ F: G \to \mathbb{R}$ are harmonic, where $(x^1, \ldots, x^d)$ is some basis for V^*. This notion is clearly independent of basis.

In what follows, we will assume that the probability density $K: G \to \mathbb{R}$, with $K \geq 0$ and $\int_G K = 1$, is continuous, compactly supported, symmetric (that is, $\widetilde{K} = K$), and that $\inf\{K(g): g \in U_0\} > 0$ for some neighborhood U_0 of the identity of G.

Our basic theorem is the following. Note that the simply connected abelian Lie group $N/[N, N] \cong \mathbb{R}^d$ can be identified with a vector space.

THEOREM 1.1. *Fix a density K on G, as above. Let $\pi: N \to N/[N, N]$ be the canonical homomorphism. There exists a compact factor M' of G such that the map $\pi \circ \Psi_{M'}: G \to N/[N, N]$ is harmonic with respect to K.*

Theorem 1.1 will be obtained from its special case Theorem 1.2, where N is abelian.

THEOREM 1.2. *Suppose $N \cong \mathbb{R}^d$ is abelian. Fix a density K on G, as above. Then there exists a unique compact factor M' of G such that $\Psi_{M'}: G \to N$ is harmonic with respect to K.*

Note that some results broadly analogous to Theorems 1.2 and 1.1 were obtained in [**10**] and [**9**], for realizations of lattice graphs in Euclidean spaces or in nilpotent groups.

Before stating an application of Theorem 1.1 to analysis, we fix some notation. Let $K^{(n)} = K * K * \cdots * K$ be the n-th convolution power of K, for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Denote by ∂_z the difference operator $I - R(z)$, $z \in G$, where $R = R_G$ is the right regular representation of G:

$$(R(h)f)(g) = f(gh), \quad g, h \in G,$$

for a function $f: G \to \mathbb{R}$. Fix a compact neighborhood $U = U^{-1}$ of the identity e of G and define $\rho = \rho_U: G \to \mathbb{N}$ by

$$\rho(g) = \inf\{n \in \mathbb{N}: g \in U^n\},$$

where U^n is the set of all products $u_1 \cdots u_n$ with $u_i \in U$. Note that G has polynomial volume growth of some order $D \in \mathbb{N}$: that is, $c^{-1}n^D \leq dg(U^n) \leq cn^D$ for all $n \in \mathbb{N}$ (the group N has polynomial growth of the same order D). In general, c, b, c' and so on, denote positive constants whose value may change from line to line when convenient.

Under our assumptions on K, one has (see [**8**]) Gaussian estimates

$$K^{(n)}(g) \leq cn^{-D/2} e^{-b\rho(g)^2/n}, \quad \left|\left(\partial_z K^{(n)}\right)(g)\right| \leq cn^{-(D+1)/2} e^{-b\rho(g)^2/n} \tag{2}$$

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for all $n \in \mathbb{N}$, $g \in G$, and $z \in U$. Moreover (see [**3**]) the first order Riesz transform $\partial_z H^{-1/2}$ is bounded in $L^p := L^p(G; dg)$ for all $z \in G$ and $1 < p < \infty$. In general these results do not extend to second or higher order difference operators: the estimate $\|\partial_{z_1}\partial_{z_2}K^{(n)}\|_\infty = O(n^{-(D+2)/2})$, $n \in \mathbb{N}$, may fail, and the transform $\partial_{z_1}\partial_{z_2}H^{-1}$ may fail to be bounded (cf. [**3**, p.122]; also see [**1**, **7**] for related results).

This failure can occur when z_1, z_2 are elements of a compact factor M''. But if the compact factor is chosen as in Theorem 1.1 we have the following positive result.

THEOREM 1.3. *Let G, K, and M' be as in Theorem* 1.1*, so that $\pi \circ \Psi_{M'}$ is harmonic. Then one has an estimate*

$$\left|\partial_m K^{(n)}(g)\right| + \left|\partial_z \partial_m K^{(n)}(g)\right| \le c n^{-(D+2)/2} e^{-b\rho(g)^2/n}$$

for all $n \in \mathbb{N}$, $g \in G$, $m \in M'$, and $z \in U$. Moreover, for any $m \in M'$ and $z \in G$, the Riesz transforms $\partial_m H^{-1}$ and $\partial_z \partial_m H^{-1}$ are bounded in L^p for $1 < p < \infty$.

The proof of Theorem 1.3 will be an extension of the analysis of Alexopoulos [**3**]. He obtains precise *Berry-Esseen* estimates which show that the convolution powers $K^{(n)}$ are asymptotically close, for large n, to the heat kernel p_n of a sublaplacian operator on N. We will improve these estimates when $\pi \circ \Psi_{M'}$ is harmonic.

To state our final theorem, given a compact factor M', we define a Lie group $G_N = G_N(M')$ with underlying manifold G and group product $*_N$ such that

$$m_1 *_N m_2 = m_1 m_2, \quad x_1 *_N x_2 = x_1 x_2, \quad x_1 *_N m_1 = m_1 *_N x_1 = x_1 m_1,$$

for all $x_1, x_2 \in N$ and $m_1, m_2 \in M'$. Observe that G_N is isomorphic to $N \times M' \cong N \times (G/N)$. To emphasize that the precise definition of G_N depends on the choice of compact factor M', we write $G_N = G_N(M')$.

Define the difference operators $\widetilde{\partial}_z = I - R_{G_N}(z)$, $z \in G$, where R_{G_N} denotes the right regular representation of the group G_N.

THEOREM 1.4. *Adopt the hypotheses of Theorem* 1.3*, and consider $G_N = G_N(M')$ where $\pi \circ \Psi_{M'}$ is harmonic. Then one has an estimate*

$$\left|\widetilde{\partial}^g_{z_1}\widetilde{\partial}^g_{z_2} K^{(n)}(g^{-1}h)\right| \le c n^{-(D+2)/2} e^{-b\rho(g^{-1}h)^2/n}$$

for all $n \in \mathbb{N}$, $g, h \in G$ and $z_1, z_2 \in U$ (the superscript g indicates that $\widetilde{\partial}_{z_i}$ act with respect to the variable g). Moreover, the transform $\widetilde{\partial}_{z_1}\widetilde{\partial}_{z_2}H^{-1}$ is bounded in L^p, $1 < p < \infty$, for all $z_1, z_2 \in G$.

We finish this section with a number of remarks.

(a) Theorems 1.3 and 1.4 are not valid without the hypothesis that $\pi \circ \Psi_{M'}$ is harmonic.

More precisely, one can show, for example, that if M'' is a compact factor such that $\|\partial_m K^{(n)}\|_\infty = O(n^{-(D+2)/2})$, $n \in \mathbb{N}$, for each $m \in M''$, then $\pi \circ \Psi_{M''}$ must be harmonic. We will omit the proof (one can prove it by a straightforward extension of the analysis of Section 4 below).

(b) See Theorem 4.4 in Section 4 below for a Berry-Esseen estimate involving the differences ∂_m and $\widetilde{\partial}_{z_1}\widetilde{\partial}_{z_2}$.

(c) The theorems in this paper could be generalized to any Lie group of polynomial volume growth. In this more general setting, roughly speaking one has $G = SM$ with M a compact subgroup and S a solvable normal subgroup, and to approximate G with a nilpotent group one defines the *nilshadow* S_N of the solvable group S (for details see [**2**, **3**, **6**]). However, for simplicity, in this paper we restrict ourselves to groups $G = NM$.

(d) For a *sublaplacian* on a Lie group of polynomial growth, the author [**5**] has obtained results comparable with Theorems 1.3 and 1.4. For the results of [**5**] one needs to choose a harmonic realization (in a sense analogous to Theorem 1.1), though the relationship with harmonic maps is not explicitly stated in [**5**].

Let us state the analogue for a sublaplacian of Theorem 1.1. The proof is omitted, but is actually essentially contained in the arguments of [**6**, pp. 139–140].

THEOREM 1.5. *Let* $\widetilde{H} = -\sum_{i=1}^{d'} A_i^2$ *be a sublaplacian on* $G = NM$, *where* $A_1, \ldots, A_{d'}$ *is a list of left invariant vector fields on* G *which satisfy the Hörmander condition* (*for background, see* [**12**] *for instance*). *Then, there exists a compact factor* M' *such that* $\pi \circ \Psi_{M'}$ *is harmonic with respect to* $\widetilde{H}$; *that is, given linear coordinates* $\{x^i\}_{i=1}^d$ *on* $N/[N, N] \cong \mathbb{R}^d$, *one has* $\widetilde{H}(x^i \circ \pi \circ \Psi_{M'}) = 0$ *for all* $i \in \{1, \ldots, d\}$.

(e) Observe that the results of Theorem 1.3 for $\partial_z \partial_m$ follow trivially from the results for ∂_m.

(f) The compact factor M' in Theorem 1.2 is unique. But M' in Theorem 1.1 is not necessarily unique; indeed, it is easy to see that

$$\pi \circ \Psi_{M'} = \pi \circ \Psi_{yM'y^{-1}}, \qquad \text{for any } y \in [N, N].$$

Conversely, one can prove (we will omit the details) that if M', M'' are any compact factors such that $\pi \circ \Psi_{M'}$ and $\pi \circ \Psi_{M''}$ are harmonic, then there exists $y \in [N, N]$ with $M'' = yM'y^{-1}$.

(g) The following Gaussian estimate for higher order differences is proved in [**4**], and can also be obtained by applying difference operators to the Taylor expansions of [**3**]. Given any $k \in \mathbb{N}$, one has an estimate

$$\left|\partial_{x_1} \ldots \partial_{x_k} \partial_z K^{(n)}(g)\right| \leq c n^{-(k+1)/2} n^{-D/2} e^{-b\rho(g)^2/n} \tag{3}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_k \in U \cap N$ and $z \in U$.

Then we can explain the situation for second-order differences as follows. Choose M' with $\pi \circ \Psi_{M'}$ harmonic, and suppose $m_1, m_2 \in M'$, $x_1, x_2 \in N$. Then by Theorem 1.3 and (3), the functions $\|\partial_{m_1} \partial_{m_2} K^{(n)}\|_\infty$, $\|\partial_{x_1} \partial_{x_2} K^{(n)}\|_\infty$ and $\|\partial_{x_1} \partial_{m_1} K^{(n)}\|_\infty$ are of order $O(n^{-(D+2)/2})$. (This assertion can also be derived from Theorem 1.4.)

On the other hand, in general $\|\partial_{m_1} \partial_{x_1} K^{(n)}\|_\infty$ is only $O(n^{-(D+1)/2})$. To see this, write

$$\partial_{m_1} \partial_{x_1} = \partial_{x_1} \partial_{m_1} + R(m_1 x_1) \partial_y, \quad \text{with } y = x_1^{-1} m_1^{-1} x_1 m_1,$$

and note that if M' acts non-trivially on N then $\|\partial_y K^{(n)}\|_\infty$ is only $O(n^{-(D+1)/2})$ in general.

This problem does not arise for G_N-invariant difference operators, since $\widetilde{\partial}_{m_1} = \partial_{m_1}$ *commutes* with $\widetilde{\partial}_{x_1}$ for all $m_1 \in M'$ and $x_1 \in N$.

Theorems 1.2 and 1.1 will be proved in Sections 2 and 3 respectively; Theorems 1.3 and 1.4 are proved in Section 4.

2. Proof of Theorem 1.2. To prove Theorem 1.2, in this section we consider $G = NM$ where M is a fixed compact factor and $N \cong \mathbb{R}^d$.

To motivate the proof, suppose temporarily that $N = \mathbb{R}^d$ and write $\Psi_M = (x^1, \ldots, x^d)$ where $x^i : G \to \mathbb{R}$. In general the "coordinates" x^i are not harmonic. But it turns out that one can solve the d equations

$$H\chi_i = Hx^i \tag{4}$$

for some functions $\chi_i : G \to \mathbb{R}$ which are constant in the direction of N, that is, they are lifts of functions on $G/N \cong M$. Then $z^i := x^i - \chi_i$ is harmonic. To show that $(z^1, \ldots, z^d) = \Psi_{M'}$ for some compact factor M', we will essentially rewrite (4) as an abstract linear equation in the vector space N (see (8) below).

Let us fix some notation. The action $T: M \to \mathrm{Aut}(N) = GL(N)$, $T(m)x := mxm^{-1}$, $m \in M$, $x \in N$, is a representation of M in the vector space N. The group product of G is given by

$$g_1 g_2 = (x_1 m_1)(x_2 m_2) = (x_1 + T(m_1)x_2)(m_1 m_2) \tag{5}$$

for $g_i = x_i m_i \in G$, $m_i \in M$, $x_i \in N$, $i = 1, 2$. (We often use $+$ to denote the product within N.) It is convenient to extend T to a representation $T: G \to \mathrm{Aut}(N)$ of G, by setting $T(xm) = T(m)$ for $m \in M$, $x \in N$.

Since M is compact, we may choose a positive-definite inner product $\langle \cdot, \cdot \rangle$ on N which is T-invariant, that is, $\langle T(g)x, T(g)y \rangle = \langle x, y \rangle$ for all $g \in G$, $x, y \in N$. Defining V_1 as the orthogonal complement in N of $V_0 := \{x \in N : T(g)x = x \text{ for all } g \in G\}$, we have

$$N = V_0 \oplus V_1, \tag{6}$$

where the vector subspaces V_i are invariant under T. Note that V_0, V_1 are normal subgroups of G and V_0 is contained in the centre of G; moreover, $G \cong V_0 \times (V_1 M)$.

For each $y \in N$, we define a compact factor $M^y := yMy^{-1}$ and set $\Psi^{(y)} := \Psi_{M^y} : G \to N$. In view of (6) we can write

$$\Psi^{(y)}(g) = \Psi_0^{(y)}(g) + \Psi_1^{(y)}(g),$$

where $\Psi_i^{(y)} : G \to V_i$, $i = 0, 1$. In case $y = e$ we have $M^e = M$, and to simplify the notation we will write $\Psi^{(e)} = \Psi$ and $\Psi_i^{(e)} = \Psi_i$.

THEOREM 2.1. *Fix the density K on G. There exists a unique element $y \in V_1$ such that the map $\Psi^{(y)}$ is harmonic with respect to K.*

Theorem 2.1 yields the existence statement of Theorem 1.2. To get the uniqueness statement of Theorem 1.2, we also need the following lemma.

LEMMA 2.2. *Any two compact factors of G are conjugate via N; that is, for any compact factor M', there is $z \in N$ with $M' = zMz^{-1}$.*

One can prove Lemma 2.2 using standard Lie algebra results about the conjugacy of Levi subalgebras and Cartan subalgebras. We omit the details (but see, for example, [**6**, p. 81] for a similar proof).

To prove uniqueness in Theorem 1.2, let M' be any compact factor such that $\Psi_{M'}$ is harmonic. By Lemma 2.2, $M' = M^z$ for some $z \in N$. Writing $z = z_0 + z_1$, $z_i \in V_i$, then clearly $M^z = M^{z_1}$. Theorem 2.1 implies that $z_1 = y$, so that $M' = M^y$ and uniqueness is proved.

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It remains to prove Theorem 2.1. In what follows, y will denote an arbitrary element of N.

Abusing notation slightly, we regard the right regular representation $R = R_G$ as acting also on functions $F: G \to N$, and denote also by H the operator defined by $HF = \int_G dg\, K(g)(I - R(g))F$. Then $F: G \to N$ is harmonic if and only if $HF = 0$.

We first derive the "change-of-coordinates" formulae relating $\Psi^{(y)}$ to $\Psi = \Psi^{(e)}$. For $g = xm$, $m \in M$, $x \in N$, observe that

$$g = x(mym^{-1})y^{-1}(ymy^{-1}) = (x - y + T(m)y)(ymy^{-1}),$$

which implies that

$$\Psi^{(y)}(g) = x - y + T(m)y = \Psi(g) - y + T(g)y \tag{7}$$

for all $g \in G$. Taking components in V_0 and V_1, and observing that $y - T(g)y \in V_1$, we find that

$$\Psi_0^{(y)} = \Psi_0, \quad \Psi_1^{(y)}(g) = \Psi_1(g) - y + T(g)y$$

for all $y \in N$, $g \in G$.

We claim that $\Psi_0^{(y)} = \Psi_0$ is harmonic. Indeed, note from (5) that $\Psi_0: G \to N$ is a group homomorphism, and apply the following lemma.

LEMMA 2.3. *A smooth homomorphism χ of G into a vector space $W \cong \mathbb{R}^s$ is harmonic. In particular, $\Psi_0^{(y)} = \Psi_0$ is harmonic.*

Proof. By a change of variable $g \to g^{-1}$ and the symmetry $K(g^{-1}) = K(g)$, we have

$$\int dg\, K(g)\chi(g) = \int dg\, K(g^{-1})\chi(g^{-1}) = -\int dg\, K(g)\chi(g),$$

so that $\int dg\, K(g)\chi(g) = 0$. Then $\int dg\, K(g)(\chi(h) - \chi(hg)) = -\int dg\, K(g)\chi(g) = 0$ for all $h \in G$, and χ is harmonic. $\square$

The next lemma establishes that $\Psi^{(y)}$ is harmonic if and only if it is harmonic at the identity e, that is, if and only if $(H\Psi^{(y)})(e) = 0$.

LEMMA 2.4. *For all $h \in G$ and $y \in N$,*

$$\big(H\Psi^{(y)}\big)(h) = -T(h)\left[\int dg\, K(g)\Psi^{(y)}(g)\right] \in V_1.$$

In particular, $\Psi^{(y)}$ is harmonic if and only if $\int dg\, K(g)\Psi^{(y)}(g) = 0$.

Proof. Suppose $g = x_1m_1$, $h = x_2m_2$, where $m_1, m_2 \in M^y$ and $x_1, x_2 \in N$. Since $hg = (x_2 + (m_2x_1m_2^{-1}))(m_2m_1)$, we calculate that

$$\begin{aligned}\Psi^{(y)}(h) - \Psi^{(y)}(hg) &= x_2 - \big(x_2 + \big(m_2x_1m_2^{-1}\big)\big)\\ &= -\big(m_2x_1m_2^{-1}\big)\\ &= -T(h)\big(\Psi^{(y)}(g)\big).\end{aligned}$$

Therefore

$$\big(H\Psi^{(y)}\big)(h) = \int dg\, K(g)\big[\Psi^{(y)}(h) - \Psi^{(y)}(hg)\big] = -\int dg\, K(g)T(h)\Psi^{(y)}(g),$$

which proves the first equality of the lemma. Since $\Psi_0^{(y)}$ is harmonic, $H\Psi^{(y)} = H\Psi_1^{(y)}$ takes values in V_1, and the lemma follows. □

From Lemma 2.4, together with (7), we obtain the following criterion.

LEMMA 2.5. *Let $y \in N$. The map $\Psi^{(y)}$ is harmonic if and only if y satisfies*

$$\int dg\, K(g)(I - T(g))y = \int dg\, K(g)\Psi(g). \tag{8}$$

By Lemma 2.4, the right side of equation (8) is in V_1. To complete the proof of Theorem 2.1 a final lemma is needed.

LEMMA 2.6. *The linear transformation $H_T := \int dg\, K(g)(I - T(g))$ of N restricts to a bijection $H_T\colon V_1 \to V_1$. Hence there is a unique $y \in V_1$ satisfying equation (8).*

Proof. We show that the restriction of H_T to V_1 is injective. Let $x \in V_1$ with $H_T x = 0$. Observe (using $K(g^{-1}) = K(g)$) that

$$H_T = 2^{-1}\int dg\, K(g)(I - T(g^{-1}))(I - T(g)),$$

so that

$$0 = \langle H_T x, x\rangle = 2^{-1}\int dg\, K(g)\langle (I - T(g))x, (I - T(g))x\rangle.$$

Since K is strictly positive in a neighborhood of the identity of G, it follows that $x = T(g)x$ for all g in some neighborhood of the identity. Because T is a representation of G, then $T(g)x = x$ for all $g \in G$, in other words, $x \in V_0 \cap V_1 = \{0\}$. This proves the lemma and completes the proof of Theorems 2.1 and 1.2. □

3. Proof of Theorem 1.1 from Theorem 1.2. In this section we derive Theorem 1.1 from Theorem 1.2.

Let G be as in Theorem 1.1. Define $\overline{G} = G/[N, N]$, $\overline{N} = N/[N, N] \subseteq \overline{G}$, and let $\pi\colon G \to \overline{G}$ be the canonical map. Observe that $\overline{N} \cong \mathbb{R}^d$ is abelian.

Let $d\overline{g}$ be a Haar measure on $\overline{G}$, and consider the probability density $\overline{K} := \pi(K)\colon \overline{G} \to \mathbb{R}$ satisfying $\int_G dg\, K(g) f(\pi(g)) = \int_{\overline{G}} d\overline{g}\, \overline{K}(\overline{g}) f(\overline{g})$ for all continuous functions $f\colon \overline{G} \to \mathbb{R}$. The discrete Laplacians H, $\overline{H}$, corresponding respectively to K and $\overline{K}$, are related by

$$(\overline{H} f) \circ \pi = H(f \circ \pi). \tag{9}$$

Let M be a compact factor of G, and observe that $\pi(M)$ is a compact factor of $\overline{G}$, that is, $\overline{G} = \overline{N}(\pi(M))$.

Applying Theorem 1.2 to $\overline{G}$ yields a compact factor M'' of $\overline{G}$ such that $\Psi_{M''}\colon \overline{G} \to \overline{N}$ is harmonic with respect to $\overline{K}$. By Lemma 2.2 applied to $\overline{G}$, there is a $z \in \overline{N}$ such that $M'' = z(\pi(M))z^{-1}$.

Choose $y \in N$ with $\pi(y) = z$, and consider the compact factor $M' = yMy^{-1}$. Clearly $\pi(M') = M''$. Now let $x^i\colon \overline{N} \to \mathbb{R}$, for $i \in \{1, \dots, d\}$, be some linear coordinates on $\overline{N}$. By applying (9) with $f = x^i \circ \Psi_{M''}$, we find that

$$x^i \circ \Psi_{M''} \circ \pi = x^i \circ \pi \circ \Psi_{M'}\colon G \to \mathbb{R}$$

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is harmonic with respect to K. Thus $\pi \circ \Psi_{M'}$ is harmonic, and Theorem 1.1 follows. $\square$

4. Proofs of Theorems 1.3 and 1.4. For the proofs of Theorems 1.3 and 1.4, in this section we fix a compact factor M' of G such that $\pi \circ \Psi_{M'}: G \to N/[N, N]$ is harmonic with respect to K.

Our analysis is an extension of the analysis of Alexopoulos [**3**], and we will need to refer to [**3**] at some points.

Let us fix Haar measures dm and dx on the groups M' and N respectively, such that $dm(M') = 1$ and $\int_G dg\, f(g) = \int_{M'} dm \int_N dx\, f(xm)$ for all $f \in C_c(G)$.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{n}$ and $\mathfrak{m}'$ be the subalgebras of $\mathfrak{g}$ corresponding to the subgroups N and M'. The lower central series $\mathfrak{n}_i$, $i \in \mathbb{N}$, of $\mathfrak{n}$ is given by by $\mathfrak{n}_1 = \mathfrak{n}$, $\mathfrak{n}_{i+1} = [\mathfrak{n}, \mathfrak{n}_i] \subseteq \mathfrak{n}_i$, and since $\mathfrak{n}$ is nilpotent there is an $r \geq 1$ such that $\mathfrak{n}_{r+1} = \{0\}$ and $\mathfrak{n}_r \neq \{0\}$.

Because $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, then $[\mathfrak{m}', \mathfrak{n}_i] \subseteq \mathfrak{n}_i$ for all i. We can then choose subspaces $\mathfrak{a}_i \subseteq \mathfrak{n}$ such that $[\mathfrak{m}', \mathfrak{a}_i] \subseteq \mathfrak{a}_i$ and $\mathfrak{n}_i = \mathfrak{a}_i \oplus \mathfrak{n}_{i+1}$ for each $i \in \{1, \dots, r\}$. In particular, $\mathfrak{n} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$. Moreover, for each i we can decompose $\mathfrak{a}_i = \mathfrak{a}_i^0 \oplus \mathfrak{a}_i^1$, where $\mathfrak{a}_i^0 = \{x \in \mathfrak{a}_i : [\mathfrak{m}', x] = \{0\}\}$ and $\mathfrak{a}_i^1 = \operatorname{span}\{[m', x] : m' \in \mathfrak{m}', x \in \mathfrak{a}_i\}$.

Set $d = \dim(\mathfrak{n})$ and $d_i = \dim(\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_i) = d - \dim(\mathfrak{n}_{i+1})$ for $i \in \{0, 1, \dots, r\}$. Now fix a basis $x_1, \dots, x_d$ of $\mathfrak{n}$ such that $\mathfrak{a}_i$ is the linear span of $\{x_j : d_{i-1} < j \leq d_i\}$ for all $i \in \{1, \dots, r\}$, and such that $\mathfrak{a}_i^0$ and $\mathfrak{a}_i^1$ are linearly spanned by the x_j's which they contain. If $j \in \{1, \dots, d\}$ with $d_{i-1} < j \leq d_i$, then we set $\sigma(j) = i$. Denote by X_j the left invariant vector field on N corresponding to x_j.

As in [**3**], one defines the *homogenized sublaplacian* associated with K: it is a left invariant sublaplacian on the nilpotent group N, of the form

$$L = -\sum_{1 \leq j,k \leq d_1} q_{jk} X_j X_k$$

with (q_{jk}) a real, positive-definite matrix of constants. Let $p_t = p_t(x, y)$, $t > 0$, $x, y \in N$, be the heat kernel of L, that is, the kernel of the semigroup e^{-tL}.

Given a kernel S on N, that is, $S: N \times N \to \mathbb{R}$, and an operator P acting on functions on N, then PS will denote the kernel $(PS)(x, y) := P^x S(x, y)$ where P acts with respect to the first variable x. We use a similar convention for kernels and operators on G.

Also, given $S : N \times N \to \mathbb{R}$ we define $S^\sharp: G \times G \to \mathbb{R}$ by $S^\sharp(g, h) = S(\Psi_{M'} g, \Psi_{M'} h)$, $g, h \in G$.

Define the Gaussian $G_{b,t}: G \times G \to \mathbb{R}$ by $G_{b,t}(g, h) = t^{-D/2} e^{-b\rho(g^{-1}h)^2/t}$, for $b, t > 0$. The Gaussian estimates for heat kernels on nilpotent Lie groups (see [**12**, Chapter IV] or [**2**]) yield, for any $n \geq 0$ and $j_1, \dots, j_n \in \{1, \dots, d\}$, an estimate

$$\left| \left(X_{j_1} X_{j_2} \dots X_{j_n} p_t \right)^\sharp (g, h) \right| \leq c t^{-(\sigma(j_1) + \cdots + \sigma(j_n))/2} G_{b,t}(g, h) \tag{10}$$

for all $t \geq 1$, $g, h \in G$.

Define an operator $\Phi := \int dg\, K(g) R(g)$ acting on functions on G, so that $H = I - \Phi$. Observe that Φ^n acts by

$$(\Phi^n f)(g) = \int_G dh\, K_n(g, h) f(h),$$

for $g \in G$, $n \in \mathbb{N}$, where we have set $K_n(g, h) := K^{(n)}(g^{-1}h)$, $g, h \in G$.

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The Berry-Esseen estimate [**3**, Theorem 1.9.1] states that

$$\|K_n - p_n^\sharp\|_\infty \le cn^{-(D+1)/2}$$

for all $n \in \mathbb{N}$ (where $\|\cdot\|_\infty$ denotes the norm in $L^\infty(G \times G)$). Consider the kernel U_t defined by

$$U_t(g,h) = p_t^\sharp(g,h) + \sum_{1 \le j \le d_2} \chi_j(g)(X_j p_t)^\sharp(g,h) + \sum_{1 \le j,k \le d_1} \chi_{jk}(g)(X_j X_k p_t)^\sharp(g,h) \quad (11)$$

for $g, h \in G$, where the smooth, bounded functions $\chi_j, \chi_{jk} : G \to \mathbb{R}$ are the *correctors* as defined in [**3**, Section 10.2]. Note that, because of estimates (10), the Berry-Esseen estimate is equivalent to an estimate $\|K_n - U_n\|_\infty \le cn^{-(D+1)/2}$, $n \in \mathbb{N}$.

In this section, by a *difference operator* of order k, $k \in \mathbb{N}$, we mean an operator of the form $P = \partial_{z_1} \ldots \partial_{z_k}$ or of the form $P = \widetilde{\partial}_{z_1} \ldots \widetilde{\partial}_{z_k}$ for some $z_1, \ldots, z_k \in G$ (the G_N-invariant operators $\widetilde{\partial}_z$ are defined as in Section 1). If $A \subseteq G$ and $z_1, \ldots, z_k \in A$, we will say that P has support in A. If $\mathcal{D}$ is a set of difference operators, all having support in a common compact set $A \subseteq G$, and all of order less than l for some $l \in \mathbb{N}$, then we call $\mathcal{D}$ a *bounded family* (of difference operators).

The following result is essentially a generalization of Theorem 1.9.5 and Corollary 1.9.6 of [**3**].

PROPOSITION 4.1. *Let $\mathcal{D}$ be a bounded family and suppose $\delta \in [1/2, 1)$ is such that, for some $b, c > 0$,*

$$|PK_n| \le cn^{-\delta} G_{b,n}$$

for all $n \in \mathbb{N}$, $P \in \mathcal{D}$. Then there exists $c' > 0$ with

$$\|PK_n - PU_n\|_\infty \le c' n^{-1/2} n^{-\delta} n^{-D/2}$$

for all $n \in \mathbb{N}$ and $P \in \mathcal{D}$. Moreover, given any $\varepsilon > 0$, there exist $c'', b'' > 0$ such that

$$|PK_n - PU_n| \le c'' n^{-(1/2)+\varepsilon} n^{-\delta} G_{b'',n}$$

for all $n \in \mathbb{N}$, $P \in \mathcal{D}$.

Proof. This result follows from [**3**, p. 146] in the case where $\delta = 1/2$ and $\mathcal{D} = \{\partial_z : z \in A\}$ where $A \subseteq G$ is compact. The general case is proved similarly, with obvious changes. In particular, note that the estimate

$$\sum_{1 \le i < [n/2]} i^{-1/2}(n-i-1)^{-(D+3)/2} \le cn^{-1} n^{-D/2}, \quad n \in \mathbb{N},$$

generalizes to

$$\sum_{1 \le i < [n/2]} i^{-\delta}(n-i-1)^{-(D+3)/2} \le c_\delta n^{-1/2} n^{-\delta} n^{-D/2}, \quad n \in \mathbb{N}.$$

□

So far in this section, we have not utilized the assumption that $\pi \circ \Psi_{M'}$ is harmonic. However, this assumption is crucial for the next part of the analysis. As in [**3**], define

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"polynomials" $\mathcal{P}_i : G \to \mathbb{R}$ on G by setting

$$\mathcal{P}_i(\exp(t_d x_d) \dots \exp(t_2 x_2) \exp(t_1 x_1) m) = t_i$$

for all $t_1, \dots, t_d \in \mathbb{R}$ and $m \in M'$, where $i \in \{1, \dots, d_1\}$.

LEMMA 4.2. *The function $\mathcal{P}_i$ is harmonic with respect to K, for each $i \in \{1, \dots, d_1\}$. Therefore, the correctors χ_i satisfy $\chi_i = 0$ for all $i \in \{1, \dots, d_1\}$.*

Proof. Consider the group $\overline{N} := N/[N, N] \cong \mathbb{R}^{d_1}$ and note that the elements $y_j := \pi_*(x_j)$, $j \in \{1, \dots, d_1\}$, form a basis for the Lie algebra of $\overline{N}$. Because $\pi \circ \Psi_{M'} : G \to \overline{N}$ is harmonic, the first statement of the lemma follows from the equality

$$\mathcal{P}_i = y^i \circ \pi \circ \Psi_{M'}, \qquad \text{for } i \in \{1, \dots, d_1\},$$

where $y^i : \overline{N} \to \mathbb{R}$ are the linear coordinates on $\overline{N}$ defined by $y^i(\exp(t_{d_1} y_{d_1}) \dots \exp(t_1 y_1)) = t_i$, for $t_1, \dots, t_{d_1} \in \mathbb{R}$.

The first statement of the lemma implies that $\int_G dg\, \mathcal{P}_i(g) K(g) = 0$, and then the definition of the correctors in [**3**, p. 140] implies that $\chi_i = 0$ for $i \in \{1, \dots, d_1\}$. □

REMARK. Conversely, one may show from the definitions in [**3**], that if the correctors χ_i vanish for all $i \in \{1, \dots, d_1\}$, then $\mathcal{P}_i$ is harmonic for such i and $\pi \circ \Psi_{M'}$ is harmonic. We omit the details.

THEOREM 4.3. *Suppose $\mathcal{D}$ is a bounded family such that $|P(p_t^\sharp)| \le ct^{-1} G_{b,t}$ for all $t \ge 1$ and $P \in \mathcal{D}$. Then*

$$|PK_n| \le c' n^{-1} G_{b',n}$$

for all $n \in \mathbb{N}$, $P \in \mathcal{D}$, and for each $\varepsilon > 0$ there exist $c'', b'' > 0$ with

$$|PK_n - PU_n| \le c'' n^{-(3/2)+\varepsilon} G_{b'',n} \tag{12}$$

for all $n \in \mathbb{N}$, $P \in \mathcal{D}$.

Proof. Estimates in this proof are understood to hold uniformly for all $P \in \mathcal{D}$. Since $\chi_i = 0$ for $i \in \{1, \dots, d_1\}$, it follows from the hypothesis, the definition (11) of U_t, and the estimates (10), that

$$|PU_t| \le ct^{-1} G_{b,t}$$

for all $t \ge 1$. Also, because $\mathcal{D}$ is a bounded family, the bounds (2) imply that $|PK_n| \le cn^{-1/2} G_{b,n}$, $n \in \mathbb{N}$.

Now suppose we have proved, for some $\delta \in [1/2, 1)$, an estimate of form $|PK_n| \le cn^{-\delta} G_{b,n}$, $n \in \mathbb{N}$. Then setting $\delta' = \min\{\delta + (1/4), 1\}$ and using Proposition 4.1, we get

$$|PK_n| \le |PU_n| + |PK_n - PU_n| \le c' n^{-\delta'} G_{b',n}$$

for $n \in \mathbb{N}$. By applying this argument with $\delta = 1/2$, then again with $\delta = 3/4$, we find that $|PK_n| \le cn^{-1} G_{b,n}$. Applying Proposition 4.1 again, with $\delta = 1 - (\varepsilon/2)$, yields the desired estimate of $PK_n - PU_n$. □

Proof of Theorem 1.3. It follows from the definition of $\sharp$ that $\partial_m p_t^\sharp = 0$ for all $m \in M'$. Then applying Theorem 4.3 to the family $\mathcal{D} = \{\partial_m, \partial_z \partial_m : m \in M', z \in U\}$ yields the Gaussian estimates of Theorem 1.3.

Next, let $P = \partial_m$ where $m \in M'$. Formally, we have $H^{-1} - I = (I - \Phi)^{-1} - I = \sum_{n=1}^{\infty} \Phi^n$. Therefore, the operator $PH^{-1} - P = P(H^{-1} - I)$ has integral kernel K' given by

$$\begin{aligned} K'(g,h) &= \sum_{n=1}^{\infty} PK_n(g,h) \\ &= S(g,h) + \sum_{n=1}^{\infty} (Pp_n^{\sharp})(g,h) + \sum_{d_1 < j \le d_2} P\{\chi_j(g)Q_j(g,h)\} \\ &\quad + \sum_{1 \le j,k \le d_1} P\{\chi_{jk}(g)Q_{jk}(g,h)\}, \end{aligned} \tag{13}$$

where we have defined kernels

$$S := \sum_{n=1}^{\infty} (PK_n - PU_n), \quad Q_j := \sum_{n=1}^{\infty} (X_j p_n)^{\sharp}, \quad Q_{jk} := \sum_{n=1}^{\infty} (X_j X_k p_n)^{\sharp},$$

and used the fact that $\chi_j = 0$ for $j \in \{1, \ldots, d_1\}$. Now $Pp_n^{\sharp} = 0$. Also, one deduces from (12) that there exists $\sigma > 0$ such that

$$|S(g,h)| \le c\rho(g^{-1}h)^{-(D+\sigma)}, \quad g, h \in G,$$

and hence the operator acting with integral kernel S is bounded in L^p for all $p \in [1, \infty]$.

Next, we claim that the operators acting with integral kernel Q_j, $d_1 < j \le d_2$, or Q_{jk}, $1 \le j, k \le d_1$, are bounded in L^p, $1 < p < \infty$. Indeed, it follows straightforwardly from (10) that these kernels satisfy standard Calderon-Zygmund estimates on G. One can use an almost-orthogonality argument to establish the boundedness of the operators in L^2, and then Calderon-Zygmund theory yields the boundedness in L^p (see, for example, [**3**, Section 17] and [**11**, pp. 623–625] for arguments of this type).

The operator P, and the operators of multiplication by χ_j, χ_{jk}, are trivially bounded in L^p. From (13) we now see that $PH^{-1} - P$, hence also PH^{-1}, is bounded in L^p, $1 < p < \infty$.

The boundedness of $\partial_z \partial_m H^{-1}$ follows from that of $\partial_m H^{-1}$, and the proof of Theorem 1.3 is complete. □

Remark. In general, the kernel K' of the operator $\partial_m H^{-1} - \partial_m$ ($m \in M'$) does *not* satisfy Calderon-Zygmund estimates on G, so one cannot apply the Calderon-Zygmund theory directly to this kernel. A similar problem occurs for the first order Riesz transforms $\partial_z H^{-1/2} - \partial_z$ considered in [**3**].

Proof of Theorem 1.4. Define $G_N = G_N(M')$ and consider the bounded family $\mathcal{D} = \{\widetilde{\partial}_{z_1}\widetilde{\partial}_{z_2} : z_1, z_2 \in U\}$ of G_N-invariant difference operators. If $z_i = w_i m_i$ ($i = 1, 2$) with $w_i \in N$ and $m_i \in M'$, then

$$\widetilde{\partial}_{z_1}\widetilde{\partial}_{z_2}(p_t^{\sharp}) = \widetilde{\partial}_{w_1}\widetilde{\partial}_{w_2}(p_t^{\sharp}) = \{(I - R_N(w_1))(I - R_N(w_2))p_t\}^{\sharp},$$

where R_N denotes the right regular representation of N. It easily follows, by using (10), that $|Pp_t^{\sharp}| \le ct^{-1}G_{b,t}$ for all $t \ge 1$ and $P \in \mathcal{D}$. Therefore, Theorem 4.3 applies and yields $|PK_n| \le c'n^{-1}G_{b',n}$ for all $n \in \mathbb{N}$, $P \in \mathcal{D}$, which is the desired Gaussian estimate.

Next, fix $z_1, z_2 \in G$ and let $P = \widetilde{\partial}_{z_1}\widetilde{\partial}_{z_2}$. Since one has an estimate $|Pp_t^{\sharp}| \le ct^{-1}G_{b,t}$, $t \ge 1$, we can apply Theorem 4.3 to P.

Then a repetition of the proof of Theorem 1.3 shows that PH^{-1} is bounded in L^p, $1 < p < \infty$. The only new step is to show that the operator T with integral kernel

$$K''(g, h) := \sum_{n=1}^{\infty} (Pp_n^{\sharp})(g, h)$$

is bounded in L^p, $1 < p < \infty$. But since K'' satisfies standard Calderon-Zygmund estimates (use again (10)), the boundedness of T can be established by the same reasoning used to prove the boundedness for the kernels Q_j, Q_{jk} in the proof of Theorem 1.3. Then the proof of Theorem 1.4 is complete. □

Finally, the following Berry-Esseen estimate is of some interest. It follows from Theorem 4.3 and the proofs of Theorems 1.3 and 1.4.

THEOREM 4.4. *Assume that* $\pi \circ \Psi_{M'}$ *is harmonic. Then for each* $\varepsilon > 0$*, there exist* $c, b > 0$ *such that*

$$\left|\widetilde{\partial}_{z_1}\widetilde{\partial}_{z_2} K_n - \widetilde{\partial}_{z_1}\widetilde{\partial}_{z_2} U_n\right| + |\partial_m K_n - \partial_m U_n| \leq c n^{-(3/2)+\varepsilon} G_{b,n}$$

for all $n \in \mathbb{N}$*,* $z_1, z_2 \in U$ *and* $m \in M'$*.*

By refining our arguments one could probably obtain this estimate also for $\varepsilon = 0$, but we do not need this improvement.

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