

On the Lie symmetries of characteristic function hierarchy in compressible turbulence

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We compute the Lie symmetries of characteristic function (CF) hierarchy of compressible turbulence, ignoring the effects of viscosity and heat conductivity. In the probability density function (PDF) hierarchy, a typical non-local nature is observed, which is naturally eliminated in the CF hierarchy. We observe that the CF hierarchy retains all the symmetries satisfied by compressible Euler equations. Broadly speaking, four types of symmetries can be discerned in the CF hierarchy: (i) symmetries corresponding to coordinate system invariance, (ii) scaling/dilation groups, (iii) projective groups and (iv) statistical symmetries, where the latter define measures of intermittency and non-gaussianity. As the multi-point CFs need to satisfy additional constraints such as the reduction condition, the projective symmetries are only valid for monatomic gases, that is, the specific heat ratio, $\gamma = 5/3$. The linearity of the CF hierarchy results in the statistical symmetries due to the superposition principle. For all of the symmetries, the global transformations of the CF and various key compressible statistics are also presented.

Keywords: Characteristic function hierarchy, compressible turbulence, symmetry analysis

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1 Introduction

Compressible flows constitute a majority of the naturally occurring fluid flows in the universe and exhibit intricate physics. In the incompressible regime, pressure acts as a Lagrange multiplier that enforces the velocity field to be solenoidal. The role of pressure is significantly altered in the compressible regime, where pressure is governed by internal energy dynamics. The non-zero dilatation in compressible turbulence facilitates the internal kinetic energy exchange across the scales, which is interspersed with nonlinear energy transfer between scales. Consequently, the thermodynamic quantities such as density, pressure and temperature exhibit significant fluctuations and necessitate statistical treatment, in addition to the velocity field.

To investigate compressible turbulence statistics, we utilise the following governing equations of the compressible fluid, assuming an ideal gas:

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = -\rho \frac{\partial u_j}{\partial x_j}, \quad (1.1a)$$



$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\gamma M^2} \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j}, \quad (1.1b)$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = -(\gamma - 1)T \frac{\partial u_k}{\partial x_k} + \frac{\gamma}{Re Pr} \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\kappa \frac{\partial T}{\partial x_j} \right) + \frac{\gamma(\gamma - 1)M^2}{Re} \frac{\tau_{ij}}{\rho} \frac{\partial u_i}{\partial x_j}, \quad (1.1c)$$

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial u_k}{\partial x_k}, \quad (1.1d)$$

$$p = \rho T. \quad (1.1e)$$

Here, ρ , \mathbf{u} , p , T and $\boldsymbol{\tau}$ denote density, velocity, pressure, temperature and viscous stress tensor, respectively. Einstein summation convention is followed. The coefficient of viscosity is denoted by μ and bulk viscosity is taken to be zero. The ratio of specific heats (C_p/C_v) is denoted by γ . The equations are normalised with respect to the reference quantities (represented with a subscript 0), leading to three dimensionless parameters: (i) Reynolds number (Re), (ii) Mach number (M) and (iii) Prandtl number (Pr), defined as:

$$Re \equiv \frac{\rho_0 u_0 l_0}{\mu_0}, \quad M \equiv \frac{u_0}{\sqrt{\gamma R T_0}}, \quad Pr \equiv \frac{\mu_0 C_p}{\kappa_0}. \quad (1.2a)$$

A straightforward way to obtain the statistics would be to analyse governing equations (1.1), upon averaging. However, a quick examination would reveal the closure problem, with each term in the continuity, momentum and energy equations requiring its own infinite hierarchy of equations. As a result, the moment hierarchy in incompressible turbulence is more involved than its incompressible counterpart. Probability density function (PDF) approach has been utilised to examine statistics of compressible non-reacting and reacting flows [5, 4, 7, 16]. The PDF approach circumvents the issue of multiple hierarchies and gives rise to a single hierarchy of multi-point PDF equations [16]. In a similar manner, characteristic function (CF), the Fourier transform of the PDF, gives rise to an alternate hierarchy of infinite multi-point CF equations [16]. Characteristic functional can also be utilized for the examination of statistics, and the resulting equation is a functional differential equation [7].

Lie group theory has been successfully applied to several areas of physics, including turbulence. In the incompressible regime, symmetry methods have been applied to glean insights in (i) the multi-point moment [12] and (ii) the PDF approaches [20]. In the symmetry analysis of the statistical approaches, it is seen that the symmetries of the instantaneous equations are retained; and two additional symmetries, namely the ‘shape’ and ‘intermittency’ symmetries, are admitted. All of the symmetry groups derived have been instrumental in obtaining scaling laws for various canonical flows and have been verified against direct numerical simulations and/or experimental data [1, 17, 18, 19, 11]. The goal of this study is to investigate the behaviour of statistics in compressible turbulence, by performing Lie symmetry analysis on the CF hierarchy. The advantages of performing the analysis on CF hierarchy will be discussed in the next section, where the governing CF equations are introduced.

This paper is organised as follows. We introduce the governing equations of the CF hierarchy and the additional conditions/constraints imposed on CFs in Section 2. In Section 3, the symmetries of CF hierarchy are derived. The global form of each of the symmetry transformations and their physical significance are presented in Section 4. The resulting global moment transformations are discussed in Section 5. The key findings of this study are summarised in Section 6.

2 CF hierarchy in compressible turbulence

For a random variable, ‘ U ’, its CF (φ) is defined as follows [15, 8]:

$$\varphi(s) \equiv \int_{-\infty}^{\infty} \exp(isV)f(V)dV \equiv \langle \exp(isU) \rangle, \tag{2.1}$$

where s is the spectral parameter and $f(U)$ is the PDF of U . In other words, CF represents the expectation of $\exp(isU)$ and is complex in nature. Like the PDF, CF contains all of the information about the statistics of the random variable, U . The moments of U can be evaluated from the CF by computing its derivatives, that is,

$$\langle U \rangle = \frac{1}{i} \frac{\partial \varphi}{\partial s} \Big|_{s=0}, \quad \langle U^2 \rangle = \frac{1}{i^2} \frac{\partial^2 \varphi}{\partial s^2} \Big|_{s=0}, \dots, \quad \langle U^n \rangle = \frac{1}{i^n} \frac{\partial^n \varphi}{\partial s^n} \Big|_{s=0}. \tag{2.2}$$

Compared to the PDF approach, the reconstruction of the CF from the moments is also more straightforward. Let us consider the Taylor series expansion of CF about $s = 0$ [21]:

$$\varphi(s) = 1 + \frac{\partial \varphi}{\partial s} \Big|_{s=0} s + \dots + \frac{1}{n!} \frac{\partial^n \varphi}{\partial s^n} \Big|_{s=0} s^n + \dots = 1 + i \langle U \rangle s + \dots + i^n \frac{1}{n!} \langle U^n \rangle s^n + \dots. \tag{2.3}$$

This implies that all the moments of U can be pieced together to form the CF.

In compressible turbulence, as it is critical to examine density, pressure and temperature statistics in addition to velocity, we utilise the following definition for the multi-point CF [16]:

$$\begin{aligned} \varphi_n \left(\overset{(1)}{\underline{\varrho}}, \overset{(1)}{\underline{\mathbf{v}}}, \overset{(1)}{\underline{\theta}}, \overset{(2)}{\underline{\varrho}}, \overset{(2)}{\underline{\mathbf{v}}}, \overset{(2)}{\underline{\theta}}, \dots, \overset{(n)}{\underline{\varrho}}, \overset{(n)}{\underline{\mathbf{v}}}, \overset{(n)}{\underline{\theta}}; t, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)} \right) = \\ \left\langle \exp \left[i \sum_{k=1}^n \left(\overset{(k)}{\underline{\varrho}} \rho(\mathbf{x}^{(k)}, t) + \overset{(k)}{\underline{\mathbf{v}}}_i u_i(\mathbf{x}^{(k)}, t) + \overset{(k)}{\underline{\theta}} T(\mathbf{x}^{(k)}, t) \right) \right] \right\rangle. \end{aligned} \tag{2.4}$$

where φ_n represents the n -point CF. $\overset{(k)}{\underline{\varrho}}$, $\overset{(k)}{\underline{\mathbf{v}}}$ and $\overset{(k)}{\underline{\theta}}$ are the spectral parameters of density, velocity and temperature, respectively, at point $\mathbf{x}^{(k)}$ and k takes values from 1 to n . Going forward, we simplify the notation for spectral parameters by dropping the underline and represent them as $\overset{(k)}{\varrho}$, $\overset{(k)}{\mathbf{v}}$ and $\overset{(k)}{\theta}$. The governing equation for φ_n , derived in [16], is given below:

$$\begin{aligned} \left[\frac{\partial}{\partial t} + \frac{1}{i} \sum_{k=1}^n \frac{\partial^2}{\partial x_j^{(k)} \partial \overset{(k)}{\mathbf{v}}_j} \right] \prod_{l=1}^n \frac{\partial}{\partial \overset{(l)}{\varrho}} \varphi_n \\ = - \sum_{k=1}^n \frac{\overset{(k)}{\varrho}}{i} \left[\frac{\partial}{\partial x_j^{(n+1)}} \prod_{l=1}^n \frac{\partial}{\partial \overset{(l)}{\varrho}} \frac{\partial^2 \varphi_{n+1}}{\partial \overset{(n+1)}{\mathbf{v}}_j \partial \overset{(k)}{\varrho}} \Big|_{\overset{(n+1)}{\varrho}, \overset{(n+1)}{\mathbf{v}}, \overset{(n+1)}{\theta} = 0} \right]_{\mathbf{x}^{(n+1)} = \mathbf{x}^{(k)}} \\ - \sum_{k=1}^n \frac{\overset{(k)}{\mathbf{v}}_j}{\gamma M^2} \left[\frac{\partial}{\partial x_i^{(n+1)}} \prod_{l=1, l \neq k}^n \frac{\partial}{\partial \overset{(l)}{\varrho}} \frac{\partial^2 \varphi_{n+1}}{\partial \overset{(n+1)}{\varrho} \partial \overset{(n+1)}{\theta}} \Big|_{\overset{(n+1)}{\varrho}, \overset{(n+1)}{\mathbf{v}}, \overset{(n+1)}{\theta} = 0} \right]_{\mathbf{x}^{(n+1)} = \mathbf{x}^{(k)}} \\ - \sum_{k=1}^n \frac{\gamma - 1}{i} \overset{(k)}{\theta} \left[\frac{\partial}{\partial x_j^{(n+1)}} \prod_{l=1}^n \frac{\partial}{\partial \overset{(l)}{\varrho}} \frac{\partial^2 \varphi_{n+1}}{\partial \overset{(k)}{\theta} \partial \overset{(n+1)}{\mathbf{v}}_j} \Big|_{\overset{(n+1)}{\varrho}, \overset{(n+1)}{\mathbf{v}}, \overset{(n+1)}{\theta} = 0} \right]_{\mathbf{x}^{(n+1)} = \mathbf{x}^{(k)}} \end{aligned} \tag{2.5}$$

In the above multi-point CF equation, we ignore the terms arising due to viscosity and heat conductivity. As the viscous and heat conductivity terms are symmetry breaking, these terms are not considered in this study. For the complete equation, the reader is referred to [16]. As the CF is a probability measure, additional conditions also need to be imposed [9]. These restrictions, called the ‘side conditions’, are listed below:

- (1) *Reduction*: The order of the CF is reduced, when we set the spectral parameters to zero, that is,

$$\begin{aligned} \varphi_{n+1}({}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta, \dots, {}^{(n)}\varrho, {}^{(n)}\mathbf{v}, {}^{(n)}\theta, \mathbf{0}, \mathbf{0}, \mathbf{0}; t, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}, \mathbf{x}^{(n+1)}) = \\ \varphi_n({}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta, \dots, {}^{(n)}\varrho, {}^{(n)}\mathbf{v}, {}^{(n)}\theta; t, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}). \end{aligned} \tag{2.6a}$$

In the above equation, we demonstrate that the $(n + 1)$ –point CF reduces to n –point CF, when the spectral parameters corresponding to $\mathbf{x}^{(n+1)}$, that is, ${}^{(n+1)}\varrho$, ${}^{(n+1)}\mathbf{v}$ and ${}^{(n+1)}\theta$, are set to be zero. For the case of a single-point CF, reduction condition implies that:

$$\varphi_1({}^{(1)}\varrho = 0, {}^{(1)}\mathbf{v} = \mathbf{0}, {}^{(1)}\theta = 0; t, \mathbf{x}^{(1)}) = 1. \tag{2.6b}$$

- (2) *Coincidence*: When two points considered in the CF are infinitesimally closer to one another,

$$\begin{aligned} \lim_{|\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}| \rightarrow 0} \varphi_{n+1}(\dots, {}^{(n)}\varrho, {}^{(n)}\mathbf{v}, {}^{(n)}\theta, {}^{(n+1)}\varrho, {}^{(n+1)}\mathbf{v}, {}^{(n+1)}\theta; t, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}, \mathbf{x}^{(n+1)}) \\ = \varphi_n(\dots, {}^{(n)}\varrho + {}^{(n+1)}\varrho, {}^{(n)}\mathbf{v} + {}^{(n+1)}\mathbf{v}, {}^{(n)}\theta + {}^{(n+1)}\theta; t, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}), \end{aligned} \tag{2.7}$$

the order of CF is effectively reduced and the spectral parameters at those points are added to one another.

- (3) *Permutation*: The order in which the points in the multi-point CF are considered has no effect on the value of φ_n . This is expressed as:

$$\begin{aligned} \varphi_n(\dots, {}^{(l)}\varrho, {}^{(l)}\mathbf{v}, {}^{(l)}\theta, \dots, {}^{(k)}\varrho, {}^{(k)}\mathbf{v}, {}^{(k)}\theta; t, \dots, \mathbf{x}^{(l)}, \dots, \mathbf{x}^{(k)}, \dots) \\ = \varphi_n(\dots, {}^{(k)}\varrho, {}^{(k)}\mathbf{v}, {}^{(k)}\theta, \dots, {}^{(l)}\varrho, {}^{(l)}\mathbf{v}, {}^{(l)}\theta; t, \dots, \mathbf{x}^{(k)}, \dots, \mathbf{x}^{(l)}, \dots). \end{aligned} \tag{2.8}$$

- (4) *Separation*: When one of the points is infinitely separated from all other points considered in the multi-point CF, then,

$$\lim_{|\mathbf{x}^{(n+1)} - \mathbf{x}^{(k)}| \rightarrow \infty} \varphi_{n+1} = \varphi_1({}^{(n+1)}\varrho, {}^{(n+1)}\mathbf{v}, {}^{(n+1)}\theta; t, \mathbf{x}^{(n+1)}) \varphi_n, \quad \forall k = 1 - n \tag{2.9}$$

the statistics at the infinitely separated point are independent from all other points, and as a result, multiplication ensues.

All of the above ‘side conditions’ will also be included as a part of the symmetry analysis, in the subsequent sections, in addition to the governing equation (2.5).

3 Symmetry analysis

In this section, we first demonstrate the symmetry groups exhibited by the compressible Euler equations. Then, we perform Lie group analysis on the compressible multi-point CF hierarchy. To this end, we first compute the symmetry groups of single- and two-point CF equations, while imposing the side conditions. The resulting infinitesimals are generalised for a multi-point CF.

3.1 Lie groups of the compressible Euler equation

Euler equations can be obtained from the set of equations (1.1) upon neglecting the effects of viscosity and heat conductivity, that is, by setting $\mu = \kappa = 0$:

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = -\rho \frac{\partial u_j}{\partial x_j}, \quad (3.1a)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\gamma M^2} \frac{1}{\rho} \frac{\partial p}{\partial x_i}, \quad (3.1b)$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = -(\gamma - 1)T \frac{\partial u_k}{\partial x_k}, \quad (3.1c)$$

$$p = \rho T. \quad (3.1d)$$

For an arbitrary value of the specific heat ratio (γ), the following are the symmetry transformations of Euler equations [6, 14]:

- (1) Time translation symmetry (T_{a_t}): The equations (3.1) are invariant, when time is translated by an arbitrary constant, $a_t \in \mathbb{R}$. The global form of symmetry transformation, T_{a_t} , is

$$T_{a_t} : \quad t^* = t + a_t, \quad \mathbf{x}^* = \mathbf{x}, \quad \rho^* = \rho, \quad \mathbf{u}^* = \mathbf{u}, \quad T^* = T, \quad p^* = p \quad (3.2a)$$

with the symmetry operator:

$$X = \frac{\partial}{\partial t}. \quad (3.2b)$$

- (2) Spatial translation symmetry ($T_{a_{x,i}}$): The three symmetry groups represented by $T_{a_{x,i}}$, $i = 1, 2, 3$ indicate the invariance of the system (3.1) with respect to arbitrary spatial translation by $a_{x,i} \in \mathbb{R}$, $i = 1, 2, 3$. The global transformation is

$$T_{a_{x,i}} : \quad t^* = t, \quad x_i^* = x_i + a_{x,i}, \quad \rho^* = \rho, \quad \mathbf{u}^* = \mathbf{u}, \quad T^* = T, \quad p^* = p, \quad (3.2c)$$

and the symmetry operator is

$$X = \frac{\partial}{\partial x_1}, \quad X = \frac{\partial}{\partial x_2}, \quad X = \frac{\partial}{\partial x_3}, \quad (3.2d)$$

for $a_{x,1}$, $a_{x,2}$ and $a_{x,3}$, respectively.

- (3) Galilean symmetry ($T_{a_{g,i}}$): The Euler equations are invariant in a coordinate frame that is moving at constant velocity, $a_{g,i} \in \mathbb{R}$, $i = 1, 2, 3$:

$$T_{a_{g,i}} : \quad t^* = t, \quad x_i^* = x_i + a_{g,i}t, \quad \rho^* = \rho, \quad u_i^* = u_i + a_{g,i}, \quad T^* = T, \quad p^* = p, \quad (3.2e)$$

with the symmetry operators given by:

$$X = t \frac{\partial}{\partial x_1} + \frac{\partial}{\partial u_1}, \quad X = t \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_2}, \quad X = X = t \frac{\partial}{\partial x_3} + \frac{\partial}{\partial u_3}, \quad (3.2f)$$

for $a_{g,1}$, $a_{g,2}$ and $a_{g,3}$, respectively.

- (4) Rotational symmetry ($T_{a_{r,i}}$): This symmetry signifies the invariance of Euler equations (3.1) under rotation, represented by three symmetry groups, $a_{r,i} \in \mathbb{R}$, $i = 1, 2, 3$. For example, rotation about x_1 axis, written in terms of the group parameter, $a_{r,1}$, is given by:

$$\begin{aligned}
 T_{a_{r,1}} : \quad & t^* = t, x_2^* = (\cos a_{r,1}) x_2 - (\sin a_{r,1}) x_3, x_3^* = (\sin a_{r,1}) x_2 + (\cos a_{r,1}) x_3, \\
 & \rho^* = \rho, u_2^* = (\cos a_{r,1}) u_2 - (\sin a_{r,1}) u_3, u_3^* = (\sin a_{r,1}) u_2 + (\cos a_{r,1}) u_3, \quad (3.2g) \\
 & x_1^* = x_1, \quad u_1^* = u_1, \quad T^* = T, \quad p^* = p,
 \end{aligned}$$

with

$$X = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} + u_2 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial u_2}. \quad (3.2h)$$

Similarly, the rotation about x_2 axis, represented by $a_{r,2}$, is given by:

$$\begin{aligned}
 T_{a_{r,2}} : \quad & t^* = t, x_1^* = (\cos a_{r,2}) x_1 - (\sin a_{r,2}) x_3, x_3^* = (\sin a_{r,2}) x_1 + (\cos a_{r,2}) x_3, \\
 & \rho^* = \rho, u_1^* = (\cos a_{r,2}) u_1 - (\sin a_{r,2}) u_3, u_3^* = (\sin a_{r,2}) u_1 + (\cos a_{r,2}) u_3, \quad (3.2i) \\
 & x_2^* = x_2, \quad u_2^* = u_2, \quad T^* = T, \quad p^* = p,
 \end{aligned}$$

with

$$X = -x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3} - u_3 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_3}. \quad (3.2j)$$

Finally, rotation about x_3 axis can be written, using the group parameter $a_{r,3}$, as:

$$\begin{aligned}
 T_{a_{r,3}} : \quad & t^* = t, x_1^* = (\cos a_{r,3}) x_1 - (\sin a_{r,3}) x_2, x_2^* = (\sin a_{r,3}) x_1 + (\cos a_{r,3}) x_2, \\
 & \rho^* = \rho, u_1^* = (\cos a_{r,3}) u_1 - (\sin a_{r,3}) u_2, u_2^* = (\sin a_{r,3}) u_1 + (\cos a_{r,3}) u_2, \quad (3.2k) \\
 & x_3^* = x_3, \quad u_3^* = u_3, \quad T^* = T, \quad p^* = p,
 \end{aligned}$$

and the symmetry operator is

$$X = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - u_2 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}. \quad (3.2l)$$

Consequently, the Euler equations are invariant under any linear combination of the above three rotation groups, leaving the equations invariant under arbitrary rotation.

- (5) Scaling of time ($T_{a_{s,t}}$): When time, velocity, temperature and pressure are scaled as shown below, the Euler equations are invariant:

$$T_{a_{s,t}} : \quad t^* = e^{a_{s,t}} t, x_i^* = x_i, \rho^* = \rho, u_i^* = e^{-a_{s,t}} u_i, T^* = e^{-2a_{s,t}} T, p^* = e^{-2a_{s,t}} p, \quad (3.2m)$$

where, the group parameter, $a_{s,t} \in \mathbb{R}$, and the corresponding symmetry operator is

$$X = t \frac{\partial}{\partial t} - u_i \frac{\partial}{\partial u_i} - 2T \frac{\partial}{\partial T} - 2p \frac{\partial}{\partial p}. \quad (3.2n)$$

- (6) Scaling of space ($T_{a_{s,x}}$): Similarly, the following scaling transformation of space and dependent variables leaves the Euler equations invariant:

$$T_{a_{s,x}} : \quad t^* = t, x_i^* = e^{a_{s,x}} x_i, \rho^* = \rho, u_i^* = e^{a_{s,x}} u_i, T^* = e^{2a_{s,x}} T, p^* = e^{2a_{s,x}} p, \quad (3.2o)$$

with $a_{s,x} \in \mathbb{R}$. The symmetry operator is given by:

$$X = x_i \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u_i} + 2T \frac{\partial}{\partial T} + 2p \frac{\partial}{\partial p}. \quad (3.2p)$$

(7) Scaling of density ($T_{a_{s,\rho}}$): The scaling symmetry that is unique to compressible flows is the scaling of density and pressure, represented by the group parameter, $a_{s,\rho} \in \mathbb{R}$:

$$T_{a_{s,\rho}} : t^* = t, x_i^* = x_i, \rho^* = e^{a_{s,\rho}} \rho, u_i^* = u_i, T^* = T, p^* = e^{a_{s,\rho}} p \tag{3.2q}$$

whose symmetry operator is

$$X = \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}. \tag{3.2r}$$

In addition to the above symmetry groups, monatomic gases, that is, gases with specific heat ratio $\gamma = 5/3$, satisfy an additional ‘projective’ symmetry transformation ($T_{a_{pr}}$):

$$T_{a_{pr}} : t^* = \frac{t}{1 - a_{pr}t}, x_i^* = \frac{x_i}{1 - a_{pr}t}, \rho^* = \rho(1 - a_{pr}t)^3, T^* = T(1 - a_{pr}t)^2, p^* = p(1 - a_{pr}t)^5, u_i^* = a_{pr}x_i + u_i(1 - a_{pr}t), \tag{3.3a}$$

where, $a_{pr} \in \mathbb{R}$, and the resulting symmetry operator is

$$X = t^2 \frac{\partial}{\partial t} + x_i t \frac{\partial}{\partial x_i} - 3\rho t \frac{\partial}{\partial \rho} - 2Tt \frac{\partial}{\partial T} - 5pt \frac{\partial}{\partial p} + (x_i - tu_i) \frac{\partial}{\partial u_i}. \tag{3.3b}$$

The projective group is demonstrative of the power of Lie group analysis in identifying the relevant physics exhibited by the differential equations. For more discussion on this group, the reader is referred to Chapter 12 of [3].

When viscosity and heat conductivity are considered, the scaling of space and time groups merge into a single scaling group. As a result, in compressible flows with constant viscosity and heat conductivity, there are two scaling groups, as opposed to Euler equations with three scaling groups. In compressible flows, μ and κ are functions of temperature, leading to more symmetry breaking. Therefore, in this paper, we perform symmetry analysis of CF hierarchy upon neglecting the viscosity and heat conduction terms. In turbulent flows, in the limit of zero viscosity ($\mu \rightarrow 0$), the assumption is not too restrictive. When $\mu \rightarrow 0$, the action of viscosity is limited to the smallest scales, thus justifying inviscid consideration for the rest of the scales. Furthermore, viscosity and heat conductivity also break projection symmetry.

3.2 Lie group analysis of the CF hierarchy

Let us begin our investigation of the CF hierarchy by looking at the single-point CF equation. The equation for single-point CF, obtained from equation (2.5) by substituting $n = 1$, is given below:

$$\begin{aligned} \left[i \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x_j^{(1)} \partial^{(1)} v_j} \right] \frac{\partial}{\partial^{(1)} \varrho} \varphi_1 = & - {}^{(1)}\varrho \left[\frac{\partial}{\partial x_j^{(2)}} \frac{\partial^3 \varphi_2}{\partial^{(2)} v_j \partial^{(1)} \varrho^2} \Big|_{(2)\varrho, (2)\mathbf{v}, (2)\theta=0} \right]_{\mathbf{x}^{(2)}=\mathbf{x}^{(1)}} \\ & - i \frac{{}^{(1)}v_i}{\gamma M^2} \left[\frac{\partial}{\partial x_i^{(2)}} \frac{\partial^2 \varphi_2}{\partial^{(2)} \varrho \partial^{(2)} \theta} \Big|_{(2)\varrho, (2)\mathbf{v}, (2)\theta=0} \right]_{\mathbf{x}^{(2)}=\mathbf{x}^{(1)}} \\ & - (\gamma - 1)^{(k)} \theta \left[\frac{\partial}{\partial x_j^{(2)}} \frac{\partial^3 \varphi_2}{\partial^{(1)} \varrho \partial^{(1)} \theta \partial^{(2)} v_j} \Big|_{(2)\varrho, (2)\mathbf{v}, (2)\theta=0} \right]_{\mathbf{x}^{(2)}=\mathbf{x}^{(1)}}. \end{aligned} \tag{3.4}$$

The advantage of using the CF formulation, over the conventional PDF formulation [16], can easily be seen in the above equation. The equation (3.4) is local, where as the PDF hierarchy is non-local as it includes integral terms. In equation (3.4), φ_1 and φ_2 are the dependent variables and $t, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, {}^{(1)}\varrho, {}^{(2)}\varrho, {}^{(1)}\mathbf{v}, {}^{(2)}\mathbf{v}, {}^{(1)}\theta$ and ${}^{(2)}\theta$ are the independent variables.

To compute the symmetries, we make use of one of the infinite classes of the ‘generalised form of the symmetry operators’ that are equivalent to Lie point groups [13]. We specifically consider the ‘evolutionary’ form of the generalised symmetry operators in which the infinitesimals for the independent variables are zero. This precludes us from considering the transformation of independent variables explicitly. Specifically, the infinitesimal transformation of the CFs, φ_1^* and φ_2^* , are represented as:

$$\varphi_1^* = \varphi_1 + a\tilde{\eta}^{\varphi_1} + \mathcal{O}(a^2), \quad \varphi_2^* = \varphi_2 + a\tilde{\eta}^{\varphi_2} + \mathcal{O}(a^2), \tag{3.5a}$$

where, a is the group parameter and,

$$\begin{aligned} \tilde{\eta}^{\varphi_1} = & \eta^{\varphi_1} - \xi^t \frac{\partial \varphi_1}{\partial t} - \xi^{x_1^{(1)}} \frac{\partial \varphi_1}{\partial x_1^{(1)}} - \xi^{x_2^{(1)}} \frac{\partial \varphi_1}{\partial x_2^{(1)}} - \xi^{x_3^{(1)}} \frac{\partial \varphi_1}{\partial x_3^{(1)}} - \xi^{(1)\varrho} \frac{\partial \varphi_1}{\partial {}^{(1)}\varrho} \\ & - \xi^{(1)v_1} \frac{\partial \varphi_1}{\partial {}^{(1)}v_1} - \xi^{(1)v_2} \frac{\partial \varphi_1}{\partial {}^{(1)}v_2} - \xi^{(1)v_3} \frac{\partial \varphi_1}{\partial {}^{(1)}v_3} - \xi^{(1)\theta} \frac{\partial \varphi_1}{\partial {}^{(1)}\theta}, \end{aligned} \tag{3.5b}$$

and

$$\begin{aligned} \tilde{\eta}^{\varphi_2} = & \eta^{\varphi_2} - \xi^t \frac{\partial \varphi_2}{\partial t} - \xi^{x_1^{(1)}} \frac{\partial \varphi_2}{\partial x_1^{(1)}} - \xi^{x_2^{(1)}} \frac{\partial \varphi_2}{\partial x_2^{(1)}} - \xi^{x_3^{(1)}} \frac{\partial \varphi_2}{\partial x_3^{(1)}} - \xi^{(1)\varrho} \frac{\partial \varphi_2}{\partial {}^{(1)}\varrho} \\ & - \xi^{(1)v_1} \frac{\partial \varphi_2}{\partial {}^{(1)}v_1} - \xi^{(1)v_2} \frac{\partial \varphi_2}{\partial {}^{(1)}v_2} - \xi^{(1)v_3} \frac{\partial \varphi_2}{\partial {}^{(1)}v_3} - \xi^{(1)\theta} \frac{\partial \varphi_2}{\partial {}^{(1)}\theta} \\ & - \xi^{x_1^{(2)}} \frac{\partial \varphi_2}{\partial x_1^{(2)}} - \xi^{x_2^{(2)}} \frac{\partial \varphi_2}{\partial x_2^{(2)}} - \xi^{x_3^{(2)}} \frac{\partial \varphi_2}{\partial x_3^{(2)}} - \xi^{(2)\varrho} \frac{\partial \varphi_2}{\partial {}^{(2)}\varrho} \\ & - \xi^{(2)v_1} \frac{\partial \varphi_2}{\partial {}^{(2)}v_1} - \xi^{(2)v_2} \frac{\partial \varphi_2}{\partial {}^{(2)}v_2} - \xi^{(2)v_3} \frac{\partial \varphi_2}{\partial {}^{(2)}v_3} - \xi^{(2)\theta} \frac{\partial \varphi_2}{\partial {}^{(2)}\theta} \end{aligned} \tag{3.5c}$$

are the characteristics of the symmetry transformation [13]. Here, η^{φ_1} and η^{φ_2} are the infinitesimals for φ_1, φ_2 ; and $\xi^t, \xi^{x_i^{(k)}}, \xi^{(k)\varrho}, \xi^{(k)v_i}$ and $\xi^{(k)\theta}$ are the infinitesimals for time, space and spectral parameters for density, velocity and temperature, respectively; and $i = 1 - 3, k = 1, 2$. It should be noted that all of the infinitesimals, $\eta^{\varphi_1}, \eta^{\varphi_2}, \xi^t, \xi^{x_i^{(k)}}, \xi^{(k)\varrho}, \xi^{(k)v_i}$ and $\xi^{(k)\theta}$ are explicit functions of t, x_i and the dependent variables. This implies that we do not consider the infinitesimals to depend on the derivatives of the dependent variables, φ_n , for all n . In the evolutionary form of generalised symmetries, the derivatives of φ_1 and φ_2 transform as:

$$\left. \frac{\partial \varphi_1}{\partial t} \right|^* = \frac{\partial \varphi_1}{\partial t} + a \frac{D}{Dt} \tilde{\eta}^{\varphi_1} + \mathcal{O}(a^2), \tag{3.6a}$$

and

$$\left. \frac{\partial^2 \varphi_1}{\partial {}^{(1)}\varrho \partial t} \right|^* = \frac{\partial^2 \varphi_1}{\partial {}^{(1)}\varrho \partial t} + a \frac{D}{Dt} \frac{D}{D{}^{(1)}\varrho} \tilde{\eta}^{\varphi_1} + \mathcal{O}(a^2), \tag{3.6b}$$

etc. This implies that the infinitesimals for the derivatives of the dependent variables (say time derivative of φ_1) in the evolutionary form of the generalised symmetry formulation can be obtained by taking the total derivatives (here, with respect to time) of the infinitesimal of the

dependent variable ($\tilde{\eta}^{\varphi_1}$). As the CF equations contain high-order derivatives (the order of the highest derivative increases as the order of CF is increased), the handling of derivatives is simplified in the evolutionary form of symmetry operators. This operator has also been used to derive the symmetries of integro-differential equations [20], as the limits on the integral do not undergo any transformation.

To compute the symmetries of CF equation (3.4), we apply the evolutionary form of the symmetry operator with the infinitesimals as defined above. Thus, the obtained determining equation is of the form:

$$\begin{aligned}
 i \frac{D^2 \tilde{\eta}^{\varphi_1}}{DtD^{(1)}\varrho} + \frac{D^3 \tilde{\eta}^{\varphi_1}}{D^{(1)}v_j Dx_j^{(1)} D^{(1)}\varrho} &= -^{(1)}\varrho \left[\frac{D}{Dx_j^{(2)}} \left(\frac{D^3 \tilde{\eta}^{\varphi_2}}{D^{(2)}v_j D^{(1)}\varrho^2} \right)_{(2)\varrho, (2)v, (2)\theta=0} \right]_{\mathbf{x}^{(2)}=\mathbf{x}^{(1)}} \\
 - i \frac{^{(1)}v_i}{\gamma M^2} \left[\frac{D}{Dx_i^{(2)}} \left(\frac{D^2 \tilde{\eta}^{\varphi_2}}{D^{(2)}\varrho D^{(2)}\theta} \right)_{(2)\varrho, (2)v, (2)\theta=0} \right]_{\mathbf{x}^{(2)}=\mathbf{x}^{(1)}} & \quad (3.7) \\
 - ^{(1)}\theta(\gamma - 1) \left[\frac{D}{Dx_j^{(2)}} \left(\frac{D^3 \tilde{\eta}^{\varphi_2}}{D^{(2)}v_j D^{(1)}\varrho D^{(1)}\theta} \right)_{(2)\varrho, (2)v, (2)\theta=0} \right]_{\mathbf{x}^{(2)}=\mathbf{x}^{(1)}} & .
 \end{aligned}$$

The point evaluations such as $^{(2)}\varrho, ^{(2)v}, ^{(2)}\theta = 0$ and $\mathbf{x}^{(2)} = \mathbf{x}^{(1)}$ do not need to be transformed as we utilise the evolutionary form of the symmetry operator. Similarly, the determining equation for the governing equation of two-point CF can also be derived.

Finally, upon utilising the computer algebra system, Maple [10], for part of the symmetry calculations, the following are the infinitesimals of the symmetries admitted by the single-point CF equation (3.4):

$$\xi^t = \frac{a_{pr}}{2} t^2 + a_{s,t} t + a_t, \tag{3.8a}$$

$$\xi^{x_1^{(1)}} = \frac{a_{pr}}{2} x_1^{(1)} t + a_{s,x} x_1^{(1)} + a_{g,1} t - a_{r,3} x_2^{(1)} - a_{r,2} x_3^{(1)} + a_{x,1}, \tag{3.8b}$$

$$\xi^{x_2^{(1)}} = \frac{a_{pr}}{2} x_2^{(1)} t + a_{s,x} x_2^{(1)} + a_{g,2} t - a_{r,1} x_3^{(1)} + a_{r,3} x_1^{(1)} + a_{x,2}, \tag{3.8c}$$

$$\xi^{x_3^{(1)}} = \frac{a_{pr}}{2} x_3^{(1)} t + a_{s,x} x_3^{(1)} + a_{g,3} t + a_{r,2} x_1^{(1)} + a_{r,1} x_2^{(1)} + a_{x,3}, \tag{3.8d}$$

$$\xi^{(1)\varrho} = \frac{a_{pr}}{\gamma - 1} ^{(1)}\varrho t - a_{s,\rho} ^{(1)}\varrho, \tag{3.8e}$$

$$\xi^{(1)v_1} = \frac{a_{pr}}{2} ^{(1)}v_1 t - a_{s,x} ^{(1)}v_1 + a_{s,t} ^{(1)}v_1 - a_{r,3} ^{(1)}v_2 - a_{r,2} ^{(1)}v_3, \tag{3.8f}$$

$$\xi^{(1)v_2} = \frac{a_{pr}}{2} ^{(1)}v_2 t - a_{s,x} ^{(1)}v_2 + a_{s,t} ^{(1)}v_2 - a_{r,1} ^{(1)}v_3 + a_{r,3} ^{(1)}v_1, \tag{3.8g}$$

$$\xi^{(1)v_3} = \frac{a_{pr}}{2} ^{(1)}v_3 t - a_{s,x} ^{(1)}v_3 + a_{s,t} ^{(1)}v_3 + a_{r,2} ^{(1)}v_1 + a_{r,1} ^{(1)}v_2, \tag{3.8h}$$

$$\xi^{(1)\theta} = a_{pr} ^{(1)}\theta t + 2(a_{s,t} - a_{s,x}) ^{(1)}\theta, \tag{3.8i}$$

and

$$\eta^{\varphi_1} = F_1(t, \mathbf{x}^{(1)}, {}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta) + i\varphi_1 \left[\frac{a_{pr}^{(1)}}{2} v_i x_i^{(1)} + a_{g,1}^{(1)} v_1 + a_{g,2}^{(1)} v_2 + a_{g,3}^{(1)} v_3 + \frac{a_\varphi}{i} + \frac{5 - 3\gamma}{2i(\gamma - 1)} a_{prt} \right]. \tag{3.9}$$

In the above infinitesimals for independent (3.8) and dependent (3.9) variables, it can be seen that the Euler equation symmetries are replicated. These are represented by the same group parameters as Section 3.1, that is, a_t for time translation, $a_{x,i}$ for space translation, $a_{g,i}$ for Galilean symmetry, $a_{s,t}$ for scaling of time, $a_{s,x}$ for scaling of space and $a_{s,\rho}$ for scaling of density and a_{pr} for projective symmetries. We also notice two additional groups, denoted by F_1 and a_φ , that appear in the statistical CF equations. These are as a result named, ‘statistical symmetries’, a convention that is adopted in incompressible turbulence as well [12, 21]. The function F_1 appears in the infinitesimal, η^{φ_1} due to *principle of superposition*, as single-point CF equation (3.4) is a linear differential equation. F_1 is an arbitrary function of the independent variables (in the CF formulation), which could lead to a translation type of symmetry in the single-point CF, φ_1 . On the other hand, a_φ is a dilation/scaling symmetry group, which will lead to the scaling of φ_1 .

It is interesting to note that the single-point CF equation (3.4) admits the projection symmetry group for all values of γ . However, as seen in the previous section, the CFs, including the transformed ones (φ_1^*), need to satisfy additional constraints (2.6a)–(2.9). In the following section, the side conditions are enforced on the transformed CFs (φ_n^*) by imposing restrictions on their infinitesimals (η^{φ_n}).

3.2.1 Imposing side conditions on the infinitesimals

Let us now consider the restrictions on the infinitesimals that are imposed as a result of side conditions. We apply the evolutionary form of the generalised symmetry operator on the reduction side condition (2.6b). This results in:

$$\tilde{\eta}^{\varphi_1} |_{(1)\varrho=0, (1)\mathbf{v}=0, (1)\theta=0} = 0, \tag{3.10a}$$

implying,

$$\left[\eta^{\varphi_1} - \xi^t \frac{\partial \varphi_1}{\partial t} - \xi^{x_1^{(1)}} \frac{\partial \varphi_1}{\partial x_1^{(1)}} - \xi^{x_2^{(1)}} \frac{\partial \varphi_1}{\partial x_2^{(1)}} - \xi^{x_3^{(1)}} \frac{\partial \varphi_1}{\partial x_3^{(1)}} - \xi^{(1)\varrho} \frac{\partial \varphi_1}{\partial (1)\varrho} - \xi^{(1)v_1} \frac{\partial \varphi_1}{\partial (1)v_1} - \xi^{(1)v_2} \frac{\partial \varphi_1}{\partial (1)v_2} - \xi^{(1)v_3} \frac{\partial \varphi_1}{\partial (1)v_3} - \xi^{(1)\theta} \frac{\partial \varphi_1}{\partial (1)\theta} \right]_{(1)\varrho=0, (1)\mathbf{v}=0, (1)\theta=0} = 0 \tag{3.10b}$$

that is,

$$\left[\eta^{\varphi_1} - i \left(\langle \rho \rangle \xi^{(1)\varrho} + \langle u_1 \rangle \xi^{(1)v_1} + \langle u_2 \rangle \xi^{(1)v_2} + \langle u_3 \rangle \xi^{(1)v_3} + \langle T \rangle \xi^{(1)\theta} \right) \right]_{(1)\varrho=0, (1)\mathbf{v}=0, (1)\theta=0} = 0. \tag{3.10c}$$

As η^{φ_1} , $\xi^{(1)\varrho}$, $\xi^{(1)v_i}$ and $\xi^{(1)\theta}$ do not depend on $\langle \rho \rangle$, $\langle \mathbf{u} \rangle$ and $\langle T \rangle$, this implies

$$\eta^{\varphi_1} |_{(1)\varrho=0, (1)\mathbf{v}=0, (1)\theta=0} = 0, \quad \xi^{(1)\varrho} |_{(1)\varrho=0, (1)\mathbf{v}=0, (1)\theta=0} = 0, \tag{3.11}$$

$$\xi^{(1)v_i} |_{(1)\varrho=0, (1)\mathbf{v}=0, (1)\theta=0} = 0, \quad \xi^{(1)\theta} |_{(1)\varrho=0, (1)\mathbf{v}=0, (1)\theta=0} = 0, \quad i = 1 - 3.$$

Applying similar techniques on the coincidence condition (2.7), we obtain

$$\lim_{|\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}| \rightarrow 0} \tilde{\eta}^{\varphi_{n+1}} = \tilde{\eta}^{\varphi_n} \left(\dots, {}^{(n+1)}\varrho + {}^{(n)}\varrho, {}^{(n+1)}\mathbf{v} + {}^{(n)}\mathbf{v}, {}^{(n+1)}\theta + {}^{(n)}\theta, \dots \right). \tag{3.12}$$

Substituting the expressions for $\tilde{\eta}^{\varphi_n}$ and $\tilde{\eta}^{\varphi_{n+1}}$ in the above equation (2.7) in terms of $\eta^{\varphi_n}, \xi^t, \xi^{x_i^{(k)}}, \xi^{(k)}\varrho, \xi^{(k)}v_i$ and $\xi^{(k)}\theta$, we obtain:

$$\begin{aligned} \lim_{|\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}| \rightarrow 0} \eta^{\varphi_{n+1}} \left(t, \mathbf{x}^{(1)}, {}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta, \dots, \mathbf{x}^{(n+1)}, {}^{(n+1)}\varrho, {}^{(n+1)}\mathbf{v}, {}^{(n+1)}\theta, \dots \right) \\ = \eta^{\varphi_n} \left(\dots, {}^{(n+1)}\varrho + {}^{(n)}\varrho, {}^{(n+1)}\mathbf{v} + {}^{(n)}\mathbf{v}, {}^{(n+1)}\theta + {}^{(n)}\theta, \dots \right). \end{aligned} \tag{3.13}$$

The permutation side condition (2.8) results in:

$$\tilde{\eta}^{\varphi_n} \left(\dots, {}^{(l)}\varrho, {}^{(l)}\mathbf{v}, {}^{(l)}\theta, \dots, {}^{(k)}\varrho, {}^{(k)}\mathbf{v}, {}^{(k)}\theta, \dots \right) = \tilde{\eta}^{\varphi_n} \left(\dots, {}^{(k)}\varrho, {}^{(k)}\mathbf{v}, {}^{(k)}\theta, \dots, {}^{(l)}\varrho, {}^{(l)}\mathbf{v}, {}^{(l)}\theta, \dots \right), \tag{3.14}$$

implying that $\tilde{\eta}^{\varphi_n}$ does not change regardless of order in which spatial locations and their corresponding spectral parameters are considered. The above equation, upon simplification, results in:

$$\eta^{\varphi_n} \left(\dots, {}^{(l)}\varrho, {}^{(l)}\mathbf{v}, {}^{(l)}\theta, \dots, {}^{(k)}\varrho, {}^{(k)}\mathbf{v}, {}^{(k)}\theta, \dots \right) = \eta^{\varphi_n} \left(\dots, {}^{(k)}\varrho, {}^{(k)}\mathbf{v}, {}^{(k)}\theta, \dots, {}^{(l)}\varrho, {}^{(l)}\mathbf{v}, {}^{(l)}\theta, \dots \right). \tag{3.15}$$

The separation side condition (2.9) leads to the following restrictions on the infinitesimals:

$$\lim_{|\mathbf{x}^{(n+1)} - \mathbf{x}^{(k)}| \rightarrow \infty} \tilde{\eta}^{\varphi_{n+1}} = \varphi_1 \tilde{\eta}^{\varphi_n} + \varphi_n \tilde{\eta}^{\varphi_1}. \tag{3.16}$$

The separation side condition is not applied in the previous studies on incompressible turbulence [12, 21], and the current study. We consider a arbitrarily large domain that is not infinite in extent, in this study. As a result, the above separation condition does not need to be strictly enforced. In addition, in most practical turbulence applications, where scaling laws are derived using symmetry principles, infinite domains are not considered.

Let us now impose the reduction side condition (3.11) on the infinitesimals in (3.8) and (3.9). It is seen that the infinitesimals $\xi^{(1)}\varrho, \xi^{(1)}v_i$ and $\xi^{(1)}\theta$ automatically satisfy the reduction side condition (3.11), as $\xi^{(1)}\varrho = \xi^{(1)}v_i = \xi^{(1)}\theta = 0$ when ${}^{(1)}\varrho = {}^{(1)}\mathbf{v} = {}^{(1)}\theta = 0$. Hence, we turn our attention to η^{φ_1} :

$$\eta^{\varphi_1} |_{({}^{(1)}\varrho=0, {}^{(1)}\mathbf{v}=0, {}^{(1)}\theta=0)} = 0 \Rightarrow F_1(t, \mathbf{x}^{(1)}, 0, \mathbf{0}, 0) + i \left[\frac{a_\varphi}{i} + \frac{5 - 3\gamma}{2i(\gamma - 1)} a_{pr}t \right] = 0. \tag{3.17}$$

It can be seen that the only symmetry groups that remain after imposing ${}^{(1)}\varrho = {}^{(1)}v_i = {}^{(1)}\theta = 0$ are the projection (a_{pr}) group, superposition group (F_1) and the dilation group (a_φ). In the above equation, we consider one group at a time, as they are single-parameter symmetry transformations. We begin with the projective group, a_{pr} :

$$\eta^{\varphi_1} |_{({}^{(1)}\varrho=0, {}^{(1)}\mathbf{v}=0, {}^{(1)}\theta=0)} = 0 \Rightarrow i \left[\frac{5 - 3\gamma}{2i(\gamma - 1)} a_{pr}t \right] = 0 \Rightarrow \gamma = \frac{5}{3}, \tag{3.18}$$

implying that the reduction side condition is satisfied by the projective group only for the case of monatomic gases. This is consistent with what is seen in the Euler equations. In CF formulation,

however, the restriction to monatomic gases is imposed by the reduction side condition. The imposition of reduction side condition on the F_1 group results in

$$F_1(t, \mathbf{x}^{(1)}, {}^{(1)}\varrho = 0, {}^{(1)}\mathbf{v} = \mathbf{0}, {}^{(1)}\theta = 0) = 0, \tag{3.19}$$

implying that F_1 must reduce to zero, when spectral parameters are zero. This also implies that F_1 cannot be a constant (other than zero), as it would violate reduction condition. Finally, the reduction condition on the dilation group, a_φ , leads to the restriction that $a_\varphi = 0$. However, one could conceive the following transformation for φ_1 , by combining F_1 and a_φ groups:

$$\eta^{\varphi_1} = -a_\varphi + \varphi_1 a_\varphi. \tag{3.20}$$

In this case, we consider $F_1 = -a_\varphi$ and as a result, the reduction condition is automatically satisfied. The groups, a_φ and $F_1 = -a_\varphi$ (as seen above, that F_1 cannot be a constant), each considered separately do not satisfy the reduction side condition. However, when considered together as shown in equation (3.20), the reduction constraint is upheld. The final form for the single-point CF infinitesimal is

$$\eta^{\varphi_1} = F_1(t, \mathbf{x}^{(1)}, {}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta) - a_\varphi + i\varphi_1 \left[\frac{a_{pr}}{2} {}^{(1)}v_i x_i^{(1)} + a_{g,1} {}^{(1)}v_1 + a_{g,2} {}^{(1)}v_2 + a_{g,3} {}^{(1)}v_3 + \frac{a_\varphi}{i} \right], \tag{3.21}$$

Similarly, utilising MAPLE [10] for part of the analysis of the two-point CF (upon imposing the reduction side condition) results in:

$$\begin{aligned} \eta^{\varphi_2} = & F_2(t, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, {}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta, {}^{(2)}\varrho, {}^{(2)}\mathbf{v}, {}^{(2)}\theta) - a_\varphi \\ & + i\varphi_2 \left[\frac{a_\varphi}{i} + \frac{a_{pr}}{2} \left({}^{(1)}v_i x_i^{(1)} + {}^{(2)}v_i x_i^{(2)} \right) + a_{g,1} \left({}^{(1)}v_1 + {}^{(2)}v_1 \right) \right. \\ & \left. + a_{g,2} \left({}^{(1)}v_2 + {}^{(2)}v_2 \right) + a_{g,3} \left({}^{(1)}v_3 + {}^{(2)}v_3 \right) \right]. \end{aligned} \tag{3.22}$$

Finally, using one- and two-point CF infinitesimals, we generalise the expression for an n -point CF infinitesimal, which is given by:

$$\begin{aligned} \eta^{\varphi_n} = & F_n(t, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}, {}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta, \dots, {}^{(n)}\varrho, {}^{(n)}\mathbf{v}, {}^{(n)}\theta) - a_\varphi \\ & + i\varphi_n \left[\frac{a_\varphi}{i} + \frac{a_{pr}}{2} \sum_{k=1}^n {}^{(k)}v_i x_i^{(k)} + a_{g,1} \sum_{k=1}^n {}^{(k)}v_1 + a_{g,2} \sum_{k=1}^n {}^{(k)}v_2 + a_{g,3} \sum_{k=1}^n {}^{(k)}v_3 \right]. \end{aligned} \tag{3.23}$$

In the above equations (3.22) and (3.23), the functions F_2 and F_n appear as a result of the superposition principle. It is important to note that in equations (3.8), (3.21), (3.22) and (3.23), a_{pr} is non-zero only for monatomic gases, that is, when $\gamma \neq 5/3$, $a_{pr} = 0$. All the other Euler symmetry groups uphold the side conditions without any restrictions. We impose the coincidence (2.7) and permutation (2.8) side conditions on statistical groups in the next section.

4 Physics represented by each of the groups

In this section, we will derive the global forms each of the symmetry transformations and discuss their relevant physics. The kinematic symmetries – namely space translation ($a_{x,i}$), time translation (a_t) and rotation groups ($a_{r,i}$) – which pertain to coordinate system independence are not discussed in this section.

4.1 Projective group

We begin with the projective transformation represented by the group parameter, a_{pr} . This transformation is a symmetry of CF hierarchy only for monatomic gases, $\gamma = 5/3$. Under this group, the infinitesimals for the dependent and independent variables are given by:

$$\begin{aligned} \xi^t &= t^2, & \xi^{x_1^{(1)}} &= x_1^{(1)}t, & \xi^{x_2^{(1)}} &= x_2^{(1)}t, & \xi^{x_3^{(1)}} &= x_3^{(1)}t, & \xi^{(1)\varrho} &= 3^{(1)}\varrho t, \\ \xi^{(1)v_1} &= {}^{(1)}v_1 t, & \xi^{(1)v_2} &= {}^{(1)}v_2 t, & \xi^{(1)v_3} &= {}^{(1)}v_3 t, & \xi^{(1)\theta} &= 2^{(1)}\theta t \\ \eta^{\varphi_1} &= \varphi_1 \left[i x_i^{(1)} {}^{(1)}v_i \right]. \end{aligned} \quad (4.1)$$

Below is the global transformation, that is obtained utilising Lie's first theorem [2]:

$$t^* = \frac{t}{1 - a_{pr}t}, \quad x_i^{(1)*} = \frac{x_i^{(1)}}{1 - a_{pr}t}, \quad (4.2a)$$

$${}^{(1)}\varrho^* = \frac{{}^{(1)}\varrho}{(1 - a_{pr}t)^3}, \quad {}^{(1)}\theta^* = \frac{{}^{(1)}\theta}{(1 - a_{pr}t)^2}, \quad {}^{(1)}v_i^* = \frac{{}^{(1)}v_i}{1 - a_{pr}t} \quad (4.2b)$$

$$\varphi_1^* = \varphi_1 \exp\left(i a_{pr} \frac{{}^{(1)}v_i x_i^{(1)}}{1 - a_{pr}t} \right). \quad (4.2c)$$

For a general multi-point CF, the infinitesimal and the global transformation take the form:

$$\eta^{\varphi_n} = i\varphi_n \left[x_i^{(1)} {}^{(1)}v_i + \dots + x_i^{(n)} {}^{(n)}v_i \right], \quad \varphi_n^* = \varphi_n \exp\left(i a_{pr} \frac{{}^{(1)}v_i x_i^{(1)} + \dots + {}^{(n)}v_i x_i^{(n)}}{1 - a_{pr}t} \right). \quad (4.3)$$

4.2 Dilation groups

Let us now investigate the dilation group, $a_{s,t}$, also known as 'scaling in time'. The non-trivial infinitesimals resulting from this symmetry transformation are

$$\xi^t = t, \quad \xi^{(1)v_i} = {}^{(1)}v_i, \quad \xi^{(1)\theta} = 2^{(1)}\theta. \quad (4.4a)$$

All other dependent and independent variables do not transform under this symmetry group. The global form of the transformation is given by:

$$t^* = e^{a_{s,t}t}, \quad {}^{(1)}v_i^* = e^{a_{s,t}} {}^{(1)}v_i, \quad {}^{(1)}\theta^* = e^{2a_{s,t}} {}^{(1)}\theta. \quad (4.4b)$$

For ‘scaling in space’ group, $a_{s,x}$, the infinitesimals and global form of the symmetry transformation are given below:

$$\xi_i^{x(1)} = x_i^{(1)}, \quad \xi^{(1)v_i} = -^{(1)}v_i, \quad \xi^{(1)\theta} = -2^{(1)}\theta. \tag{4.5a}$$

$$x_i^{(1)*} = e^{a_{s,x}} x_i^{(1)}, \quad ^{(1)}v_i^* = e^{-a_{s,x}} ^{(1)}v_i, \quad ^{(1)}\theta^* = e^{-2a_{s,x}} ^{(1)}\theta. \tag{4.5b}$$

The third dilation group that exists only in compressible fluids is ‘scaling of density’, represented by the parameter, $a_{s,\rho}$. The only non-trivial infinitesimal in the CF formulation is that of density spectral parameter, that is, $\xi^{(1)\varrho}$. The expressions for the infinitesimal and the corresponding global transformation are given below:

$$\xi^{(1)\varrho} = -^{(1)}\varrho, \quad ^{(1)}\varrho^* = e^{-a_{s,\rho}} ^{(1)}\varrho. \tag{4.6}$$

4.3 Galilean invariance

For the case of the Galilean invariance groups, $a_{g,\alpha}$, where $\alpha = 1 - 3$, the non-zero infinitesimals are

$$\xi^{x_\alpha(1)} = t, \quad \eta^{\varphi_1} = i^{(1)}v_\alpha \varphi_1. \tag{4.7}$$

The global transformation takes the form:

$$x_\alpha^{(1)*} = x_\alpha^{(1)} + a_{g,\alpha} t, \quad \varphi_1^* = e^{i^{(1)}v_\alpha a_{g,\alpha}} \varphi_1. \tag{4.8}$$

In the above equation, repeated α indices do not imply summation.

4.4 Statistical groups

We now turn our attention to the new symmetry groups in the CF formulation that are not seen in the Euler equations. We begin with the function, F_1 , seen in the η^{φ_1} equation (3.21):

$$\eta^{\varphi_1} = F_1(t, \mathbf{x}^{(1)}, ^{(1)}\varrho, ^{(1)}\mathbf{v}, ^{(1)}\theta), \tag{4.9}$$

and all other infinitesimals are trivial in nature. As a result, the global form of this symmetry takes the form:

$$\varphi_1^* = \varphi_1 + aF_1(t, \mathbf{x}^{(1)}, ^{(1)}\varrho, ^{(1)}\mathbf{v}, ^{(1)}\theta). \tag{4.10}$$

For the multi-point CF, the global transformation due to superposition principle (from equation (3.23)), takes the form:

$$\varphi_n^* = \varphi_n + aF_n(t, \mathbf{x}^{(1)}, ^{(1)}\varrho, ^{(1)}\mathbf{v}, ^{(1)}\theta, \dots, \mathbf{x}^{(n)}, ^{(n)}\varrho, ^{(n)}\mathbf{v}, ^{(n)}\theta). \tag{4.11}$$

To establish how the infinite hierarchy of functions F_1, \dots, F_n are related to one another, we impose the reduction side condition, that is, we set the spectral parameters at $\mathbf{x}^{(n)}$ to zero. The equation then becomes

$$\varphi_{n-1}^* = \varphi_{n-1} + aF_n(t, \mathbf{x}^{(1)}, ^{(1)}\varrho, ^{(1)}\mathbf{v}, ^{(1)}\theta, \dots, \mathbf{x}^{(n)}, \mathbf{0}, \mathbf{0}, \mathbf{0}). \tag{4.12}$$

As the dependence on $\mathbf{x}^{(n)}$ is not exhibited by φ_{n-1}^* and φ_{n-1} , the reduction condition implies that F_n at ${}^{(n)}\varrho = {}^{(n)}\mathbf{v} = {}^{(n)}\theta = 0$ does not depend on $\mathbf{x}^{(n)}$ too. In addition $F_n(\dots, {}^{(n)}\varrho = {}^{(n)}\mathbf{v} = {}^{(n)}\theta = 0)$ should also identically satisfy the governing equation for $(n - 1)$ -point CF. Therefore, $F_n(\dots, {}^{(n)}\varrho = {}^{(n)}\mathbf{v} = {}^{(n)}\theta = 0)$ reduces to the same family of functions that are represented by F_{n-1} . The set of functions, F_n , to satisfy the permutation side condition (2.8), when:

$$F_n(\dots, {}^{(l)}\varrho, {}^{(l)}\mathbf{v}, {}^{(l)}\theta, \dots, {}^{(k)}\varrho, {}^{(k)}\mathbf{v}, {}^{(k)}\theta, \dots) = F_n(\dots, {}^{(k)}\varrho, {}^{(k)}\mathbf{v}, {}^{(k)}\theta, \dots, {}^{(l)}\varrho, {}^{(l)}\mathbf{v}, {}^{(l)}\theta, \dots), \tag{4.13}$$

that is, the function value remains unchanged regardless of the order in which the spatial points are considered. Coincidence side condition (2.7) is upheld when:

$$\lim_{|\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}| \rightarrow 0} F_{n+1} = F_n(\dots, {}^{(n+1)}\varrho + {}^{(n)}\varrho, {}^{(n+1)}\mathbf{v} + {}^{(n)}\mathbf{v}, {}^{(n+1)}\theta + {}^{(n)}\theta, \dots). \tag{4.14}$$

In the case of separation side condition (2.9),

$$\begin{aligned} \lim_{|\mathbf{x}^{(n+1)} - \mathbf{x}^{(k)}| \rightarrow \infty} F_{n+1} &= \varphi_1({}^{(n+1)}\varrho, {}^{(n+1)}\mathbf{v}, {}^{(n+1)}\theta; t, \mathbf{x}^{(n+1)}) F_n \\ &+ \varphi_n F_1(t, \mathbf{x}^{(n+1)}, {}^{(n+1)}\varrho, {}^{(n+1)}\mathbf{v}, {}^{(n+1)}\theta) \end{aligned} \tag{4.15}$$

needs to be satisfied. As discussed previously, the separation side condition (2.9) is not imposed in this study.

As an example, let us consider a set of functions F_n that do not have any spatial and temporal dependence, that is,

$$\eta^{\varphi_n} = F_n({}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta, \dots, {}^{(n)}\varrho, {}^{(n)}\mathbf{v}, {}^{(n)}\theta). \tag{4.16}$$

The transformed CFs, φ_n^* , given by:

$$\varphi_n^* = \varphi_n + a F_n({}^{(1)}\varrho, {}^{(1)}\mathbf{v}, {}^{(1)}\theta, \dots, {}^{(n)}\varrho, {}^{(n)}\mathbf{v}, {}^{(n)}\theta). \tag{4.17}$$

automatically satisfy the CF hierarchy. This symmetry is a more general compressible analogue of the ‘shape’ symmetry demonstrated in [21]. As established in the previous section, the functions F_n cannot be constants as they would then violate the reduction side condition. As evidenced in the case of shape symmetry in incompressible flows [21], the separation side condition is not satisfied. The discussion pertinent to not strictly enforcing this side condition is presented in Section 3.2.1.

Finally, we investigate the symmetry group a_φ . The resulting infinitesimals and the global transformation are

$$\eta^{\varphi_1} = \varphi_1 - 1, \quad \varphi_1^* = 1 + e^{a\varphi}(\varphi_1 - 1), \tag{4.18a}$$

$$\eta^{\varphi_n} = \varphi_n - 1, \quad \varphi_n^* = 1 + e^{a\varphi}(\varphi_n - 1). \tag{4.18b}$$

The above global form of the transformation satisfies all the required conditions, except for the separation side condition (2.9). This symmetry is the compressible counterpart of the ‘intermittency’ symmetry, representing the external intermittency seen in turbulent flows [21]. The implications of the both of the above statistical symmetries is investigated in detail in the next section.

5 Global symmetry transformations of compressible turbulence moments

One of the major advantages of computing the symmetries of the CF hierarchy is the ease with which these symmetries could be translated to any velocity and thermodynamics moments. In this section, we explore the symmetry transformations of various key statistics in compressible turbulence, such as kinetic and internal energies, various order moments of velocity, density, temperature and pressure. The advantage of utilising the CF formulation for symmetry investigation becomes clearly evident in this section. Eventually, this will also have important implications on the development of statistical turbulence models for compressible flows.

5.1 Averaged kinetic energy

We begin our investigation with averaged kinetic energy (k), which in compressible turbulence is defined as:

$$k \equiv \left\langle \frac{1}{2} \rho u_i u_i \right\rangle. \tag{5.1}$$

As discussed in the introduction, the moments can be obtained from the CF, upon taking derivatives. In the case of kinetic energy, this would imply that we compute

$$\left\langle \frac{1}{2} \rho u_i u_i \right\rangle = \frac{i}{2} \frac{\partial^3 \varphi_1}{\partial^{(1)\varrho} \partial^{(1)v_i} \partial^{(1)v_i}} \Big|_{(1)\varrho=(1)v_i=(1)\theta=0}. \tag{5.2}$$

Computing the transformation of k in equation (5.1) under the projective group (4.2) is a bit involved. We first look at the above triple derivative of CF in transformed variables, that is,

$$\frac{\partial^3 \varphi_1^*}{\partial^{(1)\varrho^*} \partial^{(1)v_i^*} \partial^{(1)v_i^*}} = \frac{\partial^3}{\partial^{(1)\varrho^*} \partial^{(1)v_i^*} \partial^{(1)v_i^*}} \left[\varphi_1 \exp \left(i a_{pr} \frac{(1)v_i x_i^{(1)}}{1 - a_{prt}} \right) \right]. \tag{5.3}$$

We now obtain the expressions for each of the derivatives with respect to the transformed variables in terms of original variables:

$$\begin{aligned} \frac{\partial}{\partial^{(1)\varrho^*}} &= \frac{\partial t}{\partial^{(1)\varrho^*}} \frac{\partial}{\partial t} + \frac{\partial x_i^{(1)}}{\partial^{(1)\varrho^*}} \frac{\partial}{\partial x_i^{(1)}} + \frac{\partial^{(1)\varrho}}{\partial^{(1)\varrho^*}} \frac{\partial}{\partial^{(1)\varrho}} + \frac{\partial^{(1)v_i}}{\partial^{(1)\varrho^*}} \frac{\partial}{\partial^{(1)v_i}} + \frac{\partial^{(1)\theta}}{\partial^{(1)\varrho^*}} \frac{\partial}{\partial^{(1)\theta}} \\ &= \frac{1}{(1 + a_{prt^*})^3} \frac{\partial}{\partial^{(1)\varrho}} = (1 - a_{prt})^3 \frac{\partial}{\partial^{(1)\varrho}} \end{aligned} \tag{5.4a}$$

and

$$\begin{aligned} \frac{\partial}{\partial^{(1)v_j^*}} &= \frac{\partial t}{\partial^{(1)v_j^*}} \frac{\partial}{\partial t} + \frac{\partial x_i^{(1)}}{\partial^{(1)v_j^*}} \frac{\partial}{\partial x_i^{(1)}} + \frac{\partial^{(1)\varrho}}{\partial^{(1)v_j^*}} \frac{\partial}{\partial^{(1)\varrho}} + \frac{\partial^{(1)v_i}}{\partial^{(1)v_j^*}} \frac{\partial}{\partial^{(1)v_i}} + \frac{\partial^{(1)\theta}}{\partial^{(1)v_j^*}} \frac{\partial}{\partial^{(1)\theta}} \\ &= \frac{1}{1 + a_{prt^*}} \frac{\partial}{\partial^{(1)v_j}} = (1 - a_{prt}) \frac{\partial}{\partial^{(1)v_j}}. \end{aligned} \tag{5.4b}$$

As a result, we obtain

$$\begin{aligned} \frac{\partial^3 \varphi_1^*}{\partial^{(1)\varrho} \partial^{(1)v_i^*} \partial^{(1)v_i^*}} &= (1 - a_{prt})^5 \frac{\partial^3 \varphi_1^*}{\partial^{(1)\varrho} \partial^{(1)v_i} \partial^{(1)v_i}} \\ &= (1 - a_{prt})^5 \frac{\partial^3}{\partial^{(1)\varrho} \partial^{(1)v_i} \partial^{(1)v_i}} \left[\varphi_1 \exp \left(i a_{pr} \frac{(1)v_i x_i^{(1)}}{1 - a_{prt}} \right) \right] \\ &= (1 - a_{prt})^5 \left[\frac{\partial^3 \varphi_1}{\partial^{(1)\varrho} \partial^{(1)v_i} \partial^{(1)v_i}} + \frac{2i a_{pr} x_i^{(1)}}{1 - a_{prt}} \frac{\partial^2 \varphi_1}{\partial^{(1)\varrho} \partial^{(1)v_i}} \right. \\ &\quad \left. - \frac{x_i^{(1)} x_i^{(1)} a_{pr}^2}{(1 - a_{prt})^2} \frac{\partial \varphi_1}{\partial^{(1)\varrho}} \right] \exp \left(i a_{pr} \frac{(1)v_i x_i^{(1)}}{1 - a_{prt}} \right). \end{aligned} \tag{5.4c}$$

Finally, the transformed kinetic energy under the projective group is given by:

$$\left\langle \frac{1}{2} \rho u_i u_i \right\rangle^* = (1 - a_{prt})^5 \left\langle \frac{1}{2} \rho u_i u_i \right\rangle + a_{pr} x_i^{(1)} (1 - a_{prt})^4 \langle \rho u_i \rangle + \frac{a_{pr}^2}{2} x_i^{(1)} x_i^{(1)} (1 - a_{prt})^3 \langle \rho \rangle. \tag{5.4d}$$

In all of the scaling groups, the global transformation for k is straightforward, as these groups lead to the scaling of kinetic energy. For ‘scaling of time’ group (4.4), the global symmetry transformation of k looks like:

$$\left\langle \frac{1}{2} \rho u_i u_i \right\rangle^* = e^{-2a_s t} \left\langle \frac{1}{2} \rho u_i u_i \right\rangle. \tag{5.5}$$

For the ‘scaling of space’ group (4.5):

$$\left\langle \frac{1}{2} \rho u_i u_i \right\rangle^* = e^{2a_s x} \left\langle \frac{1}{2} \rho u_i u_i \right\rangle. \tag{5.6}$$

Finally, the ‘scaling of density’ group (4.6):

$$\left\langle \frac{1}{2} \rho u_i u_i \right\rangle^* = e^{a_s \rho} \left\langle \frac{1}{2} \rho u_i u_i \right\rangle. \tag{5.7}$$

We now shift our focus to two statistical symmetries that are present specifically in the case of CF hierarchy and not seen in Euler equations: shape and intermittency groups ((4.17) and (4.18)). For the shape symmetry (4.17), the global transformation of k , takes the form:

$$\begin{aligned} \left\langle \frac{1}{2} \rho u_i u_i \right\rangle^* &= \left\langle \frac{1}{2} \rho u_i u_i \right\rangle + \frac{i}{2} a \frac{\partial^3 F_1}{\partial^{(1)\varrho} \partial^{(1)v_i} \partial^{(1)v_i}} \Big|_{(1)\varrho=(1)v_i=(1)\theta=0} \\ &= \left\langle \frac{1}{2} \rho u_i u_i \right\rangle + a_k, \end{aligned} \tag{5.8}$$

where a_k is a constant. Therefore, the addition of a constant to averaged kinetic energy in compressible turbulence leaves its governing equation invariant. In the case of the intermittency group (4.18), the kinetic energy transforms as:

$$\left\langle \frac{1}{2} \rho u_i u_i \right\rangle^* = e^{a_\varphi} \left\langle \frac{1}{2} \rho u_i u_i \right\rangle. \tag{5.9}$$

5.2 Averaged internal energy

The second measure of considerable importance in compressible turbulence is the averaged internal energy (e), defined as:

$$e = \frac{\langle p \rangle}{\gamma - 1}. \tag{5.10}$$

As the ratio, $1/(\gamma - 1)$ remains a constant for a given gas, we focus on the average pressure, $\langle p \rangle$ in the remainder of the paper, when we are investigating averaged internal energy. Based on our definition of CF, $\langle p \rangle$ can be obtained by:

$$\langle p \rangle = - \frac{\partial^2 \varphi_1}{\partial^{(1)\varrho} \partial^{(1)\theta}} \Big|_{(1)\varrho=(1)v_i=(1)\theta=0}, \tag{5.11}$$

where, we have used ideal gas law, $p = \rho T$, in its non-dimensional form. As a result, $\langle p \rangle$ is obtained by taking derivatives of φ_1 with respect to $^{(1)}\varrho$ and $^{(1)}\theta$.

Let us now begin our investigation of transformation of internal energy under symmetries of CF. For the projective group (4.2), we obtain

$$\langle p \rangle^* = - \frac{\partial^2 \varphi_1^*}{\partial^{(1)\varrho^*} \partial^{(1)\theta^*}} \Big|_{(1)\varrho^*=(1)v_i^*=(1)\theta^*=0}. \tag{5.12}$$

We perform similar calculations as shown for k in the previous Subsection 5.1, to obtain the transformed e :

$$\begin{aligned} \frac{\partial}{\partial^{(1)\theta^*}} &= \frac{\partial t}{\partial^{(1)\theta^*}} \frac{\partial}{\partial t} + \frac{\partial x_i^{(1)}}{\partial^{(1)\theta^*}} \frac{\partial}{\partial x_i^{(1)}} + \frac{\partial^{(1)\varrho}}{\partial^{(1)\theta^*}} \frac{\partial}{\partial^{(1)\varrho}} + \frac{\partial^{(1)v_i}}{\partial^{(1)\theta^*}} \frac{\partial}{\partial^{(1)v_i}} + \frac{\partial^{(1)\theta}}{\partial^{(1)\theta^*}} \frac{\partial}{\partial^{(1)\theta}} \\ &= (1 - a_{prt})^2 \frac{\partial}{\partial^{(1)\theta}}, \end{aligned} \tag{5.13a}$$

leading to,

$$\begin{aligned} \frac{\partial^2 \varphi_1^*}{\partial^{(1)\varrho^*} \partial^{(1)\theta^*}} &= (1 - a_{prt})^5 \frac{\partial^2}{\partial^{(1)\varrho} \partial^{(1)\theta}} \left[\varphi_1 \exp \left(i a_{pr} \frac{(1)v_i x_i^{(1)}}{1 - a_{prt}} \right) \right] \\ &= (1 - a_{prt})^5 \frac{\partial^2 \varphi_1}{\partial^{(1)\varrho} \partial^{(1)\theta}} \exp \left(i a_{pr} \frac{(1)v_i x_i^{(1)}}{1 - a_{prt}} \right), \end{aligned} \tag{5.13b}$$

and finally,

$$\langle p \rangle^* = (1 - a_{prt})^5 \langle p \rangle. \tag{5.13c}$$

In the case of ‘scaling of time’ symmetry (4.4), the global transformation of averaged internal energy is

$$\langle p \rangle^* = e^{-2a_{s,t}} \langle p \rangle. \tag{5.14}$$

In ‘scaling of space’ (4.5),

$$\langle p \rangle^* = e^{2a_{s,x}} \langle p \rangle. \tag{5.15}$$

‘Scaling of density’ (4.6) leads to:

$$\langle p \rangle^* = e^{a_{s,\rho}} \langle p \rangle. \tag{5.16}$$

The global transformation of e , in the case of shape symmetry (4.17), is given by:

$$\langle p \rangle^* = \langle p \rangle - a \frac{\partial^2 F_1}{\partial^{(1)}\varrho \partial^{(1)}\theta} \Big|_{(1)\varrho=(1)v_i=(1)\theta=0} = \langle p \rangle + a_e. \tag{5.17}$$

Finally, the intermittency group (4.18) leads to:

$$\langle p \rangle^* = e^{a_\varphi} \langle p \rangle. \tag{5.18}$$

5.3 Higher-order velocity and thermodynamic moments

We now investigate the higher-order moments of density, temperature, pressure and velocity. Since we are interested in multi-point moments, we need to consider the derivatives of the multi-point CF. For example, the multi-point density moments are obtained from the CF as follows:

$$\left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle = \frac{1}{i^n} \left[\prod_{l=1}^n \frac{\partial}{\partial^{(l)}\varrho} \varphi_n \right]_{(1)\varrho=(1)v_i=(1)\theta=0 \dots (n)\varrho=(n)v_i=(n)\theta=0} \tag{5.19}$$

We now write down the global transformations of each of the higher-order moments with respect to the symmetry transformations of the CF hierarchy given in Section 3.2, starting with the n th-order moment of density:

$$a_{pr} : \left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle^* = (1 - a_{prt})^{3n} \left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.20a}$$

$$a_{s,\rho} : \left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{na_{s,\rho}} \left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.20b}$$

$$F_n : \left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle^* = \left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle + a_{\rho,n} \tag{5.20c}$$

$$a_\varphi : \left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{a_\varphi} \left\langle \prod_{l=1}^n \rho(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.20d}$$

Now the multi-point temperature moments transform as:

$$a_{pr} : \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle^* = (1 - a_{prt})^{2n} \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.21a}$$

$$a_{s,t} : \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{-2na_{s,t}} \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.21b}$$

$$a_{s,x} : \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{2na_{s,x}} \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.21c}$$

$$F_n : \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle^* = \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle + a_{T,n} \tag{5.21d}$$

$$a_\varphi : \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{a_\varphi} \left\langle \prod_{l=1}^n T(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.21e}$$

In the case of pressure,

$$a_{pr} : \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle^* = (1 - a_{pr}t)^{5n} \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.22a}$$

$$a_{s,t} : \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{-2na_{s,t}} \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.22b}$$

$$a_{s,x} : \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{2na_{s,x}} \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.22c}$$

$$a_{s,\rho} : \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{na_{s,\rho}} \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.22d}$$

$$F_n : \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle^* = \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle + a_{p,n} \tag{5.22e}$$

$$a_\varphi : \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{a_\varphi} \left\langle \prod_{l=1}^n p(\mathbf{x}^{(l)}, t) \right\rangle \tag{5.22f}$$

The multi-point velocity moments constitute an n th-order tensor that can be obtained from the CF as:

$$\langle u_{i_1}(\mathbf{x}^{(1)}, t) u_{i_2}(\mathbf{x}^{(2)}, t) \dots u_{i_n}(\mathbf{x}^{(n)}, t) \rangle = \frac{1}{i^n} \frac{\partial^n \varphi_n}{\partial^{(1)}v_{i_1} \partial^{(2)}v_{i_2} \dots \partial^{(n)}v_{i_n}} \Big|_{(1)0=..=(n)0=0}, \tag{5.23}$$

where i_1, i_2, \dots, i_n are indices of the multi-point velocity moment tensor taking values from 1 to 3. The global transformations of the multi-point velocity moment tensor under the CF symmetries are given below:

$$\begin{aligned} a_{pr} : \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle^* &= (1 - a_{pr}t)^n \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle \\ &+ (1 - a_{pr}t)^{n-1} \sum_{k=1}^n \left[a_{pr} x_{i_k}^{(k)} \left\langle \prod_{l=1, l \neq k}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle \right] \\ &+ \dots + (1 - a_{pr}t) \sum_{k=1}^n \left[a_{pr}^{n-1} \langle u_{i_k}(\mathbf{x}^{(k)}, t) \rangle \prod_{l=1, l \neq k}^n x_{i_l}^{(l)} \right] \\ &+ a_{pr}^n x_{i_1}^{(1)} x_{i_2}^{(2)} \dots x_{i_n}^{(n)} \end{aligned} \tag{5.24a}$$

$$a_{s,t} : \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{-na_{s,t}} \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle \quad (5.24b)$$

$$a_{s,x} : \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{na_{s,x}} \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle \quad (5.24c)$$

$$F_n : \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle^* = \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle + a_{i_1 i_2 \dots i_n} \quad (5.24d)$$

$$a_\varphi : \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle^* = e^{a_\varphi} \left\langle \prod_{l=1}^n u_{i_l}(\mathbf{x}^{(l)}, t) \right\rangle. \quad (5.24e)$$

Thus, all the flow and thermodynamics moments can be obtained from the CF hierarchy and as a result, the process of finding scaling laws via Lie group analysis is considerably simplified.

6 Conclusions

We derive the symmetries of the CF hierarchy of compressible turbulence. As the infinite set of CF equations encompasses all the flow and thermodynamic moment hierarchies, it is an effective framework, in the next step, for obtaining scaling laws for compressible statistics. The effects of viscosity and heat conductivity are neglected, as they lead to symmetry breaking in the instantaneous equations. The symmetry groups of the CF hierarchy obtained can be broadly classified into two groups:

- (1) *Euler symmetries*: The symmetries exhibited by the compressible Euler equations are replicated in the CF framework, as anticipated. These include (i) the three dilation groups: scaling in time, space and density; (ii) groups corresponding to invariance with respect to coordinate system; and (iii) projective group which is solely restricted to monatomic gases ($\gamma = 5/3$), that is, gases with only translational degrees of freedom, due to the imposition of reduction side condition.
- (2) *Statistical symmetries*: In addition to the Euler symmetries that are replicated in the CF hierarchy, additional symmetries – the so called ‘statistical’ symmetries – manifest, mainly as a result of the linearity of the governing equations. Due to the principle of superposition, a general infinite set of functions, F_n , could be added to the CF hierarchy, subject to additional constraints. Depending on the form of F_n , two types of statistical symmetry groups emerge that are also evident in the moment and PDF hierarchies in incompressible turbulence:
 - (a) *Shape symmetry*: When the functions, F_n , do not exhibit any spatial and temporal dependence, then the symmetry analogous to that of the ‘shape’ symmetry in incompressible turbulence results. Just like its incompressible counterpart, shape symmetry in the compressible CF hierarchy implies the invariance of the moments to translation.
 - (b) *Intermittency symmetry*: Like its counterpart in incompressible turbulence, this symmetry indicates the external intermittency seen in turbulent flows. This symmetry group results in the scaling of compressible moments by an arbitrary constant.

The findings of this paper will be helpful in deriving scaling laws for compressible turbulence statistics in various canonical flows. They are also useful in invariant modelling of compressible turbulence.

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Conflict of interest

None.

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