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RICE THEOREMS FOR Σ_n^{-1} SETS

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1. Introduction. In [**3**] Hay proves generalizations of Rice's Theorem and the Rice-Shapiro Theorem for differences of recursively enumerable sets (d.r.e. sets). The original Rice Theorem [**5**, p. 364, Corollary B] says that the index set of a class C of r.e. sets is recursive if and only if C is empty or C contains all r.e. sets. The Rice-Shapiro Theorem conjectured by Rice [**5**] and proved independently by McNaughton, Shapiro, and Myhill [**4**] says that the index set of a class C of r.e. sets is r.e. if and only if C is empty or C consists of all r.e. sets which extend some element of a canonically enumerable class of finite sets. Since a d.r.e. set is a difference of r.e. sets, a d.r.e. set has an index associated with it, namely, the pair of indices of the r.e. sets of which it is the difference. Thus we may speak of the index set of a class of d.r.e. sets. When generalized to d.r.e. sets, Rice's Theorem [**3**, p. 354] becomes: *The index set of a class of* d.r.e. *sets is* r.e. *if and only if C is empty or C consists of all* d.r.e. *sets*. The Rice-Shapiro Theorem [**3**, p. 355] becomes: *The index set of a class C of* d.r.e. *sets is* d.r.e. *if and only if C is empty or C consists of all* d.r.e. *sets which extend a single finite set.*

Since both the r.e. and d.r.e. sets occur as levels 1 and 2 of the hierarchy generated by Boolean combinations of r.e. sets (the finite Ershov hierarchy, see Ershov [**1**]) we prove in Sections 3 and 4 of this paper generalizations of the Rice Theorems for the higher levels of this hierarchy. The first Rice Theorem generalizes in the expected way and hold for all levels $n \geqq 1$. The generalized Rice-Shapiro Theorem on the other hand cannot be stated in such a uniform fashion, but does hold for $n \geqq 3$. In Section 2 we explicitly define this hierarchy and the index sets at each level and state some properties which are necessary for the proofs in the later sections. In Section 5 we give an example of a single fixed class whose index sets are complete at each level of the hierarchy, and prove that if the index sets of a class C are complete at the first n levels of the hierarchy where $n > 2$, then fail to be complete, the index sets for levels greater than n jump to degree at least $\underset{\sim}{0}''$.

2. Preliminaries. Ershov [**1**] defines a hierarchy on the finite Boolean combinations of r.e. sets. The levels of the hierarchy may be defined in several ways. The following definition is the characterization given by Ershov in Proposition 1 of [**1**, p. 29].

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Definition. a) $X \in \Sigma_n^{-1}$, $n \geqq 1$, if and only if there exist r.e. sets $R_1, R_2, \ldots, R_n$ such that

$$X = \bigcup_{i=1}^{[(n+1)/2]} (R_{2i-1} - R_{2i}), \quad (R_{n+1} = \emptyset).$$

b) $X \in \Pi_n^{-1}$, $n \geqq 1$, if and only if $\bar{X} \in \Sigma_n^{-1}$.

These classes of sets satisfy the usual hierarchy properties, e.g.

$$\Sigma_n^{-1} \cup \Pi_n^{-1} \subsetneqq \Sigma_{n+1}^{-1} \cap \Pi_{n+1}^{-1}, \quad \Sigma_n^{-1} - \Pi_n^{-1} \neq \emptyset$$

(see Ershov [**1**]), and we also may make the following definition of a complete Σ_n^{-1} set.

Definition. Let $n \geqq 1$. A set $S \in \Sigma_n^{-1}$ is Σ_n^{-1}-*complete* if for each $X \in \Sigma_n^{-1}$ there exists a recursive function f such that $x \in X \leftrightarrow f(x) \in S$. If such a function exists we write $X \leqq_m S$ and if f is one-one we write $X \leqq_1 S$.

It has been shown by Ershov [**1**] that for each $n \geqq 1$ there exist Σ_n^{-1}-complete sets, and if $S \in \Sigma_n^{-1}$ is Σ_n^{-1}-complete then $X \in \Sigma_n^{-1}$ if and only if $X \leqq_m S$.

Definition. Let $K = \{x | x \in W_x\}$, the complete Σ_1^0 set. For each $n \geqq 1$, let

$$K_n = \{\langle x_1, x_2, \ldots, x_n\rangle \mid \text{card}\,\{x_1 \mid x_i \in K\} \text{ is odd}\}.$$

FACT 2.1 (Hay [**2**]) *For each $n \geqq 1$, K_n is Σ_n^{-1}-complete.*

FACT 2.2 (Hay [**2**]) *For each $n \geqq 1$, $K_n \nleqq_m \bar{K}_n$.*

FACT 2.3 (Hay [**2**]) *For each $n \geqq 1$, if $X \in \Sigma_n^{-1}$ then $K_n \nleqq_1 \bar{X}$.*

FACT 2.4 (Hay [**2**]) *For each $n > 1$, $K_{n-1} \leqq_1 \bar{K}_n$.*

(These facts also follow from the first fact and the Ershov properties of the hierarchy.)

Let $\{W_e\}_{e \geqq 0}$, be a fixed acceptable enumeration of the r.e. sets. This may be used to define an enumeration of the Σ_n^{-1} sets as follows:

Definition. Let $n \geqq 1$ and let $X \in \Sigma_n^{-1}$. Suppose

$$X = \bigcup_{i=1}^{[(n+1)/2]} (W_{x_{2i-1}} - W_{x_{2i}}), \quad (W_{x_{n+1}} = \emptyset).$$

If $x = \langle x_1, x_2, \ldots, x_n\rangle$, then x is a Σ_n^{-1} index of X, and if we write V_x (or $V_{x,n}$) to indicate the xth element of Σ_n^{-1}, then $\{V_x\}_{x \geqq 0}$ is an *enumeration* of Σ_n^{-1}.

Now we may define the Σ_n^{-1} index set of a class C.

Definition. Let C be a class of sets. For $n \geqq 1$, $\theta_n(C) = \{x | V_x \in C \cap \Sigma_n^{-1}\}$ is the Σ_n^{-1} *index set of* C.

We say a class C of Σ_n^{-1} sets is *trivial* if C is empty or C consists of all Σ_n^{-1} sets. $\theta_n(C)$ is *trivial* if $\theta_n(C)$ is empty or is equal to the set of natural numbers.

We denote by D_u the finite set with canonical index u. K' denotes the complete Σ_2^0 set and Fin denotes $\{x|W_x \text{ is finite}\}$; as is well known, Fin is recursively isomorphic to K'.

Finally, if $S \leqq_m \theta_n(C)$ via f, then since $f(x) = \langle f_1(x), \ldots, f_n(x)\rangle$ where $f_i(x)$ is an index of an r.e. set, each f_i may be made one-one using the standard technique [**6**, p. 133], thus $S \leqq_m \theta_n(C)$ implies $S \leqq_1 \theta_n(C)$.

3. The first Rice theorem. In this section we prove the generalization of Rice's Theorem. It is obtained as a corollary (Corollary 3.5) of the following theorem.

THEOREM 3.1. *Let $n \geqq 2$. If C is a non-trivial class of Σ_n^{-1} sets then $K_n \leqq_1 \theta_n(C)$ or $K_n \leqq_1 \theta_n(C)$.*

For the proof we will require the following two lemmas.

LEMMA 3.2. *Suppose there is some $D_u \in C$ and some r.e. set $B \notin C$ such that $D_u \subset B$. Then $K_n \leqq_1 \overline{\theta_n(C)}$. (This holds for $n \geqq 1$.)*

Proof. We will show that there is a recursive function f such that $x \in K_n$ if and only if $f(x) \in \overline{\theta_n(C)}$. We wish to define f so that

$$x \in K_n \rightarrow V_{f(x)} = B \notin C$$

and

$$x \notin K_n \rightarrow V_{f(x)} = D_u \in C.$$

Since $x \in K_n$ if and only if card $\{x_j|x_j \in K\}$ is odd we want:

$$\text{card } \{x_j|x_j \in K\} \text{ odd} \rightarrow V_{f(x)} = B \notin C$$

and

$$\text{card } \{x_j|x_j \in K\} \text{ even} \rightarrow V_{f(x)} = D_u \in C.$$

Since $0 \leqq \text{card } \{x_j|x_j \in K\} \leqq n$, we want to define f so that we may "change our minds" n times, beginning with $V_{f(x)} = D_u$ if card $\{x_j|x_j \in K\} = 0$, and ending with $V_{f(x)} = D_u$ if n is even and $V_{f(x)} = B$ if n is odd.

So, for $1 \leqq i \leqq [(n + 1)/2]$, we define f_{2i-1} and f_{2i} by:

$$W_{f_{2i-1}(x)} = \begin{cases} D_u & \text{if card } \{x_j|x_j \in K\} < 2i - 1 \\ B & \text{otherwise} \end{cases}$$

$$W_{f_{2i}(x)} = \begin{cases} \emptyset & \text{if card } \{x_j|x_j \in K\} < 2i \\ B - D_u & \text{if card } \{x_j|x_j \in K\} = 2i \\ B & \text{otherwise.} \end{cases}$$

(If n is odd, we do not define f_{n+1}.)

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Thus:

$$W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \begin{cases} D_u & \text{if card } \{x_j | x_j \in K\} < 2i - 1 \\ B & \text{if card } \{x_j | x_j \in K\} = 2i - 1 \\ D_u & \text{if card } \{x_j | x_j \in K\} = 2i \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $f = \langle f_1, \ldots, f_n \rangle$.

1. Suppose $x \in K_n$. Then card $\{x_j | x_j \in K\} = 2i_0 - 1$ for some i_0, $1 \leqq i_0 \leqq [(n + 1)/2]$. From the above we then have:

$$W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \begin{cases} D_u & \text{if } i_0 < i \\ B & \text{if } i_0 = i \\ \emptyset & \text{if } i_0 > i. \end{cases}$$

Since $D_u \subset B$ we have:

$$\bigcup_{i=1}^{[(n+1)/2]} (W_{f_{2i-1}(x)} - W_{f_{2i}(x)}) = B.$$

Thus $x \in K_n \rightarrow V_{f(x)} = B \notin C$.

2. Now suppose $x \notin K_n$. Then card $\{x_j | x_j \in K\} = 2i_0$ for some i_0, $0 \leqq i_0 \leqq [(n + 1)/2]$. From the above we then have:

$$W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \begin{cases} D_u & \text{if } i_0 \leqq i \\ \emptyset & \text{if } i_0 > i. \end{cases}$$

So

$$\bigcup_{i=1}^{[(n+1)/2]} (W_{f_{2i-1}(x)} - W_{f_{2i}(x)}) = D_u.$$

Thus $x \notin K_n \rightarrow V_{f(x)} = D_u \in C$. Hence we have $x \in K_n \leftrightarrow f(x) \in \overline{\theta_n(C)}$.

LEMMA 3.3. *Let $n \geqq 2$. Suppose there is some $D_u \in C$ such that every r.e. extension of D_u is in C but there is some $V_a \supset D_u$ such that $V_a \in \Sigma_n^{-1} - C$. Then $K' \leqq_1 \theta_n(C)$.*

Proof. It suffices to show that Fin $\leqq_1 \theta_n(C)$. Let

$$V_a = \bigcup_{i=1}^{[(n+1)/2]} (W_{a_{2i-1}} - W_{a_{2i}}), \quad (W_{a_{n+1}} = \emptyset).$$

For each i, $1 \leqq i \leqq [(n + 1)/2]$, we define f_{2i-1} to be the constant function $f_{2i-1}(x) = a_{2i-1}$ and f_{2i} by:

$$W_{f_{2i}(x)} = \begin{cases} W_{a_{2i}} & \text{if } W_x \text{ is infinite} \\ \hat{W}_{a_{2i}} & \text{if } W_x \text{ is finite} \end{cases}$$

where $\hat{W}_{a_{2i}}$ is a finite subset of $W_{a_{2i}}$. (If n is odd we do not define f_{n+1}.)

Let $f = \langle f_1, \ldots, f_n \rangle$.

1) Suppose W_x is infinite. Then:

$$\begin{aligned} V_{f(x)} &= \bigcup_{i=1}^{[(n+1)/2]} (W_{f_{2i-1}(x)} - W_{f_{2i}(x)}) \\ &= \bigcup_{i=1}^{[(n+1)/2]} (W_{a_{2i-1}} - W_{a_{2i}}) = V_a \notin C. \end{aligned}$$

Thus W_x infinite $\rightarrow f(x) \notin \theta_n(C)$.

2) Suppose W_x is finite. Then:

$$V_{f(x)} = \bigcup_{i=1}^{[(n+1)/2]} (W_{f_{2i-1}(x)} - W_{f_{2i}(x)}) = \bigcup_{i=1}^{[(n+1)/2]} (W_{a_{2i-1}} - \hat{W}_{a_{2i}}).$$

Since $\hat{W}_{a_{2i}}$ is finite, $W_{a_{2i-1}} - \hat{W}_{a_{2i}}$ is r.e. so $V_{f(x)}$ is r.e. But $V_a \subset V_{f(x)}$ and $D_u \subset V_a$, so $V_{f(x)} \in C$ since it is an r.e. extension of D_u. Thus W_x finite $\rightarrow f(x) \in \theta_n(C)$.

We thus have Fin $\leqq_1 \theta_n(C)$, and hence $K' \leqq_1 \theta_n(C)$.

Proof of Theorem 3.1. Either $\emptyset \in C$ or $\emptyset \in \bar{C}$. Since $\theta_n(C)$ is non-trivial we apply Lemma 3.2 or 3.3 to whichever of C or $\bar{C}$ contains $\emptyset$. If Lemma 3.3 is applied, note that the result follows from the fact that $K_n \leqq_1 K'$ since $K_n \in \Sigma_2^0$.

COROLLARY 3.4. *Let $n > 2$. If C is a class of Σ_n^{-1} sets such that $\theta_n(C) \in \Sigma_n^{-1}$, then either $\theta_n(C)$ is trivial or $\theta_n(C) \equiv K_n$.*

Proof. Suppose $\theta_n(C) \in \Sigma_n^{-1}$. By Fact 2.1, $\theta_n(C) \leqq_1 K_n$ and by Fact 2.3, $K_n \nleqq_1 \overline{\theta_n(C)}$. If $\theta_n(C)$ is non-trivial, then by Theorem 3.1 either $K_n \leqq_1 \theta_n(C)$ or $K_n \leqq_1 \overline{\theta_n(C)}$. But $K_n \nleqq_1 \overline{\theta_n(C)}$, so we have $K_n \leqq_1 \theta_n(C) \leqq_1 K_n$, hence $\theta_n(C) \equiv K_n$.

COROLLARY 3.5. *Let $n > 2$. If C is a class of Σ_n^{-1} sets then $\theta_n(C) \in \Sigma_{n-1}^{-1}$ if and only if C is trivial.*

Proof. If C is trivial then $\theta_n(C) \in \Sigma_{n-1}^{-1}$. Conversely, if $\theta_n(C) \in \Sigma_{n-1}^{-1}$, then $\theta_n(C) \in \Sigma_n^{-1}$, so by Corollary 3.4 either $\theta_n(C)$ is trivial or $\theta_n(C) \equiv K_n$. Now if $\theta_n(C) \equiv K_n$, then $\theta_n(C) \notin \Sigma_{n-1}^{-1}$ else we would have $K_n \leqq_1 K_{n-1}$. Since $K_{n-1} \leqq_1 \bar{K}_n$ by Fact 2.4 we would then have $K_n \leqq_1 \bar{K}_n$, a contradiction to Fact 2.2.

Notice in Theorem 3.1 and Corollary 3.4 we may replace $n > 2$ by $n \geqq 1$ and in Corollary 3.5 if we understand Σ_0^{-1} to mean recursive sets, we may also replace $n > 2$ by $n \geqq 1$. Thus the "first Rice Theorem" may be stated for all $n \geqq 1$.

4. The Rice-Shapiro theorem. The following theorem is the generalization of the Rice-Shapiro Theorem and it holds for all $n > 2$.

THEOREM 4.1. *For each $n > 2$, if C is a class of Σ_n^{-1} sets, then $\theta_n(C) \in \Sigma_n^{-1}$*

if and only if C *is trivial or there exists a natural number* a *such that* $C = \{V_{x,n} | a \in V_{x,n}\}$.

Proof. If C is trivial then $\theta_n(C) \in \Sigma_n^{-1}$. If $C = \{V_{x,n} | a \in V_{x,n}\}$ then:

$$\theta_n(C) = \{x | a \in V_{x,n}\} = \{x | a \in \bigcup_{i=1}^{[(n+1)/2]} (W_{x_{2i-1}} - W_{x_{2i}})\}$$
$$= \bigcup_{i=1}^{[(n+1)/2]} \{x | a \in W_{x_{2i-1}} - W_{x_{2i}}\}$$
$$= \bigcup_{i=1}^{[(n+1)/2]} (\{x | a \in W_{x_{2i-1}}\} - \{x | a \in W_{x_{2i}}\}).$$

Thus $\theta_n(C) \in \Sigma_n^{-1}$.

Conversely, suppose $\theta_n(C) \in \Sigma_n^{-1}$. If $\theta_n(C)$ is not trivial then we must show there exists a such that $C = \{V_{x,n} | a \in V_{x,n}\}$. For this we will need the following sequence of lemmas.

LEMMA 4.2. *Let* $n \geqq 1$. *If* $V_b \in \Sigma_n^{-1}$ *is infinite then there exists a recursive function* f *such that*

$$W_x \text{ infinite} \rightarrow V_{f(x)} = V_b$$

and

$$W_x \text{ finite} \rightarrow V_{f(x)} \text{ is a finite subset of } V_b.$$

Proof. For $n = 1$ this is well known. For $n = 2$ this appears in the proof of Lemma 5.3 [**3**, p. 356].

Let n be the least n such that $V_b \in \Sigma_n^{-1}$. Let

$$V_b = \bigcup_{i=1}^{[(n+1)/2]} (W_{b_{2i-1}} - W_{b_{2i}}), \quad (W_{b_{n+1}} = \emptyset).$$

Since n is the least, we may assume each of the sets $W_{b_{2i-1}} - W_{b_{2i}}$ is infinite, hence $W_{b_{2i-1}}$ is infinite for each i.

Thus we may use the well known construction to obtain for each i, $1 \leqq i \leqq [(n + 1)/2]$, a recursive function f_{2i-1} such that

$$W_x \text{ infinite} \rightarrow W_{f_{2i-1}(x)} = W_{b_{2i-1}}$$

and

$$W_x \text{ finite} \rightarrow W_{f_{2i-1}(x)} = \hat{W}_{b_{2i-1}}$$

where $\hat{W}_{b_{2i-1}}$ is a finite subset of $W_{b_{2i-1}}$.

We also define f_{2i} for $1 \leqq i \leqq [(n + 1)/2]$ to be the constant function $f_{2i}(x) = b_{2i}$.

Thus, for $1 \leqq i \leqq [(n + 1)/2]$

$$W_x \text{ infinite} \rightarrow W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = W_{b_{2i-1}} - W_{b_{2i}}$$

and

$$W_x \text{ finite} \rightarrow W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \hat{W}_{b_{2i-1}} - W_{b_{2i}}$$

which is a finite subset of $W_{b_{2i-1}} - W_{b_{2i}}$.

Let $f = \langle f_1, \ldots, f_n \rangle$. Then

$$W_x \text{ infinite} \rightarrow V_{f(x)} = \bigcup_{i=1}^{[(n+1)/2]} (W_{f_{2i-1}(x)} - W_{f_{2i}(x)})$$
$$= \bigcup_{i=1}^{[(n+1)/2]} (W_{b_{2i-1}} - W_{b_{2i}}) = V_b$$

and

$$W_x \text{ finite} \rightarrow V_{f(x)} = \bigcup_{i=1}^{[(n+1)/2]} (W_{f_{2i-1}(x)} - W_{f_{2i}(x)})$$
$$= \bigcup_{i=1}^{[(n+1)/2]} (\hat{W}_{b_{2i-1}} - W_{b_{2i}})$$

which is a finite subset of V_b.

LEMMA 4.3. *Let $n \geqq 1$. If there is an infinite $V_{b,n} \in C$ such that no finite subset of $V_{b,n}$ is in C then $K' \leqq_1 \overline{\theta_n(C)}$.*

Proof. This follows from Lemma 4.2.

LEMMA 4.4. *Let $n \geqq 2$. Suppose there exist distinct finite sets $D_u, D_v \in C$ such that* card D_u = card $D_v = 1$ *and* $\emptyset \notin C$. *Then $K_n \leqq_1 \overline{\theta_n(C)}$.*

Proof. Define f_1 by:

$$W_{f_1(x)} = \begin{cases} D_u & \text{if } 0 \leqq \text{card } \{x_j | x_j \in K\} \leqq 1 \\ D_u \cup D_v & \text{otherwise.} \end{cases}$$

For $1 < i \leqq [(n+1)/2]$, define f_{2i-1} by:

$$W_{f_{2i-1}(x)} = \begin{cases} \emptyset & \text{if card } \{x_j | x_j \in K\} < 4i - 4 \\ D_u & \text{if } 4i - 4 \leqq \text{card } \{x_j | x_j \in K\} \leqq 4i - 3 \\ D_u \cup D_v & \text{otherwise} \end{cases}$$

and define f_{2i} by:

$$W_{f_{2i}(x)} = \begin{cases} \emptyset & \text{if card } \{x_j | x_j \in K\} < 4i - 3 \\ D_u & \text{if } 4i - 3 \leqq \text{card } \{x_j | x_j \in K\} \leqq 4i - 2 \\ D_u \cup D_v & \text{otherwise.} \end{cases}$$

(If n is odd f_{n+1} is not defined). So:

$$W_{f_1(x)} - W_{f_2(x)} = \begin{cases} D_u & \text{if card } \{x_j | x_j \in K\} = 0 \\ \emptyset & \text{if card } \{x_j | x_j \in K\} = 1 \\ D_v & \text{if card } \}x_j | x_j \in K\} = 2 \\ \emptyset & \text{otherwise} \end{cases}$$

and for $i > 1$,

$$W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \begin{cases} \emptyset & \text{if card } \{x_j | x_j \in K\} < 4i - 4 \\ D_u & \text{if card } \{x_j | x_j \in K\} = 4i - 4 \\ \emptyset & \text{if card } \{x_j | x_j \in K\} = 4i - 3 \\ D_v & \text{if card } \{x_j | x_j \in K\} = 4i - 2 \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $f = \langle f_1, \ldots, f_n \rangle$.

1) If $x \in K_n$ then card $\{x_j | x_j \in K\}$ is odd. From the above, we see that $W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \emptyset$ except when card $\{x_j | x_j \in K\} \equiv 0$ or $2 \pmod 4$, hence

$$V_{f(x)} = \bigcup_{i=1}^{[(n+1)/2]} (W_{f_{2i-1}(x)} - W_{f_{2i}(x)}) = \emptyset \notin C.$$

Thus $x \in K_n \rightarrow f(x) \notin \theta_n(C)$.

2) If $x \notin K_n$ then card $\{x_j | x_j \in K\}$ is even. Thus

$$V_{f(x)} = \begin{cases} D_u & \text{if card } \{x_j | x_j \in K\} \equiv 0 \pmod 4 \\ D_v & \text{if card } \{x_j | x_j \in K\} \equiv 2 \pmod 4 \end{cases}$$

Since $D_u, D_v \in C$, $x \notin K_n \rightarrow f(x) \in \theta_n(C)$. By 1) and 2), $K_n \leqq_1 \overline{\theta_n(C)}$.

LEMMA 4.5. *Let $n > 2$. Suppose $D_u \in C$,* card $D_u \geqq 1$ *and there exist D_{u_1}, $D_{u_2} \notin C$ such that $D_u = D_{u_1} \cup D_{u_2}$ and $D_{u_1} \cap D_{u_2} = \emptyset \notin C$. Then $K_n \leqq_1 \overline{\theta_n(C)}$.*

Proof. Define f_1 by:

$$W_{f_1(x)} = D_u$$

and for $1 < i \leqq [(n + 1)/2]$, define f_{2i-1} by:

$$W_{f_{2i-1}(x)} = \begin{cases} \emptyset & \text{if card } \{x_j | x_j \in K\} < 4i - 6 \\ D_{u_1} & \text{if } 4i - 6 \leqq \text{card } \{x_j | x_j \in K\} \leqq 4i - 5 \\ D_u & \text{otherwise.} \end{cases}$$

For $1 \leqq i \leqq [(n + 1)/2]$, define f_{2i} by:

$$W_{f_{2i}(x)} = \begin{cases} \emptyset & \text{if card } \{x_j | x_j \in K\} < 4i - 3 \\ D_{u_1} & \text{if } 4i - 3 \leqq \text{card } \{x_j | x_j \in K\} \leqq 4i - 2 \\ D_u & \text{otherwise.} \end{cases}$$

(If n is odd f_{n+1} is not defined).

For $i = 1$, we have:

$$W_{f_1(x)} - W_{f_2(x)} = \begin{cases} D_u & \text{if card } \{x_j | x_j \in K\} = 0 \\ D_{u_2} & \text{if } 1 \leqq \text{card } \{x_j | x_j \in K\} \leqq 2 \\ \emptyset & \text{otherwise} \end{cases}$$

and for $1 < i \leqq [(n+1)/2]$,

$$W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \begin{cases} \emptyset & \text{if card } \{x_j | x_j \in K\} < 4i - 6 \\ D_{u_1} & \text{if } 4i - 6 \leqq \text{card } \{x_j | x_j \in K\} \leqq 4i - 5 \\ D_u & \text{if card } \{x_j | x_j \in K\} = 4i - 4 \\ D_{u_2} & \text{if } 4i - 3 \leqq \text{card } \{x_j | x_j \in K\} \leqq 4i - 2 \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $f = \langle f_1, \ldots, f_n \rangle$.

1) If $x \notin K_n$ then card $\{x_j | x_j \in K\}$ is even. If card $\{x_j | x_j \in K\} = 4i_0 - 4$, then:

$$W_{f_{2i_0-1}(x)} - W_{f_{2i_0}(x)} = D_u \in C,$$

hence $V_{f(x)} = D_u \in C$ (since $\emptyset$, D_{u_1}, $D_{u_2} \subset D_u$).

If card $\{x_j | x_j \in K\} = 4i_0 - 2$ then:

$$W_{f_{2i_0-1}(x)} - W_{f_{2i_0}(x)} = D_{u_2}$$

and

$$W_{f_{2(i_0+1)-1}(x)} - W_{f_{2(i_0+1)}(x)} = D_{u_1},$$

hence

$$V_{f(x)} = D_{u_1} \cup D_{u_2} = D_u \in C.$$

Thus $x \notin K_n \rightarrow f(x) \in \theta_n(C)$.

2) If $x \in K_n$, then card $\{x_j | x_j \in K\}$ is odd.

If card $\{x_j | x_j \in K\} = 4i_0 - 3$, then:

$$W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \begin{cases} D_{u_2} & \text{if } i = i_0 \\ \emptyset & \text{if } i \neq i_0. \end{cases}$$

Thus

$$V_{f(x)} = D_{u_2} \notin C.$$

If card $\{x_j | x_j \in K\} = 4i_0 - 1$ then:

$$W_{f_{2i-1}(x)} - W_{f_{2i}(x)} = \begin{cases} D_{u_1} & \text{if } i = i_0 + 1 \\ \emptyset & \text{if } i \neq i_0 + 1 \end{cases}$$

hence

$$V_{f(x)} = D_{u_1} \notin C.$$

Thus $x \in K_n \rightarrow f(x) \notin \theta_n(C)$. By 1) and 2), $K_n \leqq_1 \overline{\theta_n(C)}$.

Now we may proceed to prove Theorem 4.1.

Proof of Theorem 4.1. If $\theta_n(C) \in \Sigma_n^{-1}$ then $\theta_n(C) \leqq_1 K_n$. If $\theta_n(C)$ is non-trivial then there exists some $V_{e,n} \in C$. Suppose no finite subset of $V_{e,n}$ is in C. Then by Lemma 4.3, $K' \leqq_1 \overline{\theta_n(C)}$, but since $\theta_n(C) \leqq_1 K_n$ we have that $K' \leqq_1 \bar{K}_n$. But $K_n \equiv_T K$ so we obtain $K' \leqq_T K$, a contradiction. Thus some finite subset of $V_{e,n}$ is in C.

Let D_{v0} be a set of minimal cardinality in $\{D_u \in C | D_u \subset V_{e,n}\}$. If card $D_{v0} > 1$ then by Lemma 4.5, $K_n \leqq_1 \overline{\theta_n(C)}$. But then since $\theta_n(C) \leqq_1 K_n$ we would have $K_n \leqq_1 \bar{K}_n$, a contradiction. Thus we may assume that the cardinality of D_{v0} is either 1 or 0.

Next we show that every Σ_n^{-1} extension of D_{v0} is in C. If some r.e. extension of D_{v0} is not in C, then by Lemma 3.2, $K_n \leqq_1 \overline{\theta_n(C)}$, a contradiction. If every r.e. extension of D_{v0} is in C but some Σ_n^{-1} extension of D_{v0} is not in C then by Lemma 3.3, $K' \leqq_1 \theta_n(C)$, a contradiction. Thus all Σ_n^{-1} extensions of D_{v0} are in C.

Now if $D_{v0} = \emptyset$, then $C = \Sigma_n^{-1}$, so $\theta_n(C)$ is trivial. Since we are assuming $\theta_n(C)$ is non-trivial, we know that card $D_{v0} = 1$.

We wish to show that $C = \{V_{e,n} | D_{v0} \subset V_{e,n}\}$. We know $\{V_{e,n} | D_{v0} \subset V_{e,n}\} \subset C$. Suppose there is some $V_{e',n} \in C$ such that $D_{v0} \not\subset V_{e',n}$. By an argument similar to the above, we know there exists a $D_{v1} \in C$ such that card $D_{v1} = 1$ and $D_{v1} \subset V_{e',n}$. Since $D_{v0} \not\subset V_{e',n}$, it follows that $D_{v1} \neq D_{v0}$ which implies that $D_{v0} \cap D_{v1} = \emptyset$. Hence by Lemma 4.4, $K_n \leqq_1 \overline{\theta_n(C)}$, but this is again a contradiction. Thus every set in C extends D_{v0}. If $D_{v0} = \{a\}$ then we have $C = \{V_{x,n} | a \in V_{x,n}\}$ which completes the proof.

Corollary 4.6. *Let C be any class of sets. Then for each $n > 2$, $\theta_n(C)$ is Σ_n^{-1}-complete if and only if there exists an a such that $C \cap \Sigma_n^{-1} = \{V_{e,n} \in \Sigma_n^{-1} | a \in V_{e,n}\}$.*

Proof. Since $\theta_n(C) = \theta_n(C \cap \Sigma_n^{-1})$, the result follows from Theorem 4.1 and Corollary 3.4.

5. Conclusions. In view of the last corollary of Section 4, and the r.e. and d.r.e. Rice-Shapiro Theorems, we see that if we let $C_a = \{X | a \in X\}$ then for each $n \geqq 1$, $\theta_n(C_a)$ is Σ_n^{-1}-complete and $\overline{\theta_n(C_a)}$ is Π_n^{-1}-complete, so we have a single class C_a whose index sets are complete at all the levels of the finite Ershov hierarchy. The situation is illustrated by the following diagram where "$\rightarrow$" means "$<_1$".

$$\begin{tikzcd}
\Sigma_1^{-1} & \Sigma_2^{-1} & \Sigma_3^{-1} & \cdots \\
\theta_1(C_a) \arrow[r] \arrow[dr] & \theta_2(C_a) \arrow[r] \arrow[dr] & \theta_3(C_a) \arrow[r] \arrow[dr] & \cdots \\
\theta_1(C_a) \arrow[r] \arrow[ur] & \theta_2(C_a) \arrow[r] \arrow[ur] & \theta_3(C_a) \arrow[r] \arrow[ur] & \cdots \\
\pi_1^{-1} & \pi_2^{-1} & \pi_3^{-1} & \cdots
\end{tikzcd}$$

We note that because of Corollary 4.6, only a class consisting of all sets which contain a single fixed element has this "uniform completeness" property.

Lemma 5.1. *Let C be any non-empty class of sets. For each $n \geqq 1$, $\theta_n(C) \leqq_1 \theta_{n+1}(C)$.*

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Proof. Let $x \in \theta_n(C)$ and let

$$V_{x,n} = \bigcup_{i=1}^{[(n+1)/2]} (W_{x2i-1} - W_{x2i}), \quad (W_{xn+1} = \emptyset),$$

where $x = \langle x_1, \ldots, x_n \rangle$. Let $x' = \langle x_1, \ldots, x_n, x_{n+1} \rangle$. Then x' is a Σ_{n+1}^{-1} index of

$$\bigcup_{i=1}^{[(n+2)/2]} (W_{x2i-1} - W_{x2i}), \quad (W_{xn+2} = \emptyset),$$

and $V_{x,n} = V_{x',n+1}$ Thus:

$$\begin{aligned} x \in \theta_n(C) &\leftrightarrow V_{x,n} \in C \\ &\leftrightarrow V_{x',n+1} \in C \\ &\leftrightarrow x' \in \theta_{n+1}(C). \end{aligned}$$

Now suppose there is a class C and some $n > 2$ such that $\theta_n(C)$ is Σ_n^{-1}-complete while $\theta_{n+1}(C)$ is not Σ_{n+1}^{-1}-complete. By Corollary 4.6, since $\theta_n(C)$ is Σ_n^{-1}-complete there exists an a such that $C \cap \Sigma_n^{-1} = \{V_{e,n} \in \Sigma_n^{-1} | a \in V_{e,n}\}$. Moreover, if $1 \leqq m \leqq n$, we have also that $C \cap \Sigma_m^{-1} = \{V_{e,m} \in \Sigma_m^{-1} | a \in V_{e,m}\}$ since $\Sigma_m^{-1} \subset \Sigma_n^{-1}$. Thus for $1 \leqq m \leqq n$, $\theta_m(C)$ is Σ_m^{-1}-complete.

Since we are assuming $\theta_{n+1}(C)$ is not Σ_{n+1}^{-1}-complete, then

$$C \cap \Sigma_{n+1}^{-1} \neq \{V_{e,n+1} \in \Sigma_{n+1}^{-1} | a \in V_{e,n+1}\}.$$

We know already that $\{a\} \in C \cap \Sigma_{n+1}^{-1}$ (since $\{a\} \in C \cap \Sigma_n^{-1}$) so either there exists a set $V_{e,n+1} \in \Sigma_{n+1}^{-1} - C$ with $a \in V_{e,n+1}$, or there exists a set $V_{e,n+1} \in C$ with $a \notin V_{e,n+1}$.

In the first case, by Lemma 3.3 we have $K' \leqq_1 \theta_{n+1}(C)$, hence by Lemma 5.1, $K' \leqq_1 \theta_m(C)$ for all $m \geqq n + 1$. In the second case since $\theta_n(C) \in \Sigma_n^{-1}$, every finite (hence Σ_n^{-1}) set in C must extend $\{a\}$. Since $a \notin V_{e,n+1}$, no finite subset of $V_{e,n+1}$ may be in C. So by Lemma 4.3 we have $K' \leqq_1 \overline{\theta_{n+1}(C)}$, hence by Lemma 5.1, $K' \leqq_1 \overline{\theta_m(C)}$ for each $m \geqq n + 1$. Thus we have for $m \geqq n + 1$ either $K' \leqq_1 \theta_m(C)$ or $K' \leqq_1 \overline{\theta_m(C)}$. So for such a class the index sets are complete for an initial segment of the finite Ershov hierarchy then "jump" to at least degree $\underset{\sim}{0}''$. This proves our final theorem.

THEOREM 5.2. *Let C be any non-empty class of sets. If there is some $n > 2$ such that $\theta_n(C)$ is Σ_n^{-1}-complete, but $\theta_{n+1}(C)$ is not Σ_{n+1}^{-1}-complete then:*

a) *For $1 \leqq m \leqq n$, $\theta_m(C)$ is Σ_m^{-1}-complete, and*

b) *for $m > n + 1$, either $K' \leqq_1 \theta_m(C)$ or $K' \leqq_1 \overline{\theta_m(C)}$.*

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