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Euclidean Sections of Direct Sums of Normed Spaces

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Abstract. We study the dimension of "random" Euclidean sections of direct sums of normed spaces. We compare the obtained results with results from [LMS], to show that for the direct sums the standard randomness with respect to the Haar measure on Grassmanian coincides with a much "weaker" randomness of "diagonal" subspaces (Corollary 1.4 and explanation after). We also add some relative information on "phase transition".

0 Introduction

Since the Dvoretzky theorem, the structure of Euclidean sections of finite-dimensional normed spaces is the best understood subject of the Asymptotic Theory of normed spaces. In spite of that some interesting observations are still left unnoticed. In this note we study the largest integer k such that a "generic" k-dimensional subspace of an N-dimensional normed space is Euclidean, up to a factor 4, say. Usually "generic" means for us "with high probability", for some natural probability distribution on the Grassmanian $G_{N,k}$. However in some cases one can introduce another natural probability distribution. Of course, the meaning of the word "generic" will be different in different cases, thus the different answers can be naturally expected. Surprisingly, in the case we study these answers essentially coincide (Corollary 1.4).

Our note is closely related to [LMS], where several instances of a phase transition behavior were discovered. We recall some of them and, summarizing some old and new facts, add more phase transitions to the behavior of the distance function to the Euclidean space of "generic" k-dimensional subspaces of the family of ℓ_q^n subspaces.

1 Direct Sum of Normed Spaces

Given an integer m we denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the canonical Euclidean norm on $\mathbb{R}^m$ and the canonical inner product. $G_{m,k}$ denotes the Grassmanian of all k-dimensional subspaces of $\mathbb{R}^m$ and $\mu = \mu_{G_{m,k}}$ denotes the canonical normalized Haar measure on the Grassmanian. By $e_1, \ldots, e_m$ we denote the canonical orthonormal basis.

By g_i, g_{ij}, we always denote the independent standard Gaussian random variables.

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Given an m-dimensional space $Z = (\mathbb{R}^m, \|\cdot\|, |\cdot|)$ and $q > 1$ we denote

$$b(Z) := \max_{x \neq 0} \|x\|/|x| = \|\operatorname{Id} : \ell_2^m \to Z\|,$$

$$M_q := \Big(\int_{S^{m-1}} \|x\|^q \, d\nu \Big)^{1/q},$$

where $d\nu$ is normalized Lebesgue measure on S^{m-1}, and

$$E_q(Z) = \Big(\mathbf{E} \Big\| \sum_{i=1}^{m} g_i e_i \Big\|^q \Big)^{1/q}.$$

Let A and B be some parameters or functions. We denote $A \approx B$ if there exist positive absolute constants c and C such that $cA \leq B \leq CA$. It is well-known (and can be directly computed) that

$$E_q(Z) \approx \sqrt{m+q}\, M_q(Z). \tag{1}$$

As usual $d(X, Y)$ denotes the Banach-Mazur distance between spaces X and Y, *i.e.*

$$d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\| \mid T \colon X \to Y \text{ is an invertible linear operator}\};$$

d_X denotes $d(X, \ell_2^k)$, where $k = \dim X$. We also denote the maximal dimension of a "random" Euclidean section of Z by $k(Z)$, *i.e.* $k(Z) =$

$$\max\Big\{ k \mid \mu\Big(\{E \in G_{m,k} \mid (M_1/2)|x| \leq \|x\| \leq 2M_1|x| \text{ for all } x \in E\} \Big) > 1/2 \Big\}.$$

It was proved in [MS2] that $k(Z) \approx \big(E_1(Z)/b(Z)\big)^2$. Note that it is known that changing $k(Z)$ to $ck(Z)$ for some absolute constant $c > 0$ we increase the measure μ of such "almost" Euclidean subspaces to $1 - e^{-k}$.

We also recall the following result from [LMS].

Lemma 1.1 *Let $1 \leq q \leq m$. There exist absolute positive constants c, C such that*

$$\max\Big\{ M_1, c\frac{b\sqrt{q}}{\sqrt{m}} \Big\} \leq M_q \leq \max\Big\{ 2\,M_1, C\frac{b\sqrt{q}}{\sqrt{m}} \Big\}.$$

In other words

(i) $M_q(Z) \approx M_1(Z)$, *for* $1 \leq q \leq k(Z)$,

(ii) $M_q(Z) \approx b(Z)\sqrt{\frac{q}{m}}$, *for* $k(Z) \leq q \leq m$,

(iii) $M_q(Z) \approx b(Z)$, *for* $q > m$.

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Fix now an n-dimensional normed space $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$. Let

$$Y = Y_q = \bigoplus_1^t X$$

be nt-dimensional space with the norm defined by

$$\|y\|_Y = \|y\|_q = \Big(\sum_{i=1}^t \|x_i\|^q/t\Big)^{1/q},$$

where $y = (x_1, x_2, \dots, x_t) \in Y$. Below by $\log t$ we always mean the logarithm with the fixed base a, where $a > 1$ is an absolute positive constant, which will be specified later in the proof of Theorem 1.2, Case 3. Clearly, if $q \geq \log t$ then

$$\|y\|_{\log t} \leq \|y\|_q \leq \|y\|_\infty = \max_i \|x_i\| \leq a\|y\|_{\log t}.$$

So we consider the case $q \leq \log t$ only.

For simplicity we denote $b(X)$, $M_q(X)$, $E_q(X)$ by b, M_q, E_q correspondingly.

The main computation we would like to present is combined in the following

Theorem 1.2 *Let t be an integer, $q \in [1, \log t]$ and $\alpha = 1/\max\{2, q\}$. Then we have*

$$k(Y) \approx t^{2\alpha} \max\{k(X), q\}.$$

The main interest of this formula lies in comparison with the following result from [LMS]:

Theorem 1.3 *Let $q > 1$. Let $t_q = t_q(X)$ be the smallest integer such that there are orthogonal transformations $u_1, \dots, u_t \in O(n)$ with*

$$\frac{M_q}{2}|x| \leq \Big(\frac{1}{t}\sum_{i=1}^t \|u_i x\|^q\Big)^{1/q} \leq 2M_q|x|, \quad \textit{for all } x \in \mathbb{R}^n. \tag{2}$$

Then for $q \leq n$ one has

$$t_q^{2\alpha} \approx \frac{n}{\max\{k(X), q\}},$$

$\alpha = 1/\max\{2, q\}$. Moreover the "random" choice of orthogonal transformations gives, with the probability exponentially close to one, the same estimate as the best one, i.e. there exists an absolute constant c_0 such that for a random choice of independent rotations $u_1, \dots, u_t$ with $t^{2\alpha} \geq c_0 n/\max\{k(X), q\}$ one has (2).

The following corollary is immediately implied by Theorems 1.2, 1.3 (the restriction $q \leq cn$ is needed to satisfy condition $q \leq \log t$ in the case $q \geq k(X)$).

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Corollary 1.4 *Let X, Y be defined as above. Let t be such that $k(Y) = n$. Then*

$$t^{2\alpha} \approx t_q^{2\alpha}$$

for $q \le cn$, where $\alpha = 1/\max\{2, q\}$ and c is an absolute constant.

The meaning of the equivalence in this corollary should be explained. It shows that in some sense the randomness with respect to the Haar measure on Grassmanian $G_{tn,n}$ coincides with a much "weaker" randomness of "diagonal" subspaces. More precisely, given n-dimensional space X let $Y = Y_q$ be as above and let $\bar{u} = (u_1, \dots, u_n) : Y \to Y$, $u_i \in O(n)$, be the linear operator defined by $\bar{u}y = (u_1 x_1, \dots, u_t x_t)$. By a "diagonal of $\bar{u}Y$" we mean the subspace of all vectors $(u_1 x, u_2 x, \dots, u_t x) \in Y$, $x \in X$. We are looking for t such that for a random $\bar{u} \in \prod_1^t O(n)$ this diagonal is equivalent to the Hilbert space, *i.e.* for every x

$$\|y\|_q = \Big(\sum_{i=1}^{t} \|u_i x\|^q / t\Big)^{1/q} \approx M_q |x|.$$

The two previous theorems show that the answer to this question is the same as the answer to the question for what t we have $k(Y) = n$, which means that $G_{tn,n}$-random subspace is Euclidean. Let us emphasize again that in the first question (Theorem 1.3) we take t "random" operators and "diagonal of Y", but in the second (Theorem 1.2), in fact, we take the random operator on the group $O(tn)$.

To prove Theorem 1.2 we need the following lemma.

Lemma 1.5 *Let t be an integer, $q \in [1, \infty)$ and $\alpha = 1/\max\{2, q\}$. Then we have*

(i) $b(Y) = t^{-\alpha} b$,
(ii) $E_1 \le E_1(Y) \le E_q(Y) = E_q$,
(iii) $M_1 \le c_1 \sqrt{t} M_1(Y) \le c_2 \sqrt{t + q/n}\, M_q(Y) \approx \sqrt{1 + q/n}\, M_q$, *where c_1, c_2 are absolute positive constants.*

Proof Let $y = (x_1, x_2, \dots, x_t) \in Y$. Then by the definition of the norm on Y and of the $b = b(X)$ we have

$$\|y\| \le t^{-1/q} b \Big(\sum |x_i|^q\Big)^{1/q} \le t^{-\alpha} b \Big(\sum |x_i|^2\Big)^{1/2} = t^{-\alpha} b |y|.$$

Thus $b(Y) \le t^{-\alpha} b$. To get the equality it is enough to take $y = (x_0, x_0, \dots, x_0)$ if $q \le 2$ and $y = (x_0, 0, \dots, 0)$ if $q \ge 2$, where $x_0 \in X$ is such that $\|x_0\| = b|x_0|$.

Denote by $\{e_{ij}\}$, $i \le n$, $j \le t$ the canonical basis of $\mathbb{R}^{nt} = \bigoplus_1^t \mathbb{R}^n$. Clearly,

$$E_1 = \mathbf{E}\Big\|\sum_{i=1}^{n} g_i e_i\Big\| = \mathbf{E}\sum_{j=1}^{t} \frac{1}{t}\Big\|\sum_{i=1}^{n} g_{ij} e_{ij}\Big\| \le \mathbf{E}\Big(\sum_{j=1}^{t} \frac{1}{t}\Big\|\sum_{i=1}^{n} g_{ij} e_{ij}\Big\|^q\Big)^{1/q}$$

$$= E_1(Y) \le E_q(Y) = \Big(\mathbf{E}\sum_{j=1}^{t} \frac{1}{t}\Big\|\sum_{i=1}^{n} g_{ij} e_{ij}\Big\|^q\Big)^{1/q} = E_q(X).$$

The last inequality follows from (1). ■

Proof of Theorem 1.2

Case 1 $q \le \max\{k(X), 2\}$ (Then, by Lemma 1.1, $E_q \approx E_1$.)
In this case we have

$$\begin{aligned} k(Y) &\approx \big(E_1(Y)/b(Y)\big)^2 \le \big(E_q(Y)/b(Y)\big)^2 \\ &= t^{2\alpha}(E_q/b)^2 \approx t^{2\alpha}(E_1/b)^2 \approx t^{2\alpha}k(X). \end{aligned}$$

On the other hand

$$k(Y) \approx \big(E_1(Y)/b(Y)\big)^2 \ge \big(E_1/b(Y)\big)^2 = t^{2\alpha}(E_1/b)^2 \approx t^{2\alpha}k(X).$$

We turn now to the cases when $q \ge \max\{2, k(X)\}$. Then $\alpha = 1/q$.

Case 2 $k(X) < q \le k(Y)$ (Then, by Lemma 1.1, $E_q(Y) \approx E_1(Y)$.)
We obtain

$$\begin{aligned} k(Y) &\approx \big(E_1(Y)/b(Y)\big)^2 \approx \big(E_q(Y)/b(Y)\big)^2 = t^{2\alpha}(E_q/b)^2 \\ &\approx t^{2\alpha}(q+n)(M_q/b)^2 \approx t^{2\alpha}(q+n)\frac{\min\{q,n\}}{n} \approx t^{2\alpha}q. \end{aligned}$$

Case 3 $k(Y) < q \le \log t$
We show that this case is impossible for an appropriate choice of the base of the logarithm. Indeed, using Lemma 1.1 we obtain

$$M_q \approx \sqrt{\frac{\min\{q,n\}}{n}}\,b \quad \text{thus} \quad E_q \approx \sqrt{q+n}M_q \approx \sqrt{q}b,$$

and

$$M_q(Y) \approx \sqrt{\frac{q}{nt}}\,b(Y) \approx \sqrt{\frac{q}{nt}}\,t^{-\alpha}b \quad \text{thus} \quad E_q(Y) \approx \sqrt{q}t^{-\alpha}b.$$

But $E_q = E_q(Y)$, therefore $t^\alpha \le c$, *i.e.* $t \le c^q$ for some absolute constant $c > 1$. Letting $a > c$ we obtain a contradiction with the condition $q \le \log t = \log_a t$. ■

Finally we would like to reformulate Theorem 1.3. The theorem, in particular, shows that "randomly" defined t_q has, up to an absolute constant, the same bounds as t_q. *i.e.* a random choice of independent rotations gives "almost" the same result as the best possible one. The theorem below provides the estimates. Note that t_q in it is defined slightly differently.

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Theorem 1.6 *Let $2 < q \le n$, X be an n-dimensional normed space, $b = b(X)$, and $M_q = M_q(X)$. Let $1 < A < b/M_q$ and $t_q = t_q(X, A)$ be the smallest integer such that there are orthogonal transformations $u_1, \dots, u_t \in O(n)$ with*

$$\Big(\frac{1}{t} \sum_{i=1}^{t} \|u_i x\|^q \Big)^{1/q} \le A M_q |x| \quad \text{for all } x \in \mathbb{R}^n.$$

Let c, C be the constants from Lemma 1.1.

There exists an absolute constant $c_0 > 1$ such that if $t_q \ge (c_0 b/M_2)^2$ and $q \le n/(eC^2A^2)$ then with high probability a random choice of t_q independent rotations $u_1, \dots, u_{t_q} \in O(n)$ gives

$$c_1 M_q |x| \le \Big(\frac{1}{t_q} \sum_{i=1}^{t_q} \|u_i x\|^q \Big)^{1/q} \le 2c_0 A M_q |x| \quad \text{for all } x \in \mathbb{R}^n,$$

where

$$1/c_1 = (3C/c) \sqrt{1 + 2 \frac{\ln(c_0 CA/c)}{\ln\big(n/(qC^2A^2)\big)}}.$$

Moreover, if $q \le k(X)$ then c_1 can be replaced with an absolute positive constant.

Remark 1 The restriction $q \le n/(eC^2A^2)$ seems to be reasonable, since otherwise, by Lemma 1.1, we have $b \le (2C/c)AM_q$, *i.e.* $\|x\| \le (2C/c)AM_q|x|$ for every $x \in \mathbb{R}^n$.

Remark 2 In particular, if $C^3A^3q \le n$ then we can substitute the constant c_1 with an absolute positive constant. More precisely, if $(CA)^{2+\varepsilon} q \le n$, $\varepsilon \in (0, 1]$ then

$$1/c_1 \le (9C/c) \sqrt{\frac{\ln(c_0/c)}{\varepsilon}}.$$

The theorem follows immediately from results proven in [LMS]. For completeness we show the proof.

Proof First we define c_0. Let $c_0 \ge \max\{4, C^2\}$ be such that given $1 \le p \le n$ one can apply "moreover" part of Theorem 1.3 for

$$t^{2\alpha} = t^{2/\max\{2,p\}} \ge c_0 \min\{(b/M_2)^2, n/p\}$$

rotations. Such c_0 exists, since $k(X) \approx (M_2/b)^2 n$.

Now let s be the largest number such that

$$t_q \ge (c_0 b/M_s)^s.$$

(Of course we may assume that s exists and that $t_q = (c_0 b/M_s)^s$.)

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Clearly, $t_q = (c_0 b/M_s)^s$ increases when s grows. Since $t_q \geq (c_0 b/M_2)^2$ we have $s \geq 2$. Thus, by Lemma 1.1,

$$t_q^{2/\max\{2,s\}} = t_q^{2/s} = c_0^2 (b/M_s)^2$$

is larger than $(c_0^2/4)(b/M_2)^2$ for small s (namely $s \leq k(X)$) and is larger than $(c_0^2/C^2)(n/s)$ for large s. Hence, by the choice of c_0, we can apply "moreover" part of Theorem 1.3 and obtain that random choice of t_q rotations satisfies

$$\frac{M_s}{2}|x| \leq \Big(\frac{1}{t_q}\sum_{i=1}^{t_q} \|u_i x\|^s\Big)^{1/s} \leq 2M_s|x| \quad \text{for all } x \in \mathbb{R}^n.$$

If $s > q$ we are done. Assume $s \leq q$. Then we have

$$\Big(\frac{1}{t_q}\sum_{i=1}^{t_q} \|u_i x\|^s\Big)^{1/s} \leq \Big(\frac{1}{t_q}\sum_{i=1}^{t_q} \|u_i x\|^q\Big)^{1/q} \leq t_q^{1/s-1/q}\Big(\frac{1}{t_q}\sum_{i=1}^{t_q} \|u_i x\|^s\Big)^{1/s}.$$

Thus to prove the theorem it is enough to show that

$$c_1 M_q \leq M_s/2 \quad \text{and} \quad t_q^{1/s-1/q} M_s \leq cAM_q.$$

By Theorem 2.3.1 of [LMS] and definition of t_q we obtain

$$t_q = \Big(\frac{c_0 b}{M_s}\Big)^s \geq \Big(\frac{b}{AM_q}\Big)^q. \tag{3}$$

This immediately implies the upper estimate. The lower estimate follows from Lemma 1.1. Indeed, if $q \leq k(X)$ then $M_q \approx M_2 \approx M_s$. Let $q \geq k(X)$. By Lemma 1.1 and (3) we observe

$$\Big(\frac{1}{CA}\sqrt{\frac{n}{q}}\Big)^q \leq \Big(\frac{c_0}{c}\sqrt{\frac{n}{s}}\Big)^s.$$

Denote $c_2 = c_0/c$, $C_A = CA$ and $a = q/s$. Then we have

$$a \ln\big(n/(qC_A^2)\big) \leq \ln(a c_2^2 n/q),$$

which implies

$$a \leq \ln a + \frac{\ln(c_2^2 n/q)}{\ln\big(n/(qC_A^2)\big)} = \ln a + 1 + 2\frac{\ln(c_2 C_A)}{\ln\big(n/(qC_A^2)\big)}.$$

Thus $a \leq 2\big(1 + 2\frac{\ln(c_2 C_A)}{\ln(n/(qC_A^2))}\big)$. Applying Lemma 1.1 again we obtain

$$\frac{M_q}{M_s/2} \leq (2C/c)\sqrt{a}.$$

That concludes the proof. ∎

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2 More on Euclidean Sections of ℓ_q

Lemma 1.1 and Theorems 1.2 and 1.3 provide a few cases of a so-called "phase transition" phenomenon in high-dimensional theory. Functions which describe behavior of some important parameters of the space are changing their analytic description at specific values. Of course, in the Asymptotic Theory all functions are described in an isomorphic form, *i.e.* up to some universal factors. In this section we will interpret a result from [GGMP] on distances of k-dimensional "random" subspaces of ℓ_q to the Euclidean space to emphasize phase transition of the distance function. This complements, in our mind, phase transitions we studied above for ℓ_q-sum of spaces. The following theorem combines some classical well-known facts with new information from [GGMP].

Theorem 2.1 *Let $2 \le q \le (\ln n)/2$. There are absolute positive constants c_1, c_2, c_3 such that for every $k \le n$ a "random" k-dimensional subspace $F \subset \ell_q^n$ satisfies*

(i)
$$d_F \le 3$$
for $k \le c_1 q n^{1/q}$,

(ii)
$$d_F \le c_3 \frac{\sqrt{k}}{n^{1/q}\sqrt{q}}$$
for $c_1 q n^{1/q} \le k \le c_2 e^{-q} q n$,

(iii)
$$d_F \le c_3 \frac{\sqrt{k}}{n^{1/q}\sqrt{\ln(2n/k)}}$$
for $c_2 e^{-q} q n \le k$.

Let us note that the case (i) is well-known (see *e.g.* [MS1]). The estimates with some constant C_q depending on q only instead of $\sqrt{q}$ (in the case (ii)) or $\sqrt{\ln(2n/k)}$ (in the case (iii)) were also known earlier ([MS1]).

Remark We would like to emphasize that the estimates are sharp up to absolute constants. Moreover, each subspace of ℓ_q (not only "random") satisfies the lower estimates of the same order. (For the case (ii) see *e.g.* [CP, GGMP, MS1], the case (iii) follows, since for any k-dimensional subspace $E \subset \ell_\infty^n$ one has $d_E \ge c\frac{\sqrt{k}}{\sqrt{\ln(2n/k)}}$ (see *e.g.* [BLM, CP, G1]). Indeed, let $\bar{E}$ be a k-dimensional subspace of $\mathbb{R}^n$, E be $\bar{E}$ endowed with $\|\cdot\|_\infty$, and F be $\bar{E}$ endowed with $\|\cdot\|_q$. Then, since $n^{-1/q}\|x\|_q \le \|x\|_\infty \le \|x\|_q$ for every $x \in \mathbb{R}^n$, we have

$$d_E \le d(E,F) d_F \le n^{1/q} d_F,$$

which implies the estimate.)

Taking into account the remark above we can reformulate the previous theorem in the following way

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Theorem 2.2 *Let c_1, c_2, c_3 be the positive constants from Theorem 2.1. Let n, k be integers satisfying $c_1e^2\ln n \le k \le 2c_1 n$. Let q_0 and q_1 be numbers defined by equations*

$$k = c_1 q_0 n^{2/q_0} \quad \textit{and} \quad k = c_2 e^{-q_1} q_1 n,$$

thus

$$\frac{2\ln n}{\ln\big(k/(c_1\ln n)\big) + \ln\ln\big(k/(c_1\ln n)\big)} \le q_0 \le \frac{2\ln n}{\ln\big(k/(c_1\ln n)\big)}$$

and

$$\ln(c_2k/n) \le q_1 \le \ln(c_2k/n) + \ln\ln(c_2k/n)^2.$$

Then there is a positive constant c_4 such that for a "random" k-dimensional subspace $F \subset \ell_q^n$ we have

(i)

$$1 \le d_F \le 3$$

for $1 \le q \le q_0$,

(ii)

$$c_4\frac{\sqrt{k}}{n^{1/q}\sqrt{q}} \le d_F \le c_3\frac{\sqrt{k}}{n^{1/q}\sqrt{q}}$$

for $q_0 \le q \le q_1$,

(iii)

$$c_4\frac{\sqrt{k}}{n^{1/q}\sqrt{(2n/k)}} \le d_F \le c_3\frac{\sqrt{k}}{n^{1/q}\sqrt{\ln(2n/k)}}$$

for $q_1 \le q \le (\ln n)/2$.

Let us note that the restriction $q \le (\ln n)/2$ can be omitted, since for larger q the space ℓ_q^n is equivalent to the space ℓ_∞^n (in fact, $d(\ell_q^n, \ell_\infty^n) \le e^2$) and for ℓ_∞^n the inequality in the item (iii) is well known ([G2]). Therefore, the distance function d_F for a "random" k-dimensional subspace of ℓ_q, as a function by q, $1 \le q$, has two points of phase transition q_0 and q_1.

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