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SOME NEW CHARACTERISATIONS OF FINITE p-SUPERSOLUBLE GROUPS

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Abstract

Let G be a finite group. A subgroup H of G is said to be E-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{eG}$, where H_{eG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormally embedded in G. In this paper, some new characterisations of p-supersolubility of finite groups are given under the assumption that some primary subgroups are E-supplemented.

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1. Introduction

All groups considered in this paper are finite. Most of the notation is standard and can be found in [4, 9]. By G we always mean a group; $|G|$ is the order of G, $O_p(G)$ is the maximal normal p-subgroup of G, $\Phi(G)$ is the Frattini subgroup of G and $F_p(G)$ is the p-Fitting subgroup of G, that is, $F_p(G) = O_{p'p}(G)$.

Recall that a subgroup H of a group G is said to be S-quasinormal (or S-permutable) (see [2]) in G if H permutes with every Sylow subgroup of G. Recently, many new generalised S-quasinormal subgroups were introduced. For example, Ballester-Bolinches and Pedraza-Aguilera called H S-quasinormally embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-quasinormal subgroup of G (see [1]). In 2007, Skiba [11] gave the concept of S-supplemented (or weakly S-supplemented) subgroups. A subgroup H of G is said to be S-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormal in G. To generalise and unify the above-mentioned subgroups, the first author introduced the following embedding property of subgroups in [6].

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Definition 1.1. A subgroup H of G is said to be E-supplemented in G if there is a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{eG}$, where H_{eG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormally embedded in G.

In [6], the first author strengthened a nice result of Skiba which gives a positive answer to an open question of Shemetkov. Now, we will continue to study the influence of E-supplemented subgroups on the structure of finite groups. A group G is called p-supersoluble if it is p-soluble and all its G-chief p-factors are cyclic. A group G is called p-nilpotent if it is p-soluble and all its G-chief p-factors are central in G. Obviously, a p-nilpotent group is also a p-supersoluble group. In this paper, we present some sufficient conditions for a group to be p-supersoluble under the assumption that some special subgroups are E-supplemented.

2. Preliminaries

Lemma 2.1 [6, Lemma 2.3]. *Let H be an E-supplemented subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is E-supplemented in L.*
(2) *If $N \lhd G$ and $N \leq H \leq G$, then H/N is E-supplemented in G/N.*
(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is E-supplemented in G/N.*

Lemma 2.2 [8, Lemma 2.4]. *Suppose that P is a p-subgroup of a group G contained in $O_p(G)$. If P is S-quasinormally embedded in G, then P is S-quasinormal in G.*

Lemma 2.3. *Suppose that P is a p-subgroup of a group G contained in $O_p(G)$. If P is E-supplemented in G, then P is S-supplemented in G.*

Proof. Suppose that there is a subgroup T of G such that $G = PT$ and $P \cap T \leq P_{eG}$. If $P_{eG} = 1$, then P_{eG} is obviously S-supplemented in G. We now assume that $P_{eG} \neq 1$. Let $U_1, U_2, \ldots, U_s$ be all the nontrivial subgroups of P which are S-quasinormally embedded in G. Since U_i satisfies $U_i \leq P \leq O_p(G)$, we have that U_i is S-quasinormal in G by Lemma 2.2. Hence $P_{eG} = \langle U_1, U_2, \ldots, U_s \rangle$ is S-quasinormal in G. It follows that $P \cap T \leq P_{sG}$. □

Lemma 2.4 [10, Lemma A]. *If P is an S-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.5 [12, Lemma 2.8]. *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Lemma 2.6 [13, Lemma 2.1]. *Let G be a group and p a prime dividing $|G|$ with* $(|G|, p - 1) = 1$.

(1) *If $M \leq G$ and $|G : M| = p$, then $M \trianglelefteq G$.*
(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*
(3) *If G is p-supersoluble, then G is p-nilpotent.*

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Using similar arguments as in the proofs of [6, Theorems 3.2 and 3.3] and Lemma 2.6, we have following lemma.

LEMMA 2.7. *Let p be a prime dividing the order of a group G, $(|G|, p-1) = 1$ and P a Sylow p-subgroup of G. Suppose that one of the following conditions is satisfied.*

(1) *Every maximal subgroup of P not having a p-nilpotent supplement in G is E-supplemented in G.*

(2) *Every cyclic subgroup of P with prime order or order four not having a p-nilpotent supplement in G is E-supplemented in G.*

Then G is p-nilpotent.

LEMMA 2.8 [7, Lemma 2.6]. *Let H be a soluble normal subgroup of a group G $(H \neq 1)$. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

3. Main results

THEOREM 3.1. *Let p be a prime dividing the order of a group G and H a p-soluble normal subgroup of G such that G/H is p-supersoluble. Suppose that every maximal subgroup of $F_p(H)$ containing $O_{p'}(H)$ is E-supplemented in G. Then G is p-supersoluble.*

PROOF. Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) $O_{p'}(H) = 1$.

Assume that $O_{p'}(H) \neq 1$. We consider the factor group $G/O_{p'}(H)$. First,

$$(G/O_{p'}(H))/(H/O_{p'}(H)) \cong G/H$$

is p-supersoluble. Now $O_{p'}(H/O_{p'}(H)) = 1$ and

$$F_p(H/O_{p'}(H)) = F_p(H)/O_{p'}(H).$$

Let $M/O_{p'}(H)$ be a maximal subgroup of $F_p(H/O_{p'}(H))$. Then M is a maximal subgroup of $F_p(H)$ containing $O_{p'}(H)$. Since M is E-supplemented in G, we have $M/O_{p'}(H)$ is E-supplemented in $G/O_{p'}(H)$ by Lemma 2.1(2). Thus $G/O_{p'}(H)$ satisfies the hypotheses of the theorem. The minimal choice of G implies that $G/O_{p'}(H)$ is p-supersoluble, and so is G, which is a contradiction.

(2) $H \cap \Phi(G) = 1$.

Write $R = H \cap \Phi(G)$. Assume $R \neq 1$ and consider the factor G/R. By [5, III, 3.5], we have $F(H/R) = F(H)/R$, and so $F(H/R) = F_p(H)/R = O_p(H)/R$ by step (1). On the other hand, writing $K/R = O_{p'}(H/R)$ and letting S be a Hall p'-subgroup of K we have $K = SR$, and by the Frattini argument $G = KN_G(S) = RN_G(S) = N_G(S)$ and $S \lhd G$. Therefore $S = 1$ and $O_{p'}(H/R) = 1$. This shows that $F_p(H/R) = O_p(H/R) = O_p(H)/R = F_p(H)/R$. If P_1/R is a maximal subgroup of $F_p(H/R)$, then P_1 is maximal

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in $F_p(H)$. By the hypothesis of the theorem, P_1 is an E-supplemented subgroup of G. Hence P_1/R is E-supplemented in G/R by Lemma 2.1(2). Now the minimal choice of G implies that G is p-supersoluble, and then so is G, which is a contradiction.

(3) Every minimal normal subgroup of G contained in $F(H)$ is cyclic of order p.

Since H is p-soluble and $O_{p'}(H) = 1$, we have $C_H(O_p(H)) \le O_p(H)$ by [3, Theorem 6.3.2]. Now $\Phi(H) = 1$ implies that $F(H) = O_p(H)$ is a nontrivial elementary abelian p-group by [5, III, 4.5]. Thus $C_H(F(H)) = F(H)$. Write $O_p(H) = P$ and take a minimal normal subgroup N of G contained in P. Since $N \nsubseteq \Phi(G)$ by step (2), there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Let M_p be a Sylow p-subgroup of M and $G_p = M_pN$. Then G_p is a Sylow p-subgroup of G. Let G_1 be a maximal subgroup of G_p containing M_p and $P_1 = G_1 \cap P$. Then

$$|P : P_1| = |P : G_1 \cap P| = |PG_1 : G_1| = |G_p : G_1| = p$$

and so P_1 is a maximal subgroup of P. We also have that

$$P_1M_p = (G_1 \cap P)M_p = G_1 \cap PM_p = G_1 \cap G_p = G_1$$

and $P_1 \cap M_p = P \cap G_1 \cap M = P \cap M_p$. By the hypothesis, P_1 is E-supplemented in G. Hence there exists a subgroup T of G such that $G = P_1T$ and $P_1 \cap T \le (P_1)_{eG}$. Since $(P_1)_{eG} \le O_p(E) \le O_p(G)$, $(P_1)_{eG}$ is S-quasinormal in G by Lemma 2.2. In view of Lemma 2.4, $O^p(G) \le N_G((P_1)_{eG})$. On the other hand, for any $x \in G_p$ we have $((P_1)_{eG})^x \le P_1^x = P_1 \le G_p$. Moreover $((P_1)_{eG})^x$ is E-supplemented in G since $(P_1)_{eG}$ is E-supplemented in G. Hence $((P_1)_{eG})^x = (P_1)_{eG}$, so $(P_1)_{eG}$ is normal in G. It follows that $(P_1)_{eG} = (P_1)_G$. Let $K = (P_1)_GT$. Then $G = P_1T = P_1K$ and $P_1 \cap K = P_1 \cap (P_1)_GT = (P_1)_G(P_1 \cap T) = (P_1)_G$. Since P_1 is a maximal subgroup of P, $P_1(P \cap M) = P$ or $P_1(P \cap M) = P_1$. If the former holds, then $G = PM = P_1(P \cap M)M = P_1M$ and so

$$P = P \cap P_1M = P_1(P \cap M) = P_1(P \cap G_1 \cap M) = P_1(P_1 \cap M) = P_1,$$

which is a contradiction. Hence $P_1(P \cap M) = P_1$, and so $P \cap M \le P_1$. Since $P \cap M \lhd G$ by Lemma 2.5, $P \cap M \le (P_1)_G = P_1 \cap K$. Assume that $K < G$. Let K_1 be a maximal subgroup of G containing K. Then $P \cap K_1 \lhd G$ by Lemma 2.5. Hence $(P \cap K_1)M$ is a subgroup of G. Since M is a maximal subgroup of G, $(P \cap K_1)M = G$ or $(P \cap K_1)M = M$. If $(P \cap K_1)M = G = PM$, then $P = P \cap (P \cap K_1)M = (P \cap K_1)(P \cap M) = P \cap K_1$ since $P \cap M \le (P_1)_G = P_1 \cap K \le P \cap K_1$. It follows that $P \le K_1$ and hence $G = PK \le PK_1 = K_1$, which is a contradiction. If $(P \cap K_1)M = M$, then $P \cap K_1 \le M$ and so

$$P_1 \cap K \le P \cap K \le P \cap K_1 = P \cap K_1 \cap M \le P \cap M \le P_1 \cap K.$$

Hence $P_1 \cap K = P \cap K$. Since $G = PK = P_1K$,

$$|G : P| = |PK : P| = |K : (P \cap K)| = |K : (P_1 \cap K)| = |P_1K : P_1| = |G : P_1|,$$

which is impossible. Thus $K = G$. It follows that $P_1 \cap K = P_1 = (P_1)_G \lhd G$. Consequently, $P_1 \cap N \lhd G$. But since $G_p = NM_p = NG_1$ and G_1 is a maximal subgroup of G_p containing M_p, we have $N \nleq P_1 = G_1 \cap P$. The minimal normality

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of N implies that $P_1 \cap N = 1$. Hence

$$|N| = |N : (P_1 \cap N)| = |NP_1 : P_1| = |N(P \cap G_1) : P_1|$$
$$= |(P \cap NG_1) : P_1| = |P \cap G_p : P_1| = |P : P_1| = p.$$

(4) The final contradiction.

By Lemma 2.8 and step (3), we have $F(H) = O_p(H) = N_1 \times N_2 \times \cdots \times N_r$, where N_i is minimal normal in G of order p and $\mathrm{Aut}(N_i)$ is cyclic. Since for each i the quotient group $G/C_G(N_i)$ is a subgroup of $\mathrm{Aut}(N_i)$, $G/C_G(N_i)$ is abelian. Since G/H is p-supersoluble, it follows that $G/(H \cap C_G(N_i)) = G/C_H(N_i)$ is p-supersoluble. Therefore $G/\bigcap_{i=1}^{r} C_H(N_i)$ is p-supersoluble, and thus $G/F(H)$ is p-supersoluble because

$$\bigcap_{i=1}^{r} C_H(N_i) = C_H(F(H)) = F(H).$$

But all chief factors of G below $F(H)$ are cyclic of order p and hence G is p-supersoluble, which is a contradiction. □

Corollary 3.2. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a p-soluble normal subgroup of G such that G/H is p-supersoluble. Suppose that every maximal subgroup of $F_p(H)$ containing $O_{p'}(H)$ is E-supplemented in G. Then G is p-nilpotent.*

Proof. Applying Theorem 3.1, G is p-supersoluble. Since $(|G|, p-1) = 1$, we have G is p-nilpotent by virtue of Lemma 2.6(3). □

Corollary 3.3. *Let G be a p-soluble group and p the smallest prime divisor of $|G|$. If every maximal subgroup of $F_p(G)$ containing $O_{p'}(G)$ is E-supplemented in G, then G is p-nilpotent.*

Theorem 3.4. *Let p be a prime dividing the order of a group G and H a p-soluble normal subgroup of G such that G/H is p-supersoluble. Suppose that every maximal subgroup of any Sylow p-subgroup of $F_p(H)$ is E-supplemented in G. Then G is p-supersoluble.*

Proof. Let V be an arbitrary maximal subgroup of $F_p(H)$ containing $O_{p'}(H)$. Since $F_p(H)$ is p-nilpotent, there exists a maximal P_1 of some Sylow p-subgroup of $F_p(H)$ such that $V = P_1 O_{p'}(H)$. By the hypothesis, P_1 is E-supplemented in G. Hence there exists a subgroup T of G such that $G = P_1 T$ and $P_1 \cap T \le (P_1)_{eG}$. Since $(|G : T|, |O_{p'}(H)|) = 1$ and $O_{p'}(H)$ is normal in G, we have $O_{p'}(H) \le T$. Consequently, $G = VT$ and $V \cap T = P_1 O_{p'}(H) \cap T = (P_1 \cap T) O_{p'}(H) \le (P_1)_{eG} O_{p'}(H) \le (P_1 O_{p'}(H))_{eG} = V_{eG}$. Thus V is E-supplemented in G. Applying Theorem 3.1, G is p-supersoluble, which concludes the proof. □

Corollary 3.5. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a p-soluble normal subgroup of G such that G/H is p-supersoluble. Suppose*

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that every maximal subgroup of any Sylow p-subgroup of $F_p(H)$ is E-supplemented in G. Then G is p-nilpotent.

Corollary 3.6. *Let G be a p-soluble group and p the smallest prime divisor of $|G|$. If every maximal subgroup of any Sylow p-subgroup of $F_p(G)$ is E-supplemented in G, then G is p-nilpotent.*

Theorem 3.7. *Let H be a normal subgroup in G such that G/H is p-supersoluble and let P be a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|H|, p-1) = 1$. Suppose that one of the following conditions is satisfied.*

(1) *Every maximal subgroup of P not having a p-supersoluble supplement in G is E-supplemented in G.*

(2) *Every cyclic subgroup of P with prime order or order four not having a p-supersoluble supplement in G is E-supplemented in G.*

Then G is p-supersoluble. In particular, if $(|G|, p-1) = 1$, then G is p-nilpotent.

Proof. Suppose that the theorem is false and let (G, H) be a counterexample for which $|G||H|$ is minimal.

(1) H is p-nilpotent.

Suppose that P has a subgroup V which has a p-supersoluble supplement T in G. Then V has a p-supersoluble supplement $T \cap H$ in H. Since $(|T \cap H|, p-1) = 1$, $T \cap H$ is also p-nilpotent by Lemma 2.6(3). Hence either all maximal subgroups of P not having a p-nilpotent supplement in H or all cyclic subgroups of P with prime order or order four not having a p-nilpotent supplement in H are E-supplemented in H by Lemma 2.1(1). In view of Lemma 2.7, H is p-nilpotent.

(2) $P = H$.

From step (1), we have that $O_{p'}(H)$ is the normal Hall p'-subgroup of H. We assume that $O_{p'}(H) \neq 1$. It is easy to see that $O_{p'}(H)$ is normal in G, $PO_{p'}(H)/O_{p'}(H)$ is a Sylow p-subgroup of $H/O_{p'}(H)$, $(G/O_{p'}(H))/(H/O_{p'}(H)) \cong G/H$ is p-supersoluble and $(|H/O_{p'}(H)|, p-1) = 1$. Let $L/O_{p'}(H)$ be a subgroup of $PO_{p'}(H)/O_{p'}(H)$. Then there is some subgroup V of P such that $L = VO_{p'}(H)$. If V has a p-supersoluble supplement T in G, then $L/O_{p'}(H)$ has a p-supersoluble supplement $TO_{p'}(H)/O_{p'}(H)$ in $G/O_{p'}(H)$. Hence, either every maximal subgroup of $PO_{p'}(H)/O_{p'}(H)$ not having a p-supersoluble supplement in $G/O_{p'}(H)$ or every cyclic subgroup of $PO_{p'}(H)/O_{p'}(H)$ with prime order or order four not having a p-supersoluble supplement in $G/O_{p'}(H)$ is E-supplemented in $G/O_{p'}(H)$ from Lemma 2.1(3). Therefore the hypothesis of the theorem is still true for $(G/O_{p'}(H), H/O_{p'}(H))$. By the choice of (G, H), $G/O_{p'}(H)$ is p-supersoluble. Consequently, G is p-supersoluble, which is a contradiction. Hence $O_{p'}(H) = 1$, that is, $H = P$.

(3) Every G-chief factor of P is cyclic.

By virtue of Lemma 2.3, either all maximal subgroups of P not having a p-supersoluble supplement in G or all cyclic subgroups of P with prime order or order four not having a p-supersoluble supplement in G are S-supplemented in G. Applying [13, Main Theorem], every G-chief factor of P is cyclic.

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(4) The final contradiction.

Since G/P is p-supersoluble, in view of step (3) we have G is p-supersoluble, which is a contradiction. □

COROLLARY 3.8. *Let H be a normal subgroup in G such that G/H is p-supersoluble and let P be a Sylow p-subgroup of H, where p is the smallest prime divisor of $|H|$. Suppose that one of the following conditions is satisfied.*

(1) *Every maximal subgroup of P not having a p-supersoluble supplement in G is E-supplemented in G.*

(2) *Every cyclic subgroup of P with prime order or order four not having a p-supersoluble supplement in G is E-supplemented in G.*

Then G is p-supersoluble. In particular, if p is also the smallest prime divisor of $|G|$, then G is p-nilpotent.

COROLLARY 3.9. *Let H be a normal subgroup in G such that G/H is supersoluble. Suppose that for each $p \in \pi(H)$ one of the following conditions is satisfied.*

(1) *Every maximal subgroup of any noncyclic Sylow p-subgroup of H not having a p-supersoluble supplement in G is E-supplemented in G.*

(2) *Every cyclic subgroup of any noncyclic Sylow p-subgroup of H with prime order or order four not having a p-supersoluble supplement in G is E-supplemented in G.*

Then G is supersoluble.

PROOF. Take the smallest prime divisor p of the order of H and a Sylow p-subgroup P of H.

(1) H is p-nilpotent.

If P is cyclic, then Lemma 2.6(2) implies that H is p-nilpotent. But if P is not cyclic, then Corollary 3.8 implies that G is p-supersoluble, and so H is p-supersoluble. In view of Lemma 2.6(3), H is also p-nilpotent.

(2) Every G-chief factor below H is cyclic.

If $H = P$, then every G-chief factor below P is cyclic from [13, Corollary 1.2.2] and Lemma 2.3. If $H \neq P$, that is, $O_{p'}(H) \neq 1$, we get (2) by induction on $O_{p'}(H)$ and $G/O_{p'}(H)$.

(3) Since G/H is supersoluble, G is supersoluble. □

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