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EXTERIOR POWERS OF THE ADJOINT REPRESENTATION

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ABSTRACT. Exterior powers of the adjoint representation of a complex semisimple Lie algebra are decomposed into irreducible representations, to varying degrees of satisfaction.

0. **Introduction.** Let G be a compact Lie group with complexified Lie algebra $\mathfrak{g}$. It is known that the de Rham cohomology of the manifold G is given by the G-invariants in the exterior algebra $\Lambda\mathfrak{g}$, where the action of G is induced by the adjoint representation. The degrees in which these invariants occur are determined by the exponents of the Weyl group W of G, as was excitingly discovered in the first half of this century. For references and a relatively short but complete treatment of this venerable tale, see [R1]. It is natural to wonder next about multiplicities of nontrivial representations in $\Lambda\mathfrak{g}$, or equivalently, about the decomposition of the space of left invariant differential forms on G under the action of G induced by right multiplication. However, there seem to be no definitive results. The dual problem of decomposing the symmetric algebra $S\mathfrak{g}$ is in better shape, thanks to Kostant's theory of harmonic polynomials $H\mathfrak{g} \subset S\mathfrak{g}$ and the Hesselink-Peterson formula. Even here however, the latter formula for the multiplicities is now known, by work of Kato, to be given by Kazhdan-Lusztig polynomials, hence is of significant combinatorial complexity. (Harmonic polynomials are briefly reviewed in Section 2 below.) For the exterior algebra we expect similar difficulties, which are perhaps mitigated by the finite dimensionality of $\Lambda\mathfrak{g}$.

Aside from the obvious symmetry of Poincaré duality in $\Lambda\mathfrak{g}$, the mitigation is as follows. Let 2ρ be the sum of the positive roots. The highest weight λ of any irreducible constituent V_λ of $\Lambda\mathfrak{g}$ lies in the root lattice and between 0 and 2ρ in the partial order on weights. If λ is in some way close to 2ρ, it is often easy to write down all highest weight vectors of weight λ, and thus quickly compute the multiplicity polynomial

$$P(V_\lambda, \Lambda\mathfrak{g}, u) := \sum_{n=0}^{\dim\mathfrak{g}} \dim \operatorname{Hom}_G(V_\lambda, \Lambda^n\mathfrak{g})u^n.$$

For example, one finds in this way $P(V_{2\rho}, \Lambda\mathfrak{g}, u) = u^\nu(1+u)^\ell$, where ν is the number of positive roots, and ℓ is the rank. This is a special case of a reduction formula (6.1) given below for parabolic subalgebras. It gives the multiplicities for only a small fraction of the irreducible representations appearing in $\Lambda\mathfrak{g}$.

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At the other extreme, $P(V_0, \Lambda\mathfrak{g}, u)$ is the Poincaré polynomial of the cohomology of G as mentioned above, and its computation seems to require arguments rooted in topology, whose applicability to more general $P(V_\lambda, \Lambda\mathfrak{g}, u)$ is not clear to me.

Despite this handicap, we can still compute many more multiplicities. We begin with the analogue of the Hesselink-Peterson formula for $P(V_\lambda, \Lambda\mathfrak{g}, u)$, which is easy to write but contains many cancellations. The comparison with the symmetric algebra is more symmetric if we consider multiplicities of the reducible modules $\mathrm{Hom}(V_\lambda, V_\mu)$, where we find an analogue of Gupta's formula. Following Kato, the terms in our formula are shown to be averages of inverse Kazhdan-Lusztig polynomials over double cosets in the affine Weyl group, and therefore inherently complicated. Nevertheless, we can use this formula to compute the following explicit multiplicity polynomials. Let I be a subset of the simple roots, and let δ_I be the sum of the roots in I. Let $c(I)$ be the number of connected components of the subgraph of the Dynkin diagram whose vertices are in I. We show in (6.3) that

$$P(V_{2\rho-\delta_I}, \Lambda\mathfrak{g}, u) = u^{\nu-|I|}(1+u)^{\ell-c(I)}(1+u^2)^{|I|-c(I)}(1+u^3)^{c(I)}.$$

Turning from large weights to small degrees, the second and third exterior powers have uniform decompositions, the latter partly coming from harmonic polynomials in degree two. This is related to the natural ring homomorphism $\Omega\colon S\mathfrak{g} \longrightarrow \Lambda^{even}\mathfrak{g}$, extending the differential $d\colon \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$. The coordinate ring R of the minimal nonzero nilpotent orbit in $\mathfrak{g}$ is a natural direct summand (as G-module) of $H\,\mathfrak{g}$, and we determine $\Omega(R)$. See Sections 2 and 5.

If we ignore degrees, the Weyl integration formula leads to an efficient and useful recursion formula for ungraded multiplicities of an irreducible module in $\Lambda\mathfrak{g}$. The recursion is trivial for "small" modules, *i.e.*, those in which twice a root is not a weight, and for small V_λ it gives

$$\dim \mathrm{Hom}_G(V_\lambda, \Lambda\mathfrak{g}) = m_\lambda^0 2^\ell,$$

where ℓ is the rank of $\mathfrak{g}$ and m_λ^0 is the dimension of the zero weight space of V_λ. For example, the adjoint representation appears in $\Lambda\mathfrak{g}$ with multiplicity $\ell 2^\ell$.

Now, Kostant has shown that $\Lambda\mathfrak{g}$ is isomorphic to 2^ℓ copies of $V_\rho \otimes V_\rho$, so the standard tensor product formulas could be applied to $V_\rho \otimes V_\rho$, but they seem impractical compared to our recursive procedure. For example, our multiplicity for small modules is equivalent to the multiplicity of V_λ in $V_\rho \otimes V_\rho$ being m_λ^0, a fact not readily seen from the tensor product formulas. However, after receiving an earlier version of this article, Kostant showed me how another old result of his (unpublished by him, but proved independently in [PRV], using Kostant's ideas) implies the ungraded multiplicity formula for small modules, as well as a strong converse. This is included in Section 4.

What about graded multiplicities for small modules? For the symmetric algebra, their multiplicity polynomials are either known, with explicit nonnegative coefficients, or reduced to invariant theory of the Weyl group. For the trivial representation this is a well-known theorem of Chevalley, for the adjoint representation it is due to Kostant, and for any small module V_λ it is a recent result of Broer [Br2] that

$$P(V_\lambda, \mathcal{H}\,\mathfrak{g}, u) = P_W(V_\lambda^0, \mathcal{H}\,, u),$$

where V_λ^0 is the zero weight space of V_λ, $\mathcal{H}$ is the space of W-harmonic polynomials on $\mathfrak{t}$ = complexified Lie algebra of a maximal torus T, and P_W is the multiplicity polynomial for W modules in $\mathcal{H}$.

One hopes for a similar description of the multiplicities in the exterior algebra. If $\mathfrak{g} = \mathfrak{sl}(n)$ and the highest weight is a partition of n (these are small), the multiplicities in $\Lambda\mathfrak{g}$ have been determined combinatorially by Stembridge [St]. For certain partitions, we show that Stembridge's formula is related to harmonic polynomials for the symmetric group (Section 7). This interpretation makes sense for any reductive Lie algebra and any small module, for which we have a conjecture generalizing the ungraded multiplicity formula.

To explain this, let us reconsider the invariants. The Weyl group acts on both factors of the manifold $G/T \times T$, and the Weyl map $G/T \times_W T \longrightarrow G$ induces an isomorphism on real cohomology [R1]. In terms of invariants, this means $(\Lambda\mathfrak{g})^G \simeq H(G/T \times T)^W$, as graded vector spaces.

We conjecture that for all small modules V_λ, there is a graded isomorphism

$$\operatorname{Hom}_G(V_\lambda, \Lambda\mathfrak{g}) \simeq \operatorname{Hom}_W\big(V_\lambda^0, H(G/T \times T)\big).$$

In other words, we propose the multiplicity formula

$$P(V_\lambda, \Lambda\mathfrak{g}, u) = \sum_{q=0}^{\ell} u^q P_W(V_\lambda^0 \otimes \Lambda^q\mathfrak{t}, \mathcal{H}\,, u^2).$$

This is false if V_λ is not small, by the converse to the ungraded multiplicity formula.

We can verify our conjecture for all small modules for $\mathfrak{g}$ of type C_2, C_3, G_2, and those $\mathfrak{sl}(n)$-modules with highest weights corresponding to partitions of the form 2^k1^{n-2k} (Section 7). At $u = 1$ it reduces to the ungraded multiplicity formula. Among additional supporting evidence is the fact that, for simple Lie algebras $\mathfrak{g}$, the adjoint representation appears in $\Lambda^3\mathfrak{g}$ if and only if $\mathfrak{g} = \mathfrak{sl}(n)$, $n \geq 3$.

In Section 8 one finds the complete decomposition of $\Lambda\mathfrak{g}$ for types A_2, A_3, C_2, C_3 and G_2. The table for C_3 was provided by the referee, along with many valuable remarks and references. I was originally reticent to extend the conjecture beyond the adjoint representation, having only the ungraded multiplicity formula as evidence for other small modules, but the referee independently suggested the broader conjecture made here, backed it up with C_3, and pointed out the analogy with [Br2]. With this encouragement, I then found more evidence in other small modules.

This paper has also been improved by informative correspondence with B. Kostant.

1. **Preliminaries.** Multiplicities in the symmetric and exterior algebras of $\mathfrak{g}$ are both expressed in the transition between two natural bases of the representation ring of G. Let Δ be the roots of $\mathfrak{t}$ in $\mathfrak{g}$, and Δ^+ a choice of positive roots, with corresponding simple roots Σ. As in the introduction, we set $\nu = |\Delta^+|$, $\ell = |\Sigma|$. Let P be the weight lattice of T, let P^+ be the dominant weights with respect to Δ^+, and let ρ be half the sum of the positive roots. To $\lambda \in P$ we associate various objects: $e_\lambda \colon T \longrightarrow \mathbb{C}^\times$ is the character whose differential is λ, Δ^+_λ is the set of positive roots orthogonal to λ, W_λ is the stabilizer in W of λ, and $W_\lambda(u) = \sum_{w\in W_\lambda} u^{n(w)}$, where $n(w)$ is the number of positive roots made negative by w. We set $W(u) = W_0(u)$. For a subset $S \subseteq \Delta^+$, let δ_S be the sum of the roots in S. Let $R = \mathbb{C}[P]^W$ be the Weyl group invariants in the character ring $\mathbb{C}[P]$ of T. We identify R with the character ring of G. Let dt be the Haar measure on T with $vol(T, dt) = 1$, and consider the following two hermitian inner products on R.

$$(f, g) = \frac{1}{|W|}\int_T f(t)\overline{g(t)} \prod_{\alpha\in\Delta} 1 - e_\alpha(t)\, dt,$$

$$\langle f, g\rangle_u = \frac{1}{|W|}\int_T f(t)\overline{g(t)} \prod_{\alpha\in\Delta} \frac{1 - e_\alpha(t)}{1 - ue_\alpha(t)}\, dt.$$

In the latter, u is an indeterminate and the inner product takes values in the formal power series ring $\mathbb{C}[[u]]$. We extend $\langle\, ,\rangle_u$ to a pairing $R[[u]] \times R[[u]] \longrightarrow \mathbb{C}[[u]]$ which is $\mathbb{R}[[u]]$-bilinear. The inner product $(\, ,)$ is, according to the Weyl integration formula, the L^2 inner product of class functions on G with respect to Haar measure of volume one. Orthogonal bases for these inner product spaces are given as follows. For $\lambda \in P$, let

$$\chi_\lambda = \frac{\sum_{w\in W} \epsilon(w) e_{w(\lambda+\rho)}}{\sum_{w\in W} \epsilon(w) e_{w\rho}} = \frac{1}{D}\sum_{w\in W} \epsilon(w) e_{w\cdot\lambda},$$

where $D = \prod_{\alpha>0} 1 - e_{-\alpha}$ and $w \cdot \lambda = w(\lambda + \rho) - \rho$. By Weyl's character formula, χ_λ is $\epsilon(\lambda)$ times the character of the irreducible G-representation V_λ with extreme weight λ, where $\epsilon(\lambda) = \epsilon(w)$ if $\lambda+\rho$ is regular (*i.e.*, $W_{\lambda+\rho} = 1$) and $w\cdot\lambda$ is dominant, and $\epsilon(\lambda) = 0$ if $\lambda+\rho$ is singular. The set of irreducible characters $\{\chi_\lambda : \lambda \in P^+\}$ forms an orthonormal basis of $R(T)^W$ with respect to $(\, ,)$.

For the other inner product, we have the polynomials

$$M^u_\lambda = \sum_{w\in W} e_{w\lambda} \prod_{\alpha>0} \frac{1 - ue_{-w\alpha}}{1 - e_{-w\alpha}}.$$

At first glance this lives in the quotient field of $R[u]$, but it is in fact in $R[u]$ itself. Macdonald [M] showed that M^u_λ is, up to a slight modification, the Satake transform of a certain spherical function on the p-adic Chevalley group whose root system is dual to that of G. As part of this work, Macdonald computed the inner product

$$\langle M^u_\lambda, M^u_\mu\rangle_u = \begin{cases} W_\lambda(u) & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.}\end{cases}$$

Using the relation $wD = \epsilon(w) e_{\rho - w\rho} D$, it is easy to see that

$$M^u_\lambda = \sum_{S\subseteq\Delta^+} (-u)^{|S|} \chi_{\lambda-\delta_S}.$$

This can be refined in case λ is singular, as follows.

1.1. PROPOSITION. *For all $\lambda \in P$, we have*

$$M^u_\lambda = W_\lambda(u) \sum_{S\subseteq\Delta^+-\Delta^+_\lambda} (-u)^{|S|}\chi_{\lambda-\delta_S}.$$

PROOF. This is proved in [M] for $\lambda = 0$. By factoring products over Δ^+ into products over Δ^+_λ and $\Delta^+ - \Delta^+_\lambda$ and applying this result to W_λ, we find

$$M^u_\lambda = W_\lambda(u) \sum_{w\in W^\lambda} w\Big[e_\lambda \frac{Q^\lambda}{D^\lambda}\Big],$$

where W^λ is the set of shortest coset representatives for W_λ in W, $Q^\lambda = \prod_{\alpha\in\Delta^+-\Delta^+_\lambda} 1-ue_\alpha$, and similarly for D^λ. Let $D_\lambda = D/D^\lambda$. The relation $wD^\lambda = \epsilon(w)e_{\rho-w\rho}\frac{D}{wD_\lambda}$ leads to

$$\begin{aligned} M^u_\lambda &= \frac{W_\lambda(u)}{|W_\lambda|}\sum_{w\in W}\epsilon(w)e_{w\cdot\lambda}\frac{w[Q^\lambda D_\lambda]}{D} \\ &= \frac{W_\lambda(u)}{|W_\lambda|}\sum_{\substack{S\subseteq\Delta^+-\Delta^+_\lambda\\ T\subseteq\Delta^+_\lambda}}(-1)^{|T|}(-u)^{|S|}\chi_{\lambda-\delta_S-\delta_T}.\end{aligned}$$

Let ρ_λ be half the sum of the roots in Δ^+_λ. Suppose there exists $\alpha \in \Delta^+_\lambda$ such that $\langle\rho_\lambda - \delta_T, \alpha\rangle = 0$. Then we have $s_\alpha\cdot(\lambda-\delta_S-\delta_T) = \lambda - \delta_{s_\alpha S} - \delta_T$. If $s_\alpha S = S$, then $\chi_{\lambda-\delta_S-\delta_T} = 0$. If $s_\alpha S \neq S$, then the sum over S in M^u_λ contains the terms $(-u)^{|S|}\chi_{\lambda-\delta_S-\delta_T} + (-u)^{|s_\alpha S|}\chi_{\lambda-\delta_{s_\alpha S}-\delta_T} = 0$. So there is no contribution to M^u_λ for T of this form. On the other hand if $\rho_\lambda - \delta_T$ is W_λ-regular, it is known that there exists a unique $x \in W_\lambda$ such that $\delta_T = \rho_\lambda - x\rho_\lambda$. Then $(-1)^{|T|} = \epsilon(x)$ and $\lambda - \delta_S - \delta_T = x\cdot(\lambda - \delta_{x^{-1}S})$. The T-th term in M^u_λ is therefore

$$\epsilon(x)\sum_{S\subseteq\Delta^+-\Delta^+_\lambda}(-u)^{|S|}\chi_{x\cdot(\lambda-\delta_{x^{-1}S})} = \sum_{S\subseteq\Delta^+-\Delta^+_\lambda}(-u)^{|S|}\chi_{\lambda-\delta_S},$$

since x permutes the subsets of $\Delta^+ - \Delta^+_\lambda$. This is independent of T and there are $|W_\lambda|$ such T. ∎

The referee informs me that (1.1) can also be proved using a sheaf cohomology result [Br1, (3.9)] plus the Borel-Weil-Bott theorem. The polynomials $M^u_\lambda/W_\lambda(u)$ are sometimes called "Hall-Littlewood polynomials". For further connections between M^u_λ, p-adic groups, and the geometry of flag manifolds, see [R2,3].

2. Review of the symmetric algebra.

We here collect and explicate known results on the symmetric algebra, for completeness, later use, and comparison with the exterior algebra.

Kato [Ka] (see also [G1, G2]) has described the transition between our two bases of R

$$\chi_\lambda = \sum_{\substack{\mu\in P^+\\ \mu\leq\lambda}}\frac{\langle\chi_\lambda, M^u_\mu\rangle_u}{W_\mu(u)}M^u_\mu,$$

by the formula

$$\langle \chi_\lambda, M_\mu^u \rangle_u = \sum_{w\in W} \epsilon(w) p(w\cdot\lambda - \mu, u),$$

where $p(\lambda, u)$ is the coefficient of e_λ in the formal power series $\prod_{\alpha>0}(1-ue_\alpha)^{-1}$. The right side is Lusztig's q-analogue of weight multiplicity. More precisely, by Kostant's weight multiplicity formula, $\langle \chi_\lambda, M_\mu^1\rangle_1$ is the multiplicity of the weight μ in the G-representation V_λ of highest weight λ. Kato and Lusztig also proved that $\langle \chi_\lambda, M_\mu^u\rangle_u$ may be expressed in terms of Kazhdan-Lusztig polynomials, as follows.

First, recall that the affine Weyl group $\tilde{W}$ is the semidirect product of W and the root lattice of T. For $x, y \in \tilde{W}$, we have Kazhdan-Lusztig polynomials $P_{y,x}(u)$. For λ belonging to the root lattice, let t_λ be its corresponding element in $\tilde{W}$. Then we have

$$\langle \chi_\lambda, M_\mu^u \rangle_u = u^{\langle \lambda-\mu, \check{\rho}\rangle} P_{w_0 t_\mu, w_0 t_\lambda}(u^{-1}),$$

where w_0 is the long word in W.

The polynomials $\langle \chi_\lambda, M_\mu^u\rangle_u$ themselves have representation-theoretic meaning. Let $S\mathfrak{g}$ be the symmetric algebra on $\mathfrak{g}^*$, and let $H\,\mathfrak{g} = \oplus H^n\mathfrak{g}$ be the harmonic polynomials in $S\mathfrak{g}$. The latter space is the annihilator in $S\mathfrak{g}$ of all G-invariant constant-coefficient differential operators on $\mathfrak{g}$ with zero constant term. For any finite dimensional representation V of G, let $P(V, H\,\mathfrak{g}, u) = \sum_{n\geq 0} \dim \mathrm{Hom}_G(V, H^n\mathfrak{g})u^n$. Kostant [Ko1] showed that this is actually a polynomial, and may be computed from the internal structure of V as follows. Let $\check{\rho} \in \mathfrak{t}$ be the unique element such that $\langle \alpha, \check{\rho}\rangle = 1$ for all simple roots α. There exist regular nilpotent elements $e, f \in \mathfrak{g}$ such that $\{e, \check{\rho}, f\}$ span a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Let $\mathfrak{a}$ be the centralizer of e in $\mathfrak{g}$. Let V_λ^i be the i-eigenspace of $\check{\rho}$ in V_λ. More generally, let $V_{\mu\lambda} = \mathrm{Hom}(V_\mu, V_\lambda)$, viewed as a G-module, and let $\mathrm{Hom}_{\mathfrak{a}}^i(V_\mu, V_\lambda)$ be those $\mathfrak{a}$-equivariant maps which send V_μ^j to V_λ^{i+j} for all j. Applying Kostant's results to the reducible representations $V_{\mu\lambda}$, we learn that $\mathrm{Hom}_{\mathfrak{a}}^i(V_\mu, V_\lambda) = 0$ unless $0 \leq i \leq \langle \lambda+\mu, \check{\rho}\rangle$, and

$$P(V_{\mu\lambda}, H\,\mathfrak{g}, u) = \sum_{i=0}^{\langle\lambda+\mu,\check{\rho}\rangle} \dim \mathrm{Hom}_{\mathfrak{a}}^i(V_\mu, V_\lambda)u^i.$$

Ginzburg [Gi] has interpreted the groups $\mathrm{Hom}_{\mathfrak{a}}^i(V_\mu, V_\lambda)$ as Ext groups in a certain derived category of complexes of sheaves on the loop group associated to G.

As first observed in [G2, Corollary 2.4], the multiplicity polynomial of $V_{\mu\lambda}$ in $H\,\mathfrak{g}$ may be expressed as the "wrong" inner product of characters. Namely, for $\lambda, \mu \in P^+$, we have

$$P(V_{\mu\lambda}, H\,\mathfrak{g}, u) = W(u)\langle \chi_\lambda, \chi_\mu\rangle_u = W(u) \sum_{\substack{\eta\in P^+\\ \eta\leq\lambda,\mu}} \frac{\langle \chi_\lambda, M_\eta^u\rangle_u \langle \chi_\mu, M_\eta^u\rangle_u}{W_\eta(u)}.$$

This formula, and its analogue for the exterior algebra in (3.2) below, are both consequences of the Weyl integration formula.

In particular, $P(V_\lambda, H\,\mathfrak{g}, u) = \langle \chi_\lambda, \chi_0\rangle_u$, a formula apparently first discovered by Peterson (see also [H]). For $u = 1$ it is Kostant's theorem that the multiplicity of V_λ in

$\mathcal{H}\,\mathfrak{g}$ equals the dimension of the zero weight space in V_λ. In case $V_\lambda = \mathfrak{g}$ is the adjoint representation, Kostant showed that $P(\mathfrak{g}, \mathcal{H}\,\mathfrak{g}, u) = u^{m_1} + \cdots + u^{m_\ell}$, where the m_i's are the exponents of W. For the computation of exponents, see [Co].

If $\mathfrak{g}$ is simple, there is one obvious submodule of each $\mathcal{H}^n\mathfrak{g}$, namely the irreducible submodule $V_{n\alpha_0}$ generated by the n-th power of a root vector e_{α_0} for the highest root α_0. These powers are harmonic since the weight $n\alpha_0$ does not appear in $S^m\mathfrak{g}$ for $m < n$. (More generally, yet another result in [Ko1] asserts that $\mathcal{H}\,\mathfrak{g}$ contains and is spanned by all powers of nilpotent elements.) There is a canonical complement to $V_{n\alpha_0}$ in $\mathcal{H}^n\mathfrak{g}$, namely the collection of harmonic polynomials vanishing on the minimal nonzero nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$. This may be seen in two ways. Let A_n be the annihilator in $\mathcal{H}^n\mathfrak{g}$ of the minimal nilpotent orbit. The Killing form induces a $G(\mathbb{C})$-invariant nondegenerate bilinear form $\langle\, ,\rangle$ on $S^n\mathfrak{g}$ which remains nondegenerate on $\mathcal{H}^n\mathfrak{g}$. For $X \in \mathfrak{g}$, and a polynomial function P of degree n (identified with an element of $S^n\mathfrak{g}$), we have $\langle X^n, P\rangle = n!\,P(X)$. This shows that A_n contains the orthogonal complement of $V_{n\alpha_0}$, and that the form is nonzero on the latter. It follows that $\mathcal{H}^n\mathfrak{g} = V_{n\alpha_0} \oplus A_n$. Alternatively, the closure of the minimal orbit is desingularized by the line bundle $L = G(\mathbb{C}) \times_P \mathbb{C}e_{\alpha_0}$, where P is the stabilizer of the line $\mathbb{C}e_{\alpha_0}$. A general result of Kempf [K] on collapsing of bundles shows that the desingularization map induces an isomorphism on rings of globally defined regular functions, and $V_{n\alpha_0}$ is realized by the functions on L which are polynomials of degree n on each fiber.

One cannot but marvel at the rich theory of the harmonic polynomials. Unfortunately, it cannot yet predict the structure of $\mathcal{H}^2\mathfrak{g}$, and the following decompositions, which shall be needed later, must be treated case by case. Let λ_i be the fundamental dominant weights for the following numberings of the Dynkin diagrams:

$$A_n\colon 12\cdots n,\quad B_n\colon 12\cdots \Rightarrow n,\quad C_n\colon 12\cdots \Leftarrow n\quad G_2\colon 1 \Lleftarrow 2,\quad F_4\colon 12 \Leftarrow 34,$$

$$D_n\colon \begin{matrix} 1\,2\cdots n-2\;n-1 \\ n \end{matrix} \qquad E_n\colon \begin{matrix} 1\,2\,3\cdots n-1 \\ n \end{matrix}$$

2.1. PROPOSITION. *The decompositions of $\mathcal{H}^2\mathfrak{g}$ into irreducible representations are given as follows.*

(1) If $\mathfrak{g} = \mathfrak{sl}(n)$ with $n \geq 4$, then

$$\mathcal{H}^2\mathfrak{g} \simeq V_{2\alpha_0} \oplus \mathfrak{g} \oplus V_{\lambda_2+\lambda_{n-2}}.$$

(2) If $\mathfrak{g} = \mathfrak{so}(V)$, where V is an nondegenerate orthogonal space of dimension at least five, then

$$\mathcal{H}^2\mathfrak{g} \simeq V_{2\alpha_0} \oplus \Lambda^4 V \oplus S_0^2 V,$$

where S_0^2V is the unique nontrivial constituent of S^2V.

(3) If $\mathfrak{g} = \mathfrak{sp}(V)$, where V is a nondegenerate symplectic space of dimension at least six, then

$$\mathcal{H}^2\mathfrak{g} \simeq V_{2\alpha_0} \oplus V_{2\lambda_2} \oplus \Lambda_0^2 V,$$

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where $\Lambda^2_0 V$ is the unique nontrivial constituent of $\Lambda^2 V$.

(4) If $\mathfrak{g}$ is of exceptional type, we have

$$\begin{aligned} H^2\mathfrak{g}_2 &= V_{2\alpha_0}(77) \oplus V_{2\lambda_1}(27)\\ H^2\mathfrak{f}_4 &= V_{2\alpha_0}(1053) \oplus V_{2\lambda_4}(324)\\ H^2\mathfrak{e}_6 &= V_{2\alpha_0}(2430) \oplus V_{\lambda_1+\lambda_5}(650)\\ H^2\mathfrak{e}_7 &= V_{2\alpha_0}(7371) \oplus V_{\lambda_5}(1539)\\ H^2\mathfrak{e}_8 &= V_{2\alpha_0}(27000) \oplus V_{\lambda_1}(3875),\end{aligned}$$

where $V_\lambda(d)$ is the irreducible module with highest weight λ and d is its dimension.

PROOF. The exceptional cases are treated with numerology. After more dimension counting in the classical cases, we need only exhibit the alleged constituents. We have already accounted for $V_{2\alpha_0}$, but the remaining two require progressively more computation.

For $\mathfrak{g} = \mathfrak{sl}(n)$, the adjoint representation appears because one exponent of the Weyl group is two. The other highest weight vector is $e_{\alpha_0-\alpha_1}e_{\alpha_0-\alpha_{n-1}} + e_{\alpha_0}e_{\alpha_0-\alpha_1-\alpha_{n-1}}$. As a polynomial function, in terms of the usual choice of root vectors, this is the determinant of the lower left 2×2 block.

For orthogonal cases, we have $\mathfrak{g} \simeq \Lambda^2 V$ as $\mathfrak{g}$-modules. The identity map $\Lambda^2 V \longrightarrow \Lambda^2 V$ extends to a surjection $S^2(\Lambda^2 V) \longrightarrow \Lambda^4 V$. If $\dim V = 2n+1$, let $e_1, \ldots, e_n, e_0, e_{-n}, \ldots, e_{-1}$ be a basis for which $\langle e_i, e_{-i}\rangle = 1$, with all other $\langle e_i, e_j\rangle = 0$. Set $e_{i,j} := e_i \wedge e_j$. Then

$$e_{1,0}^2 + 2\sum_{j=2}^{n} e_{i,j}e_{1,-j} \in S^2(\Lambda^2 V)$$

is a highest weight vector with weight that of $S^2_0 V$. For even dimensional V, omit e_0 in the above.

For symplectic cases, we have $\mathfrak{g} \simeq S^2 V$. Let $e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}$ be a symplectic basis for which only opposite signs are paired, and let $e_{i,j} = e_i e_j \in S^2 V$. Then

$$\begin{aligned} v_{2\lambda_2} &= e_{1,1}e_{2,2} - e_{1,2}^2,\\ v_{\lambda_2} &= \sum_{j=1}^{n}(e_{1,j}e_{2,-j} - e_{1,-j}e_{2,j})\end{aligned}$$

are highest weight vectors with the indicated weights. ∎

In certain cases, the structure of $\mathcal{H}\mathfrak{g}$ is related to the invariant theory of the Weyl group, as follows. Let $\mathcal{H}$ be the space of W-harmonic polynomials on $\mathfrak{t}$, that is, those polynomials killed by all constant coefficient W-invariant differential operators of positive degree on $\mathfrak{t}$. As a W-module, $\mathcal{H}$ is the regular representation of W. The W-structure of each graded piece is more subtle, but known (see [BL] and [Ki]). In particular, $\mathcal{H}^1 \simeq \mathfrak{t}$, $\mathcal{H}^\nu$ affords the sign character ϵ, and $\mathcal{H}^{\nu-n} \simeq \epsilon \otimes \mathcal{H}^n$ for all n. More generally, for

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any finite dimensional W-module E, possibly reducible, we shall consider the multiplicity polynomial

$$P_W(E, \mathcal{H}, u) = \sum_{n=0}^{\nu} \dim \mathrm{Hom}_W(E, \mathcal{H}^n) u^n.$$

One family of irreducible W-modules common to all Weyl groups is that of the exterior powers $\Lambda^q \mathfrak{t}$ of the representation of W on $\mathfrak{t}$. Solomon [So] proved that

$$P_W(\Lambda^q \mathfrak{t}, \mathcal{H}, u) = s_q(u^{m_1}, \ldots, u^{m_\ell}),$$

where $s_q(x_1, \ldots, x_\ell)$ is the elementary symmetric polynomial of degree q and as in §2, $1 + m_1, \ldots, 1 + m_\ell$ are the degrees of the homogeneous generators of the W-invariant polynomials on $\mathfrak{t}$.

Now let V_λ^0 be the zero weight space in the irreducible $\mathfrak{g}$-module V_λ. Unlike the case of the adjoint representation, it is not generally true that $P(V_\lambda, \mathcal{H}\mathfrak{g}, u)$ coincides with $P_W(V_\lambda^0, \mathcal{H}, u)$. However, a recent result of Broer [Br2] asserts this for "small" modules V_λ. We shall require several equivalent definitions of a small module:

2.2. DEFINITION. Assume that λ belongs to the root lattice. We say that V_λ is "small" if one of the following equivalent conditions holds:

(1) No nonzero weight in V_λ belongs to twice the root lattice
(2) Twice a root is not a weight of V_λ
(3) $\lambda \not\geq 2\alpha$ for any dominant root α

The implication (1) $\Rightarrow$ (2) is obvious, (2) $\Rightarrow$ (3) is "saturation of weights" *cf.* [B, p. 125 Corollary 2], and (3) $\Rightarrow$ (1) follows from the fact that any nonzero dominant weight μ in the root lattice must satisfy $\mu \geq \alpha_s$, for some shortest dominant root α_s.

The trivial and adjoint representations are small, as is V_{α_s}. For $\mathfrak{sl}(n)$, one can show that V_λ is small if and only if the highest weight of either V_λ or its dual V_λ^* comes from a partition of n. For E_n, the irreducible module $\mathcal{H}^2\mathfrak{e}_n/V_{2\alpha_0}$ (see (2.1)) is small. We shall see other small modules occurring in the exterior algebra. The result of Broer is

2.3. THEOREM (BROER, [BR2]). *If V_λ is small, then*

$$P(V_\lambda, \mathcal{H}\mathfrak{g}, u) = P_W(V_\lambda^0, \mathcal{H}, u).$$

For $\mathfrak{g} = \mathfrak{sl}(n)$, this was proved by Matsuzawa [Ma]. For other groups, one still has to compute V_λ^0, which can be difficult, but here an example. Take $G = E_n$, and consider the small module $V = \mathcal{H}^2\mathfrak{e}_n/V_{2\alpha_0}$. By (2.1), we know $P(V, \mathcal{H}\mathfrak{e}_n, u) = u^2 + \cdots$, hence V^0 has a constituent in $\mathcal{H}^2$, by (2.3). But for exceptional groups, $\mathcal{H}^2$ is an irreducible W-module, and by consulting a weight multiplicity table such as [BMP], we see that $\dim V^0 = \dim \mathcal{H}^2$ in all three cases. Hence $V^0 \simeq \mathcal{H}^2$. Since $P_W(\mathcal{H}^2, \mathcal{H}, u)$ is given in [BL], we know $P(V, \mathcal{H}\mathfrak{e}_n, u)$ completely, by (2.3).

3. **General multiplicity formulas for the exterior algebra.** Let $\Lambda\mathfrak{g} = \oplus\Lambda^q\mathfrak{g}$ denote the exterior algebra of $\mathfrak{g}$, for which the multiplicity polynomial of a finite dimensional G-module V is $P(V, \Lambda\mathfrak{g}, u) := \sum_{q\geq 0} \dim \operatorname{Hom}_G(V, \Lambda^q\mathfrak{g})u^q$. Since $\mathfrak{g}$ is a self-dual representation, so is each $\Lambda^q\mathfrak{g}$. Combining this with Poincaré duality gives the relations

$$P(V, \Lambda\mathfrak{g}, u) = P(V^*, \Lambda\mathfrak{g}, u) = u^{\dim\mathfrak{g}} P(V, \Lambda\mathfrak{g}, u^{-1}).$$

The de Rham complex on $\mathfrak{g}$ with polynomial coefficients is exact, and taking its G-invariants yields the additional relation

$$\sum_{\lambda\leq 2\rho} P(V_\lambda, \Lambda\mathfrak{g}, -u)P(V_\lambda, S\mathfrak{g}, u) = 1.$$

We shall give a formula for $P(V_\lambda, \Lambda\mathfrak{g}, u)$ in a manner analogous to the symmetric multiplicity polynomial. We begin with corresponding facts about the other inner product. Note first of all the explicit formula, following from (1.1):

$$(\chi_\mu, M^u_\lambda) = W_\lambda(u) \sum_{\substack{S\subseteq\Delta^+-\Delta^+_\lambda \\ \lambda-\delta_S\in W\cdot\mu}} \epsilon(\lambda - \delta_S)(-u)^{|S|}.$$

The first and last assertions in the following were proved in [G1]. Here we use the factorization (1.1).

3.1. PROPOSITION. *Let $\mu, \lambda \in P^+$. We have*
(1) $(\chi_\mu, M^u_\lambda) = 0$ *unless* $\mu \leq \lambda$.
(2) $(\chi_0, M^u_\lambda) = 0$ *unless* $\lambda \leq 2\rho$.
(3) $(\chi_\lambda, M^u_\lambda) = W_\lambda(u)$.

PROOF. For any $w \in W$ and $S \subseteq \Delta^+ - \Delta^+_\lambda$, we have $w\cdot(\lambda - \delta_S) = w\lambda - w\delta_S + w\rho - \rho$. Clearly $w\lambda \leq \lambda$. The negative roots in wS are of the form $-\alpha$, where $w^{-1}\alpha < 0$. Such an α also appears in $\rho - w\rho$ and hence cancels. This proves (1). Writing $w\lambda = \lambda - \gamma$, we see that $w\cdot(\lambda - \delta_S) = \lambda$ implies $\gamma = w\delta_S - w\rho + \rho = 0$, since both are ≥ 0. Hence $w\lambda = \lambda$. Since $S \subseteq \Delta^+ - \Delta^+_\lambda$, there are no negative roots in wS so $w\delta_S = \rho - w\rho = 0$. This forces $w = 1, S = \emptyset$, whence (3). Finally, if $\lambda = w\cdot 0 + \delta_S$, then $2\rho - \lambda = (2\rho - \delta_S) + (\rho - w\rho) \geq 0$. ■

Now we give a general multiplicity polynomial, which should be compared with its symmetric counterpart in Section 2.

3.2. PROPOSITION. *Let $\lambda, \mu \in P^+$. Then*

$$P(V_{\mu\lambda}, \Lambda\mathfrak{g}, -u) = (1-u)^\ell \sum_{\lambda,\mu\leq\eta\in P^+} \frac{(\chi_\lambda, M^u_\eta)(\chi_\mu, M^u_\eta)}{W_\eta(u)}$$

(Only finitely many terms in the sum are nonzero.) In particular, we have

$$P(V_\lambda, \Lambda\mathfrak{g}, -u) = (1-u)^\ell \sum_{\substack{\lambda\leq\eta\leq 2\rho,\ \eta\in P^+ \\ S\subseteq\Delta^+-\Delta^+_\eta \\ \eta-\delta_S\in W\cdot 0}} \epsilon(\eta - \delta_S)(-u)^{|S|}(\chi_\lambda, M^u_\eta),$$

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and $(1+u)^\ell$ *divides* $P(V_\lambda, \Lambda\mathfrak{g}, u)$ *for all* $\lambda \in P^+$.

PROOF. Since $\chi_\mu \prod_{\alpha\in\Delta} 1 - ue_\alpha$ belongs to $R[[u]]$, there exist $c_\eta(u) \in \mathbb{C}[[u]]$, $\eta \in P^+$, such that $\chi_\mu \prod_{\alpha\in\Delta} 1 - ue_\alpha = \sum_\eta c_\eta(u) M^u_\eta$. Then

$$\begin{aligned}(\chi_\mu, M^u_\eta) &= \langle \chi_\mu \prod_{\alpha\in\Delta} 1 - ue_\alpha, M^u_\eta\rangle_u \\ &= c_\eta(u)\langle M^u_\eta, M^u_\eta\rangle_u \\ &= c_\eta(u) W_\eta(u).\end{aligned}$$

It follows that

$$\begin{aligned}P(V_{\mu\lambda}, \Lambda\mathfrak{g}, -u) &= (1-u)^\ell (\chi_\lambda, \chi_\mu \prod_{\alpha\in\Delta} 1 - ue_\alpha) \\ &= (1-u)^\ell \sum_\eta c_\eta(u)(\chi_\lambda, M^u_\eta) \\ &= (1-u)^\ell \sum_\eta \frac{(\chi_\lambda, M^u_\eta)(\chi_\mu, M^u_\eta)}{W_\eta(u)}.\end{aligned}$$

■

Though it will not be of further use to us, it seems worth remarking that one can invert Kato's expression of $\langle \chi_\lambda, M^u_\mu\rangle_u$ as a Kazhdan-Lusztig polynomial, and show that the other inner product (χ_μ, M^u_λ) is an average of inverse Kazhdan-Lusztig polynomials. Indeed, elementary properties of the $P_{x,y}$'s show that there exist unique polynomials $Q_{y,x}$ such that $\sum_{y\in\tilde W} Q_{y,x} P_{w,w_0 y} = 1$ if $x = w$, zero otherwise, and we have

3.3. PROPOSITION. *If λ and μ are two dominant weights in the root lattice, then*

$$(\chi_\mu, M^u_\lambda) = u^{\langle \lambda-\mu, \check\rho\rangle} W_\lambda(u) \sum_{x\in Wt_\lambda W} Q_{\mu,x}(u^{-1}).$$

PROOF. It suffices to show that

$$\frac{1}{W_\lambda(u)} \sum_\mu u^{\langle \mu-\lambda, \check\rho\rangle}(\chi_\mu, M^u_\lambda) P_{w,w_0 t_\mu}(u^{-1}) = 1 \quad \text{if } w \in Wt_\lambda W, \quad \text{zero otherwise.}$$

The Hecke algebra $\mathbf{H}$ of $\tilde W$ has the $\mathbb{C}(u)$-basis $\{T_w : w \in \tilde W\}$ with the usual multiplication rules. The inverse Satake transform $f \longmapsto \check f$ maps $R(u)$ into a subalgebra of $\mathbf{H}$. For example,

$$(M^u_\lambda)^\vee = \frac{W_\lambda(u)}{W(u^{-1})} u^{\langle \lambda, \check\rho\rangle} \sum_{w\in Wt_{\bar\lambda} W} T_w,$$

where $\bar\lambda = -w_0\lambda$. Following Kato, we consider the Kazhdan-Lusztig basis element

$$C_{w_0 t_\lambda}(u) = u^{-\nu - \langle \lambda, \check\rho\rangle} \sum_{y\le w_0 t_\lambda} P_{y, w_0 t_\lambda}(u) T_y.$$

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Using Kato's result, one can write this as

$$
\begin{aligned}
C_{w_0t_\lambda}(u^{-1}) &= u^{\nu+\langle\lambda,\check\rho\rangle}\sum_{\mu\le\lambda}P_{w_0t_\mu,w_0t_\lambda}(u^{-1})\Big(\sum_{w\in Wt_\mu W}T_w\Big)\\
&= u^\nu W(u^{-1})\sum_{\mu\le\lambda}\frac{u^{\nu+\langle\lambda-\mu,\check\rho\rangle}}{W_\mu(u)}P_{w_0t_\mu,w_0t_\lambda}(u^{-1})(M^{\underline{u}}_{\bar\mu})^\vee\\
&= W(u)\check\chi_{\bar\lambda}.
\end{aligned}
$$

We can therefore express $(M^{\underline{u}}_\lambda)^\vee$ in two ways, namely

$$
\begin{aligned}
\frac{W_\lambda(u)}{W(u^{-1})}u^{\langle\lambda,\check\rho\rangle}\sum_{w\in Wt_\lambda W}T_w &= (M^{\underline{u}}_\lambda)^\vee\\
&= \sum_\mu(\chi_\mu,M^{\underline{u}}_\lambda)\check\chi_\mu\\
&= \frac{1}{W(u)}\sum_\mu(\chi_\mu,M^{\underline{u}}_\lambda)C_{w_0t_{\bar\mu}}(u^{-1})\\
&= \frac{1}{W(u)}\sum_{\substack{\mu\\ y\le w_0t_{\bar\mu}}}(\chi_\mu,M^{\underline{u}}_\lambda)u^{\nu+\langle\bar\mu,\check\rho\rangle}P_{y,w_0t_{\bar\mu}}(u^{-1})T_y.
\end{aligned}
$$

Now compare the coefficients of T_w on both sides, recalling that $(\chi_\mu,M^{\underline{u}}_\lambda)=(\chi_{\bar\mu},M^{u}_\lambda)$. ∎

4. **Ungraded multiplicities.** Long before one knew the Betti numbers of a compact Lie group, E. Cartan was able to show, using the Weyl integration formula, that $\dim H(G) = 2^\ell$. The same idea gives a recursive formula for $\dim \mathrm{Hom}_G(V_\lambda,\Lambda\mathfrak{g})$ that can easily be implemented by hand if one has a table of weight multiplicities for V_λ (*cf.* [BMP]).

Let m^μ_λ be the multiplicity of the weight μ in V_λ. For $\lambda\in P^+$, we put

$$\hat D(\lambda)=\frac{1}{|W_\lambda|}\int_T e_\lambda\prod_{\alpha\in\Delta}(1-e_\alpha)\,dt.$$

We have $\hat D(\lambda)=0$ unless λ belongs to the root lattice, and it is well known that $\hat D(0)=1$.

4.1. PROPOSITION. *For $0\ne\lambda\in P^+$, we have*

(1)

$$\sum_{\substack{\mu\le\lambda\\ \mu\in P^+}} m^\mu_\lambda\hat D(\mu)=0$$

(2)

$$\dim\mathrm{Hom}_G(V_\lambda,\Lambda\mathfrak{g})=2^\ell\sum_{\substack{\mu\le\lambda\\ \mu\in P^+}} m^\mu_\lambda\hat D\Big(\frac{\mu}{2}\Big).$$

(Here $\hat D(\frac{\mu}{2})$ is read as zero if $\frac{\mu}{2}$ is not in the root lattice.)

PROOF. Formula (1) is obtained by evaluating

$$P(V_\lambda,H\,\mathfrak{g},u)=\frac{W(u)}{|W|}\int_T\chi_\lambda\prod_{\alpha\in\Delta}\frac{1-e_\alpha}{1-ue_\alpha}\,dt$$

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at $u = 0$. For (2), we have

$$P(V_\lambda, \Lambda\mathfrak{g}, 1) = \frac{2^\ell}{|W|} \sum_{\substack{\mu \le \lambda \\ \mu \in P^+}} m_\mu^\lambda \frac{|W|}{|W_\mu|} \int_T e_\mu \prod_{\alpha \in \Delta} (1 - e_{2\alpha})\, dt.$$

The integral is zero unless μ belongs to twice the root lattice, in which case it has the same value as $\int_T e_{\mu/2} \prod_{\alpha\in\Delta}(1 - e_\alpha)\, dt$, since the squaring map on T is surjective. ■

Take $G = E_8$ as an example, with Dynkin diagram labelled as in §2. Let $\lambda = \lambda_3$ be the fundamental weight of the branch node. Using only weight multiplicity table and pencil, one can readily compute the multiplicity of V_λ in $\Lambda\mathfrak{g}$, for though there be 24 dominant weights $\mu \le \lambda$, only $\mu = 0$, $2\lambda_7$, $2\lambda_1$, $2\lambda_6$ belong to twice the root lattice. Applying (4.1)(1), we find $\hat{D}(\lambda_7) = -8$, $\hat{D}(\lambda_1) = -21$, $\hat{D}(\lambda_6) = -287$, (In general, $\hat{D}(\alpha_0) = -\ell$, where α_0 is the highest root.) Then by (4.1)(2),

$$\dim \mathrm{Hom}_G(V_\lambda, \Lambda\mathfrak{g}) = 2^8\{m_0^\lambda - 8m_{2\lambda_7}^\lambda - 21m_{2\lambda_1}^\lambda - 287m_{2\lambda_6}^\lambda\} = 2^8 \cdot 5 \cdot 7 \cdot 18671.$$

For small representations, the recursion in (4.1) is trivial by (2.2)(1), and we get

4.2. COROLLARY. *Any small module V_λ has multiplicity $m_\lambda^0 2^\ell$ in $\Lambda\mathfrak{g}$.*

After receiving an earlier version of this paper, Kostant pointed out that Corollary (4.2) is a consequence of an old result, originally proved by him but unpublished, then proved independently several years later in [PRV, Theorem 2.1]. See also [V, Chapter 4, Exercises 18-20]. It is the following

4.3. THEOREM (KOSTANT, [PRV]). *Let λ, μ, η be dominant weights, let $V_\lambda^{\eta-\mu}$ be the $\eta - \mu$-weight space in V_λ, and put*

$$Z_\lambda^{\eta-\mu} := \{v \in V_\lambda^{\eta-\mu} : X_\alpha^{1+\langle\mu,\check{\alpha}\rangle} v = 0 \qquad \text{for all } \alpha \in \Sigma\},$$

where X_α is a root vector for α. Then there is a linear isomorphism

$$Z_\lambda^{\eta-\mu} \simeq \mathrm{Hom}_G\big(V_\lambda, \mathrm{Hom}(V_\mu, V_\eta)\big).$$

For $\mu = \eta = \rho$, we have

$$Z_\lambda^0 = \{v \in V_\lambda^0 : X_\alpha^2 v = 0 \qquad \text{for all } \alpha \in \Sigma\}.$$

Thus, Z_λ^0 is all of V_λ^0 if and only if V_λ is small, by (2.2)(2).

The connection with $\Lambda\mathfrak{g}$ is provided by another old result of Kostant [Ko2, p. 357], asserting that the character of V_ρ is

$$e_\rho \prod_{\beta>0} (1 + e_{-\beta}), \tag{4.4}$$

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from which it follows that $\Lambda\mathfrak{g}$ is isomorphic, as an ungraded G-module, to 2^ℓ copies of $\mathrm{End}(V_\rho) \simeq V_\rho \otimes V_\rho$. With (4.3), this shows that

$$\dim \mathrm{Hom}_G(V_\lambda, \Lambda\mathfrak{g}) \leq m_\lambda^0 2^\ell,$$

with equality if and only if V_λ is small. Thus (4.3) and (4.4) not only imply Corollary (4.2), but also its converse.

It would be interesting to have a similar explanation of the recursive formula (4.1) for non-small modules. In the other direction, note that (4.1) gives an efficient way to decompose $V_\rho \otimes V_\rho$, or, equivalently, to find the dimension of Z_λ^0.

For ungraded multiplicities, it is perhaps more natural to replace $\Lambda\mathfrak{g}$ with its ungraded version, namely the Clifford algebra $C(\mathfrak{g})$ of $\mathfrak{g}$ with respect to a G-invariant quadratic form. In this setting Kostant has shown more [N]: As algebras, $C(\mathfrak{g}) \simeq \mathrm{End}(V_\rho) \otimes C(\mathfrak{p})$, where $C(\mathfrak{p})$ is a certain Clifford algebra on the ℓ-dimensional space $\mathfrak{p}$ of primitive G-invariants in $C(\mathfrak{g})$.

5. **Low degrees.** The lowest exterior power whose structure is not obvious is $\Lambda^2\mathfrak{g}$, which decomposes as follows. Assume $\mathfrak{g}$ is simple and not isomorphic to $\mathfrak{sl}(2)$. Let α_0 be the highest root, and let J be the set of simple roots which are not orthogonal to α_0. For each $\alpha \in J$, one knows that $\alpha_0 - \alpha$ is a root, and it is evident that $e_{\alpha_0} \wedge e_{\alpha_0-\alpha}$ is a highest weight vector in $\Lambda^2\mathfrak{g}$, and therefore generates an irreducible submodule $U_\alpha \subset \Lambda^2\mathfrak{g}$. Let U_2 be the direct sum of the U_α, for $\alpha \in J$.

Note that U_2 is irreducible, except for $\mathfrak{sl}(n)$, $n \geq 3$, when we have

$$U_2 \simeq V_{2\lambda_1+\lambda_{n-2}} \oplus V_{\lambda_2+2\lambda_{n-1}}$$

respectively. Both of these representations are small, in the sense of (2.2). More generally, from (2.2)(4), we observe that the irreducible constituents of U_2 are small if and only if $\mathfrak{g}$ is simply-laced.

Of course, the Lie bracket causes $\mathfrak{g}$ to appear in $\Lambda^2\mathfrak{g}$, with multiplicity one by Schur's lemma. Now proceeding case by case, and using the Weyl dimension formula, we find

5.1. PROPOSITION. *For $\mathfrak{g} \neq \mathfrak{sl}(2)$ we have $\Lambda^2\mathfrak{g} = \mathfrak{g} \oplus U_2$.*

This decomposition implies many nice properties of U_2, which shall be recounted elsewhere (but see §7 below).

The third exterior power also admits a uniform description, although naturally it is more complicated. We first discuss higher degree analogues of U_2.

Call a subset S of the positive roots "saturated" if $\beta \in S$, $\alpha \in \Sigma$, $\alpha + \beta \in \Delta^+$ imply $\alpha + \beta \in S$. Then the wedge product e_S of root vectors for roots in a saturated subset S is a highest weight vector in $\Lambda^{|S|}\mathfrak{g}$, of weight $\delta_S := \sum_{\beta\in S} \beta$, and e_S generates an irreducible $\mathfrak{g}$-module $U(S) \subset \Lambda^{|S|}\mathfrak{g}$. For example, we could take S to be all positive roots outside a given subset $I \subset \Sigma$, and see that $u^{\nu-|I|}$ appears in $P(V_{2\rho-\delta_I}, \Lambda\mathfrak{g}, u)$. In the next section, we shall compute this multiplicity polynomial completely, and find that we have here detected its lowest degree term.

Let U_n be the direct sum of the $U(S)$ for saturated S with $|S| = n$. Let $\partial\colon \Lambda^{n+1}\mathfrak{g} \longrightarrow \Lambda^n\mathfrak{g}$ be the Koszul boundary map, given by

$$\partial(X_0 \wedge \cdots \wedge X_n) = \sum_{i<j}(-1)^{i+j+1}[X_i, X_j] \wedge X_i \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_n.$$

Observe that if $S \subset \Delta^+$ is saturated and $h \in \mathfrak{t}$, then

$$\partial(h \wedge e_S) = \delta_S(h)e_S.$$

It follows that $U(S)$ belongs to the image of ∂, and therefore $U(S)$ also embeds in $\Lambda^{|S|+1}\mathfrak{g}$. Since U_2 is multiplicity-free we have, in particular, $U_2 \hookrightarrow \Lambda^3\mathfrak{g}$.

5.4. PROPOSITION. *Assume $\mathfrak{g} \neq \mathfrak{sl}(2)$, $\mathfrak{sl}(3)$, $\mathfrak{so}(5)$. Then there is an isomorphism of $\mathfrak{g}$-modules*

$$\Lambda^3\mathfrak{g} \simeq \mathbb{C} \oplus H^2\mathfrak{g} \oplus U_2 \oplus U_3.$$

(The decomposition of $H^2\mathfrak{g}$ was given in (2.1), and the exceptions are covered by the tables in Section 8.)

PROOF. Once again, one can compute the dimensions of each of the three terms on the right, and with the stated exceptions, it is the same on the left. Moreover, the highest weights in U_3 do not appear in the other two terms. The highest weights of U_2 appear in $S^2\mathfrak{g}$ only in the vectors $e_{\alpha_0}e_{\alpha_0-\alpha}$ for $\alpha \in J$, and these are not highest weight vectors. (In fact, they belong to the submodule $V_{2\alpha_0}$ of $H^2\mathfrak{g}$.) Since $S^2\mathfrak{g} = \mathbb{C} \oplus H^2\mathfrak{g}$, it suffices to show that $S^2\mathfrak{g}$ injects into $\Lambda^3\mathfrak{g}$.

Given a symmetric bilinear form q on $\mathfrak{g}$, consider the alternating 3-form

$$\omega_q(X, Y, Z) = q([X, Y], Z) + q([Y, Z], X) + q([Z, X], Y).$$

Let $\{h_\alpha, e_\gamma, f_\gamma : \alpha \in \Sigma,\ \gamma \in \Delta^+\}$ be a Chevalley basis of $\mathfrak{g}$, corresponding to the choice of $\mathfrak{t}$ and Σ. Define $h_\gamma = [e_\gamma, f_\gamma]$ for all $\gamma \in \Delta^+$. Let $\{H_\alpha, E_\gamma, F_\gamma\}$ be the corresponding dual basis with respect to the Killing form $\langle\,,\rangle$. Let K be the kernel of $q \longmapsto \omega_q$. This is a $\mathfrak{g}$-submodule of $S^2\mathfrak{g}$, and since every nonzero submodule of $S\mathfrak{g}$ has nonzero zero weight space, it suffices to show that the zero weight space K_0 is zero.

A typical $q \in K_0$ may be written

$$q = \sum_{\alpha\in\Sigma} A_\alpha H_\alpha^2 + \sum_{\alpha\neq\beta} B_{\alpha\beta}H_\alpha H_\beta + 2\sum_{\gamma>0} C_\gamma E_\gamma F_\gamma,$$

where $q(h_\alpha, h_\alpha) = A_\alpha$, $q(h_\alpha, h_\beta) = B_{\alpha\beta}$, $q(e_\gamma, f_\gamma) = C_\gamma$. Since $\omega_q = 0$, we have, for all $h \in \mathfrak{t}$ and $\gamma \in \Delta^+$,

$$0 = \omega_q(e_\gamma, f_\gamma, h) = q(h_\gamma, h) + 2\gamma(h)q(e_\gamma, f_\gamma),$$

so

$$q(h_\gamma, h) = -2\gamma(h)C_\gamma. \tag{5.5}$$

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This implies that the restriction map $K_0 \longrightarrow S^2\mathfrak{t}$ is injective. Moreover, taking $\gamma = \alpha \in \Sigma$ and $h = h_\alpha$ gives the relation $A_\alpha = -4C_\alpha$ for every simple root α.

Extend the Killing form $\langle\,,\rangle$ to a G-invariant inner product on $S^2\mathfrak{g}$ as follows: For $X \in \mathfrak{g}$, let D_X be the derivation of $S^2\mathfrak{g}$ extending the functional $\langle X, \cdot\rangle$. Then for $X, Y \in \mathfrak{g}$ and $P \in S^2\mathfrak{g}$ we have $\langle XY, P\rangle = D_X D_Y P$. Let $K^\perp$ be the subspace of $S^2\mathfrak{g}$ orthogonal to K.

Let $q_1 = F^2_{\alpha_0}$, where as above α_0 is the highest root, and let $\alpha \in \Sigma$ be such that $\alpha_0 - \alpha$ is a root. Then $[f_\alpha, f_{\alpha_0-\alpha}] = cf_{\alpha_0}$ for some nonzero scalar c, and $\omega_{q_1}(f_\alpha, f_{\alpha_0-\alpha}, f_{\alpha_0}) = c \neq 0$. It follows that $V_{2\alpha_0} \subseteq K^\perp$. Now the Weyl group acts transitively on long roots, so there is a long simple root α such that e^2_α belongs to $K^\perp$. Applying $ad(f_\alpha)^2$ to e^2_α, we have $2e_\alpha f_\alpha - h^2_\alpha \in K^\perp$. Hence for our $q \in K_0$ we have

$$0 = \Big\langle \sum_{\alpha\in\Sigma} A_\alpha H^2_\alpha + \sum_{\alpha\neq\beta} B_{\alpha\beta} H_\alpha H_\beta + 2\sum_{\gamma>0} C_\gamma E_\gamma F_\gamma,\ 2e_\alpha f_\alpha - h^2_\alpha \Big\rangle = 4C_\alpha - 2A_\alpha.$$

Since also $A_\alpha = -4C_\alpha$, we have $A_\alpha = C_\alpha = 0$ for this long simple α. It follows from (5.5) that $q(h_\alpha, h) = 0$ for all $q \in K_0$ and all $h \in \mathfrak{t}$. Let $q_\mathfrak{t}$ be the restriction of $q \in K_0$ to $\mathfrak{t}$, and let $q^w_\mathfrak{t}$ be the transform of $q_\mathfrak{t}$ under $w \in W$. Note that $q^w_\mathfrak{t}$ also belongs to the restriction image of K_0, because restriction $K_0 \longrightarrow S^2\mathfrak{t}$ is W-equivariant. Since $Ad(w)h_\alpha = h_{w\alpha}$, we have

$$q_\mathfrak{t}(h_{w\alpha}, h) = q^w_\mathfrak{t}\big(h_\alpha, Ad(w)^{-1}h\big) = 0$$

for all $h \in \mathfrak{t}$ and $w \in W$. Hence $q_\mathfrak{t}(h_\beta, \cdot) \equiv 0$ for all long roots β. Since the long roots span $\mathfrak{t}^*$, we have $q_\mathfrak{t} = 0$, and since restriction is injective, we have $q = 0$. ∎

(5.6) The map $q \longrightarrow \omega_q$ is a special case of the relation between symmetric and alternating forms as outlined in [Ch]. The differential $d\colon \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$ extends to an algebra homomorphism $\Omega\colon S\mathfrak{g} \longrightarrow \Lambda^{even}\mathfrak{g}$ whose image is contained in $d(\Lambda^{odd}\mathfrak{g})$. In fact there is a canonical "integral" of Ω (see [Ch]). For example, $d\omega_q = 3\Omega(q)$ for $q \in S^2\mathfrak{g}$.

Since invariant alternating forms are never coboundaries (*cf.* [CE]), the kernel of Ω contains the ideal in $S\mathfrak{g}$ generated by invariant polynomials, and therefore the image of Ω is $\Omega(\mathcal{H}\,\mathfrak{g})$. By (5.1), $\Lambda^{even}\mathfrak{g}$ is generated by $d\mathfrak{g}$ and U_2, and $\Omega(\mathcal{H}\,\mathfrak{g})$ is the subalgebra of $\Lambda^{even}\mathfrak{g}$ generated by $d\mathfrak{g}$. We shall determine the image under Ω of the submodules $V_{n\alpha_0} \in \mathcal{H}^n\mathfrak{g}$ described in Section 2. This amounts to finding the subalgebra of $\Lambda^{even}\mathfrak{g}$ generated by $\omega_0 := d(e_{\alpha_0})$. Now

$$\omega_0 = e_{\alpha_0} \wedge h + \sum C_{\alpha\beta} e_\alpha \wedge e_\beta,$$

for some $0 \neq h \in \mathfrak{t}$, where the sum is over pairs of positive roots α and β such that $\alpha + \beta = \alpha_0$, and each $C_{\alpha\beta}$ is a nonzero scalar. Say there are p such pairs of positive roots. The form $\mu := \sum C_{\alpha\beta} e_\alpha \wedge e_\beta$ may be viewed as a nondegenerate form on the $2p$-dimensional span of the root vectors it contains. Taking powers, we have

$$\omega^k_0 = k e_{\alpha_0} \wedge h \wedge \mu^{k-1} + \mu^k,$$

and $\omega^k_0 = 0$ if and only if $\mu^{k-1} = 0$, if and only if $k - 1 > p$. Thus the highest nonzero power of ω_0 is ω^{p+1}_0.

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6. **Some explicit graded multiplicity formulas.** The multiplicity formulas given in §3 involve large cancellations, and while they can be useful, one would prefer formulas with explicit nonnegative coefficients. These can be obtained for certain irreducible representations as follows.

We begin with a reduction formula. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$. Assume that $\mathfrak{t} \subseteq \mathfrak{l}$, and that the roots $\Delta_{\mathfrak{n}}$ of $\mathfrak{t}$ in $\mathfrak{n}$ are positive, so that $\Delta^+ = \Delta_{\mathfrak{l}}^+ \cup \Delta_{\mathfrak{n}}$. Correspondingly, put $\rho = \rho_{\mathfrak{l}} + \rho_{\mathfrak{n}}$. If μ belongs to the integral span of $\Delta_{\mathfrak{l}}^+$ and is dominant with respect to that choice of positive roots for $\mathfrak{l}$, let V'_μ be the $\mathfrak{l}$-module with highest weight μ.

6.1. PROPOSITION. *Suppose V'_μ appears in $\Lambda\mathfrak{l}$. Then $\mu+2\rho_{\mathfrak{n}}$ is dominant with respect to Δ^+, and we have*

$$P(V_{\mu+2\rho_{\mathfrak{n}}}, \Lambda\mathfrak{g}, u) = u^{|\Delta_{\mathfrak{n}}|} P(V'_\mu, \Lambda\mathfrak{l}, u).$$

PROOF. For the dominance assertion, note that $\mu = 2\rho_{\mathfrak{l}} - \delta$, where δ is a sum of simple roots in $\mathfrak{l}$ with non-negative coefficients, so $\mu + 2\rho_{\mathfrak{n}} = 2\rho - \delta$. Now, if α is a simple root outside $\mathfrak{l}$, then $\langle \delta, \check{\alpha} \rangle \leq 0$, so $\langle \mu + 2\rho_{\mathfrak{n}}, \check{\alpha} \rangle = \langle 2\rho, \check{\alpha} \rangle - \langle \delta, \check{\alpha} \rangle \geq 0$. If α is a simple root in $\mathfrak{l}$, then $\langle 2\rho_{\mathfrak{n}}, \check{\alpha} \rangle = 0$, since $2\rho_{\mathfrak{n}}$ is the weight of the one-dimensional $\mathfrak{l}$-module $\Lambda^{\dim \mathfrak{n}} \mathfrak{n}$. Hence $\langle \mu + 2\rho_{\mathfrak{n}}, \check{\alpha} \rangle = \langle \mu, \check{\alpha} \rangle \geq 0$ since μ is dominant for $\Delta_{\mathfrak{l}}^+$.

For the multiplicity, we compare coefficients of simple roots outside $\mathfrak{l}$, and find that every weight vector in $\Lambda\mathfrak{g}$ of weight $\mu + 2\rho_{\mathfrak{n}}$ is of the form $e_{\mathfrak{n}} \wedge v$, where $0 \neq e_{\mathfrak{n}} \in \Lambda^{|\Delta_{\mathfrak{n}}|}\mathfrak{n}$ and $v \in \Lambda\mathfrak{l}$ has weight μ. Let α be a simple root, with e_α the corresponding Chevalley basis vector. If $\alpha \in \Delta_{\mathfrak{n}}$, then $ad(e_\alpha)(e_{\mathfrak{n}}) = 0$ since $\mathfrak{n}$ is unimodular, and $ad(e_\alpha)(\mathfrak{l}) \subseteq \mathfrak{n}$, so $ad(e_\alpha)(e_{\mathfrak{n}} \wedge v) = 0$. If $\alpha \in \Sigma \cap \Delta_{\mathfrak{l}}^+$, then e_α belongs to the derived algebra of $\mathfrak{l}$ which acts trivially on $\Lambda^{|\Delta_{\mathfrak{n}}|}$, so again $ad(e_\alpha)(e_{\mathfrak{n}}) = 0$. Since $ad(e_\alpha)v \in \Lambda\mathfrak{l}$, it follows that $e_{\mathfrak{n}} \wedge v$ is a $\mathfrak{g}$-highest weight vector if and only if v is an $\mathfrak{l}$-highest weight vector, implying the proposition. ∎

Let $W_{\mathfrak{l}}$ be the Weyl group of $\mathfrak{l}$, viewed as a reflection group acting on $\mathfrak{t}$. Let $1 + n_1, \ldots, 1 + n_\ell$ be the degrees of the homogeneous generators of the $W_{\mathfrak{l}}$-invariant polynomials on $\mathfrak{t}$. Note that some n_i's will be zero. Taking $\mu = 0$ in (6.1), the Betti number formula for compact Lie groups yields

6.2. COROLLARY.

$$P(V_{2\rho_{\mathfrak{n}}}, \Lambda\mathfrak{g}, u) = u^{|\Delta_{\mathfrak{n}}|} \prod_{i=1}^{\ell} (1 + u^{2n_i+1}).$$

For example, $P(V_{2\rho}, \Lambda, u) = u^\nu (1+u)^\ell$ (here $\mathfrak{l} = \mathfrak{t}$), and if α is a simple root, we have $P(V_{2\rho-\alpha}, \Lambda, u) = u^{\nu-1}(1+u)^{\ell-1}(1+u^3)$ (here $\mathfrak{l}$ has semisimple rank one).

We next use the reduction formula (6.1) and the general formula (3.2) to compute another family of multiplicity polynomials.

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6.3. PROPOSITION. *Assume $\mathfrak{g}$ is simple. For $I \subset \Sigma$, we have*

$$P(V_{2\rho-\delta_I}, \Lambda\mathfrak{g}, u) = u^{\nu-|I|}(1+u)^{\ell-c(I)}(1+u^2)^{|I|-c(I)}(1+u^3)^{c(I)},$$

where $c(I)$ is the number of connected components of the subgraph of the Dynkin diagram of $\mathfrak{g}$ whose vertices are in I.

PROOF. Applying the reduction formula (6.1) to the Levi subalgebra generated by the roots in I, we may assume $I = \Sigma$, and must show that

$$P(V_{2\rho-\sigma}, \Lambda\mathfrak{g}, u) = u^{\nu-\ell}(1+u)^{\ell-1}(1+u^2)^{\ell-1}(1+u^3),$$

where we have written $\sigma = \delta_\Sigma$. The weights η such that $2\rho-\sigma \leq \eta \leq 2\rho$ are of the form $\eta_K = 2\rho-\delta_K$ for some $K \subseteq \Sigma$, and these are all dominant. We first compute $(\chi_{2\rho-\sigma}, M^u_{\eta_K})$ using the formula given at the beginning of Section 3.

Write $K = K_0 \cup K_d$ (disjoint union), where K_d is the set of roots in K which are orthogonal to every member of K other than themselves. Then $\Delta^+_{\eta_K} = K_d$, and $W_{\eta_K}(u) = (1+u)^{|K_d|}$. Now let $S \subseteq \Delta^+ - K_d$, and suppose there exists $w \in W$ such that

$$\eta_K - \delta_S = w \cdot (2\rho - \sigma). \tag{a}$$

Define subsets

$$A = \{\beta \in \Delta^+ : w\beta \in \Delta^+ - S\}, \quad B = \{\beta \in \Delta^+ : -w\beta \in K\},$$
$$C = \{\beta \in \Delta^+ : -w\beta \in \Delta^+ - S\}, \quad D = \{\beta \in \Delta^+ : w\beta \in K\}.$$

Note that $A \cap B = C \cap D = \emptyset$. Write $\rho - w^{-1}\rho = \sum_{\alpha\in\Sigma} m_\alpha \alpha$, m_α nonnegative integers. Then after applying w^{-1}, (a) may be written

$$\delta_A - \delta_C = w^{-1}\delta_{\Delta^+-S} = 2\rho + \sum_{\alpha\in\Sigma}(m_\alpha - 1)\alpha + w^{-1}\delta_K,$$

so

$$2\rho + \sum(m_\alpha - 1)\alpha = \delta_A + \delta_B - \delta_C - \delta_D. \tag{b}$$

It follows that $m_\alpha \leq 1$ for all $\alpha \in \Sigma$, so w is a product of mutually commuting reflections about roots in some subset $J \in \Sigma$ consisting of pairwise orthogonal roots. Moreover, (b) also shows that if $\alpha \in J$, then α cannot appear in any root in C or D. Hence $J \subseteq B$, so $J \subseteq K$. We may now rewrite (a) as

$$\delta_S = \delta_{\Sigma-K} + \sum_{\alpha\in J}(3 - \langle\sigma, \check{\alpha}\rangle)\alpha. \tag{c}$$

(As usual, $\check{\alpha} = 2\alpha/\langle\alpha,\alpha\rangle$.) I claim (c) can only hold if $J = \emptyset$. Suppose there is an $\alpha \in J$. Let $\gamma_1, \ldots, \gamma_r, \beta_1, \ldots, \beta_s$ be the roots in $\Sigma - \{\alpha\}$ which are not orthogonal to α, where $\gamma_i \in K$, $\beta_j \in \Sigma - K$ for all i, j. Also let $S - \{\alpha\} = \{\delta_1, \ldots, \delta_t\}$. By (c) we can write

$$\delta_k = n_k\alpha + \sum_{j=1}^{s} c_{kj}\beta_j + \mu_k,$$

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where $n_k \geq 0$, $\langle \mu_k, \check\alpha \rangle = 0$, and $\sum_{k=1}^t c_{kj} = 1$ for each j. Moreover, since $s_\alpha \delta_k \in \Delta^+$, we must have $n_k + \sum_j c_{kj} \langle \beta_j, \check\alpha \rangle \leq 0$. Summing these inequalities over k, we get

$$\text{(d)} \qquad \sum_{j=1}^{s} \langle \beta_j, \check\alpha \rangle + \sum_{k=1}^{t} n_k \leq 0.$$

Let $n_0 = 1$ if $\alpha \in S$, $n_0 = 0$ otherwise. Comparing the coefficient of α in both sides of (c), we find

$$n_0 + n_1 + \cdots + n_t = 3 - \langle \sigma, \check\alpha \rangle = 1 - \sum_j \langle \beta_j, \check\alpha \rangle - \sum_i \langle \gamma_i, \check\alpha \rangle.$$

Using (d) we get $1 \geq n_0 \geq 1 - \sum_i \langle \gamma_i, \check\alpha \rangle$. Since $\langle \gamma_i, \check\alpha \rangle < 0$ for all i, there can be no γ_i's, so $\alpha \in K_d$, and moreover $n_0 = 1$. On the other hand, S misses K_d, so $\alpha \notin S$, so $n_0 = 0$. This is the contradiction, so $J = \emptyset$, $w = 1$ and $\delta_S = \delta_{\Sigma - K}$.

LEMMA A. *For any subset $L \subseteq \Sigma$, we have*

$$P_L(u) := \sum_{\substack{S \subseteq \Delta^+ \\ \delta_S = \delta_L}} u^{|S|} = u^{c(L)} (1+u)^{|L| - c(L)}.$$

PROOF. If we partition $L = L_1 \cup \cdots \cup L_{c(L)}$ according to the connected components of its subgraph, we have $P_L = P_{L_1} \cdots P_{L_{c(L)}}$, so we may assume the subgraph of L is connected. If $L = \{\alpha_1, \ldots, \alpha_p\}$, every S in the sum is formed by making breaks in the chain $\alpha_1, \ldots, \alpha_p$. Hence the coefficient of u^m in $P_L(u)$ is the number of size $m-1$ subsets of the $p-1$ possible breaking points in the chain. It follows that $P_L(u) = u(1+u)^{p-1}$. ∎

Taking $L = K' := \Sigma - K$, we have shown that

$$(\chi_{2\rho-\sigma}, M^u_{\eta_K}) = (-u)^{c(K')} (1+u)^{|K_d|} (1-u)^{|K'| - c(K')}.$$

With (3.2) in mind, we again take $S \subseteq \Delta^+ - K_d$, and now suppose $w \in W$ is such that $\eta_K - \delta_S = w \cdot 0$. Hence $\rho - w\rho = \delta_K - \delta_{\Delta^+ - S} \leq \delta_{K_0}$. As before, this means w is a product of commuting reflections about a set J of mutually orthogonal roots in K_0, and therefore $\delta_{\Delta^+ - S} = \delta_{K-J}$. Setting $T = \Delta^+ - (S \cup K_d)$, (3.2) becomes

$$P(V_{2\rho-\sigma}, \Lambda\mathfrak{g}, u) = (1+u)^\ell \sum (-1)^{|J|} u^{\nu + c(K') - |T| - |K_d|} (1-u)^{|K_d|} (1+u)^{|K'| - c(K')},$$

where the sum runs over $K \subseteq \Sigma$, J an orthogonal subset of K_0, $T \subseteq \Delta^+$ such that $\delta_T = \delta_{K_0 - J}$. For fixed K and J, we can use Lemma A again, this time with $L = J' := K_0 - J$, note that $|K_d| + |J'| = |K| - |J|$, and get

$$\begin{aligned} &P(V_{2\rho-\sigma}, \Lambda\mathfrak{g}, u) \\ &\quad = (1+u)^\ell \sum_{K \subseteq \Sigma} u^{\nu + c(K') - |K|} (1-u)^{|K_d|} (1+u)^{|K'| - c(K')} \Big[\sum_J (-u)^{|J|} (1+u)^{|J'| - c(J')} \Big]. \end{aligned}$$

As in Lemma A, the inner sum is a product of similar sums for each component of K_0, so can be evaluated using

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LEMMA B.

$$\sum_J (-u)^{|J|}(1+u)^{|J'|-c(J')} = 1-u,$$

where the sum runs over those subsets $J \subseteq \Sigma$ consisting of pairwise orthogonal roots, $J' = \Sigma - J$, and $c(J')$ is the number of components of the Dynkin subdiagram with vertices in J'.

PROOF. List the simple roots $\Sigma = \{\alpha_1, \ldots, \alpha_\ell\}$ in such a way that the subdiagram with vertices $\Sigma_k := \{\alpha_1, \ldots, \alpha_k\}$ is connected, for each $k \leq \ell$. Since the assertion is easy to verify in small rank, we can assume the subdiagram with vertices $\{\alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell\}$ is of type A_3. Let R_ℓ be the polynomial on the left side of Lemma B, and write $R_\ell = R_\ell^0 + R_\ell^1$, where R_ℓ^1 is the sum over those J containing α_ℓ. Also let R_k^0, R_k^1 be the analogous polynomials for Σ_k. We will show by induction on ℓ that $R_\ell^0 = 1$ and $R_\ell^1 = -u$.

Fix $k < \ell$, and consider those J for which $\alpha_k \in J$, but $\alpha_i \notin J$ for any $i > k$. We have $c(\Sigma_k - J) = c(\Sigma - J) - 1$, so the sum of $(-u)^{|J|}(1+u)^{|J'|-c(J')}$ over such J is $(1+u)^{\ell-k-1}R_k^1$. It follows that $R_\ell^0 = (1+u)^{\ell-1} + (1+u)^{\ell-2}R_1^1 + \cdots + R_{\ell-1}^1 = 1$ by the inductive hypothesis applied to R_k^1. It remains to consider R_ℓ^1. Let $J = \{\alpha_{j_1}, \ldots, \alpha_{j_r}, \alpha_\ell\}$ be a typical subset occuring in the sum for R_ℓ^1, with $j_1 \leq \cdots \leq j_r \leq \ell - 2$. If $j_r < \ell - 2$, let $J_0 = \{\alpha_{j_1}, \ldots, \alpha_{j_r}, \alpha_{\ell-1}\}$, and if $j_r = \ell - 2$, let $J_1 = \{\alpha_{j_1}, \ldots, \alpha_{j_r}\}$. Then $c(\Sigma_{\ell-1} - J_0) = c(\Sigma_\ell - J)$ and $c(\Sigma_{\ell-2} - J_1) = c(\Sigma_\ell - J) - 1$. It follows that the sum over the J with $j_r < \ell - 2$ is $(1+u)R_{\ell-1}^1 = -u(1+u)$, and the sum over J with $j_r = \ell - 2$ is $-uR_{\ell-2}^1 = u^2$, so $R_\ell^1 = -u$. ∎

We now have

$$\begin{aligned} P(V_{2\rho-\sigma}, \Lambda\mathfrak{g}, u) &= (1+u)^\ell \sum_{K \subseteq \Sigma} u^{\nu + c(K') - |K|}(1-u)^{|K_d| + c(K_0)}(1+u)^{|K'|-c(K')} \\ &= u^{\nu-\ell}(1+u)^\ell \sum_{K \subseteq \Sigma} u^{|K'| + c(K')}(1-u)^{c(K)}(1+u)^{|K'|-c(K')}. \end{aligned}$$

It remains only to show that the last sum, call it Q_Σ, is $(1+u^2)^{\ell-1}(1-u+u^2)$. Again, this is easy to check in small rank. Label Σ as in the proof of Lemma B, and for $K = \{\alpha_{i_1}, \ldots, \alpha_{i_p}\}$, let $\hat{K} = K - \{\alpha_\ell\}$ if $i_p = \ell$, $\hat{K} = K$ otherwise. We now express $c(\Sigma_{\ell-1} - \hat{K})$ in terms of $c(\Sigma - I)$, and likewise for $c(\hat{K})$ (there are four cases, for each possibility of $\{i_{p-1}, i_p\} \cap \{\ell - 1, \ell\}$), and find that $Q_\Sigma = (1+u^2)Q_{\Sigma_{\ell-1}}$. This completes the Proof of 6.3. ∎

7. **Graded multiplicities for small modules.** From now on we assume $\mathfrak{g}$ to be simple, and consider $\mathfrak{g}$-modules V_λ which are small, in the sense of (2.2). We have seen in (4.2) that the total multiplicity of V_λ in $\Lambda\mathfrak{g}$ is $m_\lambda^0 2^\ell$, and we now seek the multiplicity of V_λ in $\Lambda^n\mathfrak{g}$ for each n. We shall state a conjecture for this, followed by a collection of evidence in favor of it.

To introduce our conjectural formula for the small multiplicity polynomials, recall that $\mathcal{H}$ is the space of W-harmonic polynomials on $\mathfrak{t}$, the multiplicity polynomial of a W-module E is

$$P_W(E, \mathcal{H}, u) = \sum_{n=0}^{\nu} \dim \mathrm{Hom}_W(E, \mathcal{H}^n)u^n,$$

and for $E = \Lambda^q \mathfrak{t}$, we have Solomon's formula

$$P_W(\Lambda^q \mathfrak{t}, \mathcal{H}, u) = s_q(u^{m_1}, \ldots, u^{m_\ell}),$$

where $m_1 \leq \cdots \leq m_\ell$ are the exponents of W.

7.1. CONJECTURE. If V_λ is small, as in (2.2), then

$$P(V_\lambda, \Lambda\mathfrak{g}, u) = \sum_{q=0}^{\ell} u^q P_W(V_\lambda^0 \otimes \Lambda^q \mathfrak{t}, \mathcal{H}, u^2).$$

When $\lambda = 0$, both sides of (7.1) reduce to the known Poincaré polynomial of the manifold G, by Cartan's theory of invariant differential forms on the left and by Solomon's formula on the right [R1].

Let us abbreviate $\langle\,,\rangle_W := \dim \mathrm{Hom}_W(\,,\,)$. The polynomial on the right side of (7.1) begins as

$$\begin{aligned}\sum_{q=0}^{\ell} u^q P_W(V_\lambda^0 \otimes \Lambda^q \mathfrak{t}, \mathcal{H}, u^2) = {} & \langle V_\lambda^0, \mathbb{C}\rangle_W + u\langle V_\lambda^0, \mathfrak{t}\rangle_W + u^2\langle V_\lambda^0, \mathfrak{t} \oplus \Lambda^2\mathfrak{t}\rangle_W \\ & + u^3\langle V_\lambda^0, \mathbb{C} \oplus \Lambda^2\mathfrak{t} \oplus \Lambda^3\mathfrak{t} \oplus \mathcal{H}^2\rangle_W \\ & + u^4\langle V_\lambda^0, \mathfrak{t} \otimes \Lambda^2\mathfrak{t} \oplus \Lambda^4\mathfrak{t} \oplus \mathcal{H}^2\rangle_W + \cdots\end{aligned} \tag{7.2}$$

Even in degree zero, (7.1) is not obvious. However, it follows from Broer's result (2.3), or a direct proof using (4.3), that (7.1) is true in degrees zero and one. Both sides of (7.1) have the same palindromy by Poincaré duality on the left hand and because $u^\nu P_W(E, \mathcal{H}, u^{-1}) = P_W(\epsilon \otimes E, \mathcal{H}, u)$ on the right. At $u = 1$, both sides of (7.1) become $m_\lambda^0 2^\ell$ by (4.2) on the left, and on the right because $\mathcal{H}$ affords the regular representation of W.

Take V_λ to be the adjoint representation, which is small, and suppose $\mathfrak{g} \neq \mathfrak{sl}(n)$. (The case $\mathfrak{g} = \mathfrak{sl}(n)$ will be considered in more detail shortly.) Let us check (7.1) in low degrees. By (7.2), the right hand polynomial in (7.1) begins as

$$\sum_{q=0}^{\ell} u^q P_W(\mathfrak{t} \otimes \Lambda^q \mathfrak{t}, \mathcal{H}, u^2) = u + u^2 + u^4 + cu^5 + (\text{higher powers}),$$

where $c = \dim \mathrm{End}_W(\mathcal{H}^2)$. This follows from the fact that no exponent equals two for $\mathfrak{g} \neq \mathfrak{sl}(n)$. We note that $c = 1$ if and only if $\mathfrak{g}$ is exceptional ($\mathfrak{sl}(3)$ is excluded!). On the other hand, we can show

7.3. LEMMA. *If $\mathfrak{g}$ is not $\mathfrak{sl}(n)$, then*

$$P(\mathfrak{g}, \Lambda\mathfrak{g}, u) = u + u^2 + c_4u^4 + c_5u^5 + (\text{higher powers}),$$

with $1 \leq c_4 \leq c_5$.

PROOF. By (5.2) the adjoint representation appears in $\Lambda^3\mathfrak{g}$ as it appears in $\mathcal{H}^2\mathfrak{g}$, where its multiplicity is the number of exponents equal to two. But no exponent is two for $\mathfrak{g} \neq \mathfrak{sl}(n)$.

Now, given any $\mu \in \Lambda\mathfrak{g}$ which is not in the top degree, we have $\mu\wedge\mathfrak{g} \neq 0$, from which it follows that $\dim \operatorname{Hom}_G(\mathfrak{g}, \Lambda^n\mathfrak{g}) \geq \dim(\Lambda^{n-1}\mathfrak{g})^G$ for every $n < \dim\mathfrak{g}$. In particular, for $\mathfrak{g} \neq \mathfrak{sl}(2)$, we have $\mathfrak{g} \simeq \omega \wedge \mathfrak{g} \subset \Lambda^4\mathfrak{g}$, where $\omega \in (\Lambda^3\mathfrak{g})^G$ is the invariant form $\langle [X, Y], Z\rangle$ on $\mathfrak{g}$. Since the invariant forms represent all the cohomology of $\mathfrak{g}$ [CE, 19.1], the differential d is exact on nontrivial isotypic components. Since $\mathfrak{g}$ does not appear in $\Lambda^3\mathfrak{g}$, it follows that d does not kill any copy of $\mathfrak{g}$ in $\Lambda^4\mathfrak{g}$. ∎

From (7.3), (4.2) and Poincaré duality, we find

$$P\big(\mathfrak{sp}(4), \Lambda\mathfrak{so}(5), u\big) = u(1+u)(1+u^3)(1+u^4),$$
$$P(\mathfrak{g}_2, \Lambda\mathfrak{g}_2, u) = u(1+u)(1+u^3)(1+u^8),$$

which agree with (7.1). All multiplicity polynomials for $Sp(4)$, G_2, along with $Sp(6)$ are given in Section 8, from which one can also verify (7.1) for the remaining small modules afforded by these groups.

For additional examples, take G of type D_n or E_n, and $V_\lambda = U_2 = \Lambda^2\mathfrak{g}/\mathfrak{g}$ (see (5.1)). These are exactly the groups for which U_2 is irreducible and small. Here the left side of (7.1) begins as $u^2 + u^3 + \cdots$, by (5.1) and (5.4). Using (5.1), it is not hard to determine the W-action on the zero weight space U_2^0, and see that the right side of (7.1) begins the same way. Given what we already know, the validity of (7.1) in degree two is equivalent to U_2 being the only small module containing $\Lambda^2\mathfrak{t}$ in its zero weight space.

For the remainder of this section, we consider (7.1) for $\mathfrak{g} = \mathfrak{sl}(n)$. Actually, it is more convenient to consider $\mathfrak{gl}(n)$, whose multiplicity polynomials are those for $\mathfrak{sl}(n)$ multiplied by $(1+u)$. However, "zero weight space" still refers to the invariants under a maximal torus of $SL(n)$.

We identify highest weights with partitions, for which unexplained notation follows [M2]. Let $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n]$ be a partition of n. Some of the λ_i's may be zero. The irreducible $\mathfrak{gl}(n)$-module V_λ is small, and Stembridge has found the following explicit formula for $P(V_\lambda, \mathfrak{gl}(n), u)$. The boxes in the Young diagram of λ are left-justified, with λ_i boxes in the i-th row from the top, in which the j-th box from the left is labelled (i, j). The hook length of box (i, j) is the number $h(i, j)$ of boxes directly to the right or directly below (i, j), including (i, j) itself.

7.4. THEOREM (STEMBRIDGE, [ST]). *For $\mathfrak{g} = \mathfrak{gl}(n)$, and λ a partition of n, we have*

$$P(V_\lambda, \Lambda\mathfrak{g}, u) = (1-u^2)(1-u^4)\cdots(1-u^{2n}) \prod_{(i,j)\in\lambda} \frac{u^{2j-2} + u^{2i-1}}{1-u^{2h(i,j)}}.$$

We turn now to the right side of (7.1). Let χ_λ be the irreducible representation of S_n corresponding to the partition λ. For example, $\chi_{n-k,1^k}$ is the k-th exterior power of the reflection representation $\chi_{n-1,1}$. In general, χ_λ appears with multiplicity one in the induced representation $\psi_\lambda := \operatorname{Ind}_{S_\lambda}^{S_n} \mathbb{C}$, where $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n}$. It is known that

$V_\lambda^0 \simeq \chi_{\lambda'}$, where λ' is the partition dual to λ. We first compute the right side of (7.1) with V_λ^0 replaced by ψ_λ. For any S_n-module E, we abbreviate

$$P_n(E,u) := P_{S_n}(E, H, u),$$
$$R(E,u) := \sum_{q=0}^{n} u^q P_n(E \otimes \Lambda_n^q, u^2).$$

Here $\Lambda_n^q := \Lambda^q \mathfrak{t}$, where $\mathfrak{t}$ is a Cartan subalgebra of $ll\mathfrak{g}(n)$.

7.5. PROPOSITION. *For any partition λ of n, we have*

$$R(\psi_\lambda, u) = (1-u^2)(1-u^4)\cdots(1-u^{2n}) \prod_{i=1}^{n} \left(\frac{1+u^{2i-1}}{1-u^{2i}} \right)^{\lambda'_i},$$

where λ' is the partition dual to λ.

PROOF. The restriction of Λ_n^q to S_λ is given by

$$\Lambda_n^q|_{S_\lambda} = \bigoplus_{\substack{\bar p \\ |\bar p|=q}} \Lambda_{\lambda_1}^{p_1} \otimes \cdots \otimes \Lambda_{\lambda_n}^{p_n},$$

where each $\bar p = (p_1, \cdots, p_n)$ is an ordered n-tuple of non-negative integers and $|\bar p| := p_1 + \cdots + p_n$. It follows that

$$\psi_\lambda \otimes \Lambda_n^q = \bigoplus_{\substack{\bar p \\ |\bar p|=q}} \operatorname{Ind}_{S_\lambda}^{S_n} (\Lambda_{\lambda_1}^{p_1} \otimes \cdots \otimes \Lambda_{\lambda_n}^{p_n}).$$

Let

$$P^\lambda(u) := (1-u)(1-u^2)\cdots(1-u^n) \prod_{i=1}^{n} (1-u^i)^{-\lambda'_i}$$

denote the Poincaré polynomial of S_n divided by that of S_λ. By Frobenius reciprocity, we have

$$P_n(\psi_\lambda \otimes \Lambda_n^q, u) = P^\lambda(u) \sum_{|\bar p|=q} \prod_{i=1}^{n} P_{\lambda_i}(\Lambda_{\lambda_i}^{p_i}, u).$$

The exponents of S_{λ_i} acting on $\mathbb{C}^{\lambda_i}$ are $0, 1, 2, \ldots, \lambda_i - 1$, so Solomon's formula gives

$$P_{\lambda_i}(\Lambda_{\lambda_i}^{p_i}, u) = s_{p_i}(1, u, \ldots, u^{\lambda_i - 1}).$$

Thus

$$\begin{aligned}
R(\psi_\lambda, u) &= P^\lambda(u^2) \sum_{\bar p} u^{|\bar p|} \prod_{i=1}^{n} s_{p_i}(1, u^2, \ldots, u^{2\lambda_i - 2}) \\
&= P^\lambda(u^2) \prod_{i=1}^{n} \sum_{p \geq 0} u^p s_p(1, u^2, \ldots, u^{2\lambda_i - 2}) \\
&= P^\lambda(u^2) \prod_{i=1}^{n} \prod_{j=1}^{\lambda_i} (1 + u^{2j-1}) \\
&= P^\lambda(u^2) \prod_{i=1}^{n} (1 + u^{2i-1})^{\lambda'_i}. \qquad \blacksquare
\end{aligned}$$

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We can write

$$\chi_\lambda = \sum_{\mu \geq \lambda} L_{\lambda\mu} \psi_\mu,$$

where the matrix $[L_{\lambda\mu}]$ is inverse to the Kostka matrix $[K_{\lambda\mu}]$ (see [M2, I.6]). Thus, conjecture (7.1) is equivalent to a combinatorial identity involving Kostka numbers.

For example, it is now easy to verify conjecture (7.1) for highest weights corresponding to partitions of the form $\lambda = 2^k 1^{n-2k}$. Indeed, we then have $V_\lambda^0 \simeq \chi_{[n-k,k]}$, and the decomposition formulas for ψ_μ's imply that

$$\chi_{[n-k,k]} = \psi_{[n-k,k]} - \psi_{[n-k+1,k-1]},$$

so $R(V_\lambda^0, u)$ can be computed from (7.5), and it agrees with Stembridge's formula (7.4) for $P(V_\lambda, \Lambda\mathfrak{g}, u)$.

8. **Tables.** The above results and remarks, plus a bit more (see below), suffice to determine the complete decomposition of $\Lambda\mathfrak{g}$ for some small groups. With Dynkin diagrams labelled (as in §2)

$$1–2, \quad 1–2–3, \quad 1 \Leftarrow 2, \quad 1–2 \Leftarrow 3 \quad 1 \Lleftarrow 2,$$

we abbreviate $P(V_{a_1\lambda_1+a_2\lambda_2+\cdots}, \Lambda\mathfrak{g}, u)$ by $P(a_1, a_2, \cdots)_d$, where d is the dimension of the irreducible module $V_{a_1\lambda_1+a_2\lambda_2+\cdots}$.

$$A_2$$

$$P(2,2)_{27} = u^3(1+u)^2$$
$$P(0,3)_{10} = P(3,0)_{10} = u^2(1+u)(1+u^3)$$
$$P(1,1)_8 = u(1+u)(1+u^2)(1+u^3)$$
$$P(0,0)_1 = (1+u^3)(1+u^5)$$

$$A_3$$

$$P(2,2,2)_{729} = u^6(1+u)^3$$
$$P(3,0,3)_{300} = P(0,3,2)_{280} = P(2,3,0)_{280} = u^5(1+u)^2(1+u^3)$$
$$P(0,4,0)_{105} = u^4(1+u)(1+u^3)^2$$
$$P(1,1,3)_{256} = P(3,1,1)_{256} = u^4(1+u)^2(1+u^2)(1+u^3)$$
$$P(1,2,1)_{175} = u^3(1+u)^2(1+u^2)^2(1+u^3)$$
$$P(2,0,2)_{84} = u^3(1+u)^2(1+u^3)(1+u^2+u^4)$$
$$P(4,0,0)_{35} = P(0,0,4)_{35} = u^3(1+u)(1+u^3)(1+u^5)$$
$$P(0,2,0)_{20} = u^3(1+u)^2(1+u^3)(1+u^4)$$
$$P(2,1,0)_{45} = P(0,1,2)_{45} = u^2(1+u)(1+u^3)^2(1+u^2+u^4)$$
$$P(1,0,1)_{15} = u(1+u)(1+u^3)(1+u^5)(1+u^2+u^4)$$
$$P(0,0,0)_1 = (1+u^3)(1+u^5)(1+u^7)$$

C_2

$$P(2,2)_{81} = u^4(1+u)^2$$
$$P(0,1)_5 = P(0,2)_{14} = P(0,3)_{30} = P(4,0)_{35} = u^3(1+u)(1+u^3)$$
$$P(2,1)_{35} = u^2(1+u)(1+u^2)(1+u^3)$$
$$P(2,0)_{10} = u(1+u)(1+u^3)(1+u^4)$$
$$P(0,0)_1 = (1+u^3)(1+u^7)$$

C_3

$$P(2,2,2)_{19683} = u^9(1+u)^3$$
$$P(5,0,1)_{2079} = P(0,3,2)_{7700} = P(4,0,2)_{4914} = P(3,0,3)_{8190}$$
$$= P(2,4,0)_{9450} = u^8(1+u)^2(1+u^3)$$
$$P(3,2,1)_{11319} = P(1,1,3)_{7168} = u^7(1+u)^2(1+u^2)(1+u^3)$$
$$P(0,5,0)_{3528} = u^7(1+u)(1+u^3)^2$$
$$P(4,2,0)_{3900} = u^6(1+u)^2(1+u^4)(1+u^3)$$
$$P(2,1,2)_{5720} = u^6(1+u)^2(1+u^3)(1+u+u^2+u^3+u^4)$$
$$P(2,0,2)_{1078} = u^6(1+u)^2(1+u^3)(2+u^2+2u^4)$$
$$P(1,3,1)_{7168} = u^6(1+u)^2(1+u^2)^2(1+u^3)$$
$$P(1,0,3)_{1386} = u^6(1+u)(1+u+u^2)(1+u^3)^2$$
$$P(0,4,0)_{1274} = u^6(1+u)(1+u^3)(1+2u+2u^4+u^5)$$
$$P(0,0,4)_{1001} = u^6(1+u)(1+u^3)(1+u^5)$$
$$P(6,0,0)_{462} = u^5(1+u)(1+u^3)(1+u^7)$$
$$P(3,1,1)_{3072} = u^5(1+u)(1+u^2)(1+u+u^2)(1+u^3)^2$$
$$P(2,3,0)_{3276} = u^5(1+u)(1+u^3)^2(1+u+u^2+u^3+u^4)$$
$$P(0,2,2)_{2457} = P(0,1,2)_{594} = u^5(1+u)(1+u^3)(1+u^2+2u^3+2u^4+u^5+u^7)$$
$$P(0,0,2)_{84} = u^5(1+u)(1+u^3)^2(1+u^4)$$
$$P(4,1,0)_{924} = u^4(1+u)(1+u^2)(1+u^3)^2(1+u^4)$$
$$P(2,2,0)_{924} = u^4(1+u)^3(1+u^3)^2(1+u^4)$$
$$P(1,2,1)_{2205} = u^4(1+u+u^2)^2(1+u^3)^3$$
$$P(1,1,1)_{512} = u^4(1+u)^2(1+u^2)^2(1+u^3)(1+u^4)$$
$$P(1,0,1)_{70} = u^4(1+u)(1+u^3)(1+u+u^2+u^4+u^5+u^7+u^8+u^9)$$
$$P(4,0,0)_{126} = u^3(1+u)(1+u^3)^2(1+u^4+u^8)$$
$$P(3,0,1)_{525} = u^3(1+u+u^2)(1+u^3)^3(1+u^4)$$
$$P(0,3,0)_{385} = u^3(1+u)(1+u^3)(1+u^3+2u^4+u^5+u^6+2u^7+u^8+u^{11})$$
$$P(0,2,0)_{90} = u^3(1+u)(1+u^3)^2(1+u^4)^2$$
$$P(0,1,0)_{14} = u^3(1+u)(1+u^3)(1+u^4)(1+u^7)$$

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$$P(2,1,0)_{189} = u^2(1+u)(1+u^3)(1+u^2+u^3+u^4+u^5+2u^6+2u^7+u^8+u^9+u^{10}+u^{11}+u^{13})$$
$$P(2,0,0)_{21} = u(1+u)(1+u^3)(1+u^7)(1+u^4+u^8)$$
$$P(0,0,0)_1 = (1+u^3)(1+u^7)(1+u^{11})$$

G_2

$$P(2,2)_{729} = u^6(1+u)^2$$
$$P(1,0)_7 = P(1,2)_{286} = P(0,3)_{273} = P(5,0)_{378} = u^5(1+u)(1+u^3)$$
$$P(1,1)_{64} = P(3,1)_{448} = u^4(1+u)(1+u^2)(1+u^3)$$
$$P(2,1)_{189} = u^4(1+u)(1+u^3)(1+u+u^2)$$
$$P(4,0)_{182} = P(0,2)_{77} = P(2,0)_{27} = u^3(1+u)(1+u^3)(1+u^4)$$
$$P(3,0)_{77} = u^2(1+u)(1+u^3)(1+u^3+u^6)$$
$$P(0,1)_{14} = u(1+u)(1+u^3)(1+u^8)$$
$$P(0,0)_1 = (1+u^3)(1+u^{11})$$

Remarks on these computations will summarize the results in this article. For A_3, the highest weights below $2\alpha_0 = 202$ are covered by Stembridge's first layer formulas. Those above $2\alpha_0$ are handled by (6.2) and (6.3), so what remain are the multiplicities of $V_{2\alpha_0}$, which are obtained by default. For G_2 we have the low degree calculations of (5.1) and (5.2), the ungraded multiplicity formula (4.1) to get upper bounds on multiplicities in each degree, and lower bounds come from multiplicities of dominant weights in $\Lambda^n\mathfrak{g}$ which are maximal among those whose multiplicities are not yet determined. As already mentioned, (7.3) gives the adjoint multiplicities. Finally, more occurrences follow from the exactness of the Koszul complex $\cdots\Lambda^n\mathfrak{g}\longrightarrow\Lambda^{n+1}\mathfrak{g}\cdots$ on nontrivial isotypic components. The table for C_3 was kindly produced by the referee, using the computer program LiE (from CWI, Amsterdam). There was an obvious small error in the adjoint polynomial, which I believe is corrected here. Of the 35 polynomials in the C_3 Table, 13 are explained by (6.2) and (6.3), another four are as predicted by conjecture (7.1), and we have checked that the rest give the correct dimensions and ungraded multiplicities, using (4.1).

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