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Duality for automorphisms on a compact C^*-dynamical system

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1. *Introduction*

When considering an action α of a compact group G on a C^*-algebra A, the notion of an α-invariant Hilbert space in A has proved extremely useful [**1**, **4**, **8**, **14**, **17**, **18**]. Following Roberts [**13**] a Hilbert space in (a unital algebra) A is a closed subspace H of A such that x^*y is a scalar for all x, y in H. For example if G is abelian, and α is ergodic in the sense that the fixed point algebra A^α is trivial, then A is generated as a Banach space by a unitary in each of the spectral subspaces

$$A^\alpha(\gamma) = \{x \in A: \alpha_g(x) = \langle g, \gamma\rangle x, g \in G\}, \qquad \gamma \in \hat{G},$$

which are then invariant one dimensional Hilbert spaces. If G is not abelian, then Hilbert spaces (which are always assumed to be invariant) do not necessarily exist, even for ergodic actions. For non-ergodic actions, it is also desirable to relax the requirement to x^*y being a constant multiple of some positive element of A^α. More generally, if γ is a finite dimensional matrix representation of G and n is a positive integer, we define $A_n^\alpha(\gamma)$ to be the subspace

$$\{x \in A \otimes M_{nd}: (\alpha_g \otimes 1)x = x(1 \otimes \gamma_g), \qquad g \in G\},$$

where d is the dimension $d(\gamma)$ of γ, and M_{nd} denotes $n \times d$ complex matrices. (Usually we will denote the extended action of α_g to $\alpha_g \otimes 1$ on $A \otimes M_{nd}$ again by α_g.) Let $A^\alpha(\gamma) = \{x_i: (x_i) \in A_1^\alpha(\gamma)\}$.

If $x, y \in A_n^\alpha(\gamma)$, then $xy^* \in A^\alpha \otimes M_n$, but x^*y is not necessarily in $A^\alpha \otimes M_d$, even for ergodic actions. For ergodic actions, the situation of full multiplicity, where there exists a unitary in $A_d^\alpha(\gamma)$, has been studied by Wasserman [**18**]. Techniques exist for handling C^*-dynamical systems, where Hilbert spaces exist in this sense, or at least when there is one non-zero x in $A_n^\alpha(\gamma)$ for some n, and $\gamma \in \hat{G}$, such that $x^*x = 1$ or more generally $x^*x \in A^\alpha \otimes 1$, [**3**, **8**]. (If such x exists, the space spanned by the d column vectors of x is a Hilbert space.) Note also that Araki *et al.* [**1**, **17**] avoided such difficulties for von Neumann algebras, by stabilising for example.

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Our first result, namely theorem 2.1 can be regarded as a technique for generating Hilbert spaces. Let α be an action of a compact group G on a separable C^*-algebra A, for which there exists an α-invariant pure state ω with GNS triple (π, H, Ω). If $H^u(\gamma)$ are spectral subspaces for the induced action u of G on H, ρ the restriction of π to A^α, we let $J_\gamma^\varphi = J_\gamma$ denote the ideal $\ker(\rho | H^u(\gamma))$, if $\gamma \in \hat{G}$. Then we show in § 2 that for any $b \in A^\alpha \backslash J_\gamma$ there exists $x \in \overline{bA_d^\alpha(\gamma)}$ such that $x^*x \in (A^\alpha \backslash J_\iota) \otimes 1$, where ι denotes the trivial representation. In [**8**], a Γ-spectrum was introduced which was useful in obtaining a covariant version of Glimm's theorem on non-type I C^*-algebras. In theorem 2.5, we characterise such a Γ-spectrum in terms of the kernels $\{J_\gamma : \gamma \in \hat{G}\}$. More precisely, if there exists a pure invariant state ω as before, let Γ_ω denote

$$\{\gamma \in \hat{G} : \forall b, c \in A^\alpha \backslash J_\iota, \exists x \in \overline{bA_1^\alpha(\gamma)c}, \quad \text{such that} \quad x^*x \in A^\alpha \backslash J_\iota \otimes 1_d\}.$$

If in addition to A being separable, A^α / J_i has no minimal projections, then

$$\Gamma_\omega = \{\gamma \in \hat{G} : J_\gamma \subset J_\iota\}.$$

This could be used to compute the Γ-spectrum in certain situations, e.g. for product type actions on UHF algebras (cf. [**8**, proposition 4.1]).

Versions of Tannaka duality in an operator algebraic context have been obtained in [**1**, **17**, **10**, **15**, **2**]. Suppose σ is an automorphism of a von Neumann algebra M, on which there is an action α of a compact group G such that $\sigma | M^\alpha = id$. Then it is shown in [**1**, **17**] that if there exists an action τ of a group H which commutes with α, and is ergodic in the sense that the fixed point algebra M^τ is trivial, then there exists $g \in G$ such that $\sigma = \alpha(g)$. In particular, if $M \cap (M^\alpha)' = \mathbb{C}$, then we could take τ to be the action of the unitary group of M^α by inner automorphisms. In [**10**, **15**, **2**] C^*-versions of Tannaka duality have been obtained for an automorphism σ of a C^*-algebra A, which is trivial on the fixed point algebra A^α of an action α of a compact group G. If α commutes with an action τ which is ergodic in the sense of being topologically transitive [**10**] when G is abelian, or strongly topologically transitive [**2**] when G is not necessarily abelian, then there exists $g \in G$ such that $\sigma = \alpha(g)$. In § 3 and § 4 we prove versions of Tannaka duality in C^*-settings, partly through exploiting the techniques of § 2 in manufacturing Hilbert spaces. Suppose α is an action of a compact group G on a C^*-algebra A, and σ an automorphism of A such that $\sigma | A^\alpha = id$. Then we show that there exists $g \in G$ such that $\sigma = \alpha(g)$ in each of the following situations:

(a) (THEOREM 3.1). *A is separable and simple. There is a non-empty family P of α-invariant pure states such that if $J_P = \bigcap_{\varphi \in P} J_\iota^\varphi$, A^α / J_P contains no minimal projections and for all $\gamma \in \hat{G}$, $b, c \in A^\alpha \backslash J_p$, there exists $x \in bA_1^\alpha(\gamma)c$ such that $x^*x \in A^\alpha \backslash J_p \otimes 1$.*

(b) (THEOREM 3.4). *There exists a faithful irreducible representation π of A such that $\pi(A)'' = \pi(A^\alpha)''$.*

(c) (THEOREM 4.1). *G is abelian, A is simple, A^σ is prime, and $M(A) \cap (A^\alpha)' = \mathbb{C}1$.*

Note that under the hypotheses of theorem 3.4, the unitary group of $M(A^\alpha)$ acts strongly topologically transitive on A, and so theorem 3.4 could be deduced from

[**2**]; (see [**5**]). However the interest in our proof is that we actually manufacture Hilbert spaces (see lemma 3.7).

The C^*-algebras studied in this paper are inherently non-type I. In [**5**] a systematic study is made for abelian group actions of the relations between the covariant version of Glimm's theorem in [**8**], the existence of pure invariant states in (a), its antithesis, namely the existence of highly non-covariant representations in (b), topological transitivity of the unitary group of $M(A^\alpha)$ in (c), and duality.

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THEOREM 2.1. *Let α be an action of a compact group G on a separable C^*-algebra A. Suppose there exists an α-invariant pure state ω of A, and define a unitary representation u of G on $\mathcal{H}_\omega$ by $u_g\pi_\omega(x)\Omega_\omega = \pi_\omega \circ \alpha_g(x)\Omega_\omega$, $x \in A$. Denote by ρ the restriction of π_ω to A^α, and P_γ the spectral projection of u corresponding to $\gamma \in \hat{G}$, and let $J_\gamma = \ker(\rho | P_\gamma \mathcal{H}_\omega)$. Then for any $b \in A^\alpha \backslash J_\gamma$, there exists $x \in \overline{bA_d^\alpha(\gamma)}$ such that*

$$x^*x \in A^\alpha \backslash J_\iota \otimes 1,$$

where ι denotes the trivial representation of G, and $d = \dim(\gamma)$.

LEMMA 2.2. *Let $b \in A^\alpha \backslash J_\gamma$, and B be the hereditary C^*-subalgebra of $A \otimes M_d$ generated by $\{x^*x: x \in bA_1^\alpha(\gamma)\}$. Then $B \cap (A^\alpha \otimes \mathbb{C}1) \not\subset J_\iota$.*

Proof. We identify $a \in A$ with $a \otimes 1$ in $A \otimes M_d$. Then $A_1^\alpha(\gamma)A^\alpha \subset A_1^\alpha(\gamma)$, and so $A^\alpha B A^\alpha \subset B$.

If p is the open projection of $(A \otimes M_d)^{**}$ corresponding to B, then

$$S = \{\varphi: \text{pure state of } A \otimes M_d, \quad \varphi(p) = 0\}$$

is the set of pure states φ of $A \otimes M_d$ such that $\varphi | B = 0$. Hence B_+ coincides with

$$\{x \in (A \otimes M_d)_+: \varphi(x) = 0, \qquad \text{for all } \varphi \in S\}, \tag{$*$}$$

for if $x \in$ the set $(*)$, then $\psi[(1-p)x(1-p)] = 0$ for all states ψ on $A \otimes M_d$, and so $x(1-p) = 0$, and $x = pxp \in (A \otimes M_d) \cap p(A \otimes M_d)^{**}p = B$. Define

$$\mathrm{I} = \bigcap_{\varphi \in S} \operatorname{Ker} \pi_{\varphi | A^\alpha},$$

which is an ideal of A^α. If $x \in \mathrm{I}_+$, then $\varphi(x) = 0$ for all $\varphi \in S$, and so $x \in B$, i.e. $\mathrm{I} \subset B$. Conversely, if $x \in B \cap A^\alpha$, then $axa' \in B \cap A^\alpha$, for $a, a' \in A^\alpha$, and so $\varphi(axa') = 0$ for all $\varphi \in S$. Hence $x \in \operatorname{Ker} \pi_{(\varphi|A^\alpha)}$ i.e. $x \in \mathrm{I}$. Thus $\mathrm{I} = B \cap A^\alpha$. Suppose $\mathrm{I} \subset J_\iota$. Then $\omega | A^\alpha$ can be regarded as a state of $(\bigoplus_{\varphi \in S} \pi_{(\varphi|A^\alpha)})(A^\alpha)$. Since $\omega | A^\alpha$ is pure, it is a weak*-limit of some net φ_ν of vector states of $(\bigoplus_{\varphi \in S} \pi_{(\varphi|A^\alpha)}(A^\alpha)$ on $\bigoplus_{\varphi \in S} \mathcal{H}_{(\varphi|A^\alpha)}$, [**9**]. For each ν, there exist $\xi_\nu^\varphi \in [\pi_\varphi(A^\alpha)\Omega_\varphi]^-$ such that $\sum_{\varphi \in S} \|\xi_\nu^\varphi\|^2 = 1$, and

$$\varphi_\nu(a) = \sum_{\varphi \in S} \langle \pi_\varphi(a)\xi_\nu^\varphi, \xi_\nu^\varphi \rangle, \qquad a \in A^\alpha (= A^\alpha \otimes 1).$$

Define a state $\overline{\varphi_\nu}$ on $A \otimes M_d$ by

$$\overline{\varphi_\nu}(x) = \sum_{\varphi \in S} \langle \pi_\varphi(x)\xi_\nu^\varphi, \xi_\nu^\varphi \rangle, \qquad x \in A \otimes M_d.$$

Since for $x \in B$ and $a \in A^\alpha$, $a^*x^*xa \in B$, one has $\pi_\varphi(x)\pi_\varphi(a)\Omega_\varphi = 0$, for any $\varphi \in S$. Hence $\pi_\varphi(x)\xi_\nu^\varphi = 0$, for $x \in B$, and so $\overline{\varphi_\nu} | B = 0$. Let ψ be a weak*-limit point of $\{\varphi_\nu\}$.

Then $\psi | A^\alpha = \omega | A^\alpha$, and $\psi | B = 0$. Hence $\omega = \int (\psi | A) \circ \alpha_g \, dg$, and since ω is pure, we must have $\omega = \psi | A$. Hence there exists a state f of M_d such that $\omega \otimes f = \psi$, [**16**]. Now we show that $(\omega \otimes f) | B \neq 0$, a contradiction.

Since $bb^* \in A^\alpha \backslash J_{\bar{\gamma}}$, there are positive continuous functions h_1, h_2 on $\mathbb{R}$ such that $h_1(0) = h_2(0) = 0$, $h_1 h_2 = h_2$, and $h_1(bb^*)$, $h_2(bb^*) \in A^\alpha \backslash J_\gamma$. Since $V = [\pi_\omega(h_2(bb^*)) P_\gamma \mathcal{H}_\omega]^-$ is a non-zero u-invariant subspace of $P_\gamma \mathcal{H}_\omega$, there exists a set $(\xi_1, \ldots, \xi_d)$ of unit vectors such that $\xi_i \in V$ and

$$u_g \xi_i = \sum_{j=1}^{d} \gamma_{ji}(g) \xi_j.$$

By Kadison's transitivity theorem, there is an $x_0 \in A$ such that $\|x_0\| = 1$, $\pi_\omega(x_0)\Omega_\omega = \xi_1$, $\pi_\omega(x_0^*)\xi_1 = \Omega_\omega$ and $\pi_\omega(x_0^*)\xi_i = 0$, for $i = 2, \ldots, d$, since $(\Omega_\omega, \xi_1, \ldots, \xi_d)$ is an orthonormal family. Define

$$x_j = d \int \overline{\gamma_{j1}(g)} \alpha_g(x_0) \, dg.$$

Then $x = (x_1, \ldots, x_d) \in A_1^\alpha(\gamma)$, and

$$\pi_\omega(x_j)\Omega_\omega = \xi_j, \qquad \pi_\omega(x_j^*)\xi_i = \delta_{ij}\Omega_\omega.$$

Since $\pi_\omega(h_1(bb^*))\xi_i = \xi_i$, for $i = 1, \ldots, d$, this implies that

$$\pi_\omega(x_i^* h_1(bb^*)^2 x_j)\Omega_\omega = \delta_{ij}\Omega_\omega.$$

Thus since $y = h_1(bb^*)x \in \overline{bA_1^\alpha(\gamma)}$, one obtains that $y^*y \in B$, $(\omega \otimes f)(y^*y) = 1$, and so $(\omega \otimes f) | B \neq 0$. (In fact letting $\{z_k\}$ be a decreasing sequence of positive elements of A^α such that $\|z_k a z_k - \omega(x) z_k^2\| \to 0$ for $x \in A$ and $\omega(z_k) = 1$, [**11**], one has that $yz_k \in \overline{bA_1^\alpha(\gamma)}$, $\|z_k y^* y z_k\| \to 1$, and $(\omega \otimes f)(z_k y^* y z_k) = 1$. This implies $\|(\omega \otimes f) | B\| = 1$). This contradiction leads to the conclusion that $I \not\subset J_\iota$. □

LEMMA 2.3. *Let $b \in A^\alpha \backslash J_\gamma$, and B be the hereditary C^*-subalgebra of $A \otimes M_d$ generated by $\{x^*x : x \in bA_1^\alpha(\gamma)\}$. Then*

$$\left\{ a \otimes 1 \in A^\alpha \otimes \mathbb{C}1 : \exists x_i \in \overline{bA_1^\alpha(\gamma)}, \quad \text{such that} \quad \sum_{i=1}^{n} x_i^* x_i = a \otimes 1 \right\}$$

is dense in the positive part of $B \cap (A^\alpha \otimes \mathbb{C}1)$.

Proof. Let $a \otimes 1$ be a non-zero positive element of $B \cap (A^\alpha \otimes \mathbb{C}1)$. Then for any $\varepsilon > 0$, there exist $x_i, y_i \in bA_1^\alpha(\gamma)$ and $z_i \in A \otimes M_d$ such that

$$\left\| a \otimes 1 - \sum_{i=1}^{n} x_i^* z_i y_i \right\| < \varepsilon.$$

Define f on $\mathbb{R}$ by $f(t) = \max(t - \delta, 0)$, for $\delta \in (\varepsilon, \|a\|)$, and we shall show that $f(a) \otimes 1$ is of the form $\sum x_i^* x_i$, which completes the proof since $\|a - f(a)\| \leq \delta$. Let p be the spectral projection of a corresponding to $[\delta, \|a\|]$. Since

$$\left\| pap \otimes 1 - \sum_{i=1}^{n} (p \otimes 1) x_i^* z_i y_i (p \otimes 1) \right\| < \varepsilon$$

one has

$$pap\otimes 1\le \frac{\|a\|}{2(\delta-\varepsilon)}\left\{\sum_{i=1}^{n}(p\otimes 1)(x_i^*z_iy_i+y_i^*z_i^*x_i)(p\otimes 1)\right\}$$

$$\le C\sum_{i=1}^{n}\{p\otimes 1)x_i^*x_i(p\otimes 1)+(p\otimes 1)y_i^*y_i(p\otimes 1)\},$$

if $C=(\max_{i=1}^{n}\|z_i\|)\|a\|/2(\delta-\varepsilon)$.

Letting $g(t)=f(t)^{1/2}t^{-1/2}$ for $t>0$, and $g(t)=0$ for $t\le 0$, and multiplying $g(a)$ from both sides of the above inequality we obtain:

$$f(a)\otimes 1\le C\sum_{i=1}^{2n}(g(a)\otimes 1)(x_i^*x_i)(g(a)\otimes 1),$$

where $x_{n+i}=y_i$, for $i=1,2,\ldots,n$. Since $x_ig(a)\otimes 1\in bA_1^{\alpha}(\gamma)$, the conclusion of Lemma 2.3 follows from Lemma 2.4: □

LEMMA 2.4. *Suppose a is a positive element of A^{α}, and b an element of A^{α} such that there exist $x_i\in bA_1^{\alpha}(\gamma)$, $i=1,\ldots,n$, with $a\otimes 1\le\sum_{i=1}^{n}x_i^*x_i$. Then there exist $y_i\in bA_1^{\alpha}(\gamma)$, $i=1,\ldots,n$ such that*

$$a\otimes 1=\sum_{i=1}^{n}y_i^*y_i.$$

Proof. Let

$$x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in A_n^{\alpha}(\gamma)$$

and $x=(xx^*)^{1/2}u$ be the polar decomposition of x in $A^{**}\otimes M_{nd}$, where M_{nd} is the space of $n\times d$ matrices, uu^* is the support projection of $(xx^*)^{1/2}$ in $A^{**}\otimes M_n$, and $u\in A_n^{\alpha}(\gamma)^{**}$. Let B_1 be the hereditary C^*-subalgebra of $A\otimes M_n$ generated by xx^*, and B_2 the hereditary C^*-subalgebra of $A\otimes M_d$ generated by x^*x. We then have an isomorphism of B_1 onto B_2 defined by

$$z\in B_1\to u^*zu\in B_2.$$

If $z=(xx^*)^{1/2}y(xx^*)^{1/2}$, with $y\in A\otimes M_n$, then $u^*zu=u^*(xx^*)^{1/2}y(xx^*)^{1/2}u=x^*yx\in B_2$. Hence $u^*B_1u\subset B_2$ as $(xx^*)^{1/2}A\otimes M_n(xx^*)^{1/2}$ is dense in B_1. Similarly, one can show $uB_2u^*\subset B_1$.

Since $a\otimes 1\le x^*x$, one has $a\otimes 1\in B_2$, and

$$a\otimes 1=(a^{1/2}\otimes 1)u^*u(a^{1/2}\otimes 1).$$

Moreover, as $y=u(a^{1/2}\otimes 1)\in A_n^{\alpha}(\gamma)^{**}$, the lemma will follow, if we can show that $y\in A\otimes M_{nd}$. This follows since u is a multiplier in the sense that $uB_2\subset A\otimes M_{nd}$, and $B_1u\subset A\otimes M_{nd}$. Hence $y\in A\otimes M_{nd}$, and writing

$$y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix},$$

one obtains $a\otimes 1=\sum_{i=1}^{n} y_i^* y_i$. Since $yy^*=u(a\otimes 1)u^*\in B_1$, and $B_1\subset bAb^*\otimes M_n$, one has that $y_i y_i^*\in \overline{bAb^*}$, i.e. $y_i\in \overline{bA}\otimes M_{1d}$. □

Proof of Theorem 2.1. By lemmas 2.2 and 2.3, we see that for any $b\in A^\alpha\backslash J_\gamma$, there exists $x\in \overline{bA_n^\alpha(\gamma)}$ such that $x^*x\in (A^\alpha\backslash J_\iota)\otimes\mathbb{C}1$. Let n be the smallest possible integer for which there exists $a\in (A^\alpha\backslash J_\iota)_+$ and $x_i\in \overline{bA_1^\alpha(\gamma)}$ such that $a\otimes 1=\sum_{i=1}^{n} x_i^*x_i$. Take such a and x_i, and we may assume that there exists $a', a''\in (A^\alpha\backslash J_\iota)_+$ such that $aa'=a'$, $a'a''=a''$, $\|a\|=1$. Since $\rho(a')\rho(a'')=\rho(a'')\neq 0$, $\operatorname{Ker}(\rho(a')-1)\neq 0$, and so by Kadison's transitivity theorem, we can find v in A^α such that $\rho(v)\Omega\in \operatorname{Ker}(\rho(a')-1)$, and $\omega(v^*a'v)=1$.

For $\varphi=\omega(v^*\cdot v)$, let R_φ be the map of $A\otimes M_d$ onto M_d defined by $R_\varphi[z_{ij}]=[\varphi(z_{ij})]$, $[z_{ij}]\in A\otimes M_d$. Then

$$\sum_{i=1}^{n} R_\varphi(x_i^*x_i)=1=\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}.$$

Since φ is a pure state of A, and A is separable, there exists a decreasing sequence z_k of positive elements of A such that $z_1=a$, and the limit of z_k is the support projection of φ. We may assume that the z_k are α-invariant, and $z_kz_{k+1}=z_{k+1}$ for $k=1,2,\ldots$. Then for any $x\in A$, $\|z_kxz_k-\varphi(x)z_k^2\|\to 0$ as $k\to\infty$, **[11]**. If $\|R_\varphi(x_i^*x_i)\|<1$ for some i, then for large k, $z_kx_i^*x_iz_k<1$. But

$$z_{k+1}^2-z_{k+1}x_i^*x_iz_{k+1}\geq (1-\|z_kx_i^*x_iz_k\|)z_{k+1}^2$$

and so from

$$z_{k+1}^2=\sum_{j=1}^{n} z_{k+1}x_j^*x_jz_{k+1}$$

we deduce

$$z_{k+1}^2\leq (1-\|z_kx_i^*x_iz_k\|)^{-1}\sum_{j\neq i} z_{k+1}x_j^*x_jz_{k+1}.$$

This contradicts Lemma 2.4, as $z_{k+1}^2\in A^\alpha\backslash J_\iota$, and $x_jz_{k+1}\in bA_1^\alpha(\gamma)$.

Hence $\|R_\varphi(x_i^*x_i)\|=1$, for all $i=1,\ldots,n$. Then as $R_\varphi(x_i^*x_i)$ is a positive matrix, $\operatorname{Tr} R_\varphi(x_i^*x_i)\geq 1$, and so

$$n\leq \operatorname{Tr}\sum_{i=1}^{n} R_\varphi(x_i^*x_i)=d. \qquad \square$$

THEOREM 2.5. *Let α be an action of a compact group G on a separable C^*-algebra A. Suppose there exists an α-invariant pure state ω of A, and define J_γ, $\gamma\in\hat{G}$ as in Theorem* 2.1. *Let Γ_ω denote*

$$\{\gamma\in\hat{G}: \forall b, c\in A^\alpha\backslash J_\iota, \exists x\in \overline{bA_1^\alpha(\gamma)c} \quad\text{such that}\quad x^*x\in A^\alpha\backslash J_\iota\otimes 1_{d(\gamma)}\}.$$

Suppose that A^α/J_ι has no minimal projections. Then

$$\Gamma_\omega=\{\gamma\in\hat{G}: J_\gamma\subset J_\iota\}.$$

Proof. First we show that $\Gamma_\omega\subset\{\gamma\in\hat{G}: J_\gamma\subset J_\iota\}$. Let $\gamma\in\Gamma_\omega$, and $b\in J_\gamma$, and B the hereditary C^*-subalgebra of $A\otimes M_d$ generated by $\{x^*x: x\in bA_1^\alpha(\gamma)\}$. Then we claim that $B\cap(A^\alpha\otimes\mathbb{C}1)\subset J_\iota$, and this is enough to get the conclusion. (For if $b\notin J_\iota$, then by definition of Γ_ω, there would exist $x \in \overline{bA_1^\alpha(\gamma)b} \subset \overline{bA_1^\alpha(\gamma)}$ such that $x^*x \in A^\alpha \backslash J_\iota \otimes 1_{d(\gamma)}$, which implies that $b \notin J_\gamma$ by the above claim. Consequently $J_\gamma \subset J_\iota$).

Let $a \otimes 1 \in B \cap (A^\alpha \otimes \mathbb{C}1)$. Then a is a limit of elements of the form where $\sum_{i=1}^n x_{i1}^* b^* z_i b y_{i1}$, where $x_i = (x_{i1}, \ldots, x_{i1})$, $y_i = (y_{id}, \ldots, y_{id}) \in A_1^\alpha(\gamma)$, and $z_i \in A$. Since $\pi_\omega(y_{i1}) P_\iota \mathcal{H}_\omega \subset P_\gamma \mathcal{H}_\omega$, and $\pi_\omega(b) | P_\gamma \mathcal{H}_\omega = 0$, it follows that $\pi_\omega(a) | P_\iota \mathcal{H}_\omega = 0$, i.e. $a \in J_\iota$. For the reverse inclusion we need:

LEMMA 2.6. *Let C be a C^*-algebra, and J an ideal of C. Suppose that the quotient C/J is prime and has no minimal projections. Then for any $n = 2, 3, \ldots$, there exist $v_1, \ldots, v_n$, e in C such that $v_i^* v_j = 0$ if $i \neq j$, $v_i^* v_i e = e$, and $e \notin J$.*

Proof. Since C/J has no minimal projections, there exists a self adjoint $h \in C$ such that $h + J$ has an infinite spectrum in C/J. By using h it is shown that there exist positive $a_1, \ldots, a_n$ in $C \backslash J$, of norm one such that $a_i a_j = 0$ for $i \neq j$. We may suppose that there exists $b_1 \in (C \backslash J)_+$ such that $a_1 b_1 = b_1$, and $\|b_1\| = 1$. Let $v_1 = a_1$. Now suppose that we have defined $v_i \in \overline{a_i C} \backslash J$, $b_i \in (C \backslash J)_+$ such that $v_i^* v_i b_k = b_k$ and $\|b_i\| = 1$, for $i = 1, \ldots, k$. Since $a_{k+1} C b_k \not\subset J$, (as C/J is prime), choose a non-zero $v_{k+1} \in \overline{a_{k+1} C b_k} \backslash J$, and assume that $v_{k+1}^* v_{k+1}$ is a unit for some $b_{k+1} \in (C \backslash J)_+$ with $\|b_{k+1}\| = 1$. Then $b_{k+1} \in \overline{b_k C b_k}$, and so $v_i^* v_i$ is a unit for b_{k+1}, $i = 1, \ldots, k$. This concludes the proof with $e = b_n$. □

Proof of Theorem 2.5. It only remains to show

$$\Gamma_\omega \supset \{\gamma \in \hat{G} : J_\gamma \subset J_\iota\}.$$

Let $\gamma \in \hat{G}$ be such that $J_\gamma \subset J_\iota$. Let $b \in A^\alpha \backslash J_\iota$. Now A^α / J_ι is prime, since it has a faithful irreducible representation. Hence applying lemma 2.5 to the C^*-algebra $C = \overline{bA^\alpha b^*}$ with $J = J_\iota \cap C$, one obtains $v_1, \ldots, v_d$, $e \in \overline{bA^\alpha b^*}$, such that $v_i^* v_j = 0$ for $i \neq j$, $v_i^* v_i e = e$ and $e \in \overline{bA^\alpha b^*} \backslash J_\iota \subset A^\alpha \backslash J_\gamma$. By theorem 2.1, there exists

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \overline{eA_d^\alpha(\gamma)},$$

such that $x^*x \in A^\alpha \backslash J_\iota \otimes 1$. Define

$$y = \sum_{i=1}^{d} v_i x_i.$$

Then $y \in A_1^\alpha(\gamma)$, and $y^*y = \sum x_i^* v_i^* v_i x_i = \sum x_i^* x_i = x^*x \in A^\alpha \backslash J_\iota \otimes 1$. Thus $\gamma \in \Gamma$.

COROLLARY 2.7. *Under the assumptions of theorem 2.1, suppose in addition that A^α / J_ι, has no minimal projections. Then for any $b \in A^\alpha \backslash J_\gamma$, $\gamma \in \hat{G}$, there exists $x \in \overline{bA_1^\alpha(\gamma)}$ such that $x^*x \in A^\alpha \backslash J_\iota \otimes 1$.*

Proof. This follows from theorem 2.1 and the proof of theorem 2.5. □

COROLLARY 2.8. *Let α be an action of a compact group G on a separable C^*-algebra A. Assume that there exists an α-invariant pure state on A, and let P be a non-empty family of α-invariant pure states. Define an ideal J_γ^φ for each $\varphi \in P$, $\gamma \in \hat{G}$ as in theorem 2.1, and let $J_\iota^P = \bigcap_{\varphi \in P} J_\iota^\varphi$. Suppose that A^α is prime and has no minimal projections,* *and* $J_\iota^P = \{0\}$. *Define*

$$\Gamma_P = \{\gamma \in \hat{G}: \forall b \in A^\alpha \backslash \{0\}, \exists x \in bA_1^\alpha(\gamma) \text{ s.t. } x^*x \in A^\alpha \backslash \{0\} \otimes 1_{d(\gamma)}\}.$$

Then

$$\Gamma_P = \{\gamma \in \hat{G}: J_\gamma^P = \{0\}\}.$$

Proof. Let $\gamma \in \hat{G}$ s.t. $J_\gamma^P \neq \{0\}$, and let $b \in J_\gamma^P \backslash \{0\}$. Then by the proof of theorem 2.5, the hereditary C^*-subalgebra B of $A \otimes M_d$ generated by x^*x for $x \in bA_d^\alpha(\gamma)$ satisfies

$$B \cap (A^\alpha \otimes \mathbb{C}1) \subset J_\iota^\varphi$$

for any $\varphi \in P$ since $b \in J_\gamma^\varphi$. Hence

$$B \cap (A^\alpha \otimes \mathbb{C}1) \subset J_\iota^P = \{0\}.$$

This implies that $\gamma \notin \Gamma_P$. Conversely suppose $\gamma \in \hat{G}$, such that $J_\gamma^P = \{0\}$, and let $b \in A^\alpha \backslash \{0\}$. Then $b \notin J_\gamma^\varphi$, for some $\varphi \in P$, and by theorem 2.1, there exists $x \in bA_d^\alpha(\gamma)$ such that $x^*x \in A^\alpha \backslash J_\iota^\varphi \otimes 1 \subset A^\alpha \backslash \{0\} \otimes 1$. Thus $\gamma \in \Gamma_P$.

3

THEOREM 3.1. *Let G be a compact group and α an action of G on a separable simple C^*-algebra A. Assume that there exists an α-invariant pure state of A and let P be a non-empty family of α-invariant pure states. Define*

$$J_P = \bigcap_{\varphi \in P} \ker \pi_{(\varphi|A^\alpha)}$$

and assume that the quotient algebra A^α / J_P contains no miminal projections. Define

$$\Gamma_P = \{\gamma \in \hat{G} \mid \forall b, c \in A^\alpha \backslash J_P, \exists x \in bA_1^\alpha(\gamma)c, \text{ s.t. } x^*x \in A^\alpha \backslash J_P \otimes 1\}$$

and assume that $\Gamma_P = \hat{G}$.

Let σ be an automorphism of A such that $\sigma(x) = x$ for all $x \in A^\alpha$. Then there exists $g \in G$ such that $\sigma = \alpha_g$.

Remark. When G is abelian, P may be chosen so that $J_P = (0)$. (Let ω be an α-invariant pure state of A, and

$$P = \{\omega(a^* \circ a): a \in A^\alpha(\gamma), \qquad \gamma \in \hat{G}, \qquad \omega(a^*a) = 1\}).$$

Then the condition $\Gamma_P = \hat{G}$ is equivalent to the Connes spectrum of α being $\hat{G}$.

LEMMA 3.2. *Adopt the assumptions of theorem 3.1 and also assume that A^α is prime and that for any α-invariant hereditary C^*-subalgebra B of A one has $M(B) \cap (B^\alpha)' = \mathbb{C}1$ where $M(B)$ is the multiplier algebra of B. If σ is an automorphism of A such that $\sigma(x) = x$ for any $x \in A^\alpha$, then there exists $g \in G$ such that $\sigma = \alpha_g$.*

Proof. Let u be a finite-dimensional unitary representation of G such that for some n there exists $x \in A_n^\alpha(u)$ with $x^*x \in A^\alpha \backslash \{0\} \otimes 1$. Then we claim that there is a $d \times d$ unitary matrix $\lambda(u)$ such that $\sigma(x) = x\lambda(u)$ for any $x \in A_1^\alpha(u)$, where $\sigma(x) = (\sigma(x_1), \ldots, \sigma(x_d))$ and d is the dimension of u.

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Let $x \in A_n^\alpha(u)$ be such that $x^*x = a \otimes 1 \in A\backslash\{0\} \otimes 1$. For small $\delta > 0$ define a continuous function f on $\mathbb{R}$ by

$$f(t) = \begin{cases} 0 & t \leq \delta \\ t^{-1/2} & t \geq 2\delta \end{cases}$$

and by linearity elsewhere. Let $y = xf(a)$ and $e = f(a)af(a)$. Then $y \in A_n^\alpha(u)$ and $y^*y = e \otimes 1$. The non-zero hereditary C^*-subalgebra

$$B = \{b \in A: \quad eb = be = b\}$$

of A is α-invariant, and for $b \in B^\alpha$, one has $yby^* \in A^\alpha \otimes M_d$. Then since $yby^* = \sigma(y)b\sigma(y^*)$,

$$\begin{aligned} \sigma(y^*)yb &= \sigma(y^*)yby^*y = \sigma(y^*)\sigma(y)b\sigma(y^*)y \\ &= b\sigma(y^*)y. \end{aligned}$$

Denoting by p the open projection corresponding to B, one obtains that $\sigma(y^*)yp = p\sigma(y^*)y \in M(B) \otimes M_d \cap (B^\alpha)' \cong M_d$. Let λ be the matrix over $\mathbb{C}$ defined by $\sigma(y^*)yp = \lambda^*p$. Then for $b \in B^\alpha$ one has that $\sigma(yb) = yb\lambda$ because

$$\begin{aligned} \sigma(b^*y^*) &= \sigma(y^*yb^*y^*) = \sigma(y^*)yb^*y^* \\ &= \lambda^*b^*y^*. \end{aligned}$$

Further λ is a unitary because $\lambda\lambda^*p = y^*\sigma(y)\sigma(y^*)yp = y^*yy^*yp = p$. Define a continuous function h on $\mathbb{R}$ by

$$h(t) = \begin{cases} 0 & t \leq \delta \\ t^{1/2} & t \geq 2\delta \end{cases}$$

and by linearity elsewhere. Then since $h(a) \in B$, and

$$\|x - yh(a)\|^2 = \|a(f(a)h(a) - 1)^2\| \leq 2\delta,$$

it follows by approximation that for any $x \in A_n^\alpha(u)$ with $x^*x \in A^\alpha \otimes 1$, there exists a $d \times d$ unitary matrix λ such that $\sigma(x) = x\lambda$.

Now fix a non-zero $x \in A_n^\alpha(u)$ such that $x^*x = a \otimes 1 \in A^\alpha \otimes 1$, and let $\lambda(u)$ be the unitary matrix defined by $\sigma(x) = x\lambda(u)$. Let $y \in A_n^\alpha(u)$. Then since $ybx^* \in A^\alpha \otimes M_n$ for any $b \in A^\alpha$, it follows that $ybx^* = \sigma(y)b\lambda(u)^*x^*$. Multiplying x from the right one obtains that $yba = \sigma(y)ba\lambda(u)^*$, i.e.

$$(\sigma(y) - y\lambda(u))ba = 0$$

for any $b \in A^\alpha$. This implies that $\sigma(y) = y\lambda(u)$ because no non-zero element of A is orthogonal to the ideal of A^α generated by a as A^α is prime. Since any $y \in A_1^\alpha(u)$ can be regarded as an element of $A_n^\alpha(u)$, this proves the assertion that $\sigma(y) = y\lambda(u)$ for any $y \in A_1^\alpha(u)$.

Let $\mathcal{R}$ be the set of finite-dimensional unitary matrix representations u of G such that there is a non-zero $x \in A_n^\alpha(u)$ with $x^*x \in A^\alpha \otimes 1$ for some n. For each $u \in \mathcal{R}$ one has a unitary matrix $\lambda(u)$ such that $\sigma(x) = x\lambda(u)$ for $x \in A_1^\alpha(u)$. Now we claim that $\mathcal{R}$ is in fact the set of all finite-dimensional unitary representations of G and that

λ satisfies that

$$\lambda(u_1 \otimes u_2) = \lambda(u_1) \otimes \lambda(u_2),$$
$$\lambda(u_1 \oplus u_2) = \lambda(u_1) \oplus \lambda(u_2),$$
$$\lambda(w u_1 w^*) = w\lambda(u_1)w^*,$$

and $\lambda(\overline{u_1}) = \overline{\lambda(u_1)}$, where $u_i \in \mathcal{R}$, and w is a unitary matrix. Then by Tannaka's duality theorem (or by mimicking the proof of theorem 2.4 in [**16**] directly), one would obtain $g \in G$ such that $\lambda(u) = u_g$ for all $u \in R$. Since the set of elements x_i, with $(x_i) \in A_1^\alpha(u)$, $u \in \mathcal{R}$ is dense in A one would get the conclusion that $\sigma = \alpha_g$.

By the assumption that $\Gamma_P = \hat{G}$, $\mathcal{R}$ contains all irreducible unitary representations of G.

Let $u_i \in \mathcal{R}$ with $i = 1, 2$, and let $x_i \in A_n^\alpha(u_i)$ be such that $x_i^* x_i = a_i \otimes 1 \in A^\alpha \backslash \{0\} \otimes 1$. We may suppose that there is $b \in A^\alpha$ such that $a_1 b = b$, $b \geq 0$, and $\|b\| = 1$. Since A^α is prime, there is $c \in A^\alpha$ such that $a_2 cb \neq 0$. Let $y_1 = x_1(bc^* a_2 cb)^{1/2}$ and $y_2 = x_2 cb$. Then $y_i \in A_n^\alpha(u_i)$ and

$$y_1^* y_1 = bc^* a_2 cb = y_2^* y_2,$$

and hence $y \equiv y_1 \oplus y_2 \in A_n^\alpha(u_1 \oplus u_2)$, with $y^* y \in A^\alpha \backslash \{0\} \otimes 1$. This proves that $u_1 \oplus u_2 \in \mathcal{R}$ and that $\lambda(u_1 \oplus u_2) = \lambda(u_1) \oplus \lambda(u_2)$, since $\sigma(y) = y_1 \lambda(u_1) \oplus y_2 \lambda(u_2) = (y_1 \oplus y_2)(\lambda(u_1) \oplus \lambda(u_2))$.

Let $u \in \mathcal{R}$ and let $x \in A_n^\alpha(u)$ with $x^* x \in A^\alpha \backslash \{0\} \otimes 1$. Let w be a $d(u) \times d(u)$ unitary matrix and let $y = xw^*$. Then $y \in A_n^\alpha(wuw^*)$ and $y^* y = x^* x \in A^\alpha \backslash \{0\} \otimes 1$. Hence $wuw^* \in \mathcal{R}$ and $\lambda(wuw^*) = w\lambda(u)w^*$, since $\sigma(y) = \sigma(x)w^* = xw^* w\lambda(u)w^*$.

The above three properties in particular imply that $\mathcal{R}$ is the set of all finite dimensional unitary representations of G.

Let $u_i \in \mathcal{R}$ with $i = 1, 2$ and assume that u_i are irreducible. Let $x \in A_1^\alpha(u_1)$ be such that $x^* x = a \otimes 1 \in A^\alpha \backslash J_P \otimes 1$. We may suppose that there is $b \in A^\alpha \backslash J_P$ such that $b \geq 0$ and $ab = b$. By the assumption that $\Gamma_P = \hat{G}$, there is $y \in bA_1^\alpha(u_2)$ such that $y^* y \in A^\alpha \backslash J_P \otimes 1$. Then $xy \in A_1^\alpha(u_1 \otimes u_2)$ and $(xy)^*(xy) = y^* y \in A^\alpha \backslash \{0\} \otimes 1$. This proves that $\lambda(u_1 \otimes u_2) = \lambda(u_1) \otimes \lambda(u_2)$ since

$$\begin{aligned}(\sigma(xy))_{ij} &= \sigma(x)_i \sigma(y)_j \\ &= \sum x_k \lambda_{ki}(u_1) \sum y_l \lambda_{lj}(u_2) \\ &= \sum_{k,l} (xy)_{kl} (\lambda(u_1) \otimes \lambda(u_2))_{kl,ij}.\end{aligned}$$

When $u_i \in \mathcal{R}$ are not irreducible, we may decompose u_i into irreducible components and apply the above properties to get the conclusion that $\lambda(u_1 \otimes u_2) = \lambda(u_1) \otimes \lambda(u_2)$.

Let $u \in \mathcal{R}$ and $x \in A_1^\alpha(u)$ be non-zero. Let $y = x^{*T}$ where T denotes transposition. Then $y \in A_1^\alpha(u)$ and $\lambda(\bar{u}) = \overline{\lambda(u)}$ since $\sigma(y) = (\lambda(u)^* x^*)^T = y\overline{\lambda(u)}$.

Proof of Theorem 3.1. We have to prove that the two additional assumptions in lemma 3.2 follow automatically from the assumptions of the theorem.

Since A is separable and $\mathrm{Sp}\,(\alpha) = \hat{G}$, $\hat{G}$ must be countable. Let $\{\gamma_i\}$ be a sequence of elements of $\hat{G}$ such that each $\gamma \in \hat{G}$ appears infinitely often in $\{\gamma_i\}$ and let $\xi_i = \iota \oplus \gamma_i$ where ι is the trivial representation of G. Let β be the infinite product action $\bigotimes_{i=1}^{\infty} \mathrm{Ad}\, \xi_i$ of G on the UHF algebra $C = \otimes M_{d(\xi_i)}$ where $d(\xi_i)$ is the dimension of ξ_i. Then by theorem 3.1 in [**8**], there exists an α-invariant C^*-subalgebra B of A and a closed α^{**}-invariant projection $q \in A^{**}$ such that $q \in B'$, $qAq = Bq$, and the C^*-dynamical systems $(Bq, G, \alpha^{**}|Bq)$ and (C, G, β) are isomorphic.

Let τ be the tracial state of C and define a state ω of A by

$$\omega(x) = \tau(qxq), \; x \in A,$$

where we identified $qAq = Bq$ with C. Then we claim that $\pi_\omega(A)'' \cap \pi_\omega(A^\alpha)' = \mathbb{C}1$.

Let $e \equiv \bar{\pi}_\omega(q) \in \pi_\omega(A^\alpha)''$, and let $c(e)$ be the central support of e in $\pi_\omega(A^\alpha)''$. We first show that $c(e) = 1$.

Define a unitary representation u of G on $\mathcal{H}_\omega$ by

$$u_g \pi_\omega(x)\Omega_\omega = \pi_\omega \circ \alpha_g(x)\Omega_\omega, \qquad x \in A,$$

by using the α-invariance of ω. Then $c(e)$ commutes with u_g, $g \in G$, and if $c(e) \neq 1$, there exist $\gamma \in \hat{G}$ and a set $(\xi_1, \ldots, \xi_d)$ of orthonormal vectors in $(1 - c(e))\mathcal{H}_\omega$ such that

$$u_g \xi_i = \sum_{j=1}^{d} \gamma_{ji}(g)\xi_j,$$

where $(\gamma_{ij}(g))$ is a matrix representative of γ. Let $x' \in A$ be such that

$$\|\pi_\omega(x')\Omega_\omega - \xi_1\| < \varepsilon,$$

for small $\varepsilon > 0$ and define

$$x_j = d \int \overline{\gamma_{j1}(g)} \alpha_g(x')\, dg.$$

Then $x = (x_1, \ldots, x_d) \in A_1^\alpha(\gamma)$ and $\|\pi_\omega(x_j)\Omega_\omega - \xi_j\| \leq d\varepsilon$ since

$$\pi_\omega(x_j)\Omega_\omega - \xi_j = d \int \overline{\gamma_{j1}(g)} u_g(\pi_\omega(x')\Omega_\omega - \xi_1)\, dg.$$

Let $v_n = (v_{n1}, \ldots, v_{nd}) \in C_1^\alpha(\gamma)$ satisfy that $\{v_{ni}\}$ is a central sequence in C and

$$v_{n1}^* v_{n1} = \cdots = v_{nd}^* v_{nd} \equiv e_n,$$

$$\sum_{i=1}^{d} v_{ni} v_{ni}^* + e_n = 1,$$

(which can be chosen from the factors $M_{d(\xi_i)}$ with $\gamma_i = \gamma$). Now $v_{n1} = u_n q$, where $u_n \in B$. We define

$$u_{nj} = d \int \overline{\gamma_{j1}(g)} \alpha_g(u_n)\, dg, \qquad j = 1, \ldots, n$$

so that $(u_{n1}, \ldots, u_{nd}) \in B_1^\alpha(\gamma)$, and $u_{nj} q = v_{nj}$. Hence

$$Q_n = \sum_{j=1}^{n} x_j v_{nj}^* \in A^\alpha q$$

and

$$\bar{\pi}(v_{n1})\Omega_\omega \in e\mathcal{H}_\omega,$$

$$\bar{\pi}_\omega(Q_n)\bar{\pi}_\omega(v_{n1})\Omega_\omega = \bar{\pi}_\omega(x_1 e_n)\Omega_\omega$$

belongs to $c(e)\mathcal{H}_\omega$. Then we compute:

$$\begin{aligned}\|\bar{\pi}_\omega(x_1e_n)\Omega_\omega-\pi_\omega(x_1)\Omega_\omega\|^2\\ =\tau(e_nqx_1^*x_1qe_n)+\tau(qx_1^*x_1q)-\tau(q_1x_1^*x_1qe_n)-\tau(e_nqx_1^*x_1q),\end{aligned}$$

which converges to $d(d+1)^{-1}\tau(qx_1^*x_1q)$ because τ is a product state and $\tau(e_n)=(d+1)^{-1}$. On the other hand,

$$\begin{aligned}\|\pi_\omega(x_1e_n)\Omega_\omega-\pi_\omega(x_1)\Omega_\omega\|&\geq\|\pi_\omega(x_1e_n)\Omega_\omega-\xi_1\|-\|\xi_1-\pi_\omega(x_1)\Omega_\omega\|\\ &\geq(\|\pi_\omega(x_1e_n)\Omega_\omega\|^2+1)^{1/2}-d\varepsilon.\end{aligned}$$

Hence we obtain

$$d(d+1)^{-1}\tau(qx_1^*x_1q)\geq\{((d+1)^{-1}\tau(qx_1^*x_1q)+1)^{1/2}-d\varepsilon\}^2.$$

Since $|\tau(qx_1^*x_1q)^{1/2}-1|<d\varepsilon$, this is a contradiction for small $\varepsilon>0$, which implies that $c(e)=1$.

Let $z\in\pi_\omega(A)''\cap\pi_\omega(A^\alpha)'$. Then since $e\pi_\omega(A)''e=\pi_\omega(B)''e$ and $e\pi_\omega(A^\alpha)''e=\pi_\omega(B^\alpha)''e$, one has that $ze=ez\in\pi_\omega(B)''e\cap\{\pi_\omega(B^\alpha)''e\}'$ which is trivial by:

$$\pi_\tau(C)''\cap\pi_\tau(C^\beta)'=\mathbb{C}1.$$

To see this (see also [**6**]); note that any finite permutation automorphism among the factors in the infinite tensor product $C=\otimes_{i=1}^\infty M_{d(\xi_i)}$ which commutes with β is implemented by a unitary of C^β [**13**]. Since those automorphisms leave τ invariant, they extend to automorphisms of $\pi_\tau(C)''$. Thus any element of $\pi_\tau(C)''\cap\pi_\tau(C^\beta)'$ is fixed under those automorphisms, and it is easy to check that they act ergodically on $\pi_\tau(C)''$ by using the fact that τ is a separating factorial state and the permutation group which commutes with β acts ergodically on C.

Thus there is a $\lambda\in\mathbb{C}$ such that $ze=\lambda e$. Since the reduction $\pi_\omega(A^\alpha)'\to\pi_\omega(A^\alpha)'e$ is an isomorphism, because $c(e)=1$, one obtains that $z=\lambda 1$, i.e. $\pi_\omega(A)''\cap\pi_\omega(A^\alpha)'=\mathbb{C}1$, as claimed. □

LEMMA 3.3 [**12**, lemma 2.1]. *If $N\subset M$ are non Neumann algebras and f a projection in N, then $(N_f)'\cap M_f=(N'\cap M)_f$.*

Let B be an α-invariant hereditary C^*-subalgebra of A. Then we claim that $M(B)\cap(B^\alpha)'=\mathbb{C}$. By simplicity of A, π_ω is faithful on A, and hence so is $\rho=\pi_\omega|B$, on fH_ω where $f=\pi_\omega(e_B)$ and e_B is the open projection for B. Moreover, $\bar{\rho}$, the unique extension of ρ to B^{**} is faithful on $M(B)$. Thus

$$\begin{aligned}\bar{\rho}(M(B))\cap\rho(B^\alpha)'&\subset\bar{\rho}(B^{**})\cap\rho(B^\alpha)'\\ &=fMf\cap(fM^{\bar{\alpha}}f)'\end{aligned}$$

where $M=\pi_\omega(A)''$, and $\bar{\alpha}$ denotes the unique extension of α to M. Since $M\cap(M^\alpha)'=\mathbb{C}$, it follows from lemma 3.3, that $M(B)\cap(B^\alpha)'=\mathbb{C}$.

By using that $\pi_\omega(A)''\cap\pi_\omega(A^\alpha)'=\mathbb{C}1$ and the faithfulness of π_ω it follows that A^α is prime. This completes the proof of theorem 3.1.

THEOREM 3.4. *Let G be a compact group and α an action on a C^*-algebra A. Assume that there exists a faithful irreducible representation π of A such that $\pi(A)''=\pi(A^\alpha)''$. Let σ be an automorphism of A such that $\sigma(x)=x$ for all $x\in A^\alpha$. Then there exists $g\in G$ such that $\sigma=\alpha_g$.*

Remark. If we further assume that A is simple, separable, and unital, and that there exists an automorphism τ of A such that $\|\tau^n(x)y - y\tau^n(x)\| \to 0$ for all $x, y \in A$, then there exists an irreducible representation π of A such that $\pi(A)'' = \pi(A^\alpha)''$ (see theorem 2.1 in [7]). Hence the present theorem gives an alternative proof to the previous result in [**15**], at least when A is separable. The derivation version of the above theorem was proved in [7] as theorem 1.1, and the method there can be applied to the present situation if A is separable.

By taking $G/\ker \alpha$ instead of G, we may assume, without loss of generality, that α is faithful in the sequel.

LEMMA 3.5. *Adopt the assumptions of theorem* 3.4. *Define a representation ρ of A by the direct integral*

$$\rho = \int_G^\oplus \pi \circ \alpha_g \, dg$$

on the Hilbert space $H_\rho \equiv H_\pi \otimes L^2(G)$. Then $\rho(A)'' = B(\mathscr{H}_\pi) \otimes L^\infty(G)$.

Proof. Since $B(\mathscr{H}_\pi) \otimes \mathbb{C}1 = \rho(A^\alpha)'' \subset \rho(A)'' \subset B(\mathscr{H}_\pi) \otimes L^\infty(G)$, it suffices to prove that $\rho(A)'' \supset p \otimes L^\infty(G)$, where p is a fixed one-dimensional projection on $\mathscr{H}_\pi$.

Define a state φ of A by

$$\varphi(x)p = p\pi(x)p, \; x \in A.$$

Let $\{z_\nu\}$ be a decreasing net of positive elements of A^α such that $\lim \pi(z_\nu) = p$ (in the strong topology). The existence of such $\{z_\nu\}$ follows from the fact that $\varphi | A^\alpha$ is pure. Then defining a continuous function f_x on G, for each $x \in A$, by

$$f_x(g)p = p\pi \circ \alpha_g(x)p, \qquad g \in G,$$

it follows that $p \otimes f_x = p\rho(x)p = \lim \rho(z_\nu x z_\nu) \in \rho(A)''$. Hence it suffices to prove that $\{f_x : x \in A\}$ separates the points of G, to conclude that $\rho(A)'' \supset p \otimes L^\infty(G)$. If there are g and h in G such that $f_x(g) = f_x(h)$ for all $x \in A$, then one has that $\varphi \circ \alpha_g = \varphi \circ \alpha_h$. Thus α_{gh}^{-1} should be weakly extendible in the representation $\pi_\varphi \approx \pi$, which is impossible as $\pi(A^\alpha)$ is irreducible, unless α_{gh}^{-1} is the identity automorphism. □

LEMMA 3.6. *Under the assumptions of theorem* 3.4, *A^α is prime, and for any non-zero $b, c \in A^\alpha$, the spectrum of α restricted to bAc, written as* Sp $(\alpha | bAc)$, *is $\hat{G}$.*

Proof. Since $\pi | A^\alpha$ is a faithful irreducible representation, A^α is prime.

Let $b, c \in A^\alpha \backslash \{0\}$, and let $x \in A_1^\alpha(\gamma) \backslash \{0\}$ with $\gamma \in \hat{G}$. Since $\sum x_i^* x_i$ and $\sum x_i x_i^*$ are α-invariant, there exist $b', c' \in A^\alpha$ such that

$$bb'\left(\sum_{i=1}^d x_i x_i^*\right) \neq 0,$$

$$\left(\sum_{i=1}^d x_i^* b'^* bb' x_i\right) c'c \neq 0.$$

Thus $bb'xc'c = (bb'x_i c'c) \in bA_1^\alpha(\gamma)c$ is non-zero, and this proves that Sp $(d | bAc) =$ Sp (α). Note that lemma 3.5 immediately implies that Sp $(\alpha) = \hat{G}$.

LEMMA 3.7. *Under the assumptions of theorem* 3.4, *for any* $\gamma \in \hat{G}$ *and* $b \in A^\alpha \backslash \{0\}$, *there exists* $x \in bA_n^\alpha(\gamma)$ *such that* $x^*x \in A^\alpha \backslash \{0\} \otimes 1$, *for some* $n = 2, 3, \ldots$.

Proof. Let B be the hereditary C^*-subalgebra of $A \otimes M_d$ generated by x^*x with $x \in bA_1^\alpha(\gamma)$. It suffices to prove that $B \cap A^\alpha \otimes \mathbb{C}1 \neq \{0\}$, because the rest of the proof goes exactly as in lemma 2.2 and theorem 2.1.

To prove that $B \cap A^\alpha \otimes \mathbb{C}1 \neq \{0\}$, we have to produce a pure state ψ of A^α such that any extension $\bar{\psi}$ of ψ to a state of $A \otimes M_d (\supset A^\alpha \otimes 1)$ satisfies $\bar{\psi} | B \neq 0$.

Without loss of generality we assume that b is positive and there is a positive non-zero $a \in A^\alpha$ such that $ba = a$. Fix a one-dimensional projection p in the range of $\pi(a)$, and note that $\pi(b)p = p$.

By lemma 3.5, $p\rho(A)p$, regarded as continuous functions on G, is dense in $L^\infty(G)$ in the weak*-topology. By using the projections of A onto $A^\alpha(\gamma)$, it is shown that $p\rho(A(\gamma))p$ is dense in, and so equal to, the finite-dimensional linear space spanned by $\{\gamma_{ij}: i, j = 1, \ldots, d\}$. Thus by spectral calculations we can choose $x \in A_d^\alpha(\gamma)$ such that

$$p\rho(x_{ij})p = p \otimes \gamma_{ij}.$$

Let $\{z_\nu\}$ be a decreasing net of positive elements of A^α such that $z_\nu \leq b$ and $\lim \pi(z_\nu) = p$ as in the proof of lemma 3.5. Let $x_\nu = z_\nu^{1/2} x \in bA_d^\alpha(\gamma)$ and note that

$$\lim p\rho(x_{ij}^* z_\nu x_{kl})p = p \otimes \overline{\gamma_{ij}}\, \gamma_{kl}.$$

Let φ be the state of A defined by $\varphi(x)p = p\pi(x)p$, $x \in A$, and let $\psi = \varphi | A^\alpha$. If f is a functional in A^{**} whose support is contained in $p \in A^{**}$, one has

$$\lim_\nu \sum_k f(x_{ki}^* z_\nu x_{kj}) = \delta_{ij} \cdot 1.$$

Thus for any extension $\bar{\psi}$ of ψ to a state of $A \otimes M_d$ one has

$$\lim_\nu \bar{\psi}(x_\nu^* x_\nu) = 1.$$

Since $x_\nu^* x_\nu \in B$, this concludes the proof. □

Proof of theorem 3.4. Since $\pi(A)'' \cap \pi(A^\alpha)' = \mathbb{C}1$, it follows that $M(B) \cap (B^\alpha)' = \mathbb{C}1$ for any α-invariant hereditary C^*-subalgebra B of A, and that A^α is prime. The rest of the proof is similar to that of lemma 3.2 with

$$\{\gamma \in \hat{G}: \forall b \in A^\alpha \backslash \{0\}, \exists x \in bA_n^\alpha(\gamma) \text{ some } n, x^*x \in A^\alpha \backslash \{0\} \otimes 1\}$$

playing the role of Γ_P. □

4

THEOREM 4.1. *Let G be a compact abelian group and α an action of G on a simple C^*-algebra A. Assume that A^α is prime and $M(A) \cap (A^\alpha)' = \mathbb{C}1$. Let σ be an automorphism of A such that $\sigma(x) = x$ for $x \in A^\alpha$. Then there exists $g \in G$ such that $\sigma = \alpha_g$.*

LEMMA 4.2. *Let B be an α-invariant hereditary C^*-subalgebra of A, and let B_1 be the C^*-subalgebra of A generated by $A^\alpha B A^\alpha$ (which is a hereditary algebra). Let e_B be the open projection in A^{**} obtained as the limit of an approximate identity for B.*

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Then the map

$$M(B_1)\cap(B_1^\alpha)'\to M(B)\cap(B^\alpha)'$$

defined by multiplication by e_B is a surjective covariant isomorphism.

Proof. Note that α induces an action on $M(B_1)$ by restricting $(a|B_1)^{**}$ to $M(B_1)$. We use the same symbol α to denote this action.

Since e_B is a weak limit of an approximate identity for B^α [**4**, lemma 4.1], one has $e_B\in(B_1^\alpha)^{**}\subset A^{**}$. Thus any element z of $M(B_1)\cap(B_1^\alpha)'$ commutes with e_B and one has that $ze_B\in M(B)\cap(B^\alpha)'$, because $ze_B\cdot b=zb\in B_1\cap e_BBe_B=B$, $b\cdot ze_B=bz\in B$ for $b\in B$, and $ze_B\cdot a=za=az=aze_B$ for $a\in B^\alpha(\subset B_1^\alpha)$. Hence the map is well defined, and covariant.

Let $c(e_B)$ be the central support of e_B in $(A^\alpha)^{**}(\subset A^{**})$. Then the multiplication map by e_B:

$$c(e_B)A^{**}c(e_B)\cap(c(e_B)A^\alpha c(e_B))'\to e_BA^{**}e_B\cap(e_BA^\alpha e_B)'$$

is an isomorphism. Since $c(e_B)=e_{B_1}$, this is equivalent to saying that

$$B_1^{**}\cap(B_1^\alpha)'\to B^{**}\cap(B^\alpha)'$$

is an isomorphism. If $ze_B\in M(B)$, for $z\in B_1^{**}\cap(B_1^\alpha)'$, then we claim that $z\in M(B_1)$. For then $ze_{B_1}a=e_{B_1}az$ for $a\in A^\alpha$, as $z\in(B_1^\alpha)'$, and so for $b\in B$, $a_i\in A^\alpha$:

$$za_1ba_2=a_1(zb)a_2\in B_1 \text{ etc.}$$

This completes the proof. □

LEMMA 4.3. *Let B, B_1 be non-zero α-invariant hereditary C^*-subalgebras of A with $B_1\subset B$. Then the map*

$$M(B)\cap(B^\alpha)'\to M(B_1)\cap(B_1^\alpha)'$$

defined by multiplication by e_{B_1} is an injective covariant homomorphism.

Proof. The map is a well defined homomorphism since $e_{B_1}\in(B_1^\alpha)^{**}$. The action α is ergodic on $M(B)\cap(B^\alpha)'$, in the sense that the fixed point algebra is trivial because $M(B)^\alpha\cap(B^\alpha)'\subset M(B^\alpha)\cap(B^\alpha)'$, and $B^\alpha=A^\alpha\cap B$ is prime. Hence there are no non-trivial α-invariant ideals in $M(B)\cap(B^\alpha)'$. Multiplication e_{B_2} preserves the induced action, and so the kernel of this map is an α-invariant ideal which is either zero or the whole algebra. Since the latter is impossible, the map must be injective.

LEMMA 4.4. *Let $\gamma\in\hat G$, and x a non-zero element of $A^\alpha(\gamma)$. Let $B_1=\overline{xAx^*}$, and $B_2=\overline{x^*Ax}$, and $x=v|x|$ be the polar decomposition of x with vv^* being the range projection of x. Then* Ad (v^*) *gives a covariant isomorphism of $M(B_1)$ onto $M(B_2)$.*

Proof. See the proof of lemma 2.4, noting that $v\in A^\alpha(\gamma)^{**}$. □

LEMMA 4.5. *Let B_i be an α-invariant hereditary C^*-subalgebra of A such that $A^\alpha B_iA^\alpha\subset B_i$. Denote by $B_1\vee B_2$ the hereditary C^*-subalgebra generated by B_1 and B_2. Then*

$$\mathrm{Sp}(\alpha|M(B_1\vee B_2)\cap((B_1\vee B_2)^\alpha)')=\mathrm{Sp}(\alpha|M(B_1)\cap(B_1^\alpha)')\cap\mathrm{Sp}(\alpha|M(B_2)\cap(B_2^\alpha)').$$

Proof. Let $\gamma\in\mathrm{Sp}(\alpha|C_1)\cap\mathrm{Sp}(\alpha|C_2)$, where $C_i=M(B_i)\cap(B_i^\alpha)'$. By the ergodicity of α on C_i there are unitaries v_i in $C_i^\alpha(\gamma)$. Now B_i^α are non-zero ideals of A^α, and so if $B = B_1 \cap B_2$, then $B^\alpha = B_1^\alpha \cap B_2^\alpha$ is a non-zero ideal. Now $B_i = \overline{B_i^\alpha A B_i^\alpha}$, $B = \overline{B^\alpha A B^\alpha}$, and so e_{B_i}, e_B are central open projections of $(A^\alpha)^{**}(\subset A^{**})$. Now $v_i e_B \in C^\alpha(\gamma)$, where $C = M(B) \cap (B^\alpha)'$, and so by ergodicity, there is a number λ of modulus one such that $v_1 e_B = \lambda v_2 e_B$. Define $v \in e_{B_1 \vee B_2} A^{**} e_{B_1 \vee B_2}$, by

$$v = v_1 e_{B_1} + \lambda v_2 (e_{B_2} - e_B).$$

Now note that $e_B = e_{B_1} e_{B_2}$. Because since e_B, e_{B_1}, e_{B_2} are mutually commuting, $e_B \le e_{B_i}$ implies that $e_B \le e_{B_1} e_{B_2}$, and moreover $e_{B_1} e_{B_2} \in (B_1^\alpha)^{**}(B_2^\alpha)^{**} = (B^\alpha)^{**}$ implies $e_{B_1} e_{B_2} \le e_B$. Furthermore note that $\overline{B_1^\alpha + B_2^\alpha} = (B_1 \vee B_2)^\alpha$. Because $B_i \subset B_1 \vee B_2$ implies $B_1^\alpha + B_2^\alpha \subset (B_1 \vee B_2)^\alpha$. Moreover $B_i = \overline{B_i^\alpha A B_i^\alpha}$ are contained in the hereditary C^*-subalgebra $\overline{(B_1^\alpha + B_2^\alpha) A (B_1^\alpha + B_2^\alpha)}$, and so $B_1 \vee B_2 \subset \overline{(B_1^\alpha + B_2^\alpha) A (B_1^\alpha + B_2^\alpha)}$. Consequently $(B_1 \vee B_2)^\alpha \subset B_1^\alpha + B_2^\alpha$. Thus $(B_1 \vee B_2)^\alpha = B_1^\alpha \vee B_2^\alpha$, and in particular $e_{B_1 \vee B_2} = e_{B_1} \vee e_{B_2}$. Hence v is a unitary in $e_{B_1 \vee B_2} A^{**} e_{B_1 \vee B_2}$ and $v \in ((B_1 \vee B_2)^\alpha)'$. Finally v is a multiplier of $B_1 \vee B_2$ as:

$$vb_1 = v_1 b_1,$$
$$vb_2 = \lambda v_2 b_2,$$
$$vb_1 x b_2 = v_1 b_1 x b_2,$$
$$vb_2 x b_1 = \lambda v_2 b_2 x b_1,$$

for $b_1 \in B_1$, $b_2 \in B_2$, $x \in A$ etc. Thus $v \in M(B_1 \vee B_2) \cap ((B_1 \vee B_2)^\alpha)'$ and $\alpha_g(v) = \langle g, \gamma \rangle v$, and so $\gamma \in \mathrm{Sp}\,(\alpha \mid M(B_1 \vee B_2) \cap ((B_1 \vee B_2)^\alpha)')$. The reverse inclusion follows from lemma 4.3. □

Proof of theorem 4.1. We may assume that α is faithful. Since A is simple and A^α is prime, we know from [**11**, 8.10.4] that $\mathrm{Sp}\,(\alpha)$ is the same as the Connes spectrum $\Gamma(\alpha)$. Since the latter is a group and α is faithful we see that $\Gamma(\alpha) = \hat{G}$. Thus inspecting the proof of lemma 3.2, we see that we only have to show for any α-invariant hereditary C^*-subalgebra B of A that $M(B) \cap (B^\alpha)' = \mathbb{C}1$. Suppose there exists an α-invariant hereditary C^*-subalgebra B_0 of A such that $M(B_0) \cap (B_0^\alpha)'$ is not trivial. By lemma 4.2 we can assume $A^\alpha B_0 A^\alpha \subset B_0$, and since α is ergodic on $M(B_0) \cap (B_0^\alpha)'$, $\mathrm{Sp}\,(\alpha \mid M(B_0) \cap (B_0^\alpha)') = H$ is not trivial.

Let $\{B_i\}$ be an increasing family of α-invariant hereditary C^*-subalgebras such that $A^\alpha B_i A^\alpha \subset B_i$, $B_0 \subset B_i$, and $\mathrm{Sp}\,(\alpha \mid M(B_i) \cap (B_i^\alpha)') = H$. Let B be the hereditary C^*-subalgebra generated by B_i. Then B is α-invariant, $A^\alpha B A^\alpha \subset B$, and we claim that $\mathrm{Sp}\,(\alpha \mid M(B) \cap (B^\alpha)') = H$. Let $\gamma \in H$, and choose a unitary $v_i \in (M(B_i) \cap (B_i^\alpha)')^\alpha(\gamma)$, such that $v_i e_{B_0} = v_0$, where v_0 is a fixed unitary in $(M(B_0) \cap (B_0^\alpha)')^\alpha(\gamma)$. If $B_i \subset B_j$, then $v_i = v_j e_{B_i}$, because of the ergodicity of α. Define v by $v e_{B_i} = v_i$, for all i, in $eA^{**}e$, where e is the supremum of (e_{B_i}). Since $e = e_B$, and v is a multiplier for $\cup B_i$, it is easy to conclude that $v \in M(B) \cap (B^\alpha)'$, and $\gamma \in \mathrm{Sp}\,(\alpha \mid M(B) \cap (B^\alpha)')$. Thus $\mathrm{Sp}\,(\alpha \mid M(B) \cap B^\alpha)') = H$ using lemma 4.3.

Let B be a maximal α-invariant heredity C^*-subalgebra A such that $A^\alpha B A^\alpha \subset B$, $B_0 \subset B$ and $\mathrm{Sp}\,(\alpha \mid M(B) \cap (B^\alpha)') = H$. We claim that $B = A$, which contradicts $M(A) \cap (A^\alpha)' = \mathbb{C}1$. Note first that the hereditary C^*-subalgebra A_1 generated by $\{xB_0x^* : x \in A^\alpha(\gamma),\ \gamma \in \hat{G}\}$ is equal to A. Because, as $(\sum x_i)(\sum x_i)^* \le 2^n \sum x_i x_i^*$ for a finite sequence (x_i) of length n, it follows that $A_1 \supset xB_0x^*$ for any x in the linear

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space A_F spanned by $A^\alpha(\gamma)$, $\gamma\in\hat{G}$. Since A_F is dense in A, this implies that A_1 is equal to the ideal generated by B_0, and hence, since A is simple, it follows that $A_1=A$. Suppose $B\neq A$, and then there must exist $\gamma\in\hat{G}$, and $x\in A^\alpha(\gamma)$ such that $xB_0x^*\not\subset B$. By replacing x by xe_ν, where e_ν is an approximate identity for B_0^α, we can assume $x^*x\in B_0$, and so $B_1=\overline{x^*xB_0x^*x}\subset B_0$. Then by lemma 4.3 we have $\mathrm{Sp}\,(\alpha\,|\,M(B_1)\cap(B_1^\alpha)')\supset\mathrm{Sp}\,(\alpha\,|\,M(B_0)\cap(B_0^\alpha)')=H$. Moreover by lemma 4.4,

$$\mathrm{Sp}\,(\alpha\,|\,M(B_1)\cap(B_1^\alpha)')=\mathrm{Sp}\,(\alpha\,|\,M(\overline{xB_0x^*})\cap((xB_0x^*)^\alpha)'),$$

and by lemma 4.2

$$\mathrm{Sp}\,(\alpha\,|\,M(\overline{xB_0x^*})\cap((xB_0x^*)^\alpha)')=\mathrm{Sp}\,(\alpha\,|\,M(B_2)\cap(B_2^\alpha)'),$$

if $B_2=\overline{A^\alpha xB_0x^*A^\alpha}$. Hence $\mathrm{Sp}\,(M(B\vee B_2)\cap((B\vee B_2)^\alpha)')=H$ by lemma 4.5, which contradicts maximality of B as $B_2\not\subset B$. This contradiction implies that $M(B)\cap(B^\alpha)'=\mathbb{C}$ for any α-invariant hereditary C^*-subalgebra B of A.

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