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Quasisymmetrically Minimal Moran Sets

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Abstract. M. Hu and S. Wen considered quasisymmetrically minimal uniform Cantor sets of Hausdorff dimension 1, where at the k-th set one removes from each interval I a certain number n_k of open subintervals of length $c_k|I|$, leaving $(n_k + 1)$ closed subintervals of equal length. Quasisymmetrically Moran sets of Hausdorff dimension 1 considered in the paper are more general than uniform Cantor sets in that neither the open subintervals nor the closed subintervals are required to be of equal length.

1 Introduction

It is well known that quasiconformal homeomorphisms of a Euclidean space $\mathbb{R}^n$, $n \geqslant 2$ can distort the Hausdorff dimension of subsets. For example, the von Koch snowflake is a quasiconformal image of the circle, but has dimension $\log 4/\log 3$. While the dimensions of sets of Hausdorff dimension zero or n must be preserved, Gehring and Väisälä [3] constructed for any $\beta \in (0, n)$, a compact set $E_\beta \subset \mathbb{R}^n$ with $\dim_{\mathcal{H}} E_\beta = \beta$ and for any $\beta, \beta' \in (0, n)$, a quasiconformal map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with $E_{\beta'} = f(E_\beta)$. Bishop [1] showed that the dimension of any compact set $E \subset \mathbb{R}^n$ of positive dimension can be raised arbitrarily close to n by a quasiconformal (quasisymmetric if $n = 1$) homeomorphisms of $\mathbb{R}^n$. Then Tyson [9] showed that for $1 \leqslant \alpha \leqslant n$ there is a compact set $E \subset \mathbb{R}^n$ with Hausdorff dimension α so that $\dim_{\mathcal{H}} f(E) \geqslant \alpha$ for all quasiconformal maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$. But according to Tukia [8], a set in $\mathbb{R}$ of Hausdorff dimension 1 may not be minimal for 1-dimensional quasisymmetric maps. On the other hand, by Kovalev [7], if $0 < \dim_{\mathcal{H}} E < 1$, then for every $\varepsilon > 0$ there is an n-dimensional quasisymmetric map f such that $\dim_{\mathcal{H}} f(E) < \varepsilon$. Thus, no sets in $\mathbb{R}^n$ of $\dim_{\mathcal{H}} \in (0, 1)$ can be quasisymmetricly minimal. Recently, Hakobyan [4] and Hu and Wen [5] proved that middle interval Cantor sets and uniform Cantor sets of Hausdorff dimension 1 are all minimal. These are the known examples of minimal sets in $\mathbb{R}$ of Hausdorff dimension 1. Our results hold for those Moran sets $E := E(\{n_k\}, \{\delta_k\}, \{c_k\})$ with $\dim_H E = 1$ for which any basic interval of order $k + 1$ is smaller than any basic interval of order k. These sets include the middle interval Cantor sets and uniform Cantor sets in [4, 5]. We prove that they are also minimal for 1-dimensional quasisymmetric maps (See Theorem 3.1) and illustrate Theorem 3.1 by Example 3.2.

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2 Preliminary

Let X, Y be metric spaces and $f\colon X \to Y$ be a topological homeomorphism. The map f is called *quasisymmetric* if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{|f(x)-f(a)|}{|f(x)-f(b)|} \le \eta\Big(\frac{|x-a|}{|x-b|}\Big) \tag{2.1}$$

for all triples a, b, x of distinct points in X. When $X = Y = \mathbb{R}^n$, we also say that f is an n-dimensional quasisymmetric map. We call a set $E \subset \mathbb{R}^n$ *quasisymmetrically minimal*, if $\dim_{\mathcal{H}} f(E) \ge \dim_{\mathcal{H}} E$ for any n-dimensional quasisymmetric map f.

By the definition of quasisymmetric maps, an increasing homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ is quasisymmetric if and only if

$$M^{-1} \le \frac{|f(I)|}{|f(J)|} \le M$$

for all pairs of adjacent intervals I, J of equal length, where $M = \eta(1)$, η is as in (2.1). In this case we also say that f is M-*quasisymmetric*. The following property of M-quasisymmetric maps is very useful for us.

Lemma 2.1 ([5, 10]) *Let f be an M-quasisymmetric map. Then*

$$(1+M)^{-2}\Big(\frac{|J|}{|I|}\Big)^{q} \le \frac{|f(J)|}{|f(I)|} \le 4\Big(\frac{|J|}{|I|}\Big)^{p}$$

for all pairs J, I of intervals with $J \subset I$, where

$$0 < p = \log_2(1+M^{-1}) \le 1 \le q = \log_2(1+M). \tag{2.2}$$

We define the Moran set $E := E(\{n_k\}, \{\delta_k\}, \{c_k\})$. Let $\{n_k\}_{k=1}^{\infty}$ be a bounded sequence of positive integers. Then $\{\delta_k\}_{k=1}^{\infty} = (\delta_{k,1}, \dots, \delta_{k,n_k+1})$ and $\{c_k\}_{k=1}^{\infty} = (c_{k,1}, \dots, c_{k,n_k})$ are sequences of real numbers in $(0,1)$ with

$$\sum_{j=1}^{n_k} c_{k,j} < 1, \quad \text{and} \quad \sum_{j=1}^{n_k+1} \delta_{k,j} + \sum_{j=1}^{n_k} c_{k,j} = 1$$

for each k. Suppose $\{E_k\}_{k=0}^{\infty}$ is a nested sequence of closed sets in $[0,1]$ satisfying the following conditions:

(i) For each $k \ge 1$, E_k is a union of disjoint closed intervals, *i.e.*, $E_k = \bigcup_{i=1}^{N_k} E_{k,i}$, where $N_k = \prod_{l=1}^{k}(n_l+1)$. (We call $E_{k,i}$ $(i = 1, \dots, N_k)$ the basic interval of order k).

(ii) Let $E_0 = [0,1]$. At level k, each interval I from E_{k-1} is replaced by $n_k + 1$ subintervals whose lengths are proportional to the $\delta_{k,j}$ $(j = 1, \dots, n_k+1)$ and the gaps between that are proportional to the $c_{k,j}$ $(j = 1, \dots, n_k)$. The leftmost one and I have the same left endpoint, and the rightmost one and I have the same right endpoint.

The set $E =: E(\{n_k\}, \{\delta_k\}, \{c_k\}) = \bigcap_{k=0}^{\infty} E_k$ is called a *Moran set.*

Lemma 2.2 ([6]) *If $E = E(\{n_k\}, \{\delta_k\}, \{c_k\})$ is a Moran set, then*

$$\dim_{\mathcal{H}} E = \liminf_{k\to\infty} s_k,$$

where $\{s_k\}_{k\geq 1}$ satisfies the equality

$$\prod_{i=1}^{k} \sum_{j=1}^{n_i+1} \delta_{i,j}^{s_k} = 1. \tag{2.3}$$

Lemma 2.3 *Let $E = E(\{n_k\}, \{\delta_k\}, \{c_k\})$ be a Moran set. If $\dim_{\mathcal{H}} E = 1$. Then*

(i)

$$\lim_{k\to\infty} \Big(\prod_{i=1}^{k} \sum_{j=1}^{n_i+1} \delta_{i,j}\Big)^{\frac{1}{k}} = 1.$$

(ii)

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \Big(\sum_{j=1}^{n_i} \frac{c_{i,j}}{n_i}\Big)^{\alpha} = 0$$

for any $0 < \alpha \leq 1$.

(iii)

$$\lim_{k\to\infty} \frac{\sharp\{i : 0 \leq i \leq k,\ \sum_{j=1}^{n_i} c_{i,j} \geq n_i\varepsilon\}}{k} = 0$$

for any $\varepsilon \in (0,1)$, where # denotes the cardinality.

Proof (i) From Hölder's inequality

$$\sum_{r=1}^{n} a_r b_r \geq \Big(\sum_{r=1}^{n} a_r^k\Big)^{\frac{1}{k}} \Big(\sum_{r=1}^{n} b_r^{k'}\Big)^{\frac{1}{k'}}, \quad \Big(k \leq 1 \text{ and } \frac{1}{k} + \frac{1}{k'} = 1\Big),$$

we get

$$\begin{aligned}\sum_{j=1}^{n_i+1} \delta_{i,j} &\geq \Big(\sum_{j=1}^{n_i+1} 1\Big)^{1-\frac{1}{s_k}} \Big(\sum_{j=1}^{n_i+1} \delta_{i,j}^{s_k}\Big)^{\frac{1}{s_k}} \\ &= (n_i+1)^{1-\frac{1}{s_k}} \Big(\sum_{j=1}^{n_i+1} \delta_{i,j}^{s_k}\Big)^{\frac{1}{s_k}}, \quad (i = 1, 2, \dots, k).\end{aligned}$$

Then

$$\begin{aligned}\prod_{i=1}^{k} \sum_{j=1}^{n_i+1} \delta_{i,j} &\geq \prod_{i=1}^{k} (n_i+1)^{1-\frac{1}{s_k}} \prod_{i=1}^{k} \Big(\sum_{j=1}^{n_i+1} \delta_{i,j}^{s_k}\Big)^{\frac{1}{s_k}} \\ &= \prod_{i=1}^{k} (n_i+1)^{1-\frac{1}{s_k}} \Big(\prod_{i=1}^{k} \sum_{j=1}^{n_i+1} \delta_{i,j}^{s_k}\Big)^{\frac{1}{s_k}}.\end{aligned}$$

From (2.3), we get

$$\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j} \geq \prod_{i=1}^{k}(n_i+1)^{1-1/s_k},$$

$$\log\Big(\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j}\Big) \geq \Big(1-\frac{1}{s_k}\Big)\sum_{i=1}^{k}\log(n_i+1).$$

It follows that

$$s_k \leq \frac{\sum_{i=1}^{k}\log(n_i+1)}{\sum_{i=1}^{k}\log(n_i+1)-\log\Big(\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j}\Big)}.$$

Note that $1 \geq \dim_{\mathcal{P}} E \geq \dim_{\mathcal{H}} E = 1$. We get

$$1=\lim_{k\to\infty}s_k \leq \lim_{k\to\infty}\frac{1}{1-\frac{\log(\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j})}{\sum_{i=1}^{k}\log(n_i+1)}}.$$

We can see that

$$\lim_{k\to\infty}\frac{\log(\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j})}{\sum_{i=1}^{k}\log(n_i+1)} \geq 0.$$

The reverse inequality is obvious, so

$$\lim_{k\to\infty}\frac{\log(\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j})}{\sum_{i=1}^{k}\log(n_i+1)} = 0.$$

Because $\{n_k\}$ is bounded, we can set $N=1+\sup_k\{n_k\}$. One has $\prod_{i=1}^{k}(n_i+1)\leq N^k$; it follows that

$$\lim_{k\to\infty}\frac{\log(\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j})}{k\log N} = 0.$$

Therefore,

$$\lim_{k\to\infty}\Big(\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j}\Big)^{1/k} = 1.$$

(ii) Since

$$\lim_{k\to\infty}\Big(\prod_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j}\Big)^{\frac{1}{k}} \leq \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{n_i+1}\delta_{i,j} = \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\Big(1-\sum_{j=1}^{n_i}c_{i,j}\Big)$$

$$= 1-\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}c_{i,j}.$$

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By conclusion (i), we have

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\le\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}c_{i,j}=0,$$

which together with Jensen's inequality yields

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^{\alpha}\le\lim_{k\to\infty}\Big(\frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^{\alpha}=0$$

for any $0<\alpha\le 1$.

(iii). Fix $\varepsilon\in(0,1)$. Then we have from conclusion (ii) that

$$\frac{\sharp\{i:0\le i\le k,\ \sum_{j=1}^{n_i}c_{i,j}\ge n_i\varepsilon\}}{k}\le\frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i\varepsilon}=\frac{1}{k\varepsilon}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\to 0$$

as k tends to ∞. ■

Lemma 2.4 *Let $a=1-\sqrt[4N]{\frac{4N}{4N+1}}$. One has $1-4mx\ge(1-x)^{4m+1}$ for all $x\in(0,a)$ and positive integers $m\le N$.*

Proof Let $f(x)=1-4mx-(1-x)^{4m+1}$, $(m\le N, 0<x<a)$. We consider the function

$$g(x)=\Big(\frac{4x}{4x+1}\Big)^{1/4x}.$$

Because

$$g'(x)=\Big[\frac{1}{4x^2}\ln\Big(\frac{4x+1}{4x}\Big)+\frac{1}{4x^2(4x+1)}\Big]\,g(x)>0,\quad\text{for } x>0.$$

We can get

$$\Big(\frac{4m}{4m+1}\Big)^{\frac{1}{4m}}\le\Big(\frac{4N}{4N+1}\Big)^{\frac{1}{4N}}=1-a<1-x.$$

It follows that

$$f'(x)=-4m+(4m+1)(1-x)^{4m}>-4m+(4m+1)\frac{4m}{4m+1}=0.$$

So $f(x)\ge f(0)=0$. That is, $1-4mx\ge(1-x)^{4m+1}$ for all $x\in(0,a)$ and positive integers $m\le N$. ■

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3 Main Theorem

Theorem 3.1 *Let* $E = E(\{n_k\}, \{\delta_k\}, \{c_k\})$ *be a Moran set with* $\dim_H E = 1$ *for which any basic interval of order* $k+1$ *is smaller than any basic interval of order* k. *Then* $\dim_{\mathcal{H}} f(E) = 1$ *for all* 1*-dimensional quasisymmetric maps* f.

Proof In order to prove $\dim_{\mathcal{H}} f(E) \geq 1$, it suffices to show that $\dim_{\mathcal{H}} f(E) \geq d$ for any $d \in (0, 1)$. For this purpose, given $d \in (0, 1)$, a probability measure μ on $f(E)$ will be defined so that the inequality

$$\mu(J) \leq C|J|^d \tag{3.1}$$

holds for any interval $J \subset \mathbb{R}$, where C is a positive constant independent of J. Then the mass distribution principle yields $\dim_{\mathcal{H}} f(E) \geq d$ (see [2]). Let $f\colon \mathbb{R} \to \mathbb{R}$ be an M-quasisymmetric map and $d \in (0, 1)$. Without loss of generality, assume that $f([0, 1]) = [0, 1]$.

First note that E has a tree structure where each level $k - 1$ parent interval has exactly $n_k + 1$ children, and the same is true for $f(E)$. Now we define a probability measure μ on $f(E)$ as follows. Let $\mu([0, 1]) = 1$. For every $k \geq 1$ and every component interval J_{k-1} of $f(E_{k-1})$, let $J_{k,0}, J_{k,1}, \dots, J_{k,n_k}$ denote the $n_k + 1$ component intervals of lying in J_{k-1}. Define

$$\mu(J_{k,i}) = \frac{|J_{k,i}|^d}{\|J_{k-1}\|_d} \mu(J_{k-1}), \quad i = 0, 1, \dots, n_k,$$

where $\|J_{k-1}\|_d = \sum_{i=0}^{n_k} |J_{k,i}|^d$. We are going to show that the measure μ satisfies $\mu(J) \leq C|J|^d$ for any $J \subset [0, 1]$, where C is a positive constant independent of J, we do this in two steps.

Step 1. Suppose that J is a component interval of $f(E_k)$. For every i, $0 \leq i \leq k$, let J_i be the component interval of $f(E_i)$ such that

$$J = J_k \subset J_{k-1} \subset \cdots \subset J_1 \subset J_0 = [0, 1]. \tag{3.2}$$

Then by the definition of the measure μ,

$$\frac{\mu(J)}{|J|^d} = \frac{1}{\|J_{k-1}\|_d} \frac{|J_{k-1}|^d}{\|J_{k-2}\|_d} \cdots \frac{|J_1|^d}{\|J_0\|_d} = \frac{|J_{k-1}|^d}{\|J_{k-1}\|_d} \cdots \frac{|J_1|^d}{\|J_1\|_d} \frac{|J_0|^d}{\|J_0\|_d}.$$

Let

$$r_i = \frac{\|J_i\|_d}{|J_i|^d}, \quad i = 0, 1, \dots, k - 1.$$

The above equality can be rewritten as

$$\frac{\mu(J)}{|J|^d} = \Big(\prod_{i=1}^{k} r_{i-1}\Big)^{-1}.$$

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To prove (3.1), it suffices to show

$$\lim_{k\to\infty}\prod_{i=1}^{k} r_{i-1}=\infty \quad \text{uniformly.} \tag{3.3}$$

Given an $i, 1 \le i \le k$, we are going to estimate r_{i-1}. Let J_{i-1} be the component interval of $f(E_{i-1})$ in the sequence (3.2). Recall that $J_i \subset J_{i-1}$ is a component interval of $f(E_i)$. Let $J_{i,1}, J_{i,2}, \dots, J_{i,n_i}$ be the other n_i component intervals of $f(E_i)$ lying in J_{i-1}. Let $G_{i,1}, G_{i,2}, \dots, G_{i,n_i}$ be the n_i gaps between these $n_i + 1$ intervals. Put

$$\begin{aligned} I_{i-1} &= f^{-1}(J_{i-1}),\\ I_i &= f^{-1}(J_i),\\ I_{i,j} &= f^{-1}(J_{i,j}), \quad j=1,\dots,n_i. \end{aligned}$$

Then $I_i, I_{i,1}, I_{i,2}, \dots, I_{i,n_i}$ are basic intervals of E_i lying in the basic interval I_{i-1} of E_{i-1}. Since f is M-quasisymmetric, it follows from Lemma 2.1 and the construction of E that

$$\frac{|G_{i,j}|}{|J_{i-1}|} \le 4c_{i,j}^{p}, \quad j=1,2,\dots,n_i, \tag{3.4}$$

and that

$$\begin{aligned} \max_j \frac{|J_{i,j}|}{|J_{i-1}|} &\ge \max_j\Big\{\frac{1}{(1+M)^2}\Big(\frac{|I_{i,j}|}{|I_{i-1}|}\Big)^{q}\Big\}\\ &\ge \frac{1}{(1+M)^2}\Big(\frac{1-\sum_{j=1}^{n_i} c_{i,j}}{n_i+1}\Big)^{q}\\ &\ge \frac{1}{(1+M)^2}\Big(\frac{1-\sum_{j=1}^{n_i} c_{i,j}}{N}\Big)^{q}, \end{aligned} \tag{3.5}$$

where p, q are numbers defined as in (2.2). The equality (3.4) yields

$$\begin{aligned} \frac{|J_i|+|J_{i,i}|+\dots+|J_{i,n_i}|}{|J_{i-1}|} &= \frac{|J_{i-1}|-|G_{i,1}|-\dots-|G_{i,n_i}|}{|J_{i-1}|}\\ &\ge 1-4\sum_{j=1}^{n_i} c_{i,j}^{p}. \end{aligned} \tag{3.6}$$

The equality (3.5) implies that

$$r_{i-1}=\frac{|J_i|^d+|J_{i,1}|^d+\dots+|J_{i,n_i}|^d}{|J_{i-1}|^d} \ge \alpha_1\Big(1-\sum_{j=1}^{n_i} c_{i,j}\Big)^{dq}, \tag{3.7}$$

where $\alpha_1 = (1+M)^{-2d}N^{-dq}$.

The estimate (3.7) is not enough to give (3.3); we need a more explicit form of r_{i-1} for some i. Let

$$S(k,p)=\Big\{ i : 1\le i\le k, \Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p < a \Big\}.$$

Then by conclusion (iii) of Lemma 2.3, one has

$$\lim_{k\to\infty}\frac{\sharp S(k,p)}{k}=1. \tag{3.8}$$

It is obvious that

$$\frac{x_0^d+x_1^d+\cdots+x_{n_i}^d}{(x_0+x_1+\cdots+x_{n_i})^d}>1$$

for $0<d<1, x_i>0, i=0,1,\ldots,n_i$. So

$$\frac{|J_i|^d+|J_{i,1}|^d+\cdots+|J_{i,n_i}|^d}{(|J_i|+|J_{i,1}|+\cdots+|J_{i,n_i}|)^d}=\frac{x_0^d+x_1^d+\cdots+x_{n_i}^d}{(x_0+x_1+\cdots+x_{n_i})^d}\triangleq\alpha_2>1, \tag{3.9}$$

where $x_0=\frac{|J_i|}{|J_{i-1}|}$ and $x_j=\frac{|J_{i,j}|}{|J_{i-1}|}$.

For $i\in S(k,p)$, we get from (3.6) and (3.9),

$$r_{i-1}=\frac{|J_i|^d+|J_{i,1}|^d+\cdots+|J_{i,n_i}|^d}{|J_{i-1}|^d}\ge\alpha_2\Big[1-4\sum_{j=1}^{n_i}c_{i,j}^p\Big]^d. \tag{3.10}$$

Note that $n_i<N$ and $(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i})^p\in S(k,p)$. From Hölder's inequality, we get $\sum_{j=1}^{n_i}c_{i,j}^p\le n_i(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i})^p$. Then Lemma 2.4 together with (3.10) yields

$$r_{i-1}\ge\alpha_2\Big[1-4n_i\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p\Big]^d\ge\alpha_2\Big[1-\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p\Big]^{(4n_i+1)d}. \tag{3.11}$$

Now we are in a position to prove (3.3). Using the estimate (3.7) for $i\notin S(k,p)$ and the estimate (3.11) for $i\in S(k,p)$, we get

$$\begin{aligned}\prod_{i=1}^k r_{i-1}&\ge\prod_{i\notin S(k,p)}\alpha_1\Big(1-\sum_{j=1}^{n_i}c_{i,j}\Big)^{dq}\prod_{i\in S(k,p)}\alpha_2\Big[1-\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p\Big]^{(4n_i+1)d}\\&\ge\xi_k\eta_k,\end{aligned} \tag{3.12}$$

where

$$\xi_k=\alpha_1^{k-\sharp S(k,p)}\prod_{i\notin S(k,p)}\Big(1-\sum_{j=1}^{n_i}c_{i,j}\Big)^{dq}\alpha_2^{\sharp S(k,p)},$$

$$\eta_k=\prod_{i\in S(k,p)}\Big[1-\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p\Big]^{(4n_i+1)d}.$$

It is clear that

$$\lim_{k\to\infty} \xi_k^{\frac{1}{k}} = \alpha_2 \tag{3.13}$$

due to conclusion (i) of Lemma 2.3 and equality (3.8). On the other hand, since $\log(1-x) \geq -2x$ for all $0 < x < 1$, conclusion (ii) of Lemma 2.3 together with the equality (3.8) yields

$$\begin{aligned}
\frac{1}{k}\log\eta_k &= \frac{1}{k}\log\prod_{i\in S(k,p)}\Big[1-\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p\Big]^{(4n_i+1)d}\\
&\geq \frac{4Nd}{k}\sum_{i\in S(k,p)}\log\Big[1-\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p\Big]\\
&\geq \frac{-8Nd}{k}\sum_{i\in S(k,p)}\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p\\
&\geq \frac{-8Nd}{k}\sum_{i=1}^{k}\Big(\sum_{j=1}^{n_i}\frac{c_{i,j}}{n_i}\Big)^p \to 0
\end{aligned}$$

as k tends to ∞. This implies

$$\lim_{k\to\infty} \eta_k^{\frac{1}{k}} = 1. \tag{3.14}$$

It follows from (3.12), (3.13), and (3.14) that

$$\lim_{k\to\infty}\inf\Big(\prod_{i=1}^{k} r_{i-1}\Big)^{\frac{1}{k}} \geq \alpha_2.$$

As $\alpha_2 > 1$, equality (3.3) then follows.

Step 2. It remains to prove (3.1) for any interval $J \subset [0,1]$. The length $d_{k,i}$ ($i = 1,\dots,N_k$) of each basic interval of order k can be written as $d_{k,i} = \delta_{1,i_1}\delta_{2,i_2}\cdots\delta_{k,i_k}$, $i_l = 1,\dots,n_l+1$, $l = 1,\dots,k$, and listed as follows:

$$\begin{aligned}
&d_{1,1}, d_{1,2}, \dots, d_{1,N_1},\\
&d_{2,1}, d_{2,2}, \dots, d_{2,N_2},\\
&\qquad\cdots\\
&d_{k-2,1}, d_{k-2,2}, \dots, d_{k-2,N_{k-2}},\\
&d_{k-1,1}, d_{k-1,2}, \dots, d_{k-1,N_{k-1}},\\
&d_{k,1}, d_{k,2}, \dots, d_{k,N_k},\\
&d_{k+1,1}, d_{k+1,2}, \dots, d_{k+1,N_{k+1}},\\
&\qquad\cdots
\end{aligned}$$

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We let $d_{k,m} = \min_i d_{k,i}$, $d_{k,M} = \max_i d_{k,i}$. Note that $d_{k,m} \geq d_{k+1,M}$. So for any interval $J \subset [0,1]$, let k be the unique positive integer such that $|f^{-1}(J)|$ has only two cases:

Case 1: $d_{k-1,m} \leq |f^{-1}(J)| < d_{k-1,M}$. (This case will appear when $d_{k-1,m} \neq d_{k-1,M}$).

Case 2: $d_{k,M} \leq |f^{-1}(J)| < d_{k-1,m}$.

First we discuss Case 1. Note that

$$d_{k,M} < d_{k-1,m} \leq |f^{-1}(J)| < d_{k-1,M} < d_{k-2,m}.$$

We find that the set $f^{-1}(J)$ meets at most two basic intervals of E_{k-2} and hence at most $2(n_{k-1}+1)(n_k+1)$ basic intervals of E_k. Equivalently, the set J meets at most $2(n_{k-1}+1)(n_k+1)$ component intervals of $f(E_k)$.

Let $J_1, J_2, \dots, J_h, h \leq 2(n_{k-1}+1)(n_k+1)$, be those component intervals of $f(E_k)$ meet J. Using the conclusion of Step 1, we get

$$\mu(J) \leq \mu(J_1) + \mu(J_2) + \cdots + \mu(J_h) \leq C \sum_{i=1}^{h} |J_i|^d. \tag{3.15}$$

In addition, since $d_{k,M} \leq |f^{-1}(J)|$, we easily see that $f^{-1}(J_i) \subset 3f^{-1}(J)$, $i = 1, \dots, h$, where $3f^{-1}(J)$ is the interval of length $|3f^{-1}(J)|$ concentric with $f^{-1}(J)$. So by Lemma 2.1, we have

$$|J_i| \leq |f(3f^{-1}(J))| \leq 3^q(1+M)^2|J|, \quad i = 1, 2, \dots, h,$$

where $q = \log_2(1+M)$. Let $K = 3^q(1+M)^2 > 0$ be a constant depending only on M. This together with (3.15) gives

$$\mu(J) \leq Ch(K|J|)^d \leq 2N^2CK^d|J|^d.$$

This proves $\mu(J) \leq C|J|^d$.

Now we discuss Case 2. Note that $d_{k,M} \leq |f^{-1}(J)| < d_{k-1,m} \leq d_{k-1,M}$.

We find that the set $f^{-1}(J)$ meets at most two basic intervals of E_{k-1} and hence at most $2(n_k+1)$ basic intervals of E_k. Equivalently, the set J meets at most $2(n_k+1)$ component intervals of $f(E_k)$.

Let $J_1, J_2, \dots, J_l, l \leq 2(n_k+1)$, be those component intervals of $f(E_k)$ meeting J. Using the conclusion of Step 1, we get

$$\mu(J) \leq \mu(J_1) + \mu(J_2) + \cdots + \mu(J_l) \leq C \sum_{i=1}^{l} |J_i|^d. \tag{3.16}$$

In addition, since $d_{k,M} \leq |f^{-1}(J)|$, we easily see that $f^{-1}(J_i) \subset 3f^{-1}(J)$, $i = 1, \dots, l$. So we have $|J_i| \leq |f(3f^{-1}(J))| \leq K|J|$, $i = 1, 2, \dots, l$. This together with (3.16) gives $\mu(J) \leq ClK^d|J|^d \leq 2NCK^d|J|^d$. This proves $\mu(J) \leq C|J|^d$. ■

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Exmple 3.2 We construct a class of non-uniform Moran sets

$$E = E(\{n_k\}, \{\delta_k\}, \{c_k\}).$$

Let $n_k = 2$ for all positive k. Then $N_k = 3^k$, $\{\delta_k\}_k^\infty = (\delta_{k,1}, \delta_{k,2}, \delta_{k,3})$, and $\{c_k\}_{k=1}^\infty = (c_{k,1}, c_{k,2})$. Let $0 < \lambda < \frac{1}{3}$ and $0 < \varepsilon < \lambda$. We choose $\{\delta_k\}_k^\infty$ and $\{c_k\}_{k=1}^\infty$ as follows.

(i) Let $\delta_{1,1} = \frac{1-\lambda+\varepsilon^2}{3}, \delta_{1,2} = \frac{1-\lambda}{3}, \delta_{1,3} = \frac{1-\lambda-\varepsilon^2}{3}$. And $c_{1,1}$, $c_{1,2}$ in $(0,1)$ satisfy $c_{1,1} + c_{1,2} = \lambda$.

(ii) Let

$$\delta_{2,1} = \frac{1-\lambda-\lambda^2+\varepsilon^3}{3(1-\lambda)}, \quad \delta_{2,2} = \frac{1-\lambda-\lambda^2}{3(1-\lambda)}, \quad \delta_{2,3} = \frac{1-\lambda-\lambda^2-\varepsilon^3}{3(1-\lambda)}.$$

And $c_{2,1}, c_{2,2}$ in $(0,1)$ satisfy $c_{2,1} + c_{2,2} = \frac{\lambda^2}{1-\lambda}$.

(iii) Let

$$\delta_{k,1} = \frac{1-\lambda-\lambda^2-\cdots-\lambda^k+\varepsilon^{k+1}}{3(1-\lambda-\cdots-\lambda^{k-1})},$$
$$\delta_{k,2} = \frac{1-\lambda-\lambda^2-\cdots-\lambda^k}{3(1-\lambda-\cdots-\lambda^{k-1})},$$
$$\delta_{k,3} = \frac{1-\lambda-\lambda^2-\cdots-\lambda^k-\varepsilon^{k+1}}{3(1-\lambda-\cdots-\lambda^{k-1})}.$$

And $c_{k,1}, c_{k,2}$ in $(0,1)$ satisfy $c_{k,1} + c_{k,2} = \frac{\lambda^k}{1-\lambda-\cdots-\lambda^{k-1}}$.

(iv) Let

$$\delta_{k+1,1} = \frac{1-\lambda-\lambda^2-\cdots-\lambda^{k+1}+\varepsilon^{k+2}}{3(1-\lambda-\cdots-\lambda^k)},$$
$$\delta_{k+1,2} = \frac{1-\lambda-\lambda^2-\cdots-\lambda^{k+1}}{3(1-\lambda-\cdots-\lambda^k)},$$
$$\delta_{k+1,3} = \frac{1-\lambda-\lambda^2-\cdots-\lambda^{k+1}-\varepsilon^{k+2}}{3(1-\lambda-\cdots-\lambda^k)}.$$

And $c_{k+1,1}, c_{k+1,2}$ in $(0,1)$ satisfy $c_{k+1,1} + c_{k+1,2} = \frac{\lambda^{k+1}}{1-\lambda-\cdots-\lambda^k}$.

We claim that the class of non-uniform Moran sets satisfies the conditions in Theorem 3.1.

Proof (i) We have $d_{1,1} = \delta_{1,1}, d_{1,2} = \delta_{1,2}, d_{1,3} = \delta_{1,3}$,

$$\sum_{j=1}^{3} d_{1,j} = 1-\lambda, \quad d_{1,m} = d_{1,3} = \frac{1-\lambda-\varepsilon^2}{3} > \frac{1-2\lambda}{3}.$$

(ii) Notice that

$$\begin{aligned} d_{2,1} &= d_{1,1}\delta_{2,1}, & d_{2,2} &= d_{1,1}\delta_{2,2}, & d_{2,3} &= d_{1,1}\delta_{2,3},\\ d_{2,4} &= d_{1,2}\delta_{2,1}, & d_{2,5} &= d_{1,2}\delta_{2,2}, & d_{2,6} &= d_{1,2}\delta_{2,3},\\ d_{2,7} &= d_{1,3}\delta_{2,1}, & d_{2,8} &= d_{1,3}\delta_{2,2}, & d_{2,9} &= d_{1,3}\delta_{2,3}. \end{aligned}$$

We have

$$\begin{aligned} \sum_{j=1}^{9} d_{2,j} &= (d_{1,1}+d_{1,2}+d_{1,3})(\delta_{2,1}+\delta_{2,2}+\delta_{2,3}) = (1-\lambda)\frac{1-\lambda-\lambda^2}{1-\lambda}\\ &= 1-\lambda-\lambda^2, \end{aligned}$$

and

$$\begin{aligned} d_{2,M} = d_{2,1} &= \frac{1-\lambda+\varepsilon^2}{3}\cdot\frac{1-\lambda-\lambda^2+\varepsilon^3}{3(1-\lambda)}\\ &= \frac{\Big(1-(\lambda-\varepsilon^2)\Big)\Big(1-\lambda-(\lambda^2-\varepsilon^3)\Big)}{3^2(1-\lambda)}\\ &< \frac{1}{3^2}. \end{aligned}$$

Then

$$d_{1,m}-d_{2,M} > \frac{1-2\lambda}{3}-\frac{1}{3^2} = \frac{2(1-3\lambda)}{3^2} > 0.$$

It follows that $d_{1,m} > d_{2,M}$.

(iii) Notice that

$$\begin{aligned} d_{k,1} &= d_{k-1,1}\delta_{k,1}, & d_{k,2} &= d_{k-1,1}\delta_{k,2}, & d_{k,3} &= d_{k-1,1}\delta_{k,3},\\ d_{k,4} &= d_{k-1,2}\delta_{k,1}, & d_{k,5} &= d_{k-1,2}\delta_{k,2}, & d_{k,6} &= d_{k-1,2}\delta_{k,3},\\ &\cdots & &\cdots & &\cdots\\ d_{k,3^k-2} &= d_{k-1,3^{k-1}}\delta_{k,1}, & d_{k,3^k-1} &= d_{k-1,3^{k-1}}\delta_{k,2}, & d_{k,3^k} &= d_{k-1,3^{k-1}}\delta_{k,3}. \end{aligned}$$

We have

$$\begin{aligned} \sum_{j=1}^{3^k} d_{k,j} &= \Big(\sum_{j=1}^{3^{k-1}} d_{k-1,j}\Big)(\delta_{k,1}+\delta_{k,2}+\delta_{k,3})\\ &= (1-\lambda-\cdots-\lambda^{k-1})\frac{1-\lambda-\cdots-\lambda^k}{1-\lambda-\cdots-\lambda^{k-1}}\\ &= 1-\lambda-\cdots-\lambda^k, \end{aligned}$$

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and

$$\begin{aligned}
d_{k,m} &= d_{k,3^k} = \delta_{1,3}\delta_{2,3}\cdots\delta_{k,3}\\
&= \frac{1-\lambda-\varepsilon^2}{3}\cdot\frac{1-\lambda-\lambda^2-\varepsilon^3}{3(1-\lambda)}\cdots\frac{1-\lambda\cdots-\lambda^k-\varepsilon^{k+1}}{3(1-\lambda\cdots-\lambda^{k-1})}\\
&> \frac{\begin{array}{l}(1-\lambda-\lambda^2)\cdots(1-\lambda\cdots-\lambda^{k-1})\\ \qquad\cdot(1-\lambda\cdots-\lambda^{k-1}-\lambda^k)(1-\lambda\cdots-\lambda^k-\lambda^{k+1})\end{array}}{3^k(1-\lambda)(1-\lambda-\lambda^2)\cdots(1-\lambda\cdots-\lambda^{k-1})}\\
&= \frac{(1-\lambda\cdots-\lambda^k)(1-\lambda\cdots-\lambda^k-\lambda^{k+1})}{3^k(1-\lambda)}\\
&> \frac{(1-\lambda\cdots-\lambda^k-\cdots)(1-\lambda\cdots-\lambda^k-\lambda^{k+1}-\cdots)}{3^k(1-\lambda)}\\
&= \frac{(1-2\lambda)^2}{3^k(1-\lambda)^3}.
\end{aligned}$$

Since $(1-2\lambda)-(1-\lambda)^3 = \lambda(1-3\lambda+\lambda^2) > 0$, we have $d_{k,m} > \frac{1-2\lambda}{3^k}$.

(iv) Notice that

$$\begin{array}{lll}
d_{k+1,1} = d_{k,1}\delta_{k+1,1}, & d_{k+1,2} = d_{k,1}\delta_{k+1,2}, & d_{k,3} = d_{k,1}\delta_{k+1,3},\\
d_{k+1,4} = d_{k,2}\delta_{k+1,1}, & d_{k+1,5} = d_{k,2}\delta_{k+1,2}, & d_{k,6} = d_{k,2}\delta_{k+1,3},\\
\cdots & \cdots & \cdots\\
d_{k+1,3^{k+1}-2} = d_{k,3^k}\delta_{k+1,1}, & d_{k+1,3^{k+1}-1} = d_{k,3^k}\delta_{k+1,2}, & d_{k+1,3^{k+1}} = d_{k,3^k}\delta_{k+1,3}.
\end{array}$$

We have

$$\begin{aligned}
\sum_{j=1}^{3^{k+1}} d_{k+1,j} &= \Big(\sum_{j=1}^{3^k} d_{k,j}\Big)(\delta_{k+1,1}+\delta_{k+1,2}+\delta_{k+1,3})\\
&= (1-\lambda-\cdots-\lambda^k)\frac{1-\lambda-\cdots-\lambda^{k+1}}{1-\lambda-\cdots-\lambda^k}\\
&= 1-\lambda-\cdots-\lambda^{k+1},
\end{aligned}$$

$$\begin{aligned}
d_{k+1,M} &= d_{k+1,1} = \delta_{1,1}\delta_{2,1}\cdots\delta_{k+1,1}\\
&= \frac{1-\lambda+\varepsilon^2}{3}\cdot\frac{1-\lambda-\lambda^2+\varepsilon^3}{3(1-\lambda)}\cdots\frac{1-\lambda-\cdots-\lambda^k+\varepsilon^{k+1}}{3(1-\lambda-\cdots-\lambda^{k-1})}\\
&\quad\cdot\frac{1-\lambda-\cdots-\lambda^{k+1}+\varepsilon^{k+2}}{3(1-\lambda-\cdots-\lambda^k)}\\
&< \frac{\begin{array}{l}(1-\lambda+\lambda^2)(1-\lambda-\lambda^2+\lambda^3)\cdots(1-\lambda-\cdots-\lambda^k+\lambda^{k+1})\\ \qquad\cdot(1-\lambda-\cdots-\lambda^{k+1}+\lambda^{k+2})\end{array}}{3^{k+1}(1-\lambda)\cdots(1-\lambda-\cdots-\lambda^{k-1})(1-\lambda-\cdots-\lambda^k)}
\end{aligned}$$

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$$< \frac{1 \cdot (1-\lambda) \cdots (1-\lambda-\cdots-\lambda^{k-1})(1-\lambda-\cdots-\lambda^k)}{3^{k+1}(1-\lambda)\cdots(1-\lambda-\cdots-\lambda^{k-1})(1-\lambda-\cdots-\lambda^k)}$$
$$= \frac{1}{3^{k+1}}.$$

Hence

$$d_{k,m} - d_{k+1,M} > \frac{1-2\lambda}{3^k} - \frac{1}{3^{k+1}} = \frac{2(1-3\lambda)}{3^{k+1}} > 0.$$

It follows that $d_{k,m} > d_{k+1,M}$.

Let $\mathcal{L}(E)$ denote the Lebesgue measure of E. From (1)–(4), we can get

$$\mathcal{L}(E) = 1 - \lambda - \lambda^2 - \cdots - \lambda^k - \cdots = \frac{1-2\lambda}{1-\lambda} > 0.$$

It then follows that $\dim_{\mathcal{H}} E(\{n_k\}, \{\delta_k\}, \{c_k\}) = 1$. ■

The class of non-uniform Moran sets that we constructed satisfies the conditions in Theorem 3.1. Therefore, $\dim_{\mathcal{H}} f(E) = 1$ for all 1-dimensional quasisymmetric maps f.

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