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SLIM TREES

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Abstract

A semilattice tree T with 0 is slim if there is a chain C with 0 so that the lattices $\theta(T)$ and $\theta(C)$ of semilattice congruences are isomorphic. This paper establishes elementary consequences of slimness and uses simple constructive techniques to show certain small trees slim. If T is the union of at most countably many branches, each of which has a maximum or a countable cofinal subset, then T is slim. For trees with enough maximals slimness is equivalent with not having any uncountable anti-chains. If a tree T has a countable cofinal subset then T is slim. Thus finitary trees are slim.

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1. Introduction

A meet semilattice T is called a tree if for each $x \in T$, $\{y \in T \mid y \leqslant x\}$ is a chain. For a semilattice M, $\theta(M)$ here denotes the lattice of semilattice congruences of M. Call a tree T with zero (having a least element) slim if there is some chain C with zero so that $\theta(T) \cong \theta(C)$. Slim trees are then those trees whose semilattice congruence structure is chainlike. It is the purpose of this paper to use elementary constructive techniques to establish some criteria for slimness as well as the slimness of certain small trees.

The approach used here is based on the notion of the Boolean ring $B[M]$ universal over a semilattice M (Evans (1977)). This is roughly speaking a Boolean ring generated by an independent multiplicative system M in it. For a tree T with zero the lattice $\mathscr{I}(B[T])$ of ideals of the ring $B[T]$ is isomorphic with $\theta(T)$. This allows a rephrasing of slimness for T: there is a chain C with zero so that $B[T] \cong B[C]$. Detecting slimness then boils down to asking which trees T have a generating chain in their $B[T]$. This work is then tied in with the long standing research (Mostowski and Tarski (1939), Mayer and Pierce (1960) and others) on chain generated Boolean rings. Essentially the work here consists of various elementary ways of slimming out a tree T inside $B[T]$ to produce a chain which will generate $B[T]$.

In Section 2 we review the notion of the Boolean ring universal over a semilattice and establish some terminology. In Section 3 we introduce slimness and some of its elementary consequences, finding that slim trees cannot have in them uncountable anti-chains. Section 4 introduces two constructions. The first roughly consists of taking a family of trees T_i with zero and gluing them together at zero (taking their disjoint union and then identifying the zeros) to produce a tree we denote by $\dot{\bigvee} T_i$. The other is a familiar stacking of chains construction.

Section 5 analyses trees via their branches showing that if a tree T has an at most countable family $\{Z_i\}$ of branches whose union is T then one can produce chains with zero $\{C_i\}$ so that $B[T] \cong B[\dot{\bigvee} C_i]$. If these branches $\{Z_i\}$ have top points we show that the chains $\{C_i\}$ can be stacked and the original T is shown slim. Section 6 shows first that for trees where each element is dominated by a maximal, slimness is equivalent with the tree not having any uncountable anti-chains. Then the notion of countable chain is introduced and it is shown that if a tree T has an at most countable family of branches whose union is T, and if these branches are either countable chains or have top points, then T is slim. We conclude in Section 7 by presenting the notion of a finitary tree, an idea designed to encompass the well-known binary trees (trees generally with too many branches). We show that finitary trees are slim.

All semilattices here are meet (lower) semilattices; their operation is written as multiplication. If M is a semilattice, $D \subseteq M$, $\downarrow_M D$ denotes $\{x \in M \mid \text{for some } d \in D, x \leqslant d\}$. If $D = \{d\}$ write $\downarrow_M d$ instead of $\downarrow_M \{d\}$. If the context is clear, drop the subscript M. Similar comments hold for $\uparrow_M D$, $\uparrow D$, $\uparrow_M d$, $\uparrow d$. An ordinal number as used here is thought of as the set of its predecessors. Each natural number (starting with 0, the empty set) is an ordinal and the set ω of natural numbers is an ordinal. The two element field is denoted Z_2.

2. Boolean rings universal over semilattices

Let M be a semilattice with 0, B be a Boolean ring and ϕ: $M \to B$ be a zero preserving multiplicative homomorphism. The pair (B, ϕ) is said to be *universal over* M if for any Boolean ring R and any zero preserving multiplicative homomorphism ψ: $M \to R$ there is exactly one ring homomorphism χ: $B \to R$ so that $\chi \circ \phi = \psi$. If ϕ: $M \to B$ is universal over M the map ϕ is an order embedding and B is unique up to isomorphism extending ϕ. For each semilattice M with zero there is a universal Boolean ring (and map); we denote this ring by $B[M]$.

We say for a semilattice M and a Boolean ring B that M is a *subsemilattice in B* if M is a subsemilattice of B's multiplicative reduct; we say that M is a 0-*semilattice in B* if M, beyond being a semilattice in B, contains the zero, the least element of B. If the meet semilattice T is a tree (that is, for each $x \in T$, $\downarrow x$ is a chain) we use the phrases *trees in B*, 0-*tree in B*; similar phrases are used for chains. If $D \subseteq B$, B a Boolean ring then $\langle D \rangle$ denotes the subring generated by D. If M is a semilattice in B then $\langle M \rangle$ consists precisely of those elements which are finite sums of elements of M. If M is a 0-semilattice in B then $M \setminus \{0\}$ is $\mathbf{Z}_2$ linearly independent in B if and only if whenever $m_1, \dots, m_n$, $m \in M$ and all $m_i < m$ then their join in B, $m_1 \vee \dots \vee m_n < m$ (in B).

Let M be a semilattice with 0. Construct $B[M]$ so that it contains M and so that the universal map mentioned above is inclusion. Internally characterized $B = B[M]$ is a Boolean ring such that M is a 0-semilattice in B, $M \setminus \{0\}$ is $\mathbf{Z}_2$ linearly independent in B and $\langle M \rangle = B$. If M is a tree T then these reduce to the demands: T is a 0-tree in B so that $\langle T \rangle = B$ (T generates B).

The space $\mathscr{F}_p(M)$ of proper filters (filters $\neq \emptyset, M$) of a semilattice with zero M with its intrinsic order topology (inherited from the power set of M) is homeomorphic with the Stone space (prime ideal space) of the ring $B[M]$. The lattices of open sets $\mathscr{C}(\mathscr{F}_p(M))$, $\mathscr{C}(S(B[M]))$ of these spaces are then isomorphic from which follows $\mathscr{C}(\mathscr{F}_p(M) \cong \mathscr{I}(B[M])$ the ideal lattice of the ring $B[M]$.

For any congruence σ of M, σ^e denotes the extension of σ to $B[M]$, that is the ring congruence of $B[M]$ generated by σ. For a ring congruence δ of $B[M]$, δ^c denotes the contraction of δ to M, the congruence $\delta \cap (M \times M)$ of M. For each congruence σ of M, $B[M/\sigma] = B[M]/\sigma^e$ and as a consequence $\sigma = (\sigma^e)^c$. If M is a tree with 0 then for each congruence of $B[M]$, $(\delta^c)^e = \delta$. Thus the Galois connexion of extension and contraction establishes, in the case of a semilattice tree T with 0, an order isomorphism between the lattice $\theta(T)$ of congruences of T and the lattice $\mathscr{I}(B[T])$ of ideals of the ring $B[T]$. Hence for such a T the lattices $\theta(T)$ and $\mathscr{C}(\mathscr{F}_p(T))$ are isomorphic.

We mention one final fact. If M is a semilattice in the Boolean ring B, $b \in B$ is a lower bound in B of M, then the set $b + M = \{b + m \mid m \in M\}$ is a semilattice in M and the map $M \to b + M$ whereby $m \mapsto b + m$ is a multiplicative isomorphism between M and $b + M$. Similar results hold if we assume, rather than b being a lower bound of M, that for each $m \in M$, $b \cdot m = 0$ (calculated in B).

3. Slim trees

We begin with a result of Mayer and Pierce (1960, page 930) which can be generalized to arbitrary meet semilattices M.

THEOREM 3.1. *Let M be a chain with largest element. The following statements are equivalent.*

(i) *$B[M]$ is countably complete (countable subsets have least upper bounds),*

(ii) *$B[M]$ is complete,*

(iii) *M is finite.*

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Applying this result to trees one obtains the following.

COROLLARY 3.2. *Let T be any tree with 0. The following statements are equivalent.*
(i) *$B[T]$ is conditionally complete (nonempty bounded above subsets have sups).*
(ii) *for each $t \in T$, $\downarrow_T t$ is finite,*
(iii) *There is a set S so that $B[T]$ is isomorphic with the lattice $\mathrm{Fin}(S)$ of finite subsets of S.*

PROOF. Write B for $B[T]$. Assume (i). Let $t \in T$. Note that

$$B[\downarrow_T t] = \downarrow_B t = \{b \in B \mid b \leqslant t\}.$$

Since B is conditionally complete then $B[\downarrow_T t]$ is complete. But $\downarrow_T t$ is a chain with a largest element. Thus 3.1 says $\downarrow_T t$ is finite.

Assume (ii). T is then a tree wherein every interval is finite. So by a result of Varlet (1965) the congruence lattice $\theta(T)$ is a Boolean algebra. But $\theta(T)$ is complete and dually atomic. (The latter is because $\theta(T) \cong \mathscr{I}(B[T])$ and every ideal of $B[T]$ is the intersection of maximal ideals.) Thus $\theta(T)$ is isomorphic to a power set. Hence there is a set S so that $\theta(T) \cong P(S) = \{x \mid x \subseteq S\}$. Restricting the isomorphisms $\mathscr{I}(B[T]) \cong \theta(T)$ and $\theta(T) \cong P(S)$ of these algebraic lattices to their compact elements we have

$$B[T] \cong c(\theta(T)) \cong c(P(S)) = \mathrm{Fin}(S)$$

Here $c(L)$ denotes the collection of compact elements of an algebraic lattice L. This gives (iii). Note that where $B[T] \cong \mathrm{Fin}(S)$ and T is infinite, $\# T = \# S$.

The implication (iii) to (i) is trivial.

A tree T with 0 is said to be slim if there is a chain C with 0 so that $B[T] \cong B[C]$. Internally this means that there must be a 0-chain C in $B[T]$ so that $\langle C \rangle = B[T]$. The first result on slim trees is trivial.

THEOREM 3.3. *Let T be any tree with 0. The following statements are equivalent.*
(i) *T is slim,*
(ii) *there is a chain C with 0 so that $\theta(T) \cong \theta(C)$,*
(iii) *there is a chain C with 0 so that $\mathscr{F}_p(T)$ is homeomorphic with $\mathscr{F}_p(C)$.*

THEOREM 3.4. *Let T be a slim tree. Each semilattice homomorphic image of T is slim.*

PROOF. Suppose T is a slim tree and M is a homomorphic image of T; then for some semilattice congruence α of T, $M \cong T/\alpha$. Thus M is a tree with 0. But $B[M] \cong B[T/\alpha] \cong B[T]/\alpha^e$. Since T is slim there is a chain C with 0 so that $B[T] \cong B[C]$. There is then a ring congruence β of $B[C]$ so that $B[T]/\alpha^e \cong B[C]/\beta$.

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Now there is a congruence γ of C so that $\gamma^e = \beta$. Hence

$$B[M] \cong B[T]/\alpha^e \cong B[C]/\beta = B[C]/\gamma^e \cong B[C/\gamma].$$

But C/γ is certainly a chain with 0. Hence M is a slim tree.

THEOREM 3.5. *Suppose T is a tree so that $B[T]$ is conditionally complete and $\# T > \aleph_0$. Then T is not slim.*

PROOF. Since $B[T]$ is conditionally complete and $\# T > \aleph_0$ then there is a set D so that $\# D > \aleph_0$ and $B[T] \cong \mathrm{Fin}(D)$. Suppose by way of contradiction that T is slim. Then for some chain C with 0, $B[C] \cong \mathrm{Fin}(D)$. Hence by 3.2 each interval of C is finite. Since C is a chain with 0 where each interval is finite then C is countable. But with $B[C] \cong \mathrm{Fin}(D)$, $\# D > \aleph_0$ this is impossible.

A subset D of a tree T is called an anti-chain if for any $x, y, \in D$, $x \neq y$ we have $x \nleqslant y$ and $y \nleqslant x$. Clearly by Zorn's lemma each anti-chain of T is contained in a maximal anti-chain.

COROLLARY 3.6. *If the tree T is slim then every anti-chain in T is at most countable.*

PROOF. Suppose T is slim but that T has an anti-chain A so that $\# A > \aleph_0$. Let $\mathscr{A}$ be an anti-chain maximal in T containing A. Then $\# \mathscr{A} > \aleph_0$. Note that if $t \in \uparrow_T \mathscr{A}$ there is exactly one $a_t \in \mathscr{A}$ with $t \geqslant a_t$. Observe also that $T = (\uparrow_T \mathscr{A}) \cup (\downarrow_T \mathscr{A} \setminus \mathscr{A})$, a disjoint union.

Let R denote the tree whose carrier set is $\mathscr{A} \cup \{0\}$ and whose multiplication is given by $x \cdot y = 0$ whenever $x \neq y$. Note that $\# R > \aleph_0$ and $\mathscr{A}$ is a maximal anti-chain in R. Define a map $\phi : T \to R$ as follows. For any $t \in T$,

$$\phi(t) = \begin{cases} a_t & \text{if } t \in \uparrow_T \mathscr{A}, \\ 0 & \text{if } t \in \downarrow_T \mathscr{A} \setminus \mathscr{A}. \end{cases}$$

The map ϕ decomposes the tree T into convex subsemilattices; thus ϕ is a homomorphism. Clearly ϕ is onto R. Thus R is a homomorphic image of T. So R is slim. But for each $r \in R$, $\downarrow_R r$ is certainly finite so 3.2 says $B[R]$ is conditionally complete. But then Theorem 3.5 says R cannot be slim.

Note that the construction of R given in the proof of 3.6 allows one to construct many Boolean rings which are tree generated but which cannot be chain generated.

4. Some constructions

Let $(T_i \mid i \in I)$ be a system of trees with zero. Suppose T is a tree with 0 and for each i there is an injective zero preserving homomorphism ϕ_i of T_i into T. Suppose the

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system (T, ϕ_i) satisfies

(i) If $i \neq j$ then the composite product $\phi_i[T_i] \cdot \phi_j[T_j] = \{0\}$,

(ii) $T = \bigcup \{\phi[T_i] \mid i \in I\}$.

The tree T if it exists is unique up to isomorphism. We will here denote it by $\bigvee T_i$. If $I = \{1,2\}$ we will write it as $T_1 \vee T_2$. Note that as a consequence of (i), if $i \neq j$, $\phi_i[T_i] \cap \phi_j[T_j] = \{0\}$.

For a collection $(B_i \mid i \in I)$ of Boolean rings ΣB_i denotes the subring $\{f \in \times B_i \mid \text{for almost all } i,\ f(i) = 0\}$ of the ring $\times B_i$.

For a system $(T_i \mid i \in I)$ of trees with zero and for each index $j \in I$ define a map ϕ_j: $T_j \to \Sigma B[T_i]$ as follows. If $t \in T_j$ and if $i \in I$

$$\phi_j(t)(i) = \begin{cases} t & \text{if } i = j, \\ 0_i & \text{if } i \neq j. \end{cases}$$

Then $\phi_j(t)$: $I \to \bigcup B[T_i]$ so that for each i, $\phi_j(t)(i) \in B[T_i]$ and so that for almost all i, $\phi_j(t)(i) = 0_i$. But clearly ϕ_i is an order embedding zero preserving multiplicative homomorphism. Now if $i \neq j$ and if $t \in T_i$, $s \in T_j$ then for any index $k \in I$

$$[\phi_i(t) \cdot \phi_j(s)](k) = [\phi_i(t)(k)] \cdot [\phi_j(s)(k)]$$

and one of these right-hand factors must be zero; so for any $k \in I$, $[\phi_i(t) \cdot \phi_j(s)](k) = 0$. Hence $\phi_i[T_i] \cdot \phi_j[T_j]$ calculated in $\Sigma B[T_i]$ is $\{\mathscr{C}\}$ ($\mathscr{C}$ in $\Sigma B[T_i]$). Thus if $T = \bigcup \{\phi_i[T_i] \mid i \in I\}$ then T is a 0-tree in $\Sigma B[T_i]$ and is isomorphic with $\bigvee T_i$. Let

$$D = \{f \in \Sigma B[T_i] \mid \text{for exactly one } i \in I,\ f(i) \neq 0_i\}.$$

Certainly D generates the ring $\Sigma B[T_i]$. Let $f \in D$ and suppose $i \in I$ so that $f(i) \neq 0$. Now $f(i)$ is a nonzero element of $B[T_i]$ and $T_i \setminus \{0\}$ is a base for $B[T_i]$ as a $\mathbf{Z}_2$ vector space. Thus there exist nonzero elements $t_1, \ldots, t_n \in T_i$ so that $f(i) = t_1 + \ldots + t_n$. Then without difficulty

$$f = \phi_i(t_1) + \phi_i(t_2) + \ldots + \phi_i(t_n).$$

Thus $D \subseteq \langle T \rangle$, so $\langle T \rangle = \Sigma B[T_i]$.

THEOREM 4.1. *Let $(T_i \mid i \in I)$ be a system of trees with zero. Then the tree $\bigvee T_i$ exists and $B[\bigvee T_i] \cong \Sigma B[T_i]$. Hence for trees T_1, T_2 with zero, $B[T_1 \vee T_2] \cong B[T_1] \times B[T_2]$.*

COROLLARY 4.2. *Suppose each of $(T_i \mid i \in I)$ and $(S_i \mid i \in I)$ is a system of trees with zero so that for each $i \in I$, $B[T_i] \cong B[S_i]$. Then $B[\bigvee T_i] \cong B[\bigvee S_i]$.*

PROOF. The individual isomorphisms $B[T_i] \cong B[S_i]$ induce an isomorphism $\times B[T_i] \cong \times B[S_i]$. This latter restricts to an isomorphism $\Sigma B[T_i] = \Sigma B[S_i]$.

Let n be an ordinal, $0 < n \leqslant \omega$. If T is a tree with zero then $n \cdot T$ denotes $\bigvee(T_i \mid i \in n)$ where the system $(T_i \mid i \in I)$ is determined by: for each $i \in n$, $T_i = T$.

For any chain C use C' to denote C if C has no largest element and to denote $C \setminus \{\max C\}$ if C has a largest element. For a tree T with zero and C a chain with zero write $C \lhd T$ to denote the semilattice whose carrier set is the disjoint union of C' and T and whose multiplication is given by

$$x \cdot y = \begin{cases} x \cdot y & \text{calculated in } T \text{ if } x, y \in T, \\ x \cdot y & \text{calculated in } C \text{ if } x, y \in C, \\ x & \text{if } x \in C',\ y \in T. \end{cases}$$

Notice that $C \lhd T$ is again a tree with zero. If C_0, C_1 are chains with zero and one and if T is a tree with zero we define $C_0 \lhd C_1 \lhd T$ by $(C_0 \lhd C_1) \lhd T$. Similarly define $C_0 \lhd C_1 \lhd C_2 \lhd T$ and so forth. If $(C_i)_{i \in \omega}$ is a sequence of chains with 0 and 1 then $C_0 \lhd C_1 \lhd \ldots$ (or $\lhd_{i \in \omega} C_i$) denotes the direct limit of $(C_0 \lhd \ldots \lhd C_n)_{n \in \omega}$ with appropriate inclusion maps.

THEOREM 4.3. *Suppose C is a chain with zero and* 1 *and T is a tree with zero. Then* $B[C \vee T] \cong B[C \lhd T]$.

PROOF. Let γ denote the maximum element of C. We know $B[C \vee T] \cong B[C] \times B[T]$. In the latter construct the set

$$D = \{(c, 0) \mid c \in C\} \cup \{(\gamma, t) \mid t \in T\}.$$

This D is a 0-tree in $B[C] \times B[T]$ which is isomorphic with $C \lhd T$. The elements of the set $\{(c, 0) \mid c \in C\} \cup \{(0, t) \mid t \in T\}$ generate $B[C] \times B[T]$. But for each $c \in C$, $(c, 0) \in \langle D \rangle$ and if $t \in T$ then $(0, t) = (\gamma, 0) + (\gamma, t)$ so that $(0, t) \in \langle D \rangle$. Thus $B[C] \times B[T] = D$. Hence $B[C \vee T] = B[C \lhd T]$.

Note that the assumption that C has a 1 is necessary here. For if $C = \omega$ and if $T = 1$ (a 1 element tree) then $C \vee T \cong C$ and $B[C] = \mathrm{Fin}(C)$. However $C \lhd T \cong \omega + 1$ (that is the ordered set of natural numbers followed by one point) and there is an element $t \in C \lhd T$ with $\downarrow t$ infinite. Hence $B[C \lhd T]$ is not conditionally complete. Thus $B[C \lhd T] \ncong B[C \vee T]$.

5. Branches

LEMMA 5.1. *In a tree T if $x \leqslant a$ and $x \nleqslant b$ then $x \cdot b = a \cdot b$.*

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PROOF. Certainly $x \cdot b \leqslant a \cdot b$. Since T is a tree and both x and $a \cdot b$ are below a then either $x \leqslant a \cdot b$ or $a \cdot b \leqslant x$. But if $x \leqslant a \cdot b$ then $x \leqslant b$, against our assumption. Hence $a \cdot b \leqslant x$. Thus $a \cdot b \leqslant x \cdot b$.

A *branch Z of a tree T* is a maximal chain in T. Certainly each branch Z of T is a lower end of T ($x \in Z$, $y \leqslant x$ implying $y \in Z$). Every point of T and every chain in T is contained in a branch of T.

LEMMA 5.2. *Suppose $Z_1, \ldots, Z_n$, C are distinct branches of the tree T with zero. Then the set $C \cap (Z_1 \cup \ldots \cup Z_n)$ has a largest element. Let $z = \max[C \cap (Z_1 \cup \ldots \cup Z_n)]$. If $t \in C \backslash (Z_1 \cup \ldots \cup Z_n)$ and if $s \in Z_1 \cup \ldots \cup Z_n$ then $t \cdot s = z \cdot s$. Finally $C \backslash (Z_1 \cup \ldots \cup Z_n) = (\uparrow z \backslash \{z\}) \cap C$.*

PROOF. For each $i = 1, \ldots, n$, $b_i = \max(C \cap Z_i)$ exists (take $x \in C \backslash Z_i$, $y \in Z_i \backslash C$; it is not difficult to show $x \cdot y = \max(Z_i \cap C)$). The set $\{b_1, \ldots, b_n\}$, being a subset of C, is a chain. Let $z = \max\{b_1, \ldots, b_n\}$. Then $z \in C \cap (Z_1 \cup \ldots \cup Z_n)$. If $w \in C \cap (Z_1 \cup \ldots \cup Z_n)$ then for some i, $w \in C \cap Z_i$ so that $w \leqslant b_i \leqslant z$. Hence $z = \max(C \cap (Z_1 \cup \ldots \cup Z_n))$.

Let $s \in Z_1 \cup \ldots \cup Z_n$, say $s \in Z_i$ and let $t \in C \backslash (Z_1 \cup \ldots \cup Z_n)$. Then $z \leqslant t$ (*actually strictly less than* t since each Z_i is a lower end). If $z \nleqslant s$ then Lemma 5.1 says $z \cdot s = t \cdot s$. So suppose $z \leqslant s$. Then $z \leqslant s \cdot t$. Since $s \cdot t \in C \cap Z_i$ then $s \cdot t \leqslant b_i \leqslant z$ so $s \cdot t = z = z \cdot s$.

The last assertion of the lemma is trivial.

Let n be any nonzero ordinal so that $n \leqslant \omega$ (thus either n is a natural number or n is the set of natural numbers). Suppose $\{Z_i \mid i \in n\}$ are distinct branches of a tree T with zero so that $T = \bigcup_{i \in n} Z_i$. Let $z_0 = 0$ in T and choose for each $i \in n$ ($i > 0$), $z_i = \max(Z_i \cap (\bigcup_{j<i} Z_j))$. Let $C_i = (\uparrow_T z_i) \cap Z_i$ for each $i \in n$. Note that C_i is a chain with least element z_i. Working in the ring $B = B[T]$ let $D_i = z_i + C_i$ for each $i \in n$. Note that $D_0 = C_0 = Z_0$ and that each D_i is a 0-chain in B which is order isomorphic with C_i.

We claim that if $x \in D_i$, $y \in D_j$ for $i, j \in n$, $i \neq j$ then $x \cdot y = 0$. To see this suppose $i < j$ and write $x = z_i + c_i$, $c_i \in C_i$ and $y = z_j + c_j$, $c_j \in C_j$. Now $z_i, c_i \in \bigcup_{k<j} Z_k$. If $z_j = c_j$ then $y = 0$ so $x \cdot y = 0$. So assume $z_j < c_j$. Then $c_j \in Z_j \backslash (\bigcup_{k<j} Z_k)$. So by Lemma 5.2 since $z_j = \max[Z_j \cap (\bigcup_{k<j} Z_k)]$ then $z_i c_j = z_i z_j$ and $c_i c_j = c_i z_j$. Hence $(z_i + c_i) c_j = (z_i + c_i) z_j$ so that

$$x \cdot y = (z_k + c_i)(z_j + c_j) = (z_i + c_i) z_j + (z_i + c_i) c_j = 0.$$

Thus $\bigcup_{i \in n} D_i$ is a 0-tree in B and it is order isomorphic with $\bigvee_{i \in n} C_i$. Now $Z_0 \subseteq \langle \bigcup_{i \in n} D_i \rangle$. Now suppose $Z_0, \ldots, Z_k \subseteq \langle \bigcup_{i \in n} D_i \rangle$. If $t \in Z_{k+1} \backslash (\bigcup_{j<k+1} Z_j)$ then $t \in C_{k+1}$, so $z_{k+1} + t \in D_{k+1}$. Then with $z_{k+1} \in \langle \bigcup_{j<k+1} D_i \rangle$ we have

$$t = z_{k+1} + (z_{k+1} + t)$$

so that $t \in \langle \bigcup_{i \in n} D_i \rangle$. Thus

$$B = B[T] = \langle \bigcup_{i \in n} D_i \rangle \cong B[\bigvee_{i \in n} C_i].$$

THEOREM 5.3. *Suppose n is a nonzero ordinal, $n \leqslant \omega$ and T is a tree with zero having certain distinct branches $\{Z_i\}_{i \in n}$ so that $T = \bigcup_{i \in n} Z_i$. Let $z_0 = 0$ and for $i \in n$, $i > 0$ let $z_i = \max[Z_i \cap (\bigcup_{j<i} Z_j)]$. For each $i \in n$ let $C_i = (\uparrow z_i) \cap Z_i$ and calculating in $B[T]$ let $D_i = z_i + C_i$. Then*

(i) *$\bigcup_{i \in n} D_i$ is a 0-tree in $B[T]$,*
(ii) *$\bigcup_{i \in n} D_i$ is isomorphic to $\bigvee_{i \in n} C_i$,*
(iii) *$\langle \bigcup_{i \in n} D_i \rangle = B[T]$.*

COROLLARY 5.4. *Let n be a nonzero ordinal $n \leqslant \omega$, and let T be a tree with zero having certain distinct branches $\{Z_i\}_{i \in n}$ whose union is T. Choose C_i as in Theorem 5.3. Then $\{C_i \mid i \in n\}$ is a family of chains with zero so that $B[T] \cong B[\bigvee_{i \in n} C_i]$.*

THEOREM 5.5. *Suppose n is a nonzero ordinal, $n \leqslant \omega$ and T is a tree with zero having distinct branches $\{D_i\}_{i \in n}$ so that*

(i) *$D_i \cdot D_j = \{0\}$ if $i, j \in n$, $i \neq j$,*
(ii) *each D_i has a maximum element a_i,*
(iii) *T is the union of $\{D_i\}_{i \in n}$.*

In $B = B[T]$ let $b_i = a_0 + a_1 + \ldots + a_i$ for each $i \in n$. Let $E_0 = D_0$ and if $0 < i \in n$ let $E_i = b_{i-1} + D_i$. Then

(a) *each E_i is a chain in B order isomorphic with D_i;*
(b) *$E = \bigcup_{i \in n} E_i$ is a 0-chain in B order isomorphic with $D_0 \lhd D_1 \lhd \ldots$;*
(c) *$\langle E \rangle = B$.*

PROOF. For all i, j, $a_i \cdot a_j = 0$ $(i \neq j)$ so that $b_i = a_0 \vee \ldots \vee a_i$ formed in B. If $t \in D_i$ $(i > 0)$ then $t \cdot b_{i-1} = 0$. Thus for $i > 0$, $E_i \cong D_i$. Certainly $E_0 \cong D_0$. Notice that b_{i-1} is least in E_i $(i > 0)$ and b_i is greatest in E_i. We claim that for each i, $x \in E_i$, $y \in E_{i+1}$ imply $x \leqslant y$ and that $x = y$ if and only if $x = b_i = y$. This is clear for $i = 0$. Assume this true for $j < i$ and suppose $i + 1 \in n$. Suppose $x \in E_i$, $y \in E_{i+1}$. Then $y = b_i + d_{i+1} = b_i \vee d_{i+1}$ for some $d_{i+1} \in D_{i+1}$. Since b_i is greatest in E_i we have $x \leqslant b_i \leqslant b_i \vee d_{i+1} = y$. Now if $x = y$ then $x = b_i = y$ is clear.

Thus $E = \bigcup_{i \in n} E_i$ is a 0-chain in B isomorphic with $D_0 \lhd D_1 \lhd \ldots$. Certainly $D_0 \subseteq \langle E \rangle$. Suppose for all $i < j \in n$ that $D_i \subseteq \langle E \rangle$. We show $D_j \subseteq \langle E \rangle$. Then $a_0, \ldots, a_{j-1} \in \langle E \rangle$. Let $x \in D_j$. Then $b_{j-1} + x \in E_j$ so $a_0, \ldots, a_{j-1}$, $b_{j-1} + x$ are all elements of $\langle E \rangle$ hence $x = a_0 + \ldots + a_{j-1} + b_{j-1} + x$ is in $\langle E \rangle$. So $T = \bigcup_{i \in n} D_i \subseteq \langle E \rangle$.

COROLLARY 5.6. *Suppose $(C_i \mid i \in n)$ is a system of chains with zero and one (n is a nonzero ordinal, $n \leqslant \omega$). Then $\bigvee_{i \in n} C_i$ is slim. In fact $B[\bigvee C_i] \cong B[C_0 \lhd C_1 \lhd \ldots]$.*

6. Trees with enough maximals; countable chains

We say a tree T *has enough maximals* if for each $t \in T$ there is an element m, maximal in T, so that $t \leqslant m$.

THEOREM 6.1. *Suppose a tree T has enough maximals and that the set of maximal elements of T is at most countable. Then T is slim.*

PROOF. There is an ordinal number n so that $0 < n \leqslant \omega$ and so that $(a_i \mid i \in n)$ is a listing without repetitions of the maximal elements of T. For each $i \in n$ let Z_i denote $\downarrow_T a_i$. Then the Z_i's are distinct branches of T whose union is T. Form the C_i's and the D_i's as in Theorem 5.3. Then $T_1 = \bigcup_{i \in n} D_i$ os a 0-tree in $B[T]$ whose span is $B[T]$ and which satisfies the conditions of Theorem 5.5. So applying 5.5 to the tree T_1 we produce a 0-chain E (isomorphic with $D_0 \lhd D_1 \lhd \ldots$) in $B[T] = B[T_1]$ so that $\langle E \rangle = B[T_1]$. Then $B[E] = \langle E \rangle = B[T_1] = B[T]$. Thus T is slim.

As a corollary we state the main theorem of this section.

COROLLARY 6.2. *Suppose T is a tree with zero having enough maximals. Then the following statements are equivalent.*

(i) *Each anti-chain in T is at most countable.*

(ii) *The set of maximal elements of T is at most countable.*

(iii) *T is slim.*

Thus for trees with enough maximals, slimness is equivalent with not having any too big anti-chains.

EXAMPLE. Let C denote the usual real number unit interval (the rational interval would work here also). Let T denote any of the trees $m \cdot C$, $C \lhd (n \cdot C)$, $m \cdot (C \lhd n \cdot C) \vee l \cdot (C \lhd k \cdot C)$ and so forth (here n, m, l, k are nonzero natural numbers). Then passing through the application of 5.3, 5.5 as in the proof of 6.1 we have $B[T] \cong B[C]$. If $D = [0, 1)$ (1 removed, do the same in the rational case) and if T denotes any of the trees $\omega \cdot C$, $C \lhd \omega \cdot C$, and so forth, then $B[T] \cong B[D]$.

We say a chain C is of *countable type* if C has no greatest element and if there is an increasing sequence $c_1, c_2, \ldots$ of elements of C so that for each $t \in C$, $t \leqslant c_i$, for some i. Observe that if $C_0, C_1, \ldots$ is a sequence of chains with least and greatest elements then $C_0 \lhd C_1 \lhd C_2 \lhd \ldots$ is a chain of countable type. Thus there is a chain C of countable type with $B[C] \cong B[\bigvee C_i]$.

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LEMMA 6.3. *Let n be an ordinal, $0 < n \leqslant \omega$. Suppose $\{C_i \mid i \in n\}$ is a family of countable type chains. Then there is a chain C of countable type so that $B[C] \cong B[\bigvee C_i]$.*

PROOF. For each $i \in n$ write C_i as $C_i = D_0^i \lhd D_1^i \lhd D_2^i \lhd \ldots$ where each D_j^i is an interval in C_i (and so is a chain with 0 and 1). Now $B[C_i] = B[\bigvee(D_j^i \mid j \in \omega)]$. So $B[\bigvee C_i] \cong B[\bigvee_{i \in n}(\bigvee_{j \in \omega} D_j^i)] = B[\bigvee D_j^i]$.
Since (D_j^i) is a countably infinite family of chains with 0 and 1 there is a countable type chain C so that $B[\bigvee D_j^i] = B[C]$. This completes the proof.

THEOREM 6.4. *Suppose a tree T with zero has a set of finitely or countably many branches, each of which is a countable type chain, whose union is T. Then there is a chain C of countable type so that $B[T] = B[C]$.*

PROOF. One follows through the application of 5.3 and 5.5 as in the proof of Theorem 6.1. One need only observe that if Z is a countable type chain in T and $z \in Z$ then $(\uparrow z) \cap Z$ is again a countable type chain.

EXAMPLE. Let $C = [0, 1)$. Let T be either $n \cdot C$ or $\omega \cdot C$ ($n \neq 0$, $n \in \omega$). Then $B[T] \cong B[C]$.

THEOREM 6.5. *Suppose the tree T with zero has an at most countable family of distinct branches whose union is T and so that each of these branches either has a maximum or is a countable type chain. Then T is slim.*

PROOF. Denote the collection of branches in question by $\{Z_i \mid i \in K\}$. Let $I = \{i \in K \mid Z_i \text{ has a maximum}\}$ let $J = \{i \in K \mid Z_i \text{ is of countable type}\}$. Using 5.3 we can produce a family of chains with 0 $\{C_i \mid i \in I\} \cup \{C_j \mid j \in J\}$ so that:

(i) $B[T] \cong B[\bigvee(C_i \mid i \in I \cup J)] \cong B[\bigvee_{i \in I} C_i] \times B[\bigvee_{j \in J} C_i]$.

(ii) Each C_i ($i \in I$) has a maximum.

(iii) Each C_j ($j \in J$) is of countable type.

Notice that if either I or J is empty the result is given by earlier work.

Assume $J \neq \emptyset \neq I$. Since J is at most countable there is a countable type chain C with 0 so that $B[\bigvee_{j \in J} C_j] \cong B[C]$.

Case 1. Suppose I is finite. Then there is a chain D with 0 and 1 so that $B[D] \cong B[\bigvee_{i \in I} C_i]$. Then $B[T] \cong B[D] \times B[C] \cong B[D \lhd C]$.

Case 2. Suppose I is infinite. Then there is a countable type chain E so that $B[\bigvee_{i \in I} C_i] \cong B[E]$. Then $B[T] \cong B[E] \times B[C] \cong B[C \bigvee E]$ where C, E are countable type chains. Lemma 6.3. says T is slim.

7. Applications; finitary and other trees

Let T be a tree, let B be a subset of T. We will say B is cofinal in T if for each $t \in T$, $t \leqslant b$ for some $b \in B$.

LEMMA 7.1. *Suppose C is a chain without a greatest element having a countable subset D which is cofinal in C. Then C is a countable chain.*

PROOF. Let $D = \{d_1, d_2, \ldots\}$. Choose $x_1 = d_1$ and if x_n has been chosen, choose x_{n+1} to be any element of C greater than $\max(x_n, d_{n+1})$. Then (x_n) is an increasing sequence so that for each $z \in C$, $z < x_n$ for some n.

Thus if C is a chain with no maximum having countably many elements then C is a countable chain.

LEMMA 7.2. *Suppose B is a countable subset of the tree T and that B is cofinal in T. Let C be a branch of T having no maximum. Let $U(C)$ denote the set:*

$$(B \cap C) \cup \{x \cdot b \mid x \in C,\ b \in B,\ x \nleqslant b,\ b \nleqslant x\}.$$

Then $U(C)$ is countably infinite and cofinal in C.

PROOF. Let $b \in B$. Lemma 5.1 implies that the set $\{x \cdot b \mid x \in C,\ x \nleqslant b,\ b \nleqslant x\}$ has at most one element. Hence $U(C)$ is countable. Let $x \in C$. Then there is a $b \in B$ with $x \leqslant b$. If $b \in C$ then $x \leqslant b$ for $b \in U(C)$. If $b \notin C$ then since C is a chain maximal in T there is some $y \in C$ with $b \nleqslant y$ and $y \nleqslant b$. Certainly $y \nleqslant x$. Thus $x < y$. But then $x \leqslant y \cdot b$ with $y \cdot b \in U(C)$. Hence $U(C)$ is cofinal in C. Since C has no maximum, $U(C)$ is infinite. Notice that $U(C) \subseteq C$.

LEMMA 7.3. *If T is a tree with a countable subset which is cofinal in T then each branch of T either has a maximum or is a countable chain.*

PROOF. This is an easy consequence of 7.1. and 7.2.

THEOREM 7.4. *If T is a tree with a countable subset B so that B is cofinal in T, then T is slim.*

PROOF. Each branch of T either has a maximum point or is a countable type chain. Let $B = \{b_1, b_2, \ldots\}$. Choose for each i, a branch Z_i containing the point b_i. Then T is the union of the branches $Z_1, Z_2, \ldots$. Thus T is the union of finitely or countably many branches, each of which either has a maximum or is a countable type chain. Theorem 6.5 tells us that T is slim.

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A well-ordered tree (that is, where each $\downarrow x$ is well ordered) will be said to be of *finite type* if:

F1 For each positive integer n there is an element x with $\operatorname{ord}[\downarrow x] = n$.

F2 For each positive integer n, the set $\{x \mid \operatorname{ord}[\downarrow x] = n\}$ is finite.

F3 For each x, $\operatorname{ord}[\downarrow x]$ is finite.

Here for any well-ordered set D, $\operatorname{ord}[D]$ denotes the unique ordinal number which is order isomorphic to D.

Certainly each well-ordered tree of finite type is slim.

For any tree T let $N(T) = N$ be the set

$$\{x \mid x = 0 \text{ or } x \text{ is maximal in } T \text{ or } x \in (T - \{x\})^2\}.$$

Then $N(T)$ is meet closed in T. Call a tree T *finitary* if

(i) $N(T)$ is either finite or in the inherited order it is well ordered of finite type, and

(ii) $N(T)$ is cofinal in T.

Such trees have at most countable cofinal subsets.

COROLLARY 7.5. *Each finitary tree is slim.*

Familiar examples of finitary trees are the binary and ternary trees. For a positive integer n, call a finitary tree *n-ary* if for each positive integer k the set

$$\{x \in N(T) \mid \operatorname{ord}[\downarrow_N x] = k\}$$

has precisely n^{k-1} elements. It is not difficult to show: if T is any n-ary tree so that for each x, $\downarrow x$ is order isomorphic to $[0, 1]$ then $B[T]$ is isomorphic with $B[[0, 1)]$. If $n > 1$ each n-ary tree has uncountably many branches (overall) but the tree can be expressed as the union of countably many of these.

We say a well-ordered tree is of countable type if it satisfies F1 and F3 above as well as:

C2 For each positive integer n the set $\{x \mid \operatorname{ord}[\downarrow x] = n\}$ is at most countable.

A tree T might be called *countable-ary* if $N(T)$ is cofinal in T and if $N(T)$ is either finite or a well-ordered tree of countable type. Certainly each countable-ary tree T is slim ($N(T)$ constitutes an at most countable cofinal subset). By analogy with n-ary trees, we might say a countable-ary tree is *$\aleph_0$-ary* if for each n the set

$$\{x \in N(T) \mid \operatorname{ord}[\downarrow_N x] = n\}$$

has a countably infinite number of elements. All such trees are slim.

Let A denote the class of those trees with an at most countable cofinal subset (all examples of this section fit here). Let B denote the class of trees where each anti-chain is at most countable. With S denoting the class of slim trees one has $A \leqslant S \leqslant B$.

Writing β to denote the first uncountable ordinal, β is certainly a slim tree so $A \neq S$. Just where in the interval $[A, B]$ S lies is currently open, but helpful in this direction would be a decision as to whether the tree $\beta \vee \beta$ (with β as above) is slim.

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