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On Special Fiber Rings of Modules

Cleto B. Miranda-Neto

Abstract. We prove results concerning the multiplicity as well as the Cohen–Macaulay and Gorenstein properties of the special fiber ring $\mathscr{F}(E)$ of a finitely generated R-module $E \subsetneq R^e$ over a Noetherian local ring R with infinite residue field. Assuming that R is Cohen–Macaulay of dimension 1 and that E has finite colength in R^e, our main result establishes an asymptotic length formula for the multiplicity of $\mathscr{F}(E)$, which, in addition to being of independent interest, allows us to derive a Cohen–Macaulayness criterion and to detect a curious relation to the Buchsbaum–Rim multiplicity of E in this setting. Further, we provide a Gorensteinness characterization for $\mathscr{F}(E)$ in the more general situation where R is Cohen–Macaulay of arbitrary dimension and E is not necessarily of finite colength, and we notice a constraint in terms of the second analytic deviation of the module E if its reduction number is at least three.

1 Introduction

The study of blowup rings in the classical situation of ideals has attracted the attention of several authors in the past decades, and the list of works on the subject is huge, *e.g.*, [1, 6–10, 13–18, 20, 21, 23–25, 28, 29, 31, 32, 34–36, 38]. However, as far as we know, there exist only a few papers in the literature considering the context of modules [2, 11, 12, 24, 27, 30, 33]. For instance, Goto, Hayasaka, Kurano, and Nakamura [12] studied the case of the second syzygy module of the residue field of a regular local ring and Lin and Polini [24] proved some results in the situation where the module is a direct sum of powers of the homogeneous maximal ideal of a standard graded polynomial ring. For the case of the module of logarithmic vector fields of a quasi-homogeneous hypersurface, see [27].

In the present paper we deal with the special fiber of the Rees algebra, the so-called *special fiber ring*, of finitely generated modules over a (commutative, unital) Noetherian local ring $(R, \mathbf{m})$. We are interested more precisely in the Cohen–Macaulay and the Gorenstein properties of this blowup algebra. For the former, we focus on the case of modules of finite colength in a free R-module provided that R is one-dimensional and Cohen–Macaulay. Regarding the Gorenstein property, we study it in a more general context where R is Cohen–Macaulay of arbitrary dimension and the module is not necessarily of finite colength. As will be clear, the investigation developed herein highlights, in particular, the fact that certain numerical invariants, such as analytic spread, multiplicity, and reduction number, play a crucial role in the study of the special fiber ring in the situation of modules as well.

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The structure of the paper is as follows. First, in Section 2, we fix the setup to be in force in the entire paper and we give some preliminaries, for instance, concerning the notions of Rees algebra and reduction number of modules.

Let $E \subsetneq R^e$ be a finitely generated R-module (possessing a generic, constant rank) with n-th Rees power E^n for $n \geq 0$. Consider the special fiber ring

$$\mathscr{F}(E) = \bigoplus_{n=0}^{\infty} \frac{E^n}{\mathbf{m}E^n}$$

and let $\deg(\mathscr{F}(E))$ be its Hilbert–Samuel multiplicity (or degree). In the investigation of the Cohen–Macaulay property (Section 3), we essentially derive results in terms of $\deg(\mathscr{F}(E))$. Proposition 3.2 invokes the generalization, for modules, of the useful criterion furnished by Corso, Polini, and Vasconcelos [7, Proposition 2.4] in the case of ideals (it is worth noting that a multiplicity-based Cohen–Macaulayness criterion for the special fiber ring of an ideal was first given by Shah [32, Theorem 5]). The case of reduction number 1 is considered in Corollary 3.3. Concerning the multiplicity of $\mathscr{F}(E)$, the situation of interest is when E is not of linear type, *i.e.*, the symmetric algebra of E has nontrivial R-torsion, since otherwise it is easy to see that $\deg(\mathscr{F}(E)) = 1$; see Remark 3.4 (ii), where we furnish an example of a maximal Cohen–Macaulay R-module E of rank 2 that is not of linear type but satisfies $\deg(\mathscr{F}(E)) = 1$. In Remark 3.4 (iii) we raise some issues pointing to possible developments of a well-known result due to Sancho de Salas [31] (later improved by Lipman [25]) on the connection between Cohen–Macaulayness of associated graded rings of high powers and the vanishing of certain cohomology groups of a Cohen–Macaulay blowing-up with supports in the closed fiber.

In the situation where R is one-dimensional and Cohen–Macaulay and E has finite colength in the free R-module R^e (we require the rank of E to be equal to $e \geq 1$), we establish in Theorem 3.5 (see also Corollary 3.9) an asymptotic length formula for $\deg(\mathscr{F}(E))$ that, in addition to being of independent interest, led us to the characterization given in Corollary 3.12. In Proposition 3.14 we apply our formula in order to obtain, in particular, a new expression for the Buchsbaum–Rim multiplicity of E, with the aid of results from Brennan, Ulrich, and Vasconcelos [2]. Furthermore, in a general setting, Proposition 3.16 gives a test for the detection of an $\mathscr{F}(E)$-regular element, which might presumably be helpful in trying to derive a maximal $\mathscr{F}(E)$-sequence towards a general Cohen–Macaulayness criterion.

In the last part of the paper (Section 4) we exploit the Gorensteinness of $\mathscr{F}(E)$, without any restriction on the dimension of the Cohen–Macaulay ring R and without assuming that E has finite colength in R^e. As a warming-up result, in Proposition 4.1 we study the case where the second analytic deviation of E is equal to 1; it is illustrated in Example 4.3, where we consider the module of logarithmic derivations of a Fermat-type cubic in 4 indeterminates. The main result of the section is Theorem 4.4, which provides a general Gorensteinness characterization. Furthermore, Proposition 4.8 detects a constraint to this property in terms of the second analytic deviation of the module E in the case where its reduction number is at least 3.

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2 Setup and Preliminaries

Throughout this paper, by ring we mean commutative ring with 1. We permanently fix, unless explicitly stated otherwise, a Noetherian local ring $(R, \mathbf{m})$ having Krull dimension $d \geq 1$ and infinite residue field k (this latter condition may be superfluous to some of the arguments, in virtue of the classical trick of passing to the faithfully flat extension $R[t]_{\mathbf{m}R[t]}$, but we have adopted it in order to avoid technicalities), as well as a strict embedding $E \subsetneq R^e$ of a finitely generated R-module E possessing a rank $e \geq 1$, which means, as usual, that $K \otimes_R E \simeq K^e$, where K is the total ring of fractions of R. Thus, E is torsionfree over R. We shall denote $S := \mathscr{S}_R(R^e) = \bigoplus_{n\geq 0} S_n$, the homogeneous symmetric algebra of R^e that may be regarded as a standard graded polynomial ring $S = R[t_1, \ldots, t_e]$ in indeterminates $t_1, \ldots, t_e$ over $S_0 = R$. In degree 1, we get the R-module $S_1 = \sum_{i=1}^{e} R\, t_i$ together with the natural "linearization" map $l\colon R^e \to S_1$ that sends a given $v = (x_1, \ldots, x_e) \in R^e$ to the linear form $l(v) = \sum_{i=1}^{e} x_i t_i$. Now restricting l to E yields the so-called *Rees algebra* of E, that is, the graded subalgebra $\mathscr{R}(E) = \bigoplus_{n\geq 0} E^n \subset S$, $E^n = [\mathscr{R}(E)]_n$ generated over $E^0 = R$ by $l(v_1), \ldots, l(v_m)$, for some (in fact, any) generating set $\{v_1, \ldots, v_m\}$ of E as an R-module. In other words, $\mathscr{R}(E)$ is the R-algebra generated by $E^1 \subset S_1$. Its Krull dimension is known to be $d + e$ [2, Proposition 2.1]. Each E^n is dubbed a *Rees power* of $E \subset R^e$.

An R-submodule $U \subseteq E$ is said to be a *reduction* of E if $U^1 E^r = E^{r+1}$ for $r \gg 0$. The *reduction number of E with respect to U* is defined as

$$\mathrm{r}_U(E) = \min\{n \geq 0 \mid U^1 E^n = E^{n+1}\}.$$

A *minimal reduction* of E is a reduction that is minimal with respect to inclusion. Since the residue field k of R is assumed to be infinite, minimal reductions are known to exist; moreover, they have rank e as well. The (*absolute*) *reduction number of E* is the integer $\mathrm{r}(E) = \min\{\mathrm{r}_U(E)\}$, where $U \subseteq E$ ranges over all minimal reductions of E.

Still in analogy with the case of ideals, the *special fiber ring* of the R-module $E \subset R^e$ is the special fiber of its Rees algebra, that is, the graded ring

$$\mathscr{F}(E) = \mathscr{R}(E) \otimes_R k = \bigoplus_{n\geq 0} \frac{E^n}{\mathbf{m}E^n},$$

which is standard graded over the field $[\mathscr{F}(E)]_0 = k$. Its Krull dimension is the so-called *analytic spread* of E, denoted by $\ell(E)$. Since k is infinite, it is well known that $\ell(E) = \nu(U)$ for any minimal reduction $U \subseteq E$, where $\nu(\cdot)$ stands for the minimal number of generators; moreover, if E is of *finite colength* in R^e, in the sense that $\lambda(R^e/E) < \infty$, where $\lambda(\cdot)$ denotes length function, then any minimal reduction U of E satisfies $\nu(U) = d + e - 1$ and hence $\ell(E) = d + e - 1$, so that E has maximal analytic spread in this case. In general, $e \leq \ell(E) \leq d + e - 1$.

For the basic notions and facts of commutative algebra that we have tacitly assumed in this paper, see Bruns–Herzog [3]. For the general theory of blowup algebras of modules we refer to Vasconcelos [37].

3 Cohen–Macaulayness and Multiplicity

In this section we are mainly concerned with the Cohen–Macaulay property and the Hilbert–Samuel multiplicity of special fiber rings of modules. In Subsection 3.3 we will describe a precise relation to the Buchsbaum–Rim multiplicity in a suitable setting.

3.1 A Multiplicity-based General Characterization

We begin by recalling a very useful characterization proved by Corso, Polini, and Vasconcelos [7, Proposition 2.4] for the Cohen–Macaulay property of the special fiber ring $\mathscr{F}(I) = \bigoplus_{n\geq 0} I^n/\mathbf{m}I^n$ of an ideal $I \subset R$ in terms of its multiplicity $\deg(\mathscr{F}(I))$ (denoted f_0 therein). We point out that a multiplicity-based Cohen–Macaulayness criterion for $\mathscr{F}(I)$ was first achieved by Shah [32, Theorem 5].

Proposition 3.1 *Let $I \subset R$ be an ideal and let $J \subset I$ be a minimal reduction of I. Set $r = \mathrm{r}_J(I)$. Then $\mathscr{F}(I)$ is Cohen–Macaulay if and only if* $\deg(\mathscr{F}(I)) = 1 + \sum_{i\leq r} \nu(\frac{I^i}{JI^{i-1}})$.

Proposition 3.2 provides the well-known extension of this characterization into the context of modules. We require E to possess a rank and to be embedded in a free R-module R^e in order to keep the very definition of Rees algebra as adopted in Section 2, but, specifically for this result, we do not require the rank of E to be equal to e.

Proposition 3.2 *Let $U \subset E$ be a minimal reduction of E and set $r := \mathrm{r}_U(E)$, with $r \geq 1$ (as we may assume; see Remark 3.4 (ii) below). Then $\mathscr{F}(E)$ is Cohen–Macaulay if and only if its multiplicity can be written as*

$$\deg(\mathscr{F}(E)) \;=\; 1 \;+\; \sum_{i=1}^{r} \nu\Big(\frac{E^i}{U^1E^{i-1}}\Big).$$

Proof Since U is a minimal reduction of E, the ideal $U^1\mathscr{F}(E) \subset \mathscr{F}(E)$ is generated by a homogeneous system of parameters and therefore, by the graded version of [3, Corollary 4.7.11], we have that $\mathscr{F}(E)$ is Cohen–Macaulay if and only if $\deg(\mathscr{F}(E))$ is equal to the length of $\mathscr{F}(E)/U^1\mathscr{F}(E)$. But we can write

$$\frac{\mathscr{F}(E)}{U^1\mathscr{F}(E)} = k \oplus \Big(\bigoplus_{i=1}^{\infty} \frac{E^i}{\mathbf{m}E^i + U^1E^{i-1}}\Big) = k \oplus \Big(\bigoplus_{i=1}^{r} \frac{E^i}{\mathbf{m}E^i + U^1E^{i-1}}\Big)$$

so that

$$\lambda\Big(\frac{\mathscr{F}(E)}{U^1\mathscr{F}(E)}\Big) = 1 + \sum_{i=1}^{r} \nu\Big(\frac{E^i}{U^1E^{i-1}}\Big),$$

which gives the result. ■

Corollary 3.3 *Assume that $E \subsetneq R^e$ has rank e and reduction number 1. Then $\mathscr{F}(E)$ is Cohen–Macaulay if and only if $\deg(\mathscr{F}(E)) = \nu(E) - \ell(E) + 1$. In particular, if E is of finite colength, we have that $\mathscr{F}(E)$ is Cohen–Macaulay if and only if $\deg(\mathscr{F}(E)) = \nu(E) - d - e + 2$.*

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Proof Choose any minimal reduction $U \subset E$ such that $\mathrm{r}(E) = \mathrm{r}_U(E)$. As we shall verify later, the number $\nu(E^1/U^1) = \lambda(E^1/(\mathbf{m}E^1 + U^1))$ is equal to the second analytic deviation $\nu(E) - \ell(E)$ (see the proof of Proposition 4.8, and notice that this part does not depend on the condition $\mathrm{r}_U(E) \geq 3$ imposed therein). Now the result follows from Proposition 3.2. The particular assertion also follows easily since in the finite colength case we have $\ell(E) = d + e - 1$. ∎

Remarks 3.4 (i) *Cohen–Macaulayness and independence on the minimal reduction*: as in Proposition 3.2, let $U \subset E$ be a minimal reduction of E and set $r := \mathrm{r}_U(E) \geq 1$. If $\mathscr{F}(E)$ is Cohen–Macaulay, the ideal $U^1\mathscr{F}(E) \subset \mathscr{F}(E)$ can be generated by a regular sequence and therefore the Hilbert series of $\mathscr{F}(E)$ and $\mathscr{F}(E)/U^1\mathscr{F}(E)$ are related by the equality

$$H(\mathscr{F}(E), t) = \frac{1}{(1-t)^{\ell(E)}} H\Big(\frac{\mathscr{F}(E)}{U^1\mathscr{F}(E)}, t\Big),$$

where t is a variable. Hence, the h-polynomial of $\mathscr{F}(E)$ is

$$H\Big(\frac{\mathscr{F}(E)}{U^1\mathscr{F}(E)}, t\Big) = 1 + \sum_{i=1}^{r} \lambda\Big(\frac{E^i}{\mathbf{m}E^i + U^1E^{i-1}}\Big) t^i = 1 + \sum_{i=1}^{r} \nu\Big(\frac{E^i}{U^1E^{i-1}}\Big) t^i$$

whose degree is $\mathrm{r}_U(E)$. This shows, in particular, that the reduction number of E does not depend on the choice of the minimal reduction $U \subset E$ in this situation. For ideals, this feature was observed by Huckaba–Marley [15, Theorem 3.3] through different arguments.

(ii) *The linear type case*: for E of *linear type*, in the sense that $\mathscr{R}(E) = \mathscr{S}_R(E)$, or equivalently, $\mathscr{S}_R(E)$ has trivial R-torsion, it is well known that E admits no proper reductions, that is, $\mathrm{r}(E) = 0$. In this case, $\mathscr{F}(E) = \mathscr{S}_R(E) \otimes_R k = \mathscr{S}_k(E \otimes_R k)$, which is a polynomial ring (in $\nu(E)$ indeterminates) over k, and hence $\deg(\mathscr{F}(E)) = 1$, so that the interesting situation is when $\mathrm{r}_U(E) \geq 1$ for any given minimal reduction $U \subset E$, as in Proposition 3.2.

However, it is possible for a module E that is not of linear type (hence $r \geq 1$) to satisfy $\deg(\mathscr{F}(E)) = 1$, so that $\mathscr{F}(E)$ cannot be Cohen–Macaulay as the sum of minimal numbers of generators in the statement of Proposition 3.2 must be strictly positive in this case. To illustrate this situation, consider the surface singularity $R = \mathbb{C}[x, y, z]_{(x,y,z)}/(x^3 - y^2z)$ and set $E := \mathrm{Der}_{\mathbb{C}}(R) \hookrightarrow R^3$, the (maximal Cohen–Macaulay, 4-generated, rank 2) module of $\mathbb{C}$-derivations of R. In [26, Subsection 2.4], where we dealt with the graded context, we checked that E is not of linear type (since $\mathscr{S}_R(E)$ is Cohen–Macaulay while $\mathscr{R}(E)$ is not) and that its special fiber ring admits a presentation

$$\mathscr{F}(E) \simeq \frac{\mathbb{C}[T_1, T_2, T_3, T_4]}{(T_1^2, T_1T_2)},$$

which is not Cohen–Macaulay and has multiplicity 1. It would be of interest to investigate whether this phenomenon is related to the non-normality of the domain R.

Later in Example 4.3 we shall consider a similar module (namely, the one formed by the logarithmic derivations of the Fermat cubic in four indeterminates, which in particular yields a *normal* residue ring) whose special fiber ring is Cohen–Macaulay (it will, in fact, define a hypersurface) of multiplicity 2. As a bonus from Proposition 3.2, we will get that the reduction number of this module is exactly 1. Since, moreover, we will check that its second analytic deviation is 1, this example will serve to illustrate Corollary 3.3.

(iii) *Cohen–Macaulayness and the vanishing of cohomology*: let us consider provisionally the standard situation where $e = 1$ and $E = I \subset \mathbf{m}$ is an ideal of positive height. Assume that R is a d-dimensional Cohen–Macaulay local ring that is essentially of finite type over $k = \mathbb{C}$, and that the blowup $X := \mathrm{Proj}(\mathscr{R}(I))$ (which clearly equals $\mathrm{Proj}(\mathscr{R}(I^n))$ for any given n) is a not necessarily smooth Cohen–Macaulay scheme. (See Kurano [22, Theorem 2.2] for a criterion in the case where I is *equimultiple* in the sense that its height equals $\ell(I)$.) Let $Y = f^{-1}(\mathbf{m}) \subset X$ be the closed fiber of the natural proper morphism $f\colon X \to \mathrm{Spec}(R)$. In this setting, some of the results proved by Sancho de Salas [31] can be summarized in terms of the following statement: the associated graded ring $\mathscr{G}(I^n)$ is Cohen–Macaulay for $n \gg 0$ if and only if

$$H^i_Y(X, \mathscr{O}_X) = 0, \quad \forall\, i < d,$$

where $H^i_Y(X, \mathscr{O}_X)$ denotes the i-th cohomology with supports in Y (this fact was proved in more generality by Lipman [25, Theorem 4.3] and was also invoked, precisely with the above statement, by Huckaba–Marley [16, p. 147]). For the sake of completeness recall that, in general, given a sheaf $\mathscr{E}$ of Abelian groups on a topological space Z and a closed subset $W \subset Z$, the i-th cohomology group of Z with coefficients in $\mathscr{E}$ and supports in W, denoted by $H^i_W(Z, \mathscr{E})$, was introduced by Grothendieck in the 1960s in such a way that, if Z is the spectrum of a Noetherian ring A, $\mathscr{E}$ is the sheaf of sections of an A-module E, and W is the Zariski-closed subset associated with an A-ideal $\mathfrak{a}$, then $H^i_W(Z, \mathscr{E})$ turns out to be the i-th local cohomology module $H^i_{\mathfrak{a}}(E) = \varinjlim \mathrm{Ext}^i_A(A/\mathfrak{a}^n, E)$, where the maps in the direct limit system are induced by the natural surjections $A/\mathfrak{a}^{n+1} \to A/\mathfrak{a}^n$. In particular,

$$H^0_W(Z, \mathscr{E}) = \Gamma_{\mathfrak{a}}(E) = \bigcup_{n \geq 0} 0 :_E \mathfrak{a}^n.$$

For an illustration of the aforementioned result of Sancho de Salas, let R be the two-dimensional Cohen–Macaulay local domain $\mathbb{C}[x, y, z]_{(x,y,z)}/(xy - z^2)$, and take the nonzero ideal $I = (x^2, y^2, xz, yz)R$. By Korb–Nakamura [21, Example 2.6], the Rees algebra $\mathscr{R}(I^n)$ is Cohen–Macaulay for $n \geq 2$ (so the blowup $\mathfrak{X} := \mathrm{Proj}(\mathscr{R}(I))$ is Cohen–Macaulay) and therefore so is $\mathscr{G}(I^n)$ by Huneke [17, Proposition 1.1]. Thus, considering the natural morphism $f\colon \mathfrak{X} \to \mathrm{Spec}(R)$, we get

$$H^0_{f^{-1}((x,y,z)R)}(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = H^1_{f^{-1}((x,y,z)R)}(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = 0.$$

By virtue of the above discussion, it seems natural to raise a couple of general issues.

- Is there any cohomological characterization in terms of $\mathrm{Proj}(\mathscr{R}(I))$ for the Cohen–Macaulayness of the special fiber ring $\mathscr{F}(I^n)$ for n large? A quite naive starting point might be to control the local cohomology modules of the extended ideal

$$\mathbf{m}\mathscr{G}(I^n) = \bigoplus_{j \geq 0} \frac{\mathbf{m} I^{jn}}{I^{n+jn}} \subset \mathscr{G}(I^n)$$

that is the kernel of the natural epimorphism $\mathscr{G}(I^n) \to \mathscr{G}(I^n) \otimes_R k = \mathscr{F}(I^n)$, so that we could attempt to rely back on the Sancho de Salas' situation.

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• Is there any reasonable analogue of Sancho de Salas's result in the context of (torsionless) modules? Given the embedded module $E \subsetneq R^e$, a strategy could be to consider the annihilator $I := E :_R R^e$ of the cokernel R^e/E, assumed to be of positive height, as well as the associated graded $\mathscr{G}(I^n)$-module

$$\mathscr{G}(E, I^n) := \bigoplus_{j \geq 0} \frac{I^{jn}E}{I^{n+jn}E}$$

of E with respect to I^n for $n \gg 0$. As a replacement for the blowup, we should presumably resort to $X := \operatorname{Proj}(\mathscr{R}(E))$ that, correspondingly, we assume to be Cohen–Macaulay. Thus, under this viewpoint, the transcription for modules seems to rely on the problem of characterizing the (maximal) Cohen–Macaulayness of $\mathscr{G}(E, I^n)$ for $n \gg 0$ in terms of the vanishing of cohomology groups of X with coefficients in $\mathscr{O}_X$ and supports in the closed fiber of the natural morphism $X \to \operatorname{Spec}(R)$. In case $\mathscr{G}(E, I^n)$ fails to be a good candidate, it could be useful to find out whether the Rees power E^n plays any (eventually a major) role in the formation of the ring or module replacing $\mathscr{G}(I^n)$.

3.2 Modules of Finite Colength Over One-dimensional Cohen–Macaulay Rings

In a particular setting, we are able to obtain an asymptotic length formula for the multiplicity of $\mathscr{F}(E)$.

Theorem 3.5 *Assume that R is Cohen–Macaulay and $d = 1$. If $E \subsetneq R^e$ is of finite colength and $x \in \mathbf{m}$ is a nonzero-divisor, then the limit*

$$\mathscr{L}_E(x) := \lim_{n \to \infty} \frac{1}{n^{e-1}} \lambda\Big(\frac{\mathbf{m}E^n}{xE^n}\Big)$$

exists. More precisely, it satisfies $\deg(\mathscr{F}(E)) = \lambda(\frac{R}{(x)}) - (e-1)!\mathscr{L}_E(x)$.

Proof In virtue of the natural short exact sequence

$$0 \longrightarrow \frac{E^n}{\mathbf{m}E^n} \longrightarrow \frac{S_n}{\mathbf{m}E^n} \longrightarrow \frac{S_n}{E^n} \longrightarrow 0$$

we can write $\lambda(\frac{S_n}{\mathbf{m}E^n}) = \lambda(\frac{S_n}{E^n}) + \lambda(\frac{E^n}{\mathbf{m}E^n})$. Similarly, comparing length along the short exact sequences

$$0 \longrightarrow \frac{\mathbf{m}E^n}{xE^n} \longrightarrow \frac{S_n}{xE^n} \longrightarrow \frac{S_n}{\mathbf{m}E^n} \longrightarrow 0,$$
$$0 \longrightarrow \frac{xS_n}{xE^n} \longrightarrow \frac{S_n}{xE^n} \longrightarrow \frac{S_n}{xS_n} \longrightarrow 0,$$

we obtain $\lambda(\frac{S_n}{\mathbf{m}E^n}) + \lambda(\frac{\mathbf{m}E^n}{xE^n}) = \lambda(\frac{S_n}{xS_n}) + \lambda(\frac{xS_n}{xE^n})$. Now set $\overline{R} := R/(x)$, which is an Artinian ring. Clearly, $(\frac{S_n}{xS_n}) \simeq \overline{R}[t_1, \ldots, t_e]_n$ and then $\lambda(\frac{S_n}{xS_n}) = \lambda(\overline{R})\binom{n+e-1}{e-1}$. Moreover, x is S-regular, since this element is R-regular, and thus we have

$$\lambda\Big(\frac{xS_n}{xE^n}\Big) = \lambda\Big(\frac{S_n}{E^n}\Big).$$

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Putting these facts together, we can write

$$\lambda\Big(\frac{E^n}{\mathbf{m}E^n}\Big) = \lambda(\overline{R})\binom{n+e-1}{e-1} - \lambda\Big(\frac{\mathbf{m}E^n}{xE^n}\Big).$$

Since $E \subset R^e$ is of finite colength and $d = 1$, we have that $\mathscr{F}(E)$ has Krull dimension $\ell(E) = e$. Therefore, the Hilbert polynomial of the graded ring $\mathscr{F}(E)$ has total degree $e-1$, and the multiplicity can be computed as the limit

$$\deg(\mathscr{F}(E)) = \lim_{n\to\infty}\frac{(e-1)!}{n^{e-1}}\lambda([\mathscr{F}(E)]_n) = \lim_{n\to\infty}\frac{(e-1)!}{n^{e-1}}\lambda\Big(\frac{E^n}{\mathbf{m}E^n}\Big)$$

so that

$$\deg(\mathscr{F}(E)) = \lim_{n\to\infty}\frac{(e-1)!}{n^{e-1}}\Big[\lambda(\overline{R})\binom{n+e-1}{e-1} - \lambda\Big(\frac{\mathbf{m}E^n}{xE^n}\Big)\Big].$$

Now from the elementary observation that

$$\binom{n+e-1}{e-1} = \frac{n^{e-1}+q(n)}{(e-1)!}$$

for some polynomial $q(n)$ that is either zero if $e = 1$ or of total degree $e-2$ if $e \geq 2$, we derive that

$$\lim_{n\to\infty}\frac{(e-1)!}{n^{e-1}}\binom{n+e-1}{e-1} = \lim_{n\to\infty}\Big(1+\frac{q(n)}{n^{e-1}}\Big) = 1.$$

We thus conclude that the limit $\mathscr{L}_E(x) = \lim_{n\to\infty}\frac{1}{n^{e-1}}\lambda(\frac{\mathbf{m}E^n}{xE^n})$ exists and satisfies the proposed equality $\deg(\mathscr{F}(E)) = \lambda(\overline{R}) - (e-1)!\mathscr{L}_E(x)$. ∎

Remark 3.6 Considering the classical situation where $e = 1$ and $E = \mathbf{m}$, we wish to show that if $x \in \mathbf{m}$ is R-regular, then the limit $\mathscr{L}_{\mathbf{m}}(x)$ fits into an alternative upper bound for the embedding dimension $\operatorname{edim}(R) = \nu(\mathbf{m})$ of R. To this end, use that $\mathscr{F}(\mathbf{m})$ equals the associated graded ring $\mathscr{G}(\mathbf{m}) = \bigoplus_{n\geq 0}\mathbf{m}^n/\mathbf{m}^{n+1}$, *i.e.*, the so-called (*Zariski*) *tangent cone* of R, known to possess the same multiplicity as R. Theorem 3.5 yields $\deg(R) = \lambda(R/(x)) - \mathscr{L}_{\mathbf{m}}(x)$. Of course, by the additivity of length along standard short exact sequences, this turns out to be a manifestation of the well-known fact that $\deg(R) = \nu(\mathbf{m}^n)$, $n \gg 0$, provided that $d = 1$ [3, Exercise 4.6.15(c)]. Now the classical Abhyankar bound (in the present case where $d = 1$) simply reads $\deg(R) \geq \operatorname{edim}(R)$ and thus we immediately obtain $\operatorname{edim}(R) \leq \lambda(\frac{R}{(x)}) - \mathscr{L}_{\mathbf{m}}(x)$.

In order to derive an interesting consequence of Theorem 3.5 in the case where x belongs to a suitable class of nonzero-divisors (Corollary 3.9), we invoke first a basic ingredient. Recall that if $d = 1$ then, as we are assuming that the residue field k is infinite, there exists an element $x \in R$ which is *superficial*, a well-studied property that in this situation can be characterized by the equality $\mathbf{m}^{n+1} = x\mathbf{m}^n$, $n \gg 0$ [3, Exercise 4.6.15(a)]. In other words, the principal ideal (x) is a reduction of $\mathbf{m}$; in particular, x lies in $\mathbf{m} \setminus \mathbf{m}^2$ and is necessarily a nonzero-divisor.

It is clear that $\mathscr{L}_{\mathbf{m}}(x) = 0$ if x is superficial. Thus, from the discussion in Remark 3.6 we easily recover the following standard fact, which is known to admit higher-dimensional statements.

Lemma 3.7 *Assume that R is Cohen–Macaulay and $d = 1$. If $x \in R$ is a superficial element, then* $\deg(R) = \lambda(R/(x))$.

Remark 3.8 There are examples where, for a (necessarily nonsuperficial) suitable nonzero-divisor $x \in \mathbf{m}$, the difference $\mathscr{L}_{\mathbf{m}}(x) = \lambda(R/(x)) - \deg(R)$ can take any prescribed value. In order to illustrate this easily, consider the one-dimensional regular local ring $R = k[[t]]$, a formal power series ring in an indeterminate t over an infinite field k. Let $\alpha \geq 1$ be an arbitrary positive integer, and take the element $x = t^{\alpha+1}$, which cannot be superficial since $x \in \mathbf{m}^2$. We have $\lambda(R/(x)) = \lambda(\bigoplus_{j=0}^{\alpha} kt^j) = \alpha + 1$, so that $\lambda(R/(x)) - \deg(R) = (\alpha + 1) - 1 = \alpha$.

Combining Theorem 3.5 and Lemma 3.7, we immediately obtain that the set of superficial elements of R, $\mathbf{S}(R) \subset R \setminus \bigcup_{P \in \mathrm{Ass}(R)} P$, is such that the restriction of the assignment $x \mapsto \mathscr{L}_E(x)$ to $\mathbf{S}(R)$ turns out to be constant.

Corollary 3.9 *Assume that R is Cohen–Macaulay, $d = 1$, and $E \subset R^e$ is of finite colength. If $\mathbf{S}(R)$ stands for the set of superficial elements of R, then the function $\mathbf{S}(R) \to \mathbb{Q}$ given by $x \mapsto \mathscr{L}_E(x)$ is constant; explicitly, if we denote its value by $\mathscr{L}_E$, then it satisfies* $\deg(\mathscr{F}(E)) = \deg(R) - (e-1)!\mathscr{L}_E$.

Example 3.10 Apart from the case $E = \mathbf{m} \subset R$ (for which $\mathscr{L}_{\mathbf{m}} = 0$, since $d = 1$ and superficial elements exist), the simplest instance is when $E = F \simeq R^e$ is free, *e.g.*, if $F = \bigoplus_{i=1}^{e}(x_i)$ for nonzero-divisors $x_1, \ldots, x_e \in \mathbf{m}$. In this situation,

$$\mathscr{L}_F = \frac{\deg(R) - 1}{(e-1)!}$$

and thus, since R is a one-dimensional Cohen–Macaulay ring, we have that $\mathscr{L}_F = 0$ if and only if R is a discrete valuation ring.

Remark 3.11 It would be of interest to extend Theorem 3.5 (and consequently Corollary 3.9) to the case of arbitrary $d \geq 2$. We suggest, quite vaguely, two independent approaches that could presumably be efficient.

- To extend the R-regular element $x_1 := x$ to an appropriate sort of R-sequence $\{x_1, \ldots, x_d\}$, so that our result could serve to start a proof by induction. For the formulations of sequences, *e.g.*, the so-called *superficial sequence*, that might work well in this regard, see [19, §§8.5, 8.6].
- To discover conditions under which the upper bound for $\lambda([\mathscr{F}(E)]_n)$ given in [2, Theorem 3.2] is an equality at least for $n \gg 0$. In this case the Buchsbaum–Rim multiplicity of E would turn out to be an ingredient in the formula for $\deg(\mathscr{F}(E))$ (see Proposition 3.14 for the case $d = 1$).

From Proposition 3.2 and Corollary 3.9 we readily establish a Cohen–Macaulayness criterion for $\mathscr{F}(E)$ in terms of the number $\mathscr{L}_E$. Notice, for completeness, that if $E \subset R^e$ is of finite colength, then any minimal reduction F of E is necessarily R-free since $d = 1$; in fact, $\nu(F) = \ell(E) = d + e - 1 = e = \mathrm{rank}(E) = \mathrm{rank}(F)$. In particular, any minimal reduction of an $\mathbf{m}$-primary ideal of R is necessarily principal.

Corollary 3.12 *Assume that R is Cohen–Macaulay and $d = 1$. Suppose in addition that $E \subsetneq R^e$ is of finite colength, and let F be any minimal reduction of E with $r := \mathrm{r}_F(E) \geq 1$. Then $\mathscr{F}(E)$ is Cohen–Macaulay if and only if*

$$\mathscr{L}_E = \frac{1}{(e-1)!}\Big[\deg(R) - \sum_{i=1}^{r} \lambda\Big(\frac{E^i}{\mathbf{m}E^i + F^1E^{i-1}}\Big) - 1\Big].$$

Corollary 3.13 *Assume that R is Cohen–Macaulay and $d = 1$. Let $I \subset R$ be an $\mathbf{m}$-primary ideal, and let (y) be any minimal reduction of I with $r := \mathrm{r}_{(y)}(I) \geq 1$. Then $\mathscr{F}(I)$ is Cohen–Macaulay if and only if $\mathscr{L}_I = \deg(R) - \sum_{i=1}^{r} \lambda(\frac{I^i}{\mathbf{m}I^i + yI^{i-1}}) - 1$.*

3.3 Connection to the Buchsbaum–Rim Multiplicity

Our aim now is to provide, with the aid of our results on the Hilbert–Samuel multiplicity of $\mathscr{F}(E)$ together with suitable facts from Brennan, Ulrich, and Vasconcelos [2], a new formula for the Buchsbaum–Rim multiplicity of the finite colength R-module $E \subset \mathbf{m}R^e$ (the classical reference for this numerical invariant is [4]). Along the way, as a bonus, we shall obtain information about the reduction number of E.

A fundamental result proved in [4] states that the length function $n \mapsto \lambda(S_n/E^n)$ gives a polynomial for $n \gg 0$ with the leading term given by

$$\frac{\mathrm{br}(E)}{(d+e-1)!} n^{d+e-1}$$

for an integer $\mathrm{br}(E) \geq 1$, called the *Buchsbaum–Rim multiplicity* of $E \subsetneq R^e$. This number can be expressed by means of an Euler–Poincaré characteristic of the Buchsbaum–Rim complex [4, Corollary 4.3]. The special case where R is a two-dimensional regular local ring is considered in [5].

As previously observed, if $d = 1$, then any minimal reduction F of E is necessarily a free R-module (of rank e). Therefore, F can be expressed as the image of an R-linear endomorphism $\varphi_F\colon R^e \to R^e$ that we regard as a square matrix (we consider the canonical basis of R^e) whose determinant $\det(\varphi_F) \in R$ is immediately seen to be a nonzero-divisor, so that evaluating the asymptotic length function $\mathscr{L}_E$ (Theorem 3.5) at $\det(\varphi_F)$ is meaningful. For convenience, we set $\delta_E(F) := (e-1)!\mathscr{L}_E(\det(\varphi_F))$. As will be clear from Proposition 3.14 (i), the function $F \mapsto \delta_E(F)$ (defined on the set of minimal reductions of E) is constant; this fact will justify the notation δ_E, without explicit reference to any minimal reduction of E. In particular, from the perspective of effectiveness, in trying to compute the number δ_E, we can choose a minimal reduction F corresponding to the simplest $\det(\varphi_F)$ possible.

Proposition 3.14 *Suppose that R is Cohen–Macaulay and $d = 1$, and that $E \subset \mathbf{m}R^e$ is nonfree and of finite colength.*

(i) *If $F \subset E$ is a minimal reduction, then* $\mathrm{br}(E) = \deg(\mathscr{F}(E)) + \delta_E(F)$. *In particular, the number $\delta_E := \delta_E(F)$ does not depend on the choice of minimal reduction, and* $\mathrm{br}(E) \geq \delta_E + 1$.

(ii) $(\mathrm{char}(k) = 0)$ *If* $\mathscr{F}(E)$ *is Cohen–Macaulay and* $\deg(\mathscr{F}(E)) \le 2e - \delta_E$, *then the following hold.*
 (a) $\mathrm{r}(E) = 1$.
 (b) $\mathrm{br}(E) = 2e$.
 (c) $\nu(E) = 3e - \delta_E - 1$.

Proof For simplicity, write $x := \det(\varphi_F)$, which clearly lies in $\mathbf{m}^e$ and hence cannot be superficial if $e \ge 2$. By [2, Theorem 3.1] we can write

$$\mathrm{br}(E) = \mathrm{br}(F) = \lambda\Big(\frac{R}{(x)}\Big).$$

It is clear that $\mathrm{r}(E) \ge 1$, since $d = 1$ and E is nonfree. Hence, Theorem 3.5 yields $\deg(\mathscr{F}(E)) = \mathrm{br}(E) - (e-1)!\mathscr{L}_E(x)$, which gives (i). Now let us prove (ii). Note that $\nu(R^e/E) = e$, since $E \subset \mathbf{m}R^e$. In virtue of part (i) above, the hypothesis $\deg(\mathscr{F}(E)) \le 2e - \delta_E$ means that $\mathrm{br}(E) \le 2e$. Thus, by [2, Remark 4.2], we get

$$\mathrm{r}(E) < \frac{\mathrm{br}(E)}{e} \le 2,$$

which yields $\mathrm{r}(E) = 1$ and $\mathrm{br}(E) = 2e$. In particular, part (i) forces $\deg(\mathscr{F}(E)) = 2e - \delta_E$. On the other hand, since $\mathrm{r}(E) = 1$ and $\mathscr{F}(E)$ is Cohen–Macaulay, we are in a position to apply Corollary 3.3 in order to express this multiplicity as

$$\deg(\mathscr{F}(E)) = \nu(E) - d - e + 2 = \nu(E) - e + 1,$$

and consequently $\nu(E) - e + 1 = 2e - \delta_E$, that is, $\nu(E) = 3e - \delta_E - 1$. ■

Remarks 3.15 From the proof above we immediately obtain an upper bound for the reduction number of E without requiring any prior constraint on $\mathscr{F}(E)$. Precisely, if R is Cohen–Macaulay and one-dimensional, and if $E \subset \mathbf{m}R^e$ is nonfree and of finite colength, then (if in addition $\mathrm{char}(k) = 0$, so that [2, Remark 4.2] can be employed) we have $\mathrm{r}(E) < \frac{\deg(\mathscr{F}(E)) + \delta_E}{e}$.

Under the conditions required in the second part of the proposition, the equality $\nu(E) = 3e - \delta_E - 1$ readily yields that $\delta_E \le 2(e-1)$, since $\nu(E) \ge e + 1$ as E is nonfree. In particular, if $e \ge 2$ and $F \subset E$ is any given minimal reduction, we get

$$\mathscr{L}_E(\det(\varphi_F)) = \frac{3e - \nu(E) - 1}{(e-1)!} \le \frac{2}{(e-2)!}.$$

3.4 Detecting a Regular Element in the Special Fiber Ring

Independently of the results obtained so far in this paper, but with a view towards Cohen–Macaulayness, we now easily establish a test for the detection of a non-zero-divisor in $\mathscr{F}(E)$, where $E \subsetneq R^e$ is not required to be of finite colength and R is not assumed to be Cohen–Macaulay.

As before, given $v = (x_1, \dots, x_e) \in R^e$, we set $l(v) = \sum_{i=1}^{e} x_i t_i \in S_1$. Thus, $l(v) \in E^1$ (resp. $l(v) \in \mathbf{m}E^1$) if and only if $v \in E$ (resp. $v \in \mathbf{m}E$). Now for a homogeneous element $\eta \in E^n = [\mathscr{R}(E)]_n$, we set $\eta^o := \eta + \mathbf{m}E^n \in [\mathscr{F}(E)]_n$; equivalently, η^o is the image of η under the natural homogeneous epimorphism

$$\mathscr{R}(E) \to \mathscr{R}(E)/\mathbf{m}\mathscr{R}(E) = \mathscr{F}(E).$$

Given $v \in E$ and $n \geq 0$, we conveniently introduce the R-submodule

$$\mathscr{Q}(E, v, n) := \frac{\mathbf{m}E^{n+1} :_{E^n} l(v)}{\mathbf{m}E^n} \subset [\mathscr{F}(E)]_n,$$

where $\mathbf{m}E^{n+1} :_{E^n} l(v) = \{\eta \in E^n \mid \eta l(v) \in \mathbf{m}E^{n+1}\}$. Note that $v \notin \mathbf{m}E$ if and only if $\mathscr{Q}(E, v, 0) = 0$ (thus, $v \in \mathbf{m}E$ if and only if $\mathscr{Q}(E, v, 0) = k = [\mathscr{F}(E)]_0$).

Proposition 3.16 *Let $v \in E \smallsetminus \mathbf{m}E$ be given. Then $l(v)^o$ is $\mathscr{F}(E)$-regular if and only if $\mathscr{Q}(E, v, n) = 0$, for all $n \geq 1$.*

Proof Since $l(v) \in E^1$, the inclusion $\mathbf{m}E^n \subseteq \mathbf{m}E^{n+1}:_{E^n} l(v)$ is always true for every n. Thus we need to prove that $l(v)^o$ is $\mathscr{F}(E)$-regular if and only if $\mathbf{m}E^{n+1}:_{E^n} l(v) \subseteq \mathbf{m}E^n$ for all $n \geq 1$.

Suppose that the element $l(v)^o$ is a nonzero-divisor, and pick an arbitrary $\eta \in E^n$ (for any given $n \geq 1$) satisfying $\eta l(v) \in \mathbf{m}E^{n+1}$. Then

$$\eta^o l(v)^o = (\eta l(v))^o = 0 \in [\mathscr{F}(E)]_{n+1},$$

which implies $\eta^o = 0 \in [\mathscr{F}(E)]_n$ as $l(v)^o$ is $\mathscr{F}(E)$-regular. It follows that $\eta \in \mathbf{m}E^n$.

Conversely, suppose that $\mathbf{m}E^{n+1}:_{E^n} l(v) \subseteq \mathbf{m}E^n$ for all $n \geq 1$. In order to conclude that $l(v)^o$ is $\mathscr{F}(E)$-regular, it suffices to check that it cannot be annihilated by a nonzero homogeneous element of $\mathscr{F}(E)$. Thus, take $\sigma \in E^n$ satisfying $\sigma^o l(v)^o = 0$ and let us prove that $\sigma^o = 0$; equivalently, $\sigma \in \mathbf{m}E^n$. We have $(\sigma l(v))^o = 0 \in [\mathscr{F}(E)]_{n+1}$, which implies that $\sigma l(v) \in \mathbf{m}E^{n+1}$ and therefore

$$\sigma \in E^n \cap (\mathbf{m}E^{n+1}: l(v)) = \mathbf{m}E^{n+1}:_{E^n} l(v) \subseteq \mathbf{m}E^n. \qquad \blacksquare$$

The consequence below gives a useful Cohen–Macaulayness characterization in the situation of ideals of analytic spread 1. It will be employed later in Example 4.2.

Corollary 3.17 *Let $I \subsetneq R$ be an ideal with $\ell(I) = 1$,* e.g., *if I is $\mathbf{m}$-primary and R is one-dimensional. Then $\mathscr{F}(I)$ is Cohen–Macaulay if and only if there exists $x \in I \smallsetminus \mathbf{m}I$ such that $\mathbf{m}I^{n+1} :_{I^n} x = \mathbf{m}I^n$ for all $n \geq 1$.*

Proof If there exists an element $x \in I \smallsetminus \mathbf{m}I$ satisfying $\mathbf{m}I^{n+1} :_{I^n} x = \mathbf{m}I^n$ for all $n \geq 1$, then by Proposition 3.16 its image $x^o \in [\mathscr{F}(I)]_1$ is $\mathscr{F}(I)$-regular, and hence $\mathscr{F}(I)$ must be Cohen–Macaulay since it has dimension 1 by hypothesis. Conversely, if the one-dimensional ring $\mathscr{F}(I)$ is Cohen–Macaulay, then there exists a homogeneous nonzero-divisor $y^o \in [\mathscr{F}(I)]_s$, for some $y \in I^s \smallsetminus \mathbf{m}I^s$, $s \geq 1$. Since the graded k-algebra $\mathscr{F}(I)$ is standard, we may assume that $s = 1$. Now, given any $n \geq 1$, take an arbitrary $z \in \mathbf{m}I^{n+1} :_{I^n} y$. We have $yz \in \mathbf{m}I^{n+1}$, or, equivalently, $y^o z^o = 0 \in [\mathscr{F}(I)]_{n+1}$, which yields $z^o = 0 \in [\mathscr{F}(I)]_n$, since y^o is $\mathscr{F}(I)$-regular. Thus $z \in \mathbf{m}I^n$, as needed. $\blacksquare$

Remark 3.18 Along the same lines of Remark 3.11, it seems plausible to guess that Proposition 3.16 (and hence Corollary 3.17) admits a more general statement that could be useful for the detection of maximal $\mathscr{F}(E)$-sequences, with a view to the Cohen–Macaulayness of $\mathscr{F}(E)$ regardless of any constraint on its dimension. This might be achieved by considering an appropriate sequence of homogeneous elements starting with $l(v)^o$ for a given $v \in E \smallsetminus \mathbf{m}E$.

4 The Gorenstein Property

Throughout this section, we assume that the local ring R is Cohen–Macaulay (of arbitrary dimension). As before, we suppose that the residue field $k = R/\mathbf{m}$ is infinite and we let $E \subsetneq R^e$ be a finitely generated R-module (not necessarily of finite colength) with rank $e \geq 1$.

4.1 Warming-up: Modules of Second Analytic Deviation One

By tensoring the natural epimorphism $\mathscr{S}_R(E) \to \mathscr{R}_R(E)$ with the residue field k, we easily derive $\nu(E) \geq \ell(E)$. The non-negative integer $s(E) := \nu(E) - \ell(E)$ is the *second analytic deviation* of E.

We start with a characterization of the Gorensteinness of $\mathscr{F}(E)$ in the case where E has second analytic deviation equal to 1. Notice that this condition is satisfied if, for instance, $E \subsetneq R^e$ has finite colength and is minimally generated by $d+e$ elements. As it turns out, the result, which affords a very simple proof, also characterizes when $\mathscr{F}(E)$ is a hypersurface ring in this case, extending a result for modules due to Heinzer and Kim [13, Proposition 5.4] in the standard situation of ideals.

Proposition 4.1 *Assume that $s(E) = 1$. Then the following assertions are equivalent.*

(i) *$\mathscr{F}(E)$ is Gorenstein.*
(ii) *$\mathscr{F}(E)$ is Cohen–Macaulay.*
(iii) *$\mathscr{F}(E)$ is a hypersurface ring.*

Proof Since the implications (iii) $\Rightarrow$ (i) $\Rightarrow$(ii) are clear, it remains to prove that (ii) $\Rightarrow$ (iii). Let $\{v_1, \dots, v_m\} \subset R^e$ be a minimal set of generators of E as an R-module. Consider the polynomial ring $A = k[T_1, \dots, T_m]$ in indeterminates $T_1, \dots, T_m$ over k, as well as the homomorphism $\varphi\colon A \to \mathscr{F}(E)$ defined by

$$\varphi(T_i) = l(v_i) + \mathbf{m}E^1, \quad i = 1, \dots, m,$$

where $l\colon R^e \to S_1 = [\mathscr{S}_R(R^e)]_1$ is the natural linearization map (see Section 2). Thus, φ is surjective and its kernel $\mathfrak{a} \subset A$ has height equal to $m - \ell(E) = 1$. Furthermore, $\mathfrak{a}$ is unmixed, as the ring $A/\mathfrak{a} \simeq \mathscr{F}(E)$ is Cohen–Macaulay. Since A is factorial, it follows that $\nu(\mathfrak{a}) = 1$. ∎

Example 4.2 We first want to illustrate that, as expected, Proposition 4.1 may fail if the second analytic deviation is bigger than 1. We give a simple instance in the case of an ideal whose second analytic deviation is 2. Consider the ideal

$$I = (s^4, s^5, s^6) \subset R = k[[s^4, s^5, s^6, s^7]].$$

The R-ideal $J = (s^4)$ is easily seen to be a minimal reduction of I. Thus, $\ell(I) = \nu(J) = 1$, and since $\mathscr{F}(I)$ has a presentation of the form $k[T_1, T_2, T_3]/\mathfrak{a}$, this k-algebra cannot define a hypersurface, but we claim that it is Cohen–Macaulay. Taking the element $x := s^4 \in I \setminus \mathbf{m}I$ and using that $I^3 = xI^2$ and $\mathbf{m}x = \mathbf{m}I$, we easily verify that $\mathbf{m}I^{n+1} :_{I^n} x = \mathbf{m}I^n$ for every $n \geq 1$. By Corollary 3.17, the ring $\mathscr{F}(I)$ must be Cohen–Macaulay (we shall see in Example 4.6 that $\mathscr{F}(I)$ is, in fact, Gorenstein). Note that $s(I) = \nu(I) - \ell(I) = 3 - 1 = 2$.

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Example 4.3 Take the Fermat cubic $f = x^3 + y^3 + z^3 + w^3 \in R = k[x, y, z, w]_{(x,y,z,w)}$ and consider its tangential idealizer $E := T_{R/k}(f) \subset R^4$, quite often denoted by $\mathrm{Der}(-\log f)$ and also called the *logarithmic derivation module* of $(f) \subset R$, defined as

$$E = \left\{ (g, h, p, q) \in R^4 \;\middle|\; g\frac{\partial f}{\partial x} + h\frac{\partial f}{\partial y} + p\frac{\partial f}{\partial z} + q\frac{\partial f}{\partial w} \in (f) \right\}.$$

It is easily seen to be a torsionless R-module of rank $e = 4$ that is not of finite colength in R^4 (more precisely, the cokernel R^4/E is isomorphic to the Jacobian ideal of $R/(f)$). Since f defines a quasi-homogeneous isolated complete intersection singularity (at the origin), the module structure of E [26, Lemma 2.2] yields that E is a reflexive R-module (minimally generated by the Euler derivation of R and the six Koszul syzygies of the gradient ideal of f) of projective dimension 2, so that, in particular, it cannot be of linear type by [36, Proposition 3.1.11] [27, Example 5.12]. Using, moreover, suitable facts from the theory of blowup algebras of torsionless modules applied to the present situation, we get that the Rees algebra $\mathscr{R}(E)$ turns out to be a (eight-dimensional) Cohen–Macaulay ring and its special fiber $\mathscr{F}(E)$ is a six-dimensional ring which is Cohen–Macaulay as well. Therefore $s(E) = \nu(E) - \ell(E) = 7 - 6 = 1$, which implies, by Proposition 4.1, that the k-algebra $\mathscr{F}(E)$ is a hypersurface ring; in fact, by the computation given in [27, Example 5.12], we obtain that it can be explicitly described as

$$\mathscr{F}(E) = \frac{k[T_1, T_2, T_3, T_4, T_5, T_6, T_7]}{(T_2T_4 - T_1T_5 - T_3T_6)}.$$

Notice that, since $\mathscr{F}(E)$ is Cohen–Macaulay and $\deg(\mathscr{F}(E)) = 2$, Proposition 3.2 forces $\mathrm{r}_U(E) = 1$ for any minimal reduction U of E (see Remark 3.4 (i)). Hence $\mathrm{r}(E) = 1$, thus putting this example in a position to illustrate Corollary 3.3 as well.

4.2 A General Characterization

For the proof of Theorem 4.4, which will provide a Gorensteinness characterization for $\mathscr{F}(E)$, recall that the *socle* of a graded k-algebra $A = A_0 \oplus A_+ = k \oplus A_+$, with $A_+ = \bigoplus_{i \geq 1} A_i$ the homogeneous maximal ideal, is defined by $\mathrm{Soc}(A) = 0 :_A A_+$. Thus, if A is *standard* graded over the field k, we can write $\mathrm{Soc}(A) = 0 :_A (A_1)$. Also recall that the *type* of A is, by definition, the number $\mathrm{t}(A) := \lambda(\mathrm{Ext}^q_A(k, A))$, where $q = \mathrm{depth}(A)$. Alternatively, we have $\mathrm{t}(A) = \lambda(\mathrm{Soc}(A/(\mathbf{a})))$ provided that $\mathbf{a} \subset A_+$ is a maximal A-sequence of homogeneous elements (see [3, Lemma 1.2.19], which states this standard fact in the local setting). By a well-known basic characterization, A is Gorenstein if and only if A is Cohen–Macaulay and $\mathrm{t}(A) = 1$.

Theorem 4.4 *Let $U \subset E$ be a minimal reduction and set $r := \mathrm{r}_U(E) \geq 1$. Then $\mathscr{F}(E)$ is Gorenstein if and only if the following conditions hold: $\mathscr{F}(E)$ is Cohen–Macaulay, E^r/U^1E^{r-1} is cyclic, and (in case $r \geq 2$)*

$$(\mathbf{m}E^{i+1} + U^1E^i) :_{E^i} E^1 = \mathbf{m}E^i + U^1E^{i-1} \quad i = 1, \ldots, r-1.$$

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Proof Set $\ell := \ell(E)$. Thus, $\nu(U) = \ell$. Let $\{u_1, \dots, u_\ell\} \subset R^e$ be a set of generators of U. For each $j \in \{1, \dots, \ell\}$, consider the linear form $l(u_j) \in U^1 \subset E^1$ and set

$$L_j := l(u_j) + \mathbf{m}E^1 \in [\mathscr{F}(E)]_1.$$

The crucial point is to note that if the standard graded k-algebra $\mathscr{F}(E)$ is Cohen–Macaulay, then $\mathbf{L} := \{L_1, \dots, L_\ell\} \subset \mathscr{F}(E)_+$ is a maximal $\mathscr{F}(E)$-sequence and

$$\operatorname{Soc}\Big(\frac{\mathscr{F}(E)}{(\mathbf{L})}\Big) \simeq \operatorname{Soc}(\mathscr{A}),$$

where $\mathscr{A} := \mathscr{F}(E)/U^1\mathscr{F}(E) = \bigoplus_{i=0}^{\infty} \mathscr{A}_i$, so that

$$\mathscr{A}_i = \begin{cases} k & \text{if } i = 0, \\ \dfrac{E^i}{\mathbf{m}E^i + U^1E^{i-1}} & \text{if } i = 1, \dots, r, \\ 0 & \text{if } i > r. \end{cases}$$

Since $\operatorname{Soc}(\mathscr{A}) = 0 :_{\mathscr{A}} (\mathscr{A}_1) = 0 :_{\mathscr{A}} (E^1/(\mathbf{m}E^1 + U^1))$, an easy inspection yields that this socle can be written explicitly as $\operatorname{Soc}(\mathscr{A}) = \big(\bigoplus_{i=1}^{r-1} \widetilde{\mathscr{A}_i}\big) \oplus \mathscr{A}_r$, where

$$\widetilde{\mathscr{A}_i} := \frac{(\mathbf{m}E^{i+1} + U^1E^i) :_{E^i} E^1}{\mathbf{m}E^i + U^1E^{i-1}} \subset \mathscr{A}_i, \quad i = 1, \dots, r-1.$$

Notice that $\lambda(\mathscr{A}_r) = \nu(E^r/U^1E^{r-1}) \neq 0$. We finally obtain that the ring $\mathscr{F}(E)$ is Gorenstein if and only if it is Cohen–Macaulay and $\lambda(\operatorname{Soc}(\mathscr{A})) = 1$, the latter condition being equivalent to the equalities $\nu(E^r/U^1E^{r-1}) = 1$ and $\widetilde{\mathscr{A}_i} = 0$ for $i = 1, \dots, r-1$, as asserted. ∎

In the classical situation of ideals we readily obtain the following.

Corollary 4.5 *Let $I \subsetneq R$ be an ideal and let $J \subset I$ be a minimal reduction. Set $r := \mathrm{r}_J(I) \geq 1$. Then $\mathscr{F}(I)$ is Gorenstein if and only if the following conditions hold: $\mathscr{F}(I)$ is Cohen–Macaulay, $\nu(I^r/JI^{r-1}) = 1$, and (in case $r \geq 2$)*

$$(\mathbf{m}I^{i+1} + JI^i) :_{I^i} I = \mathbf{m}I^i + JI^{i-1} \quad i = 1, \dots, r-1.$$

Example 4.6 Let $I \subset R$ be exactly as in Example 4.2. The element $x = s^4 \in I \setminus \mathbf{m}I$ is easily seen to satisfy $xI \neq I^2$ and $xI^2 = I^3$, which means $\mathrm{r}_{(x)}(I) = 2$. Moreover, $\mathbf{m}I = \mathbf{m}x \subset (x)$, which implies that $\mathbf{m}I^2 \subset xI$. As we have verified in Example 4.2, the ring $\mathscr{F}(I)$ is Cohen–Macaulay. Since clearly $xI :_I I = (x)$, we get $(\mathbf{m}I^2 + xI) :_I I = xI :_I I = (x) = \mathbf{m}x + (x) = \mathbf{m}I + (x)$. Furthermore, $I^2 = xI + (s^{11})$, so that $\nu(I^2/xI) = 1$. By Corollary 4.5, we conclude that $\mathscr{F}(I)$ is Gorenstein.

4.3 An Obstruction in the Case of Reduction Number at Least 3

We finish the paper by detecting a quite rigid constraint (on the second analytic deviation) for the Gorensteinness of the special fiber ring $\mathscr{F}(E)$ in the situation where the reduction number of E is at least 3. We shall need the following technical observation, which is well known, at least in the case of ideals [19, Proposition 8.3.3].

Lemma 4.7 *For any minimal reduction U of E, we have $U^1 \cap \mathbf{m}E^1 = \mathbf{m}U^1$.*

Proof Set $L := U^1 \cap \mathbf{m}E^1$. By virtue of the natural surjection $U^1/\mathbf{m}U^1 \to U^1/L$, we have $\lambda(U^1/L) < \infty$, so that $U^1/L \simeq k^s$ for some positive integer s, and we can write $U^1 = \sum_{i=1}^{s} Rl(u_i) + L$ for certain $u_1, \dots, u_s \in U$. We claim that the R-submodule $G := \sum_{i=1}^{s} Ru_i \subseteq U$ is a reduction of E. By hypothesis, there exists an integer n such that $U^1E^n = E^{n+1}$. Thus, noticing that $U^1 = G^1 + L \subseteq G^1 + \mathbf{m}E^1$, we get

$$E^{n+1} = U^1E^n \subseteq (G^1 + \mathbf{m}E^1)E^n.$$

Now Nakayama's Lemma yields $E^{n+1} \subseteq G^1E^n$, whence $E^{n+1} = G^1E^n$ as needed. Moreover, by the minimality of U, we must have $G = U$. Hence $s = \nu(U)$, yielding $L = \mathbf{m}U^1$. ∎

Proposition 4.8 *Let U be a minimal reduction of E such that $r := \mathrm{r}_U(E) \geq 3$. If*

$$s(E) \neq \nu\Big(\frac{E^{r-1}}{U^1E^{r-2}}\Big),$$

then $\mathscr{F}(E)$ is not Gorenstein.

Proof Suppose by contradiction that $\mathscr{F}(E)$ is Gorenstein. Set

$$\mathscr{A} := \mathscr{F}(E)/U^1\mathscr{F}(E),$$

so that $\mathscr{A} = \bigoplus_{i=0}^{r} \mathscr{A}_i$ with homogeneous components $\mathscr{A}_i$'s as in the proof of Theorem 4.4. This standard graded k-algebra is, in this situation, a zero-dimensional Gorenstein ring. As we have seen in Remark 3.4 (i), the h-polynomial of $\mathscr{F}(E)$ is

$$H(\mathscr{A}, t) = 1 + \sum_{i=1}^{r} \lambda(\mathscr{A}_i)t^i$$

and hence its h-vector is $(1, \lambda(\mathscr{A}_1), \dots, \lambda(\mathscr{A}_{r-1}), \lambda(\mathscr{A}_r))$, which, by Gorensteinness, must be symmetric; since $r \geq 3$, this implies $\lambda(\mathscr{A}_1) = \lambda(\mathscr{A}_{r-1})$ and therefore,

$$\lambda(\mathscr{A}_1) = \lambda\Big(\frac{E^{r-1}}{\mathbf{m}E^{r-1} + U^1E^{r-2}}\Big) = \nu\Big(\frac{E^{r-1}}{U^1E^{r-2}}\Big),$$

so that $\lambda(\mathscr{A}_1) \neq s(E)$. On the other hand, from Lemma 4.7 we derive

$$(\mathbf{m}E^1 + U^1)/\mathbf{m}E^1 \simeq U^1/\mathbf{m}U^1,$$

thus yielding a short exact sequence $0 \to \frac{U^1}{\mathbf{m}U^1} \to \frac{E^1}{\mathbf{m}E^1} \to \mathscr{A}_1 \to 0$ and, consequently, $\lambda(\mathscr{A}_1) = \nu(E) - \nu(U) = \nu(E) - \ell(E) = s(E)$, a contradiction. ∎

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Departamento de Matemática, Universidade Federal da Paraíba, 58051-900 João Pessoa, Paraíba, Brazil
Email: cletoneto2011@hotmail.com