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AN ALGORITHM FOR CONSTRUCTING BIORTHOGONAL MULTIWAVELETS WITH HIGHER APPROXIMATION ORDERS

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Abstract

Given a pair of biorthogonal multiscaling functions, we present an algorithm for raising their approximation orders to any desired level. Precisely, let $\Phi(x)$ and $\tilde{\Phi}(x)$ be a pair of biorthogonal multiscaling functions of multiplicity r, with approximation orders m and $\tilde{m}$, respectively. Then for some integer s, we can construct a pair of new biorthogonal multiscaling functions $\Phi^{new}(x) = [\Phi^T(x), \phi_{r+1}(x), \phi_{r+2}(x), \ldots, \phi_{r+s}(x)]^T$ and $\tilde{\Phi}^{new}(x) = [\tilde{\Phi}(x)^T, \tilde{\phi}_{r+1}(x), \tilde{\phi}_{r+2}(x), \ldots, \tilde{\phi}_{r+s}(x)]^T$ with approximation orders n ($n > m$) and $\tilde{n}$ ($\tilde{n} > \tilde{m}$), respectively. In addition, corresponding to $\Phi^{new}(x)$ and $\tilde{\Phi}^{new}(x)$, a biorthogonal multiwavelet pair $\Psi^{new}(x)$ and $\tilde{\Psi}^{new}(x)$ is constructed explicitly. Finally, an example is given.

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1. Introduction

A refinable function vector of multiplicity r is a vector $\Phi(x) = [\phi_1(x), \ldots, \phi_r(x)]^T$, which satisfies a matrix refinement equation

$$\Phi(x) = \sum_k P_k \Phi(2x - k). \tag{1.1}$$

The sequence $\{P_k\}_{k\in Z}$ of coefficient matrices is called the two-scale matrix sequence of $\Phi(x)$. We assume that only finitely many P_k are nonzero and that all $\phi_j(x)$ have compact support.

We call $\Phi(x)$ a multiscaling function with multiplicity r if it generates a multiresolution analysis (MRA) of $L^2(R)$. This means that there exists a sequence of subspaces V_j, $j \in Z$, of $L^2(R)$ with the following properties:

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(1) $\cdots \subset V_0 \subset V_1 \subset V_2 \cdots$;
(2) $\mathrm{clos}_{L^2(R)}(\bigcup_{j\in Z} V_j) = L^2(R)$;
(3) $\bigcap_{j\in Z} V_j = \{0\}$;
(4) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1},\ j \in Z$;
(5) The family $\{\phi_\ell(x-k) : 1 \le \ell \le r,\ k \in Z\}$ forms a Riesz basis of V_0.

In detail, property (5) means that there exist two constants $0 < A \le B < \infty$ so that

$$A \sum_{j\in Z} \|C_j\|_2^2 \le \left\| \sum_{j\in Z} C_j^* \Phi(x-j) \right\|_2^2 \le B \sum_{j\in Z} \|C_j\|_2^2$$

for any sequence of coefficient vectors $\{C_j\}$ with $\sum_{j\in Z} \|C_j\|_2^2 < \infty$. The superscript $*$ denotes the transpose.

Corresponding to a multiscaling function $\Phi(x)$, $\Psi(x) = [\psi_1(x), \dots, \psi_r(x)]^T$ is called a multiwavelet if $\{\psi_\ell(x-k) : 1 \le \ell \le r; k \in Z\}$ forms Riesz bases of subspace W_0 so that $V_1 = V_0 \bigoplus W_0$ and $\{2^{n/2}\psi_\ell(2^n x - k) : 1 \le \ell \le r; k, n \in Z\}$ forms a Riesz basis of $L^2(R)$.

$\Psi(x) = [\psi_1(x), \dots, \psi_r(x)]^T$ satisfies the refinement equation

$$\Psi(x) = \sum_{k\in Z} Q_k \Phi(2x-k) \tag{1.2}$$

for some $r \times r$ matrices sequence $\{Q_k\}_{k\in Z}$.

By taking Fourier transforms on both sides of (1.1) and (1.2), respectively, we have

$$\begin{aligned} \hat{\Phi}(w) &= P(e^{-iw/2})\hat{\Phi}(w/2), \quad P(z) = \frac{1}{2}\sum_{k\in Z} P_k z^k, \\ \hat{\Psi}(w) &= Q(e^{-iw/2})\hat{\Phi}(w/2), \quad Q(z) = \frac{1}{2}\sum_{k\in Z} Q_k z^k, \end{aligned} \tag{1.3}$$

where $P(z)$ and $Q(z)$ are called the two-scale matrix symbols of $\Phi(x)$ and $\Psi(x)$, respectively.

The properties of multiscaling functions and multiwavelets are discussed in many papers (see [3, 4, 6, 15, 17–19]). One of the properties of a multiscaling function which has great practical interest is the approximation order (see [2, 8, 10–12]). One known way to raise the approximation order is through the use of two-scale similarity transforms (TSTs) (see [13, 16]). In this paper, we will give a general scheme for constructing a pair of biorthogonal multiscaling functions and multiwavelets with arbitrary desired approximation orders from any given pair of biorthogonal multiscaling functions $\Phi(x)$ and $\tilde{\Phi}(x)$. In addition, we also present an explicit formula for constructing a pair of biorthogonal multiwavelets $\Psi^{\text{new}}(x)$ and $\tilde{\Psi}^{\text{new}}(x)$ associated with a new biorthogonal multiscaling function pair $\Phi^{\text{new}}(x)$ and $\tilde{\Phi}^{\text{new}}(x)$.

2. Basic concept

Two multiscaling functions $\Phi(x)$ and $\tilde{\Phi}(x)$ form a biorthogonal pair if

$$\langle \Phi(x), \tilde{\Phi}(x-k) \rangle = \delta_{0,k} I_{r\times r}, \quad k \in Z, \tag{2.1}$$

where δ is the Kronecker delta, and $I_{r\times r}$ denotes the identity matrix.

Corresponding to $\Phi(x)$ and $\tilde{\Phi}(x)$, two multiwavelets $\Psi(x) = [\psi_1(x), \dots, \psi_r(x)]^T$ and $\tilde{\Psi}(x) = [\tilde{\psi}_1(x), \dots, \tilde{\psi}_r(x)]^T$ form a biorthogonal multiwavelet pair if they satisfy the following equations:

$$\begin{aligned} &\langle \Phi(x), \tilde{\Psi}(x-k) \rangle = \langle \Psi(x), \tilde{\Phi}(x-k) \rangle = O_{r\times r}, \\ &\langle \Psi(x), \tilde{\Psi}(x-k) \rangle = \delta_{0,k} I_{r\times r}, \quad k \in Z, \end{aligned} \tag{2.2}$$

where $O_{r\times r}$ denotes the zero matrix.

Similarly, let $\tilde{P}(z)$ and $\tilde{Q}(z)$ be the two-scale matrix symbols of $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$, respectively. In terms of the two-scale matrix symbols $P(z)$, $Q(z)$, $\tilde{P}(z)$ and $\tilde{Q}(z)$, the biorthogonality of conditions (2.1)–(2.2) implies (see [1, 9, 19])

$$\begin{cases} P(z)\tilde{P}(z)^* + P(-z)\tilde{P}(-z)^* = I_{r\times r}, \\ P(z)\tilde{Q}(z)^* + P(-z)\tilde{Q}(-z)^* = O_{r\times r}, \\ \tilde{P}(z)Q(z)^* + \tilde{P}(-z)Q(-z)^* = O_{r\times r}, \\ Q(z)\tilde{Q}(z)^* + Q(-z)\tilde{Q}(-z)^* = I_{r\times r}. \end{cases} \tag{2.3}$$

LEMMA 2.1. *Let $\Phi(x)$ and $\tilde{\Phi}(x)$ be a pair of biorthogonal multiscaling functions, and let $\Psi(x)$ and $\tilde{\Psi}(x)$ be the corresponding biorthogonal multiwavelet pair, with two-scale matrix symbols $P(z)$, $\tilde{P}(z)$, $Q(z)$ and $\tilde{Q}(z)$, respectively. Suppose $Q^k(z)$, $k = 1, \dots, r$ is the kth row of $Q(z)$, and $\tilde{Q}^k(z)$, $k = 1, \dots, r$ is the kth row of $\tilde{Q}(z)$. Then*

$$\begin{cases} P(z)\tilde{Q}^k(z)^* + P(-z)\tilde{Q}^k(-z)^* = O_{r\times 1}, & k = 1, \dots, r, \\ \tilde{P}(z)Q^k(z)^* + \tilde{P}(-z)Q^k(-z)^* = O_{r\times 1}, & k = 1, \dots, r, \\ Q^j(z)\tilde{Q}^k(z)^* + Q^j(-z)\tilde{Q}^k(-z)^* = \delta_{j,k}, & j, k = 1, \dots, r. \end{cases} \tag{2.4}$$

PROOF. In terms of the biorthogonality of $\Phi(x)$, $\tilde{\Phi}(x)$, $\Psi(x)$ and $\tilde{\Psi}(x)$, we can show that $P(z)$, $Q(z)$, $\tilde{P}(z)$ and $\tilde{Q}(z)$ satisfy (2.3). Substituting $Q(z) = [Q^1(z)^*, \dots, Q^r(z)^*]^*$ and $\tilde{Q}(z) = [\tilde{Q}^1(z)^*, \dots, \tilde{Q}^r(z)^*]^*$ into (2.3), respectively, we obtain (2.4). □

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A multiscaling function $\Phi(x)$ has approximation order $m \geq 1$ if m is the largest integer for which there is a set of row vectors $\{\mathbf{a}^\ell\}_{\ell=0}^{m-1} \subset R^{1\times r}$, with $\mathbf{a}^0 \neq O_{1\times r}$ that satisfy, for $\ell = 0, 1, \ldots, m-1$,

$$\sum_{k=0}^{\ell}(-1)^k \frac{1}{2^k}\binom{\ell}{k}\mathbf{a}^{\ell-k}\sum_{j\in Z}(2j)^k P_{2j} = \frac{1}{2^\ell}\mathbf{a}^\ell,$$
$$\sum_{k=0}^{\ell}(-1)^k \frac{1}{2^k}\binom{\ell}{k}\mathbf{a}^{\ell-k}\sum_{j\in Z}(2j+1)^k P_{2j+1} = \frac{1}{2^\ell}\mathbf{a}^\ell. \tag{2.5}$$

See [8, 10, 11] for details. As is well known, if a multiscaling function $\Phi(x)$ has approximation order m, this implies that the multiwavelet $\tilde{\Psi}(x)$ has m vanishing moments, that is, $\int x^j \tilde{\psi}_k(x)\,dx = 0$, for $j = 0, 1, \ldots, m-1$; $k = 1, \ldots, r$.

By repeated application of (1.3), we have

$$\hat{\Phi}(w) = \left(\prod_{j=1}^{\infty} P\left(e^{-iw/2^j}\right)\right)\hat{\Phi}(0).$$

According to [3, 5], the infinite matrix product $\left(\prod_{j=1}^{\infty} P\left(e^{-iw/2^j}\right)\right)$ converges uniformly on compact sets to a continuous matrix-valued function if and only if $P(1)$ has eigenvalues $\lambda_1 = \cdots = \lambda_k = 1$ and $|\lambda_{k+1}|, \ldots, |\lambda_r| < 1$, with the eigenvalue 1 nondegenerate for $k \geq 1$.

A two-scale matrix symbol $P(z)$ satisfies Condition **E**, if $P(1)$ has a simple eigenvalue of 1, with all other eigenvalues less than 1 in modulus. Condition **E** is automatically satisfied if the two-scale matrix symbol $P(z)$ generates an MRA of $L^2(R)$ with compactly supported basis functions.

In order to obtain the conditions that the matrix refinement equation has an L^2-stable solution, we introduce the transition operator $\mathcal{T}_P$:

$$\mathcal{T}_P A(z^2) = P(z)A(z)P(z)^* + P(-z)A(-z)P(-z)^*,$$

where $A(z)$ is an $r \times r$ matrix with trigonometric polynomial entries. See [15] for details. It was shown in [15] that the matrix refinement equation has an L^2-stable solution if and only if the corresponding transition operator $\mathcal{T}_P$ satisfies Condition **E**, and its eigenmatrix corresponding to the eigenvalue 1 is positive definite for all $w \in R$.

3. Biorthogonal multiscaling functions

In this section, we will introduce a procedure for constructing a pair of biorthogonal multiscaling functions with multiplicity $r + s$ starting with any given pair of biorthogonal multiscaling functions with multiplicity r.

Let $H(z) = [h_{i,j}(z)]$ be the $s \times r$ matrix of Laurent polynomials with $H(z) = H(-z)$ and $H(z)H(z)^* = CI$ $(0 < C < 1, |z| = 1)$. Construct two $s \times r$ matrices $A(z)$ and $\tilde{A}(z)$ as follows:

$$A(z) = H(z)Q(z), \tag{3.1}$$

$$\tilde{A}(z) = H(z)\tilde{Q}(z). \tag{3.2}$$

LEMMA 3.1. *In the setting of Lemma* 2.1, *suppose that* $A(z)$ *and* $\tilde{A}(z)$ *are two* $s \times r$ *matrices defined in* (3.1) *and* (3.2), *respectively. Then*

$$A(z)\tilde{A}(z)^* + A(-z)\tilde{A}(-z)^* = CI_{s\times s}, \tag{3.3}$$

$$P(z)\tilde{A}(z)^* + P(-z)\tilde{A}(-z)^* = O_{r\times s}, \tag{3.4}$$

$$\tilde{P}(z)A(z)^* + \tilde{P}(-z)A(-z)^* = O_{r\times s}, \tag{3.5}$$

$$A(z)\tilde{Q}(z)^* + A(-z)\tilde{Q}(-z)^* = H(z), \tag{3.6}$$

$$\tilde{A}(z)Q(z)^* + \tilde{A}(-z)Q(-z)^* = H(z). \tag{3.7}$$

PROOF. Suppose that Equations (2.3) hold and that $H(z)$ satisfies the conditions above. Then we have

$$\begin{aligned}
&A(z)\tilde{A}(z)^* + A(-z)\tilde{A}(-z)^* \\
&\quad = H(z)Q(z)\tilde{Q}(z)^*H(z)^* + H(-z)Q(-z)\tilde{Q}(-z)^*H(-z)^* \\
&\quad = H(z)[Q(z)\tilde{Q}(z)^* + Q(-z)\tilde{Q}(-z)^*]H(-z)^* = H(z)H(-z)^* = CI_{s\times s}.
\end{aligned}$$

This implies that (3.3) holds. Similarly, applying Lemma 2.1, (3.4)–(3.7) can also be proven. □

THEOREM 3.2. *Under the condition of Lemma* 3.1, *suppose that* $B(z)$ *and* $\tilde{B}(z)$ *are two* $s \times s$ *matrices, and satisfy* $B(z)\tilde{B}(z)^* + B(-z)\tilde{B}(-z)^* = (1 - C)I_{s\times s}$, *where* $0 < C < 1$. *Define*

$$P^{\text{new}}(z) = \begin{bmatrix} P(z) & O \\ A(z) & B(z) \end{bmatrix}, \quad \tilde{P}^{\text{new}}(z) = \begin{bmatrix} \tilde{P}(z) & O \\ \tilde{A}(z) & \tilde{B}(z) \end{bmatrix}. \tag{3.8}$$

Then $P^{\text{new}}(z)\tilde{P}^{\text{new}}(z)^* + P^{\text{new}}(-z)\tilde{P}^{\text{new}}(-z)^* = I_{(r+s)\times(r+s)}$.

PROOF. By Lemmas 2.1 and 3.1, we have

$$\begin{aligned}
&P^{\text{new}}(z)\tilde{P}^{\text{new}}(z)^* + P^{\text{new}}(-z)\tilde{P}^{\text{new}}(-z)^* \\
&\quad = \begin{bmatrix} P(z) & 0 \\ A(z) & B(z) \end{bmatrix} \begin{bmatrix} \tilde{P}(z)^* & \tilde{A}(z)^* \\ 0 & \tilde{B}(z)^* \end{bmatrix}
\end{aligned}$$

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$$+\begin{bmatrix} P(-z) & 0 \\ A(-z) & B(-z) \end{bmatrix} \begin{bmatrix} \tilde{P}(-z)^* & \tilde{A}(-z)^* \\ 0 & \tilde{B}(-z)^* \end{bmatrix}$$

$$= \begin{bmatrix} P(z)\tilde{P}(z)^* + P(-z)\tilde{P}(-z)^* & P(z)\tilde{A}(z)^* + P(-z)\tilde{A}(-z)^* \\ A(z)\tilde{P}(z)^* + A(-z)\tilde{P}(-z)^* & A(z)\tilde{A}(z)^* + A(-z)\tilde{A}(-z)^* \\ & +B(z)\tilde{B}(z)^* + B(-z)\tilde{B}(-z)^* \end{bmatrix}$$

$$= \begin{bmatrix} I_{r\times r} & O_{r\times s} \\ O_{s\times r} & I_{s\times s} \end{bmatrix} = I_{(r+s)\times(r+s)}.$$

This completes the proof of Theorem 3.2. □

REMARK 1. There exist a lot of $B(z)$, $\tilde{B}(z)$ satisfying the condition

$$B(z)\tilde{B}(z)^* + B(-z)\tilde{B}(-z)^* = (1 - C)I_{s\times s}.$$

Additionally, we can choose $B(z) = \tilde{B}(z)$.

THEOREM 3.3. *Suppose that all eigenvalues of the matrices $B(1)$ and $\tilde{B}(1)$ are less than* 1 *in modulus. If both $P(z)$ and $\tilde{P}(z)$ satisfy Condition* **E**, *then both $P^{\text{new}}(z)$ and $\tilde{P}^{\text{new}}(z)$ satisfy Condition* **E**.

PROOF. Since $P^{\text{new}}(1) = \left[\begin{smallmatrix} P(1) & O \\ A(1) & B(1) \end{smallmatrix}\right]$, then

$$|\lambda I_{(r+s)\times(r+s)} - P^{\text{new}}(1)| = |\lambda I_{r\times r} - P(1)||\lambda I_{s\times s} - B(1)|.$$

Obviously, all the eigenvalues of the matrices $P(1)$ and $B(1)$ must be the eigenvalues of the matrix $P^{\text{new}}(1)$. This means that matrix $P^{\text{new}}(1)$ has a simple eigenvalue of 1, with all other eigenvalues less than 1 in modulus. That is, $P^{\text{new}}(z)$ satisfies Condition **E**. Similarly, we can prove that $\tilde{P}^{\text{new}}(z)$ also satisfies Condition **E**. This completes the proof of Theorem 3.3. □

It was shown in [7, 14] that the representation matrix of the transition operator $\mathcal{T}_{P^{\text{new}}}$ is $\mathcal{T}_{P^{\text{new}}} = [2\mathcal{A}_{2i-j}]_{i,j}$, where $\mathcal{A}_j$ is the $(r+s)^2 \times (r+s)^2$ matrix defined by $\mathcal{A}_j = \sum_k P^{\text{new}}_{k-j} \otimes P^{\text{new}}_k$.

According to the above discussion and [15], we have the following construction theorem.

THEOREM 3.4. *Let the conditions of Lemma* 3.1 *and Theorems* 3.2 *and* 3.3 *be satisfied. Further, let the transition operator $\mathcal{T}_{P^{\text{new}}}$ satisfy Condition* **E**, *and let its eigenmatrix corresponding to the eigenvalue* 1 *be positive definite for all $w \in R$. Then there are $\phi_{r+1}(x), \dots, \phi_{r+s}(x)$ and $\tilde{\phi}_{r+1}(x), \dots, \tilde{\phi}_{r+s}(x)$ such that $\Phi^{\text{new}}(x) = [\Phi^T(x), \phi_{r+1}(x), \dots, \phi_{r+s}(x)]^T$ and $\tilde{\Phi}^{\text{new}}(x) = [\tilde{\Phi}(x)^T, \tilde{\phi}_{r+1}(x), \dots, \tilde{\phi}_{r+s}(x)]^T$ are a pair of biorthogonal multiscaling functions with multiplicity $r+s$. Their two-scale matrix symbols $P^{\text{new}}(z)$ and $\tilde{P}^{\text{new}}(z)$ are given by* (3.8).

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4. Explicit formula for constructing biorthogonal multiwavelets

In the above section, we have given a method for constructing a pair of biorthogonal multiscaling functions. In this section, we will discuss the construction of the corresponding biorthogonal multiwavelet pair.

For simplicity, in this section, we suppose that matrices $B(z)$ and $\tilde{B}(z)$ of Theorem 3.2 satisfy the following conditions:

(A1) $B(z) = \tilde{B}(z)$;

(A2) $B(z)B(z)^* + B(-z)B(-z)^* = (1-C)I_{s\times s}$, where $0 < C < 1$;

(A3) $B(z)B(-z) = B(-z)B(z)$.

Clearly, if $B(z)$ is an $r \times r$ diagonal matrix, then condition (A3) must hold.

Construct the matrices $Q^{\text{new}}(z)$ and $\tilde{Q}^{\text{new}}(z)$, respectively, by

$$\begin{aligned} Q^{\text{new}}(z) &= \begin{bmatrix} X(z)Q(z) & Y(z)B(z) \\ O & (1-C)^{-1/2}z^k B(-z)^* \end{bmatrix}, \\ \tilde{Q}^{\text{new}}(z) &= \begin{bmatrix} \tilde{X}(z)\tilde{Q}(z) & \tilde{Y}(z)B(z) \\ O & (1-C)^{-1/2}z^k B(-z)^* \end{bmatrix}, \end{aligned} \tag{4.1}$$

where $X(z)$ and $\tilde{X}(z)$ are two $r \times r$ matrices, $Y(z)$ and $\tilde{Y}(z)$ are two $r \times s$ matrices, and k is an odd number.

Next we will give an explicit formula for constructing a biorthogonal multiwavelet pair corresponding to $\Phi^{\text{new}}(x)$ and $\tilde{\Phi}^{\text{new}}(x)$.

THEOREM 4.1. *Under the conditions of Theorem* 3.4, *if matrices* $X(z)$, $\tilde{X}(z)$, $Y(z)$ *and* $\tilde{Y}(z)$ *satisfy the following conditions*:

$$\begin{cases} H(z)X(z)^* + (1-C)Y(z)^* = O_{s\times r}, \\ H(z)\tilde{X}(z)^* + (1-C)\tilde{Y}(z)^* = O_{s\times r}, \\ X(z)\tilde{X}(z)^* + (1-C)Y(z)\tilde{Y}(z)^* = I_{r\times r}, \end{cases} \tag{4.2}$$

then a biorthogonal multiwavelet pair $\Psi^{\text{new}}(x)$ *and* $\tilde{\Psi}^{\text{new}}(x)$ *corresponding to* $\Phi^{\text{new}}(x)$ *and* $\tilde{\Phi}^{\text{new}}(x)$ *is given, in terms of Fourier transforms, by*

$$\hat{\Psi}^{\text{new}}(w) = Q^{\text{new}}(e^{-iw/2})\hat{\Phi}^{\text{new}}(w/2), \quad \hat{\tilde{\Psi}}^{\text{new}}(w) = \tilde{Q}^{\text{new}}(e^{-iw/2})\hat{\tilde{\Phi}}^{\text{new}}(w/2).$$

PROOF. According to our wavelet construction theorem, we only need prove that $P^{\text{new}}(z)$, $\tilde{P}^{\text{new}}(z)$, $Q^{\text{new}}(z)$ and $\tilde{Q}^{\text{new}}(z)$ satisfy the following equations:

$$P^{\text{new}}(z)\tilde{P}^{\text{new}}(z)^* + P^{\text{new}}(-z)\tilde{P}^{\text{new}}(-z)^* = I_{(r+s)\times(r+s)}, \tag{4.3}$$

$$P^{\text{new}}(z)\tilde{Q}^{\text{new}}(z)^* + P^{\text{new}}(-z)\tilde{Q}^{\text{new}}(-z)^* = O_{(r+s)\times(r+s)}, \tag{4.4}$$

$$\tilde{P}^{\mathrm{new}}(z)Q^{\mathrm{new}}(z)^* + \tilde{P}^{\mathrm{new}}(-z)Q^{\mathrm{new}}(-z)^* = O_{(r+s)\times(r+s)}, \tag{4.5}$$

$$Q^{\mathrm{new}}(z)\tilde{Q}^{\mathrm{new}}(z)^* + Q^{\mathrm{new}}(-z)\tilde{Q}^{\mathrm{new}}(-z)^* = I_{(r+s)\times(r+s)}. \tag{4.6}$$

By Theorem 3.2, (4.3) holds. Next, we only need to prove that (4.4), (4.5) and (4.6) hold. In fact

$$\begin{aligned} P^{\mathrm{new}}(z)\tilde{Q}^{\mathrm{new}}(z)^* &= \begin{bmatrix} P(z) & 0 \\ A(z) & B(z) \end{bmatrix} \begin{bmatrix} \tilde{Q}(z)^*\tilde{X}(z)^* & O \\ B(z)^*\tilde{Y}(z)^* & (1-C)^{-1/2}\bar{z}^k B(-z) \end{bmatrix} \\ &= \begin{bmatrix} P(z)\tilde{Q}(z)^*\tilde{X}(z)^* & O \\ A(z)\tilde{Q}(z)^*\tilde{X}(z)^* + B(z)B(z)^*\tilde{Y}(z)^* & (1-C)^{-1/2}\bar{z}^k B(z)B(-z) \end{bmatrix}. \end{aligned}$$

By (2.3), we have $P(z)\tilde{Q}(z)^* + P(-z)\tilde{Q}(-z)^* = O_{r\times r}$. Hence

$$[P(z)\tilde{Q}(z)^* + P(-z)\tilde{Q}(-z)^*]\tilde{X}(z)^* = O_{r\times r}.$$

Using Lemma 3.1 and the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1-C)I_{s\times s}$, we obtain

$$\begin{aligned} &[A(z)\tilde{Q}(z)^* + A(-z)\tilde{Q}(-z)^*]\tilde{X}(z)^* + [B(z)B(z)^* + B(-z)B(-z)^*]\tilde{Y}(z)^* \\ &\quad = H(z)\tilde{X}(z)^* + (1-C)\tilde{Y}(z)^* = O_{s\times r}. \end{aligned}$$

Therefore (4.4) holds. Similarly, we can prove that (4.5) holds. Finally, we prove (4.6) holds. Since

$$\begin{aligned} Q^{\mathrm{new}}(z)\tilde{Q}^{\mathrm{new}}(z)^* &= \begin{bmatrix} X(z)Q(z) & Y(z)B(z) \\ O & (1-C)^{-1/2}z^k B(-z)^* \end{bmatrix} \begin{bmatrix} \tilde{Q}(z)^*\tilde{X}(z)^* & O \\ B(z)^*\tilde{Y}(z)^* & (1-C)^{-1/2}\bar{z}^k B(-z) \end{bmatrix} \\ &= \begin{bmatrix} X(z)Q(z)\tilde{Q}(z)^*\tilde{X}(z)^* + Y(z)B(z)B(z)^*\tilde{Y}(z)^* & (1-C)^{-1/2}\bar{z}^k Y(z)B(z)B(-z) \\ (1-C)^{-1/2}z^k B(-z)^*B(z)^*\tilde{Y}(z)^* & (1-C)^{-1}z^k\bar{z}^k B(-z)^*B(-z) \end{bmatrix}, \end{aligned}$$

by (4.2), we have

$$\begin{aligned} &Q^{\mathrm{new}}(z)\tilde{Q}^{\mathrm{new}}(z)^* + Q^{\mathrm{new}}(-z)\tilde{Q}^{\mathrm{new}}(-z)^* \\ &\quad = \begin{bmatrix} X(z)\tilde{X}(z)^* + (1-C)Y(z)\tilde{Y}(z)^* & O \\ O & (1-C)^{-1}[B(z)^*B(z) + B(-z)^*B(-z)] \end{bmatrix} \\ &\quad = \begin{bmatrix} I_{r\times r} & O \\ O & I_{s\times s} \end{bmatrix}. \end{aligned}$$

This completes the proof of Theorem 4.1. □

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5. Approximation orders

In this section, we discuss the approximation orders of a pair of new biorthogonal multiscaling functions constructed in Section 3.

Let for $u = 1, \ldots, s$ and $n_u \in \mathbb{Z}_+$

$$b_u(z) = \sum_{j \in Z} b_j^u z^j = \frac{1}{2^{m-1}} \left(\frac{1+z}{2} \right)^{n_u} h_u(z), \quad h_u(1) = 1, \tag{5.1}$$

where $h_u(z)$ are Laurent polynomials.

By $b_u(z)$ defined in (5.1), construct an $s \times s$ diagonal matrix $B(z)$ by

$$B(z) = \operatorname{diag}[b_1(z), \ldots, b_s(z)]. \tag{5.2}$$

Then we have the following lemma.

LEMMA 5.1. *Let $b_u(z)$ defined in* (5.1) *be symbols of sequences* $\{b_j^u\}$. *Then*

$$2^m \sum_{j \in Z} b_{2j}^u = 2^m \sum_{j \in Z} b_{2j+1}^u = 1, \qquad u = 1, \ldots, s,$$

$$\sum_{j \in Z} (2j)^k b_{2j}^u = \sum_{j \in Z} (2j+1)^k b_{2j+1}^u, \quad k = 1, \ldots, n_u - 1.$$

Further, suppose that $B(z) = \sum_{j \in Z} B_j z^j$, *and* $L = \min\{n_1, \ldots, n_s\}$. *Then*

$$\sum_{j \in Z} (2j)^k B_{2j} = \sum_{j \in Z} (2j+1)^k B_{2j+1}, \quad k = 1, \ldots, L.$$

LEMMA 5.2. *If all* $b_u(z)$, $u = 1, \ldots, s$, *satisfy* $|b_u(z)|^2 + |b_u(-z)|^2 = 2^{-(2m-2)}$, *then*

$$B(z)B(z)^* + B(-z)B(-z)^* = \left[1 - \frac{2^{2m-2} - 1}{2^{2m-2}} \right] I_{s \times s}. \tag{5.3}$$

THEOREM 5.3. *In the setting of Theorem* 3.4, *suppose that* $\Phi(x)$ *and* $\tilde{\Phi}(x)$ *have approximation orders* m *and* $\tilde{m}$, *respectively. If the following conditions hold*:

(C1) $B(z)$ *given by* (5.2) *satisfies* (5.3),

(C2) $A(z)$, $\tilde{A}(z)$ *defined in* (3.1) *and* (3.2) *satisfy*

$$A(z)\tilde{A}(z)^* + A(-z)\tilde{A}(-z)^* = \frac{2^{2m-2} - 1}{2^{2m-2}},$$

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then $P^{\text{new}}(z)$ *and* $\tilde{P}^{\text{new}}(z)$ *given by* (3.8) *can generate a pair of new biorthogonal multiscaling functions* $\Phi^{\text{new}}(x) = [\Phi^T(x), \phi_{r+1}(x), \dots, \phi_{r+s}(x)]^T$ *and* $\tilde{\Phi}^{\text{new}}(x) = [\tilde{\Phi}(x)^T, \tilde{\phi}_{r+1}(x), \dots, \tilde{\phi}_{r+s}(x)]^T$, *which have approximation orders* $m + L$ *and* $\tilde{m} + L$, *respectively.*

PROOF. By Theorem 3.4, $P^{\text{new}}(z)$ and $\tilde{P}^{\text{new}}(z)$ can generate a new biorthogonal multiscaling function pair $\Phi^{\text{new}}(x)$ and $\tilde{\Phi}^{\text{new}}(x)$. Next, we will prove that this new biorthogonal multiscaling function pair have approximation orders of $m + L$ and $\tilde{m} + L$, respectively.

Since the approximation order of $\Phi(x)$ is m, there are $\mathbf{a}^\ell \in R^r$, $\ell = 0, 1, \dots, m-1$, with $\mathbf{a}^0 \neq O_{1\times r}$, such that, by (2.4) and (2.5),

$$\mathbf{a}^\ell \left(\sum_{j\in Z} P_{2j} - \frac{1}{2^\ell} I_{r\times r}\right) = -\sum_{k=0}^{\ell-1} (-1)^{\ell-k} \frac{1}{2^{\ell-k}} \binom{\ell}{k} \mathbf{a}^k \sum_{j\in Z} (2j)^{\ell-k} P_{2j},$$

$$\mathbf{a}^\ell \left(\sum_{j\in Z} P_{2j+1} - \frac{1}{2^\ell} I_{r\times r}\right) = -\sum_{k=0}^{\ell-1} (-1)^{\ell-k} \frac{1}{2^{\ell-k}} \binom{\ell}{k} \mathbf{a}^k \sum_{j\in Z} (2j+1)^{\ell-k} P_{2j+1}.$$

Next, we will prove the approximation order of $\Phi^{\text{new}}(x)$ is $m + L$. That is, we will find a set of row vectors $\mathbf{w}^\ell \in R^{r+s}$, $\ell = 0, 1, \dots, m + L - 1$, with $\mathbf{w}^0 \neq O_{1\times(r+s)}$ such that

$$\begin{aligned} &\mathbf{w}^\ell \left(\begin{bmatrix} \sum_{j\in Z} P_{2j} & O_{r\times s} \\ \sum_{j\in Z} A_{2j} & \sum_{j\in Z} B_{2j} \end{bmatrix} - \frac{1}{2^\ell} I_{(r+s)\times(r+s)}\right) \\ &= -\sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{\ell}{k} \mathbf{w}^k \begin{bmatrix} \sum_{j\in Z} (2j)^{\ell-k} P_{2j} & O_{r\times s} \\ \sum_{j\in Z} (2j)^{\ell-k} A_{2j} & \sum_{j\in Z} (2j)^{\ell-k} B_{2j} \end{bmatrix}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\mathbf{w}^\ell \left(\begin{bmatrix} \sum_{j\in Z} P_{2j+1} & O_{r\times s} \\ \sum_{j\in Z} A_{2j+1} & \sum_{j\in Z} B_{2j+1} \end{bmatrix} - \frac{1}{2^\ell} I_{(r+s)\times(r+s)}\right) \\ &= -\sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{\ell}{k} \mathbf{w}^k \begin{bmatrix} \sum_{j\in Z} (2j+1)^{\ell-k} P_{2j+1} & O_{r\times s} \\ \sum_{j\in Z} (2j+1)^{\ell-k} A_{2j+1} & \sum_{j\in Z} (2j+1)^{\ell-k} B_{2j+1} \end{bmatrix}. \end{aligned} \tag{5.5}$$

It is clear that $\mathbf{w}^\ell = [\mathbf{a}^\ell, 0, \dots, 0] \in R^{r+s}$, $\ell = 0, 1, \dots, m-1$, as the first m vectors satisfy (5.4) and (5.5). Hence we choose $\mathbf{w}^\ell = [\mathbf{a}^\ell, 0, \dots, 0] \in R^{r+s}$, $\ell = 0, 1, \dots, m-1$, to be the first m vectors in (5.4) and (5.5). The remaining L row vectors are denoted by $\mathbf{w}^{m+\ell} = [\mathbf{a}^{m+\ell}, c^1_{m+\ell}, c^2_{m+\ell}, \dots, c^s_{m+\ell}]$, $\ell = 0, 1, \dots, L-1$. Obviously, $\mathbf{w}^m$ must satisfy $\sum_{j=1}^s |c^j_m| \neq 0$. In fact, if all $c^j_m = 0$, then $\mathbf{w}^m = [\mathbf{a}^m, 0, \dots, 0]$. This means that the approximation order of $\Phi(x)$ is $m+1$. If we use the notation $\mathbf{w}^\ell = [\mathbf{a}^\ell, c^1_\ell, c^2_\ell, \dots, c^s_\ell]$, then $c^j_\ell = 0$ for $j = 1, \dots, s$; $\ell = 0, 1, \dots, m-1$.

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Hence (5.4) is equivalent to

$$\mathbf{a}^\ell \left(\sum_{j\in Z} P_{2j} - \frac{1}{2^\ell} I_{r\times r} \right) + [c_\ell^1, \dots, c_\ell^s] \sum_{j\in Z} A_{2j}$$
$$= - \sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{\ell}{k} \left[\mathbf{a}^k \sum_{j\in Z} (2j)^{\ell-k} P_{2j} + [c_k^1, \dots, c_k^s] \sum_{j\in Z} (2j)^{\ell-k} A_{2j} \right], \quad (5.6)$$

$$[c_\ell^1, \dots, c_\ell^s] \left[\sum_{j\in Z} B_{2j} - \frac{1}{2^\ell} I_{s\times s} \right]$$
$$= - \sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{\ell}{k} [c_k^1, \dots, c_k^s] \sum_{j\in Z} (2j)^{\ell-k} B_{2j}. \quad (5.7)$$

Since $c_\ell^j = 0$ for $j = 1, \dots, s$, $\ell = 0, 1, \dots, m-1$, then (5.7) implies the following two identities:

$$[c_m^1, \dots, c_m^s] \left[\sum_{j\in Z} B_{2j} - \frac{1}{2^m} I_{s\times s} \right] = O_{s\times s}, \quad (5.8)$$

$$[c_{m+\ell}^1, \dots, c_{m+\ell}^s] \left[\sum_{j\in Z} B_{2j} - \frac{1}{2^{m+\ell}} I_{s\times s} \right]$$
$$= - \sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{m+\ell}{\ell-k} [c_{m+k}^1, \dots, c_{m+k}^s] \sum_{j\in Z} (2j)^{\ell-k} B_{2j}, \quad (5.9)$$

for $\ell = 1, \dots, L-1$.

By Lemma 5.1, $\sum_{j\in Z} B_{2j} = 2^{-m} I_{s\times s}$. Hence

$$\sum_{j\in Z} B_{2j} - \frac{1}{2^{m+\ell}} I_{s\times s} = \frac{2^\ell - 1}{2^{m+\ell}} I_{s\times s}.$$

Therefore, for $\ell = 1, \dots, L-1$,

$$[c_{m+\ell}^1, \dots, c_{m+\ell}^s]$$
$$= - \frac{2^m}{2^\ell - 1} \sum_{k=0}^{\ell-1} (-1)^{\ell-k} 2^k \binom{m+\ell}{\ell-k} [c_{m+k}^1, \dots, c_{m+k}^s] \sum_{j\in Z} (2j)^{\ell-k} B_{2j}. \quad (5.10)$$

Similarly, applying (5.5), we have

$$[c_m^1, \dots, c_m^s] \left[\sum_{j\in Z} B_{2j+1} - \frac{1}{2^m} I_{s\times s} \right] = O_{s\times s},$$

$$[c^1_{m+\ell}, \dots, c^s_{m+\ell}]\left[\sum_{j\in Z} B_{2j+1} - \frac{1}{2^{m+\ell}} I_{s\times s}\right]$$
$$= -\sum_{k=0}^{\ell-1} \frac{(-1)^{\ell-k}}{2^{\ell-k}} \binom{m+\ell}{\ell-k} [c^1_{m+k}, \dots, c^s_{m+k}] \sum_{j\in Z} (2j+1)^{\ell-k} B_{2j+1},$$

for $\ell = 1, \dots, L-1$. Hence we have

$$[c^1_{m+\ell}, \dots, c^s_{m+\ell}] = -\frac{2^m}{2^\ell - 1} \sum_{k=0}^{\ell-1} (-1)^{\ell-k} 2^k \binom{m+\ell}{\ell-k} \times [c^1_{m+k}, \dots, c^s_{m+k}] \sum_{j\in Z} (2j+1)^{\ell-k} B_{2j+1}, \tag{5.11}$$

for $\ell = 1, \dots, L-1$. By (5.10) or (5.11), taking any $[c^1_m, \dots, c^s_m] \neq \mathbf{O}_{1\times s}$, we can obtain $[c^1_{m+\ell}, \dots, c^s_{m+\ell}]$, $\ell = 1, \dots, L-1$. And then applying (5.6), we can obtain $\mathbf{a}^{m+\ell}$. This means that the remaining $L-1$ row vectors $\mathbf{w}^{m+\ell} = [\mathbf{a}^{m+\ell}, c^1_{m+\ell}, \dots, c^s_{m+\ell}]$, $\ell = 1, \dots, L-1$ are obtained. Thereby, we prove that $\Phi^{\text{new}}(x)$ has approximation order $m + L$. Similarly, we also prove that the approximation order of $\tilde{\Phi}^{\text{new}}(x)$ is $\tilde{m} + L$. This completes the proof of Theorem 5.3. □

REMARK 2. Lemma 5.1 can guarantee that vectors $[c^1_{m+\ell}, \dots, c^s_{m+\ell}]$, $\ell = 1, \dots, L-1$, obtained by (5.10) and (5.11) are the same.

6. Example

Case of $r = s = 1$ Let $\phi_1(x)$ and $\tilde{\phi}_1(x)$ be a pair of biorthogonal scaling functions, and let $\psi_1(x)$ and $\tilde{\psi}_1(x)$ be the corresponding biorthogonal wavelet pair. Their corresponding two-scale symbols are

$$P(z) = \left[\frac{1+z}{2}\right]^2 \left(-\frac{1}{2}z^{-2} + 2z^{-1} - \frac{1}{2}\right), \quad \tilde{P}(z) = \left[\frac{1+z}{2}\right]^2 z^{-1},$$
$$Q(z) = -\frac{1}{4}z^2 + \frac{1}{2}z - \frac{1}{4} \quad \text{and} \quad \tilde{Q}(z) = -\frac{1}{8}z^3 - \frac{1}{4}z^2 + \frac{3}{4}z - \frac{1}{4} - \frac{1}{8}z^{-1}.$$

It is easy to verify that both the approximation orders of $\phi(x)$ and $\tilde{\phi}(x)$ are 2. That is, $m = \tilde{m} = 2$. Take

$$H(z) = \sqrt{\frac{2^{2m-2} - 1}{2^{2m-2}}} = \frac{\sqrt{3}}{2}.$$

Then by (3.1) and (3.2), $A(z) = (\sqrt{3}/2)Q(z)$ and $\tilde{A}(z) = (\sqrt{3}/2)\tilde{Q}(z)$. Take

$$B(z) = \frac{1}{2}\left[\frac{1+z}{2}\right]^2 \frac{(1+\sqrt{3}) + (1-\sqrt{3})z}{2}.$$

It is easy to verify that

$$A(z)\widetilde{A}(z)^* + A(-z)\widetilde{A}(-z)^* = 3/4, \quad B(z)B(z)^* + B(-z)B(-z)^* = 1 - 3/4.$$

By (3.8), we construct

$$P^{\text{new}}(z) = \begin{bmatrix} \left[\frac{1+z}{2}\right]^2 (-\frac{1}{2}z^{-2} + 2z^{-1} - \frac{1}{2}) & 0 \\ \frac{\sqrt{3}}{2}(-\frac{1}{4}z^2 + \frac{1}{2}z - \frac{1}{4}) & \frac{1}{2}\left[\frac{1+z}{2}\right]^2 \frac{(1+\sqrt{3})+(1-\sqrt{3})z}{2} \end{bmatrix}, \tag{6.1}$$

$$\tilde{P}^{\text{new}}(z) = \begin{bmatrix} \left[\frac{1+z}{2}\right]^2 z^{-1} & 0 \\ \frac{\sqrt{3}}{2}(-\frac{1}{8}z^3 - \frac{1}{4}z^2 + \frac{3}{4}z - \frac{1}{4} - \frac{1}{8}z^{-1}) & \frac{1}{2}\left[\frac{1+z}{2}\right]^2 \frac{(1+\sqrt{3})+(1-\sqrt{3})z}{2} \end{bmatrix}. \tag{6.2}$$

From [6, 14], the transition operation $\mathscr{T}_{P^{\text{new}}}$ associated with $P^{\text{new}}(z)$ is a 44×44 matrix. By calculation, the transition operation $\mathscr{T}_{P^{\text{new}}}$ satisfies condition **E**. Hence, applying Theorem 3.4, we obtain a pair of new biorthogonal multiscaling functions $\Phi^{\text{new}}(x) = [\phi_1(x), \phi_2(x)]^T$ and $\tilde{\Phi}^{\text{new}}(x) = [\tilde{\phi}_1(x), \tilde{\phi}_2(x)]^T$, with two-scale matrix symbols $P^{\text{new}}(z)$ and $\tilde{P}^{\text{new}}(z)$ given by (6.1) and (6.2), respectively.

Let $X(z) = X(z)^* = 1/2$ and $Y(z) = Y(z)^* = -\sqrt{3}$. It is easy to verify that $X(z)$, $X(z)^*$, $Y(z)$ and $Y(z)^*$ satisfy (4.2). Thus, by (4.1), and taking $k = 3$, we can construct two matrices $Q^{\text{new}}(z)$ and $\tilde{Q}^{\text{new}}(z)$. Hence, applying Theorem 4.1, the corresponding biorthogonal multiwavelet pair $\Psi^{\text{new}}(x) = [\psi_1(x), \psi_2(x)]^T$ and $\tilde{\Psi}^{\text{new}}(x) = [\tilde{\psi}_1(x), \tilde{\psi}_2(x)]^T$ can be constructed by the two scale matrix symbols $Q^{\text{new}}(z)$ and $\tilde{Q}^{\text{new}}(z)$.

Further, by Theorem 5.3, both approximation orders of the new biorthogonal multiscaling functions $\Phi^{\text{new}}(x)$ and $\tilde{\Phi}^{\text{new}}(x)$ are 4. That is, we raise the approximation orders of $\phi_1(x)$ and $\tilde{\phi}_1(x)$ from 2 to 4.

Similar to the case of $r = s = 1$, some examples can also be constructed for the settings $r > 1$ and $s > 1$.

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