An Extension of Heaviside's Operational Method of Solving Differential Equations.

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§ 1. *Introductory.*

An elegant symbolic method of solving differential equations was developed by Heaviside in his "Electrical Papers" and "Electromagnetic Theory," chiefly in connexion with problems concerning electric currents in net-works of wires. Attention has recently been called to the method by Bromwich,* who applied it to a wider range of problems and gave an extension of Heaviside's formula; another generalisation of the formula has been obtained by Carson.†

In the present paper a formula is obtained which contains the formulae of Heaviside, Bromwich and Carson as particular cases, and whose form is such that it may be readily applied to physical problems.

§ 2. *Deduction of the Formula.*

Consider a physical system which is governed by one or more differential equations; let u be one of the dependent variables and t one of the independent variables. Let p be written for the operator $\partial/\partial t$, p^2 for $\partial^2/\partial t^2$ and so on and let p be treated as a symbol obeying the ordinary laws of Algebra.

Suppose that after making this substitution a solution of the differential equation or equations can be obtained, satisfying any given boundary conditions, in the symbolic form

$$u = \frac{F(p)}{\Delta(p)} P, \quad \text{..........................(1)}$$

where $F(p)$ and $\Delta(p)$ denote polynomials in p, which are such that $\Delta(p)$ is of higher degree than $F(p)$, and P is a function of the

* *Phil Mag*, 6th Ser., 37, (1919), p. 407.

† *Physical Review*, 2nd Ser., 10, (1917), p. 217.

independent variables. The method consists in interpreting this symbolic solution by obtaining an equivalent expression which no longer involves the operator p.

If $\Delta(p)$ be of degree n, and if its zeros be $\alpha_1, \alpha_2, \ldots, \alpha_n$, which we suppose all different, it is a well-known algebraic result that

$$\frac{F(p)}{\Delta(p)} = \sum_{r=1}^{n} \frac{A_r}{p - \alpha_r}, \text{ where } A_r = \frac{F(\alpha_r)}{\Delta'(\alpha_r)} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Let us suppose in the first place that $P = Kt^k e^{\lambda t}$ where K, k and λ are independent of t. Let z be the function obtained by operating on P with the operator $(p - \alpha_r)^{-1}$, so that z is a solution of the differential equation

$$\left(\frac{\partial}{\partial t} - \alpha_r\right) z = Kt^k e^{\lambda t}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

integrating this as an ordinary differential equation for z, we obtain

$$ze^{-\alpha_r t} = C + K\int t^k e^{(\lambda - \alpha_r)t}\, dt$$

$$= C + Ke^{(\lambda - \alpha_r)t}\left[\frac{t^k}{\lambda - \alpha_r} - \frac{kt^{k-1}}{(\lambda - \alpha_r)^2} + \frac{k(k-1)t^{k-2}}{(\lambda - \alpha_r)^3} - \ldots + \frac{(-1)^k k!}{(\lambda - \alpha_r)^{k+1}}\right]$$

where C is an arbitrary constant of integration. We now take z to be that solution of (3) which reduces to zero when $t = 0$, so that

$$z = Ke^{\lambda t}\left[\frac{t^k}{\lambda - \alpha_r} - \frac{kt^{k-1}}{(\lambda - \alpha_r)^2} + \frac{k(k-1)t^{k-2}}{(\lambda - \alpha_r)^3} - \ldots + \frac{(-1)^k k!}{(\lambda - \alpha_r)^{k+1}}\right]$$
$$- \frac{(-1)^k k!\, Ke^{\alpha_r t}}{(\lambda - \alpha_r)^{k+1}}. \quad \ldots\ldots\ldots\ldots(4)$$

To obtain the result of operating on P with $F(p)/\Delta(p)$ we multiply the expression (4) by A_r and sum for all values of r from 1 to n, giving for the value of u which reduces to zero when $t = 0$ the formula

$$u = K\left[e^{\lambda t}\left\{t^k \sum_{r=1}^{n} \frac{A_r}{\lambda - \alpha_r} - kt^{k-1} \sum_{r=1}^{n} \frac{A_r}{(\lambda - \alpha_r)^2} + k(k-1)t^{k-2} \sum_{r=1}^{n} \frac{A_r}{(\lambda - \alpha_r)^3}\right.\right.$$
$$\left.\left. - \ldots + (-1)^k k! \sum_{r=1}^{n} \frac{A_r}{(\lambda - \alpha_r)^{k+1}}\right\} - (-1)^k k! \sum_{r=1}^{n} \frac{A_r e^{\alpha_r t}}{(\lambda - \alpha_r)^{k+1}}\right]. \quad \ldots(5)$$

Now suppose that $F(p)/\Delta(p)$ can be expanded in a series of positive powers of $(p-\lambda)$ in the form

$$\frac{F(p)}{\Delta(p)} = N_0 + N_1(p-\lambda) + N_2(p-\lambda)^2 + N_3(p-\lambda)^3 + \ldots\ldots$$

From equation (2) we obtain also

$$\frac{F(p)}{\Delta(p)} = \sum_{r=1}^{n} \frac{A_r}{\lambda-\alpha_r}\left(1+\frac{p-\lambda}{\lambda-\alpha_r}\right)^{-1} = \sum_{r=1}^{n}\frac{A_r}{\lambda-\alpha_r} - (p-\lambda)\sum_{r=1}^{n}\frac{A_r}{(\lambda-\alpha_r)^2}$$
$$+ (p-\lambda)^2\sum_{r=1}^{n}\frac{A_r}{(\lambda-\alpha_r)^3} - \ldots\ldots$$

Comparing the coefficients of powers of $(p-\lambda)$ in these expansions we have

$$N_0 = \sum_{r=1}^{n}\frac{A_r}{\lambda-\alpha_r},\ N_1 = -\sum_{r=1}^{n}\frac{A_r}{(\lambda-\alpha_r)^2},\ \ldots,$$
$$N_s = (-1)^s \sum_{r=1}^{n}\frac{A_r}{(\lambda-\alpha_r)^{s+1}},\ \ldots,$$

and the solution (5) can be written in the form

$$u = K\Big[\ e^{\lambda t}\{N_0 t^k + N_1 k t^{k-1} + N_2 k(k-1)t^{k-2} + \ldots + N_k k!\}$$
$$- (-1)^k k!\sum_{r=1}^{n}\frac{A_r}{(\lambda-\alpha_r)^{k+1}}\ e^{\alpha_r t}\Big]$$
$$= K\Big[\ e^{\lambda t}\left\{N_0 t^k + N_1\frac{d}{dt}t^k + N_2\frac{d^2}{dt^2}t^k + \ldots + N_k\frac{d^k}{dt^k}\ t^k\right\}$$
$$-(-1)^k k!\sum_{r=1}^{n}\frac{F(\alpha_r)e^{\alpha_r t}}{(\lambda-\alpha_r)^{k+1}\Delta'(\alpha_r)}\Big]$$

It is then easy to extend the result to the case in which P has the form $f(t)e^{\lambda t}$ where $f(t)$ is a polynomial in t of degree m, and we obtain the following general theorem :—

If u be one of the dependent variables in a differential equation or system of differential equations and t be one of the independent variables, and if by writing $\partial/\partial t = p$ and treating p as an algebraic quantity a solution satisfying the boundary conditions can be obtained in the symbolic form

$$u = \frac{F(p)}{\Delta(p)}P,$$

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where $F(p)$ and $\Delta(p)$ are polynomials in p such that $\Delta(p)$ is of higher degree than $F(p)$, and P is a function of t (and of the other independent variables) of the form

$$P = f(t)e^{\lambda t}, \text{ where } f(t) = C_0 + C_1 t + C_2 t^2 + \dots + C_k t^k;$$

then the solution which satisfies the prescribed boundary conditions and which reduces to zero when $t = 0$ is given by

$$u = e^{\lambda t}[N_0 f(t) + N_1 f'(t) + N_2 f''(t) + \dots + N_k f^k(t)]$$
$$+ \sum_{r=1}^{n} \frac{F(\alpha_r)}{\Delta'(\alpha_r)} e^{\alpha_r t}\left[\frac{C_0}{\alpha_r - \lambda} + \frac{C_1}{(\alpha_r - \lambda)^2} + \frac{C_2 . 2\,!}{(\alpha_r - \lambda)^3} + \frac{C_3 . 3\,!}{(\alpha_r - \lambda)^4} + \dots + \frac{C_k . k\,!}{(\alpha_r - \lambda)^{k+1}}\right], \quad \dots\dots(6)$$

where N_0, N_1, ... are the coefficients in the expansion of $F(p)/\Delta(p)$ in a series of powers of $(p - \lambda)$ in the form

$$\frac{F(p)}{\Delta(p)} = N_0 + N_1(p - \lambda) + N_2(p - \lambda)^2 + \dots,$$

and α_1, α_2, ... , α_n are the roots of $\Delta(p) = 0$.

The formula (6) may be written in the alternative form

$$u = e^{\lambda t}\left[N_0 + N_1 \frac{d}{dt} + N_2 \frac{d^2}{dt^2} + \dots + N_k \frac{d^k}{dt^k}\right] f(t)$$
$$+ \sum_{r=1}^{n} \frac{F(\alpha_r)}{\Delta'(\alpha_r)} e^{\alpha_r t}\left[f\left(\frac{d}{ds}\right)\left(\frac{1}{\alpha_r - s}\right)\right]_{s=\lambda}$$

or

$$u = e^{\lambda t}\left[\frac{F\left(\lambda + \frac{d}{dt}\right)}{\Delta\left(\lambda + \frac{d}{dt}\right)} f(t)\right] + \sum_{r=1}^{n} \frac{F(\alpha_r)}{\Delta'(\alpha_r)} e^{\alpha_r t}\left[f\left(\frac{d}{ds}\right)\left(\frac{1}{\alpha_r - s}\right)\right]_{s=\lambda}$$

where the expression $F\left(\lambda + \frac{d}{dt}\right)\Big/\Delta\left(\lambda + \frac{d}{dt}\right)$ is supposed to be expanded in a series of powers of $\frac{d}{dt}$.

It is easy to modify the result for the case in which the roots of $\Delta(p) = 0$ are repeated.

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Although the theorem has only been proved when $F(p)$ and $\Delta(p)$ are polynomials in p, it is readily seen that the formula may also be used when more general types of functions take the place of these polynomials, though a formal proof in such cases is more troublesome; such cases will be illustrated in the examples which follow.

Evidently the result will also apply when the function $f(t)$ is in the form of an *infinite* series of powers of t, provided that the series on the right hand side of (6) are convergent.

When P is a constant, say C, the formula reduces to Heaviside's result,

$$u = C\left[\frac{F(0)}{\Delta(0)} + \sum_{r=1}^{n} \frac{F(\alpha_r)}{\Delta'(\alpha_r)} \frac{e^{\alpha_r t}}{\alpha_r}\right].$$

When $P = Ce^{\lambda t}$, the formula becomes

$$u = C\left[\frac{F(\lambda)}{\Delta(\lambda)} e^{\lambda t} + \sum_{r=1}^{n} \frac{F(\alpha_r)}{\Delta'(\alpha_r)} \frac{e^{\alpha_r t}}{(\alpha_r - \lambda)}\right]$$

which is the result given by Carson (*loc. cit.*)

When $P = Ct$, we obtain the formula

$$u = C\left[N_0 t + N_1 + \sum_{r=1}^{n} \frac{F(\alpha_r)}{\Delta'(\alpha_r)} \frac{e^{\alpha_r t}}{\alpha_r^2}\right]$$

which is Bromwich's expression (*loc. cit.*).

§ 3. *Example 1. The interaction of Two Equal Coils.*

As a first illustration of the method we will consider the effect of switching an electromotive force E into one of two coils which, to simplify the algebra we will take to be equal in all respects. Let R be the resistance, L the self-inductance and M the mutual inductance of the two coils and let the current in the primary coil be x, that in the secondary being y; the equations to be satisfied are therefore

$$L\frac{dx}{dt} + M\frac{dy}{dt} + Rx = E,\ M\frac{dx}{dt} + L\frac{dy}{dt} + Ry = 0.$$

Writing p for $\frac{d}{dt}$ these become

$$(Lp+R)x+Mpy=E,\ Mpx+(Lp+R)y=0$$

and therefore

$$x=\frac{(Lp+R)E}{(Lp+R)^2-M^2p^2},\ y=-\frac{MpE}{(Lp+R)^2-M^2p^2}.$$

Let the electromotive force E be switched into the primary circuit at the instant $t=0$, and let it be represented by

$$E=f(t)e^{\lambda t} \text{ where } f(t)=M_0+M_1t+M_2t^2+\ldots+M_mt^m.$$

To obtain the current x we have to interpret the effect of the operator $F(p)/\Delta(p)$ acting on the function E, where

$$F(p)=Lp+R \text{ and } \Delta(p)=(Lp+R)^2-M^2p^2.$$

Writing $\xi=p-\lambda$, the operator $F(p)/\Delta(p)$ can be written in the form

$$\frac{A\xi+B}{C\xi^2+D\xi+F}=\phi(\xi) \text{ say, where}$$

$$A=L,\ B=L\lambda+R,\ C=L^2-M^2,\ D=2\{(L^2-M^2)\lambda+LR\},$$
$$F=(L\lambda+R)^2-M^2\lambda^2.$$

Expanding the function $\phi(\xi)$ in a series of powers of ξ by Maclaurin's theorem we obtain

$$\phi(\xi)=\frac{B}{F}+\frac{(AF-BD)}{F^2}\xi+\frac{(BD^2-CBF-AFD)}{F^3}\xi^2+\ldots$$

the later coefficients in the series being determined from the recurrence formula

$$F\phi^n(0)+nD\phi^{n-1}(0)+n(n-1)C\phi^{n-2}(0)=0.$$

In the notation of § 2 we have therefore

$$N_0=\frac{B}{F},\ N_1=\frac{AF-BD}{F},\ N_2=\frac{BD^2-CBF-ADF}{F},\ \ldots$$

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Again the roots of $\Delta(p)=0$ are $\alpha=-\dfrac{R}{L+M}$, $\beta=-\dfrac{R}{L-M}$ and therefore

$$\frac{F(\alpha)}{\Delta'(\alpha)}=\frac{L\alpha+R}{2L(L\alpha+R)-2M^2\alpha}=\frac{L\alpha+R}{2\{(L^2-M^2)\alpha+LR\}}=\frac{1}{2(L+M)},$$

$$\frac{F(\beta)}{\Delta'(\beta)}=\frac{L\beta+R}{2\{(L^2-M^2)\beta+LR\}}=\frac{1}{2(L-M)}.$$

Applying the formula of § 2 the current in the primary circuit is given by

$$x=e^{\lambda t}\left[N_0 f(t)+N_1\frac{d}{dt}f(t)+N_2\frac{d^2}{dt^2}f(t)+\ldots+N_m\frac{d^m}{dt^m}f(t)\right]$$

$$+\frac{e^{\alpha t}}{2(L+M)}\left[\frac{M_0}{\alpha-\lambda}+\frac{M_1}{(\alpha-\lambda)^2}+\frac{M_2\,.\,2\,!}{(\alpha-\lambda)^3}+\frac{M_3\,.\,3\,!}{(\alpha-\lambda)^4}+\ldots+\frac{M_m\,m\,!}{(\alpha-\lambda)^{m+1}}\right]$$

$$+\frac{e^{\beta t}}{2(L-M)}\left[\frac{M_0}{\beta-\lambda}+\frac{M_1}{(\beta-\lambda)^2}+\frac{M_2\,.\,2\,!}{(\beta-\lambda)^3}+\frac{M_3\,.\,3\,!}{(\beta-\lambda)^4}+\ldots+\frac{M_m\,m\,!}{(\beta-\lambda)^{m+1}}\right]$$

To obtain the current in the secondary circuit we consider the operator $F(p)/\Delta(p)$ where $F(p)=-Mp$ and $\Delta(p)=(Lp+R)^2-M^2p^2$ and obtain in this case

$$N_0'=-\frac{M\lambda}{F},\quad N_1'=-\frac{M(F-D\lambda)}{F^2},\quad N_2'=-\frac{M(D^2\lambda-CF\lambda-DF)}{F^3},\ \ldots,\ \text{and}$$

$$y=e^{\lambda t}\left[N_0'f(t)+N_1'\frac{d}{dt}f(t)+N_2'\frac{d^2}{dt^2}f(t)+\ldots+N_m'\frac{d^m}{dt^m}f(t)\right]$$

$$+\frac{e^{\alpha t}}{2(L+M)}\left[\frac{M_0}{\alpha-\lambda}+\frac{M_1}{(\alpha-\lambda)^2}+\frac{M_2\,.\,2\,!}{(\alpha-\lambda)^3}+\frac{M_3\,.\,3\,!}{(\alpha-\lambda)^4}+\ldots+\frac{M_m\,.\,m\,!}{(\alpha-\lambda)^{m+1}}\right]$$

$$-\frac{e^{\beta t}}{2(L-M)}\left[\frac{M_0}{\beta-\lambda}+\frac{M_1}{(\beta-\lambda)^2}+\frac{M_2\,.\,2\,!}{(\beta-\lambda)^3}+\frac{M_3\,.\,3\,!}{(\beta-\lambda)^4}+\ldots+\frac{M_m\,.\,m\,!}{(\beta-\lambda)^{m+1}}\right]$$

By putting $\lambda=\iota\mu$ and then taking the real parts of these expressions for x and y we should obtain the currents in the two circuits when the electromotive force in the primary circuit is of the form

$$E=f(t)\cos\mu t,\ \text{where}\ f(t)=M_0+M_1t+M_2t^2+\ldots+M_mt^m,$$

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§ 4. *Example 2. The conduction of Heat through a Block of Finite Thickness.*

Consider the conduction of heat in an infinite block of conducting material of thickness l, whose initial temperature is zero and whose faces are parallel and are maintained at temperatures $f_1(t)$ and $f_2(t)$ from the instant $t=0$ onwards.

The axis of x being taken perpendicular to the bounding planes, the temperature u satisfies the differential equation

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

where $k=K/\rho\sigma$, K being the thermal conductivity, ρ the density and σ the specific heat of the substance composing the block; subject to the boundary conditions:—$u=f_1(t)$ for $x=0$ and $u=f_2(t)$ for $x=l$, for all positive values of t.

Writing p for $\partial/\partial t$ and putting $p=kq^2$, the solution satisfying the assigned boundary conditions is obtained in the symbolic form

$$u=\frac{\sinh q(l-x)}{\sinh ql}f_1(t)+\frac{\sinh qx}{\sinh ql}f_2(t).$$

Now suppose

$$f_1(t)=(A_0+A_1t+A_2t^2)e^{\lambda_1 t},\ f_2(t)=(B_0+B_1t+B_2t^2)e^{\lambda_2 t},$$

and write $u=u_1+u_2$, where u_1 is the part of u arising from $f_1(t)$ and u_2 is the part arising from $f_2(t)$.

Consider first u_2 and, in the notation of § 2, take

$$F(p)=\sinh qx/q,\ \Delta(p)=\sinh ql/q;$$

then expanding $F(p)/\Delta(p)$ in a series of powers of $(p-\lambda_2)$ we obtain

$$F(p)/\Delta(p)=N_0+N_1(p-\lambda_2)+N_2(p-\lambda_2)^2+\dots,$$

where

$$N_0=\frac{\sinh\beta_2 x}{\sinh\beta_2 l},\ N_1=\frac{x\cosh\beta_2 x-l\coth\beta_2 l\sinh\beta_2 x}{2\sqrt{k\lambda_2}\sinh\beta_2 l}$$

$$N_2=\frac{(x^2-l^2)\sinh\beta_2 l\sinh\beta_2 x-(\beta_2^{-1}\sinh\beta_2 l+2l\cosh\beta_2 l)(x\cosh\beta_2 x-l\coth\beta_2 l\sinh\beta_2 x)}{8k\lambda_2\sinh^2\beta_2 l}$$

where $\beta_2=\sqrt{\lambda_2/k}.$

The roots $\alpha_0, \alpha_1, \alpha_2, \ldots$ of $\Delta(p) = 0$ occur when $ql = \iota r\pi$, where r is zero or a positive integer,* so that $\alpha_r = -kr^2\pi^2/l^2$; we obtain

$$\frac{F(\alpha_r)}{\Delta'(\alpha_r)} = -(-1)^r \frac{2kr\pi}{l^2} \sin\frac{r\pi x}{l}.$$

It follows from the formula of § 2, that

$$u_2 = \{N_0B_0 + N_1B_1 + 2N_2B_2 + (N_0B_1 + 2N_1B_2)t + N_0B_2t^2\}e^{\lambda_2 t}$$

$$+ \sum_{r=0}^{\infty}(-1)^r \frac{2kr\pi}{l^2} \sin\frac{r\pi x}{l} e^{\frac{-kr^2\pi^2 t}{l^2}}$$

$$\left\{ \frac{B_0}{\lambda_2 + kr^2\pi^2/l^2} - \frac{B_1}{(\lambda_2 + kr^2\pi^2/l^2)^2} + \frac{2B_2}{(\lambda_2 + kr^2\pi^2/l^2)^3} \right\}.$$

The value of u_1 is obtained from this by changing x into $l-x$ and replacing B_0, B_1, B_2 by A_0, A_1, A_2, respectively, and λ_2 by λ_1. By adding the value of u_1 so obtained to the above expression for u_2 we obtain the required temperature u.

By writing $\lambda_1 = \iota\sigma_1$ and $\lambda_2 = \iota\sigma_2$ and taking the real parts, the formulae give the temperature at any point when, from the instant $t = 0$ onwards, the bounding planes are maintained at temperatures

$$f_1(t) = (A_0 + A_1t + A_2t^2)\cos\sigma_1 t, \; f_2(t) = (B_0 + B_1t + B_2t^2)\cos\sigma_2 t.$$

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* The values for which r is a negative integer provide no new roots of $\Delta(p) = 0$.

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