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GROWTH AND OSCILLATION THEORY OF NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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Abstract We investigate the growth and the frequency of zeros of the solutions of the differential equation $f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_0(z)f = H(z)$, where $P_0(z), P_1(z), \ldots, P_{n-1}(z)$ are polynomials with $P_0(z) \not\equiv 0$, and $H(z) \not\equiv 0$ is an entire function of finite order.

Keywords: linear differential equation; non-homogeneous; entire function; order of growth; exponent of convergence

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1. Introduction

For $n \geqslant 1$ consider the non-homogeneous linear differential equation

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_0(z)f = H(z), \tag{1.1}$$

where $P_0(z), P_1(z), \ldots, P_{n-1}(z)$ are polynomials with $P_0(z) \not\equiv 0$, and $H(z) \not\equiv 0$ is an entire function of finite order. It is well known that every solution f of equation (1.1) is an entire function.

For an entire function g, let $\rho(g)$ denote the order of growth of g, and when $g \not\equiv 0$, let $\lambda(g)$ denote the exponent of convergence of the sequence of zeros of g.

For solutions f of equation (1.1), we investigate the possible values of $\rho(f)$, and the relationships between the values of $\rho(f)$, $\lambda(f)$, $\rho(H)$ and $\lambda(H)$. We give examples to illustrate the sharpness of our results.

2. Statement of results

To discuss the possible orders of solutions of equation (1.1), we need to know the possible orders of the solutions of the corresponding homogeneous equation

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_0(z)f = 0, \tag{2.1}$$

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where $P_0(z), P_1(z), \dots, P_{n-1}(z)$ are the polynomials in (1.1). In [**3**], the authors present an algorithm which generates a non-empty finite set Φ of at most n positive rational numbers that includes all the possible orders of the transcendental solutions of (2.1). This algorithm to obtain the set Φ uses only simple arithmetic with the degrees of the coefficients in (2.1). Others have obtained such a set by appealing to the Newton–Puiseux diagram (see [**6,7,10,11**]). For the reader's convenience, we now state the algorithm in [**3**].

For each polynomial $P_j(z)$ in (2.1), we set $d_j = \deg P_j$ when $P_j(z) \not\equiv 0$, and we set $d_j = -\infty$ when $P_j(z) \equiv 0$. We define a strictly decreasing finite sequence of non-negative integers,

$$s_1 > s_2 > \cdots > s_p \geqslant 0, \tag{2.2}$$

in the following manner. We choose s_1 to be the unique integer satisfying

$$s_1 = \min\left\{ j : \frac{d_j}{n-j} = \max_{0 \leqslant k \leqslant n-1} \frac{d_k}{n-k} \right\}.$$

Then, given s_j $(j \geqslant 1)$, we define s_{j+1} to be the unique integer satisfying

$$s_{j+1} = \min\left\{ i : \frac{d_i - d_{s_j}}{s_j - i} = \max_{0 \leqslant k < s_j} \frac{d_k - d_{s_j}}{s_j - k} > -1 \right\}.$$

For a certain p, the integer s_p will exist, but the integer s_{p+1} will not exist, and then the sequence $s_1, s_2, \dots, s_p$ terminates with s_p. Then $p \leqslant n$, and (2.2) holds.

Correspondingly, define for $j = 1, 2, \dots, p$,

$$\alpha_j = 1 + \frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j},$$

where we set $s_0 = n$ and $d_{s_0} = d_n = 0$. Let Φ denote the set

$$\Phi = \{\alpha_1, \alpha_2, \dots, \alpha_p\}. \tag{2.3}$$

We see that each $\alpha_j \in \Phi$ has the form

$$\alpha_j = m_j / q_j, \tag{2.4}$$

where m_j and q_j are positive integers with $q_j \leqslant n$. We also note that

$$\alpha_1 = 1 + \max_{0 \leqslant k \leqslant n-1} \frac{\deg P_k}{n-k}. \tag{2.5}$$

When $p \geqslant 2$, we have

$$\alpha_1 > \alpha_2 > \cdots > \alpha_p.$$

From Theorem 1 and Corollary 3 in [**3**], we have the following theorem.

Theorem 2.1 **(see [3]).** *The following statements hold for equation (2.1).*

(i) *If f is a transcendental solution of (2.1), then $\rho(f) \in \Phi$.*

(ii) *There exists a solution f_0 of (2.1) satisfying $\rho(f_0) = \alpha_1$.*

The next result gives the possible orders of solutions of the non-homogeneous equation (1.1).

Theorem 2.2. *Let Φ be the set in (2.3). If f is a solution of (1.1), then*

$$\text{either } \rho(f) = \rho(H), \text{ or } \rho(f) = \alpha \in \Phi, \text{ where } \alpha > \rho(H). \tag{2.6}$$

Moreover, there exists a solution f_0 of (1.1) satisfying

$$\rho(f_0) = \max\left\{\rho(H), 1 + \max_{0 \leqslant k \leqslant n-1} \frac{\deg P_k}{n-k}\right\}. \tag{2.7}$$

Regarding Theorem 2.2, it is easy to see from Lemma 3.1 in § 3 that there exist equations of the form (1.1) where $\rho(f) = \rho(H)$ for every solution f. On the other hand, there exist equations of the form (1.1) where every solution f satisfies $\rho(f) > \rho(H)$; see Example 4.1 in § 4.

In the case when $\rho(H)$ is an integer in (1.1), the following result gives a lower bound for $\rho(f)$ when $\rho(f) > \rho(H)$.

Theorem 2.3. *Let $\rho(H)$ be an integer, and let f be a solution of (1.1). If $\rho(f) > \rho(H)$, then*

$$\rho(f) \geqslant \rho(H) + 1/n. \tag{2.8}$$

The inequality (2.8) is sharp. Example 4.1 gives an equation of the form (1.1), where $\rho(H)$ is an integer, such that (2.8) becomes an equality for every solution f. Example 4.1 also gives equations of the form (1.1) where $\rho(H)$ is an integer, such that (2.8) becomes a strict inequality for every solution f. We cannot delete the condition that $\rho(H)$ be an integer from the hypothesis of Theorem 2.3; see Example 4.2.

We now consider the relationships between the values of $\rho(f)$, $\lambda(f)$, $\rho(H)$ and $\lambda(H)$ for solutions f of (1.1). Of course, it is well known that $\lambda(w) \leqslant \rho(w)$ for every entire function $w \not\equiv 0$. Gao proved the following two results.

Theorem 2.4 (see [1]). *If f is a solution of (1.1), then $\lambda(f) \geqslant \lambda(H)$.*

Theorem 2.5 (see [1]). *If $\lambda(H) = \rho(H)$ in (1.1), then every solution f of (1.1) satisfies $\lambda(f) = \rho(f)$.*

Another proof of Theorem 2.4 is given in [**4**]. We prove the following result.

Theorem 2.6. *If f is a solution of (1.1), then*

$$\rho(f) - \lambda(f) \leqslant \rho(H) - \lambda(H). \tag{2.9}$$

Theorem 2.5 is a corollary of Theorem 2.6. Since any solution f of (1.1) clearly satisfies $\rho(f) \geqslant \rho(H)$, we see that Theorem 2.4 is also a corollary of Theorem 2.6.

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The inequality (2.9) is sharp. Theorem 2.9 and Example 4.3 show that it is possible for an equality to occur in (2.9), while Examples 4.4 and 4.5 show that it is possible for a strict inequality to occur in (2.9).

The next result is analogous to Theorem 2.3, and it shows that we can say more than what is stated in Theorem 2.4 in the case when $\lambda(H)$ is an integer in (1.1).

Theorem 2.7. *Let $\lambda(H)$ be an integer, and let f be a solution of (1.1). If $\lambda(f) > \lambda(H)$, then*

$$\lambda(f) \geqslant \lambda(H) + 1/n. \tag{2.10}$$

The inequality (2.10) is sharp; see Example 4.4. Regarding Theorem 2.7, there exist equations of the form (1.1), where $\lambda(H)$ is an integer, such that every solution f satisfies (2.10); see Example 4.6. Thus there exist equations of the form (1.1) where every solution f satisfies $\lambda(f) > \lambda(H)$. We cannot delete the condition that $\lambda(H)$ be an integer in the hypothesis of Theorem 2.7; see Examples 4.2 and 4.5.

For the case when $\rho(H) > \lambda(H)$ in (1.1), the next theorem shows that there exists a positive lower bound $C > 0$, depending only on H and the order of the equation, such that for any solution f of (1.1) satisfying $\rho(f) > \lambda(f)$, we have $\rho(f) - \lambda(f) \geqslant C$.

Theorem 2.8. *Let $\rho(H) > \lambda(H)$, and let f be a solution of (1.1). If $\rho(f) > \lambda(f)$, then*

$$\rho(f) - \lambda(f) \geqslant \min\{1/n, \rho(H) - \lambda(H)\}. \tag{2.11}$$

The inequality (2.11) is sharp. Example 4.3 gives an equation of the form (1.1), where $\rho(H) - \lambda(H) < 1/n$, which possesses a solution f_0 satisfying $\rho(f_0) - \lambda(f_0) = \rho(H) - \lambda(H) > 0$, while Example 4.4 gives an equation of the form (1.1), where $\rho(H) - \lambda(H) > 1/n$, which possesses a solution f_0 satisfying $\rho(f_0) - \lambda(f_0) = 1/n$. These examples give an equality in (2.11), and also show that neither of the constants '$1/n$' and '$\rho(H) - \lambda(H)$' can be deleted or replaced with larger constants in (2.11). It is also possible for a strict inequality to occur in (2.11); see Example 4.5.

By combining Theorems 2.6 and 2.8, we obtain the following result.

Theorem 2.9. *Let $\rho(H) - \lambda(H) \leqslant 1/n$, and let f be a solution of (1.1). If $\rho(f) > \lambda(f)$, then*

$$\rho(f) - \lambda(f) = \rho(H) - \lambda(H). \tag{2.12}$$

Theorem 2.9 is sharp. Specifically, if in the hypothesis of Theorem 2.9 we replace the condition '$\rho(H) - \lambda(H) \leqslant 1/n$' with the condition '$\rho(H) - \lambda(H) \leqslant \beta$', where β is any fixed constant satisfying $\beta > 1/n$, then (2.12) does not necessarily hold; see Example 4.5.

3. Lemmas

In this section we give lemmas which are used in the proofs of our theorems and examples.

Lemma 3.1. *Every solution f of (1.1) satisfies*

$$\rho(H) \leqslant \rho(f) \leqslant \max\left\{\rho(H), 1 + \max_{0\leqslant k\leqslant n-1} \frac{\deg P_k}{n-k}\right\}. \tag{3.1}$$

Moreover, there exists a solution f_0 of (1.1) satisfying

$$\rho(f_0) = \max\left\{\rho(H), 1 + \max_{0\leqslant k\leqslant n-1} \frac{\deg P_k}{n-k}\right\}. \tag{3.2}$$

Proof. Since statement (3.1) is contained in Lemma 6 in [**4**], we need only show that statement (3.2) holds.

For convenience, we set

$$\alpha_1 = 1 + \max_{0\leqslant k\leqslant n-1} \frac{\deg P_k}{n-k}, \tag{3.3}$$

as in (2.5). If

$$\rho(H) = \max\{\rho(H), \alpha_1\},$$

then, from (3.1), $\rho(f) = \rho(H)$ for every solution f of (1.1). Hence, statement (3.2) holds in this case.

We now assume that

$$\rho(H) < \max\{\rho(H), \alpha_1\} = \alpha_1. \tag{3.4}$$

Let f be a solution of (1.1). If $\rho(f) = \alpha_1$, then, from (3.3) and (3.4), we obtain that statement (3.2) holds. Now suppose that f satisfies $\rho(f) < \alpha_1$. From Theorem 2.1 (ii), there exists a solution g of the homogeneous equation (2.1) which satisfies $\rho(g) = \alpha_1$. Then $f + g$ is a solution of (1.1), and

$$\rho(f + g) = \alpha_1. \tag{3.5}$$

Thus, from (3.3), (3.4) and (3.5), we see that statement (3.2) holds in this case also. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $H(z)$ in (1.1) have the form $H = h\mathrm{e}^Q$, where $h \not\equiv 0$ is an entire function and Q is a non-constant polynomial, such that $\lambda(h) = \rho(h) < \deg Q$. Let f be a solution of (1.1), and set $g = f\mathrm{e}^{-Q}$. Then $\lambda(g) = \rho(g)$.*

Proof. Suppose that the conclusion is not true, i.e. suppose that $\lambda(g) < \rho(g)$. Then g has the form $g = w\mathrm{e}^R$, where $w \not\equiv 0$ is an entire function and R is a non-constant polynomial, such that

$$\lambda(w) = \rho(w) < \deg R. \tag{3.6}$$

Then $f = w\mathrm{e}^{R+Q}$, and so, from (1.1), we obtain that

$$w^{(n)} + b_{n-1}(z)w^{(n-1)} + \cdots + b_0(z)w = h(z)\mathrm{e}^{-R(z)}, \tag{3.7}$$

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where $b_0(z), b_1(z), \ldots, b_{n-1}(z)$ are polynomials. From (3.6) and (3.7),

$$\rho(h\mathrm{e}^{-R}) \leqslant \rho(w) < \rho(\mathrm{e}^R).$$

It follows that h must have the form $h = v\mathrm{e}^R$, where $v \not\equiv 0$ is an entire function satisfying $\rho(v) < \deg R$. Thus, $\rho(h) = \deg R$, and so

$$\lambda(h) = \lambda(v) \leqslant \rho(v) < \deg R = \rho(h),$$

which contradicts the hypothesis that $\lambda(h) = \rho(h)$. This proves Lemma 3.2. □

We obtain the following result by combining Theorem 2.5 above with Theorem 4 of [**4**].

Lemma 3.3 (see [1, 4]). *If f is a solution of (1.1) satisfying $\rho(f) > \lambda(f)$, then $H(z)$ must have the form $H = h\mathrm{e}^Q$, where $h \not\equiv 0$ is an entire function and Q is a non-constant polynomial, such that $\lambda(h) = \rho(h) < \deg Q$. Furthermore, f must have the form $f = g\mathrm{e}^Q$, where g is an entire function, such that $\rho(g) < \deg Q$ and $\rho(f) = \rho(H) = \deg Q$.*

Lemma 3.4 (see [2]). *Let $u(r)$ and $w(r)$ be monotone non-decreasing functions on $0 \leqslant r < \infty$, such that $u(r) \leqslant w(r)$ for all $r \notin E \cup [0, 1]$, where $E \subset (1, \infty)$ is a set of finite logarithmic measure. Then, for any given constant $\alpha > 1$, there exists a constant $r_0 = r_0(\alpha) > 0$ such that $u(r) \leqslant w(\alpha r)$ for all $r \geqslant r_0$.*

In the next lemma, $M(r, g)$ denotes the standard maximum modulus function for an entire function g.

Lemma 3.5. *Let g be an entire function satisfying $\rho(g) > 0$, and let α be any fixed constant satisfying $\alpha < \rho(g)$. Then there exists a set $S \subset (1, \infty)$ that has infinite logarithmic measure, such that*

$$M(r, g) > \mathrm{e}^{r^\alpha}, \tag{3.8}$$

for all $r \in S$.

Proof. Suppose that Lemma 3.5 is not true. Then, for some constant β satisfying $\beta < \rho(g)$, there exists a set $E \subset (1, \infty)$ that has finite logarithmic measure, such that

$$M(r, g) \leqslant \mathrm{e}^{r^\beta}, \tag{3.9}$$

for all $r \notin E \cup [0, 1]$. By applying Lemma 3.4 to (3.9), we obtain that there exists a constant $r_0 > 0$ such that

$$M(r, g) \leqslant \exp\{(2r)^\beta\},$$

for all $r \geqslant r_0$. But this implies that $\rho(g) \leqslant \beta$, which contradicts $\beta < \rho(g)$. This proves Lemma 3.5. □

4. Examples

In this section we give examples to illustrate the sharpness of our theorems, and to exhibit some possibilities that can occur.

The following example shows that Theorem 2.3 is sharp, and that there exist equations of the form (1.1) where every solution f satisfies $\rho(f) > \rho(H)$.

Example 4.1. Consider the differential equation

$$f'' + z^m f = R(z)e^{z^2}, \tag{4.1}$$

where $m \geqslant 3$ is an integer, and $R(z) \not\equiv 0$ is a polynomial satisfying $\deg R < m$. Let f be any solution of (4.1), and set

$$g(z) = f(z)e^{-z^2}. \tag{4.2}$$

Then g satisfies the equation

$$g'' + 4zg' + (z^m + 4z^2 + 2)g = R(z). \tag{4.3}$$

We see from (4.3) that g cannot be a polynomial. Also from (4.3),

$$\frac{g''}{g} + 4z\frac{g'}{g} + z^m + 4z^2 + 2 = \frac{R(z)}{g}. \tag{4.4}$$

Since $m \geqslant 3$, we can apply the Wiman–Valiron theory [**9**, pp. 105–108] to equation (4.4) to obtain that $\rho(g) = 1 + m/2 > 2$. Thus, from (4.2), we have $\rho(f) = 1 + m/2$.

Since (4.1) is an equation of the form (1.1) where $\rho(H) = 2$, and since we just proved that every solution f of (4.1) satisfies $\rho(f) = 1 + m/2 > 2 = \rho(H)$, this shows that there exist equations of the form (1.1) where $\rho(f) > \rho(H)$ for every solution f. In the particular case when $m = 3$, the inequality (2.8) becomes an equality $(5/2 = 5/2)$ for every solution f of (4.1). On the other hand, in the cases when $m \geqslant 4$, the inequality (2.8) becomes a strict inequality $(1 + m/2 > 5/2)$ for every solution f of (4.1). Thus Theorem 2.3 is sharp.

The next example shows that we cannot delete the condition that $\rho(H)$ be an integer from the hypothesis of Theorem 2.3, and that we cannot delete the condition that $\lambda(H)$ be an integer from the hypothesis of Theorem 2.7. In particular, this example gives an equation of the form (1.1), where $\lambda(H) = \rho(H)$ is not an integer, which possesses a solution f_0 satisfying both

(a) $\rho(f_0) > \rho(H)$ with $\rho(f_0)$ arbitrarily close to $\rho(H)$; and

(b) $\lambda(f_0) > \lambda(H)$ with $\lambda(f_0)$ arbitrarily close to $\lambda(H)$.

Example 4.2. Let $n \geqslant 2$ be an integer, let ϵ be any fixed constant satisfying $0 < \epsilon < 1/n$, and let $G(z)$ be an entire function satisfying

$$1 + 1/n - \epsilon < \rho(G) < 1 + 1/n. \tag{4.5}$$

Let $H(z)$ be the function defined by

$$H(z) = G^{(n)}(z) + zG(z), \tag{4.6}$$

and consider the differential equation

$$f^{(n)} + zf = H(z). \tag{4.7}$$

We first show that $\rho(G) = \rho(H)$. From Theorem 2.1, all non-trivial solutions of the homogeneous equation

$$f^{(n)} + zf = 0$$

have order $\alpha_1 = 1 + 1/n$. From (4.6), G is a solution of (4.7), and so from Theorem 2.2, we obtain that either $\rho(G) = \rho(H)$ or $\rho(G) = 1 + 1/n$. Since $\rho(G) < 1 + 1/n$ from (4.5), we have $\rho(G) = \rho(H)$.

Since $\rho(G) = \rho(H)$, we obtain from (4.5) that

$$1 + 1/n - \epsilon < \rho(H) < 1 + 1/n. \tag{4.8}$$

Hence, $\rho(H)$ is not an integer. From (4.8),

$$\max\{\rho(H), 1 + 1/n\} = 1 + 1/n.$$

Thus, from Lemma 3.1, it follows that there exists a solution f_0 of (4.7) satisfying

$$\rho(f_0) = 1 + 1/n. \tag{4.9}$$

Combining (4.8) and (4.9), we obtain that f_0 is a solution of (4.7) satisfying

$$0 < \rho(f_0) - \rho(H) < \epsilon < 1/n. \tag{4.10}$$

This shows that we cannot delete the condition that $\rho(H)$ be an integer in the hypothesis of Theorem 2.3.

Since $n \geqslant 2$, we obtain from (4.9) that $\rho(f_0)$ is not an integer, and so $\lambda(f_0) = \rho(f_0)$. Also, since $\rho(H)$ is not an integer, we have $\lambda(H) = \rho(H)$. Thus, from (4.10), we obtain

$$0 < \lambda(f_0) - \lambda(H) < \epsilon < 1/n,$$

which shows that we cannot delete the condition that $\lambda(H)$ be an integer in the hypothesis of Theorem 2.7.

The following four examples show that Theorems 2.6, 2.7, 2.8 and 2.9 are all sharp.

Example 4.3. Let $n \geqslant 1$ be an integer, let $Q(z)$ be a non-constant polynomial of degree m, and let $A(z)$ be an entire function satisfying

$$m - 1/n < \lambda(A) = \rho(A) < m. \tag{4.11}$$

Let $P_0(z), P_1(z), \ldots, P_{n-1}(z)$ be polynomials with $P_0(z) \not\equiv 0$.

We now define $h(z)$ to be the function so that $f_0 = Ae^Q$ is a solution of the equation

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_0(z)f = h(z)e^{Q(z)}. \tag{4.12}$$

From (4.12), we see that h is a polynomial in $A, A', \ldots, A^{(n)}$, $P_0, P_1, \ldots, P_{n-1}$, and $Q', Q'', \ldots, Q^{(n)}$. Hence, from (4.11),

$$\rho(h) \leqslant \rho(A) < m = \deg Q. \tag{4.13}$$

Since $f_0 = Ae^Q$, it follows from (4.11) that

$$\rho(f_0) - \lambda(f_0) = m - \rho(A). \tag{4.14}$$

Hence, from (4.14) and (4.11),

$$0 < \rho(f_0) - \lambda(f_0) < 1/n. \tag{4.15}$$

In equation (4.12) we set

$$H(z) = h(z)e^{Q(z)}. \tag{4.16}$$

From (4.13) and (4.16), we have $\rho(H) > \lambda(H)$. Then, by applying Theorem 2.8 to equation (4.12) and using (4.15), we obtain that

$$\rho(f_0) - \lambda(f_0) \geqslant \min\{1/n, \rho(H) - \lambda(H)\} = \rho(H) - \lambda(H). \tag{4.17}$$

On the other hand, from (4.13), (4.14) and (4.16), we obtain that

$$\rho(f_0) - \lambda(f_0) = m - \rho(A) \leqslant m - \rho(h) \leqslant \rho(H) - \lambda(H). \tag{4.18}$$

Therefore, from (4.17) and (4.18), we have $\rho(f_0) - \lambda(f_0) = \rho(H) - \lambda(H)$. Hence, from (4.15),

$$0 < \rho(f_0) - \lambda(f_0) = \rho(H) - \lambda(H) < 1/n. \tag{4.19}$$

Thus, from (4.19), (4.16) and (4.12), we see that this example gives an equality in (2.11), and it also shows that the constant '$\rho(H) - \lambda(H)$' cannot be deleted or replaced by a larger constant in (2.11). Also from (4.19), (4.16) and (4.12), this example gives an equality in (2.9).

Example 4.4. Let $n \geqslant 2$ be an integer, and consider the equation

$$g^{(n)} + z^{n-1}g = \sin z. \tag{4.20}$$

Since $n \geqslant 2$, it follows from Lemma 3.1 that there exists a solution g_0 of (4.20) such that $\rho(g_0) = 1 + (n-1)/n = 2 - 1/n$. Since $\rho(g_0) = 2 - 1/n$ is not an integer, we have

$$\lambda(g_0) = \rho(g_0) = 2 - 1/n. \tag{4.21}$$

Set

$$f_0(z) = g_0(z)e^{z^2}. \tag{4.22}$$

Since g_0 is a solution of (4.20), it can be verified that f_0 is a solution of an equation of the form

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = (\sin z)e^{z^2}, \tag{4.23}$$

where each $a_k(z)$ is a polynomial with $a_0(z) \not\equiv 0$. From (4.22) and (4.21),

$$\lambda(f_0) = \lambda(g_0) = \rho(g_0) = 2 - 1/n < 2 = \rho(f_0). \tag{4.24}$$

Hence,

$$\rho(f_0) - \lambda(f_0) = 1/n. \tag{4.25}$$

In equation (4.23) we set

$$H(z) = (\sin z)e^{z^2}. \tag{4.26}$$

Since $n \geqslant 2$, we have $\rho(H) - \lambda(H) = 1 > 1/n$. Hence, from (4.25),

$$0 < \rho(f_0) - \lambda(f_0) = 1/n < \rho(H) - \lambda(H). \tag{4.27}$$

Thus, from (4.27), (4.26) and (4.23), we see that this example gives an equality in (2.11), and it also shows that the constant '$1/n$' cannot be deleted or replaced by a larger constant in (2.11). This example gives a strict inequality in (2.9).

We also note that when $n = 2$ in this example, then, from (4.24), we have $\lambda(f_0) = 3/2$. Since $\lambda(H) = 1$ from (4.26), this gives an equality in (2.10). On the other hand, when $n \geqslant 3$ in this example, then, from $\lambda(H) = 1$ and (4.24), we obtain a strict inequality in (2.10).

The next example gives an equation of the form (1.1), where $\rho(H)$ is an integer, that possesses a solution f_0 satisfying $\rho(f_0) = \rho(H)$ and $\lambda(f_0) > \lambda(H)$, with $\lambda(f_0)$ arbitrarily close to $\lambda(H)$.

Example 4.5. Let ϵ be any constant satisfying $0 < \epsilon < 1/2$, and let $h(z)$ be an entire function satisfying

$$\lambda(h) = \rho(h) = 3/2 - \epsilon. \tag{4.28}$$

Consider the equation

$$g'' + (z+1)g = h(z). \tag{4.29}$$

From (4.28), it follows from Lemma 3.1 that there exists a solution g_0 of (4.29) satisfying $\rho(g_0) = 3/2$. Since $\rho(g_0) = 3/2$ is not an integer, we have

$$\lambda(g_0) = \rho(g_0) = 3/2. \tag{4.30}$$

Set

$$f_0(z) = g_0(z)\exp\{\tfrac{1}{2}z^2\}. \tag{4.31}$$

Since g_0 is a solution of (4.29), we obtain that f_0 is a solution of the equation

$$f'' - 2zf' + (z^2 + z)f = h(z)\exp\{\tfrac{1}{2}z^2\}. \tag{4.32}$$

In equation (4.32) we set

$$H(z) = h(z)\exp\{\tfrac{1}{2}z^2\}. \tag{4.33}$$

From (4.33) and (4.28),

$$\lambda(H) = 3/2 - \epsilon \quad \text{and} \quad \rho(H) = 2. \tag{4.34}$$

From (4.30) and (4.31),

$$\lambda(f_0) = 3/2 \quad \text{and} \quad \rho(f_0) = 2. \tag{4.35}$$

Then, from (4.34) and (4.35),

$$\rho(f_0) - \lambda(f_0) = 1/2 < 1/2 + \epsilon = \rho(H) - \lambda(H). \tag{4.36}$$

Thus, from (4.36), (4.33) and (4.32), we see that this example gives a strict inequality in (2.9).

This example also shows that Theorem 2.9 is sharp. Indeed, if in the hypothesis of Theorem 2.9 we replace the condition '$\rho(H) - \lambda(H) \leqslant 1/n$' with the condition '$\rho(H) - \lambda(H) \leqslant \beta$', where β is any fixed constant satisfying $\beta > 1/n$, then, from (4.36), (4.33) and (4.32), we see that (2.12) does not hold in this case.

We note from (4.34) and (4.35) that

$$\lambda(f_0) = \lambda(H) + \epsilon \quad \text{and} \quad \lambda(H) = 3/2 - \epsilon.$$

Thus, like Example 4.2, this example illustrates that we cannot delete the condition that $\lambda(H)$ be an integer in the hypothesis of Theorem 2.7.

We now use this example to show that a strict inequality can occur in (2.11). By differentiating equation (4.32), we obtain that f_0 is a solution of the equation

$$f''' - 2zf'' + (z^2 + z - 2)f' + (2z + 1)f = G(z), \tag{4.37}$$

where

$$G(z) = \{h'(z) + zh(z)\}\exp\{\tfrac{1}{2}z^2\}. \tag{4.38}$$

From (4.38), (4.33) and (4.28), we obtain that $\rho(H) = \rho(G) = 2$ and $\lambda(G) \leqslant \rho(h) = \lambda(h) = \lambda(H)$. Hence, $\rho(H) - \lambda(H) \leqslant \rho(G) - \lambda(G)$, and so, from (4.36),

$$1/3 < \rho(f_0) - \lambda(f_0) = 1/2 < 1/2 + \epsilon \leqslant \rho(G) - \lambda(G). \tag{4.39}$$

Then, from (4.39) and (4.37), we obtain a strict inequality in (2.11) in this case.

The next example shows that there exist equations of the form (1.1) where every solution f satisfies $\lambda(f) > \lambda(H)$.

Example 4.6. Let f be a solution of the equation

$$f'' - 6z^2 f' + (10z^4 - 6z)f = \exp(z^3 + z^2). \tag{4.40}$$

Then the function

$$g(z) = f(z)\exp(-z^3) \tag{4.41}$$

satisfies the equation

$$g'' + z^4 g = \exp(z^2). \tag{4.42}$$

From (4.42),

$$\frac{1}{g} = \exp(-z^2)\left(\frac{g''}{g} + z^4\right). \tag{4.43}$$

We assume that the reader is familiar with the symbols and basic results of Nevanlinna theory (see [**5**, **7**, **8**]). Since equation (4.42) is a special case of equation (4.1), we obtain from the calculation in Example 4.1 that $\rho(g) = 3$. Then, from (4.43) and Nevanlinna's fundamental estimate, we obtain

$$m(r, 1/g) \leqslant m\{r, \exp(-z^2)\} + O(\log r) = O(r^2), \tag{4.44}$$

as $r \to \infty$. Since $\rho(g) = 3$, we deduce from (4.44) and the Nevanlinna theory that $\lambda(g) = \rho(g) = 3$. Thus, from (4.41), we obtain $\lambda(f) = \rho(f) = 3$. In (4.40) we set

$$H(z) = \exp(z^3 + z^2). \tag{4.45}$$

Then (4.40) is an equation of the form (1.1), where every solution f satisfies

$$\rho(H) = \rho(f) = \lambda(f) = 3 > 0 = \lambda(H). \tag{4.46}$$

Thus, from (4.46) and (4.45), we see that (4.40) is an equation of the form (1.1), where both (2.9) and (2.10) become strict inequalities for every solution f. This also shows that there exist equations of the form (1.1), where every solution f satisfies $\lambda(f) > \lambda(H)$.

5. Proof of Theorem 2.2

Since statement (2.7) follows immediately from Lemma 3.1, we need only prove statement (2.6).

Let f be a solution of (1.1) satisfying $\rho(f) \neq \rho(H)$. Then, from Lemma 3.1,

$$\rho(H) < \rho(f) < \infty. \tag{5.1}$$

Since f is an entire function of finite positive order, it is well known that f satisfies the statements (5.2) and (5.3) below (see [**7**, pp. 199–209], [**9**, pp. 105–108] or [**11**, pp. 65–67]).

Let $V(r)$ denote the central index of f, let z_r denote a point on the circle $|z| = r$ that satisfies $|f(z_r)| = M(r, f)$, and set $\alpha = \rho(f)$. Then

$$V(r) = (1 + o(1))Cr^{\alpha}, \tag{5.2}$$

as $r \to \infty$, where C is a positive constant. Furthermore, from the Wiman–Valiron theory, there exists a set $E \subset (0, \infty)$ that has finite logarithmic measure, such that for all $k = 1, 2, \dots, n$, we have

$$\frac{f^{(k)}(z_r)}{f(z_r)} = (1 + o(1))\left(\frac{V(r)}{z_r}\right)^k, \tag{5.3}$$

as $r \to \infty$, $r \notin E$.

From (1.1), we obtain

$$z_r^n \frac{f^{(n)}(z_r)}{f(z_r)} + z_r^n P_{n-1}(z_r) \frac{f^{(n-1)}(z_r)}{f(z_r)} + \cdots + z_r^n P_0(z_r) = z_r^n \frac{H(z_r)}{f(z_r)}. \tag{5.4}$$

Now, from (5.2) and (5.3), we find that if $P_k(z) \not\equiv 0$, then

$$\left| z_r^n P_k(z_r) \frac{f^{(k)}(z_r)}{f(z_r)} \right| = (1 + o(1))C_k r^{n-k+d_k+k\alpha}, \tag{5.5}$$

as $r \to \infty$, $r \notin E$, where $C_k > 0$ is a constant and $d_k = \deg P_k$. Here we set $P_n \equiv 1$.

From Lemma 3.5 and (5.1), there exists a set $S \subset (1, \infty)$ that has infinite logarithmic measure such that

$$z_r^n \frac{H(z_r)}{f(z_r)} \to 0, \tag{5.6}$$

as $r \to \infty$, $r \in S$.

We now suppose that $\alpha \notin \Phi$, and use the argument in the proof of Theorem 1 (i) in [**3**]. More specifically, if $\alpha \notin \Phi$, then, from the argument in [**3**, § 5], it can be deduced that there will exist exactly one term on the left-hand side of (5.4) which is unbounded and dominant relative to the other terms on the left-hand side of (5.4) as $r \to \infty$, $r \in S \setminus E$; in particular, there will exist exactly one value m where $P_m(z) \not\equiv 0$, such that in (5.5) we have

$$C_m > 0 \quad \text{and} \quad n - m + d_m + m\alpha > n - k + d_k + k\alpha \geqslant 0, \tag{5.7}$$

for all $k \neq m$ for which $P_k(z) \not\equiv 0$. From (5.4), (5.5), (5.6) and (5.7), we obtain a contradiction. Therefore, $\alpha = \rho(f)$ must satisfy $\alpha \in \Phi$. This proves statement (2.6), and thus completes the proof of Theorem 2.2. □

6. Proof of Theorem 2.3

Let $\rho(H)$ be an integer, and let f be a solution of (1.1) which satisfies

$$\rho(f) > \rho(H). \tag{6.1}$$

Then, from Theorem 2.2 and (2.4), we obtain that

$$\rho(f) = m/q, \tag{6.2}$$

where m and q are positive integers with $q \leqslant n$. From (6.1) and (6.2), we have $m > q\rho(H)$. Since $m > 0$, $q > 0$ and $\rho(H) \geqslant 0$ are all integers, it follows that

$$m \geqslant q\rho(H) + 1. \tag{6.3}$$

Since $0 < q \leqslant n$, we obtain from (6.2) and (6.3) that

$$\rho(f) = m/q \geqslant \rho(H) + 1/q \geqslant \rho(H) + 1/n.$$

This proves Theorem 2.3. □

7. Proof of Theorem 2.6

Suppose first that $\lambda(H) = \rho(H)$ in (1.1), and let f be a solution of (1.1). Then, from Theorem 2.5, $\lambda(f) = \rho(f)$. Hence, $\rho(f) - \lambda(f) = \rho(H) - \lambda(H)$, and so (2.9) holds.

Now suppose that $\lambda(H) < \rho(H)$, and let f be a solution of (1.1). From Theorem 2.4, $\lambda(f) \geqslant \lambda(H)$. If $\rho(f) = \rho(H)$, then (2.9) holds. If $\rho(f) \neq \rho(H)$, then from Lemma 3.1, we have $\rho(f) > \rho(H)$. Then, from Lemma 3.3, we deduce that $\lambda(f) = \rho(f)$. Thus, $\lambda(f) - \lambda(H) > \rho(f) - \rho(H)$, and so (2.9) holds. □

8. Proof of Theorem 2.7

Let $\lambda(H)$ be an integer, and let f be a solution of (1.1) which satisfies $\lambda(f) > \lambda(H)$.

Suppose first that $\lambda(H) = \rho(H)$. Then, from Theorem 2.5, $\lambda(f) = \rho(f)$. Hence,

$$\rho(f) = \lambda(f) > \lambda(H) = \rho(H). \tag{8.1}$$

Since $\lambda(H)$ is an integer, $\rho(H)$ is an integer. Then, from (8.1) and Theorem 2.3, we obtain (2.10).

Now suppose that $\lambda(H) < \rho(H)$. It follows that $H(z)$ has the form

$$H(z) = h(z)\mathrm{e}^{Q(z)}, \tag{8.2}$$

where $h(z) \not\equiv 0$ is an entire function and $Q(z)$ is a non-constant polynomial, such that

$$\lambda(h) = \rho(h) < \deg Q. \tag{8.3}$$

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Set

$$g(z) = f(z)e^{-Q(z)}. \tag{8.4}$$

Since f satisfies (1.1), we obtain from (8.4) and (8.2) that g satisfies an equation of the form

$$g^{(n)} + a_{n-1}(z)g^{(n-1)} + \cdots + a_0(z)g = h(z), \tag{8.5}$$

where each $a_k(z)$ is a polynomial. From (8.4), (8.3), (8.2) and Lemma 3.2, we obtain that $\lambda(f) = \lambda(g) = \rho(g)$. Since $\rho(g) = \lambda(f)$, and since by hypothesis, $\lambda(f) > \lambda(H)$, we obtain from (8.2) and (8.3) that

$$\rho(g) = \lambda(f) > \lambda(H) = \rho(h). \tag{8.6}$$

Since $\lambda(H)$ is an integer, $\rho(h)$ is an integer. From (8.6), we have $\rho(g) > \rho(h)$, and so it follows that in (8.5) we must have $a_k(z) \not\equiv 0$ for some k satisfying $0 \leqslant k \leqslant n-1$. This means that we can apply Theorem 2.3 to equation (8.5), regardless of whether $a_0(z) \not\equiv 0$ or $a_0(z) \equiv 0$. Thus, by applying Theorem 2.3 to equation (8.5), it can be deduced that

$$\rho(g) \geqslant \rho(h) + 1/n. \tag{8.7}$$

Then (2.10) follows from (8.7) and (8.6). □

9. Proof of Theorem 2.8

Let $\rho(H) > \lambda(H)$, and let f be a solution of (1.1) satisfying $\rho(f) > \lambda(f)$. Then, from Lemma 3.3, the function $H(z)$ in (1.1) must be of the form

$$H(z) = h(z)e^{Q(z)}, \tag{9.1}$$

where $h(z) \not\equiv 0$ is an entire function and $Q(z)$ is a non-constant polynomial, such that

$$\lambda(h) = \rho(h) < \deg Q. \tag{9.2}$$

Also from Lemma 3.3, f must have the form

$$f(z) = g(z)e^{Q(z)}, \tag{9.3}$$

where g is an entire function, such that

$$\rho(g) < \deg Q = \rho(f). \tag{9.4}$$

Since f satisfies (1.1), we obtain from (9.3) and (9.1) that g satisfies an equation of the form

$$g^{(n)} + a_{n-1}(z)g^{(n-1)} + \cdots + a_0(z)g = h(z), \tag{9.5}$$

where each $a_k(z)$ is a polynomial. Thus $\rho(g) \geqslant \rho(h)$. We consider separately the two cases $\rho(g) = \rho(h)$ and $\rho(g) > \rho(h)$.

Case 1. Suppose first that $\rho(g) = \rho(h)$. Then, from (9.1), (9.2) and (9.3),

$$\lambda(f) = \lambda(g) \leqslant \rho(g) = \rho(h) = \lambda(H). \tag{9.6}$$

Since $\rho(f) > \lambda(f)$, we obtain from Lemma 3.3 that

$$\rho(f) = \rho(H). \tag{9.7}$$

Combining (9.6) and (9.7) yields

$$\rho(f) - \lambda(f) \geqslant \rho(H) - \lambda(H).$$

Thus, (2.11) holds in this case.

Case 2. Now suppose that $\rho(g) > \rho(h)$. It follows that in (9.5) we must have $a_k(z) \not\equiv 0$ for some k satisfying $0 \leqslant k \leqslant n-1$. This means that we can apply Theorem 2.2 to equation (9.5), regardless of whether $a_0(z) \not\equiv 0$ or $a_0(z) \equiv 0$. Thus, by applying Theorem 2.2 to equation (9.5) and by using (2.4), it can be deduced that $\rho(g) = m/q$, where m and q are positive integers with $q \leqslant n$. Then, from (9.3) and (9.4),

$$\lambda(f) = \lambda(g) \leqslant \rho(g) = m/q < \rho(f),$$

and so

$$\rho(f) - \lambda(f) \geqslant \rho(f) - m/q > 0. \tag{9.8}$$

Thus, $q\rho(f) - m > 0$. Since $\rho(f)$ is a positive integer from (9.4), and since m and q are also positive integers, it follows that

$$q\rho(f) - m \geqslant 1. \tag{9.9}$$

Since $0 < q \leqslant n$, we obtain from (9.8) and (9.9) that

$$\rho(f) - \lambda(f) \geqslant \rho(f) - m/q \geqslant 1/q \geqslant 1/n.$$

Thus, (2.11) holds in this case also. This completes the proof of Theorem 2.8. □

References

1. S.-A. Gao, On the complex oscillation of solutions of nonhomogeneous linear differential equations with polynomial coefficients, *Comm. Math. Univ. Sancti Pauli* **38** (1989), 11–20.
2. G. Gundersen, Finite order solutions of second order linear differential equations, *Trans. Am. Math. Soc.* **305** (1988), 415–429.
3. G. Gundersen, E. Steinbart and S. Wang, The possible orders of solutions of linear differential equations with polynomial coefficients, *Trans. Am. Math. Soc.* **350** (1998), 1225–1247.

https://doi.org/10.1017/S0013091500020964 Published online by Cambridge University Press

4. G. GUNDERSEN, E. STEINBART AND S. WANG, Solutions of nonhomogeneous linear differental equations with exceptionally few zeros, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **23** (1998), 429–452.
5. W. K. HAYMAN, *Meromorphic functions* (Clarendon Press, Oxford, 1964).
6. W. HELMRATH AND J. NIKOLAUS, Ein elementarer Beweis bei der Anwendung der Zentralindexmethode auf Differentialgleichungen, *Complex Variables* **3** (1984), 387–396.
7. G. JANK AND L. VOLKMANN, *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen* (Birkhäuser, Basel, 1985).
8. I. LAINE, *Nevanlinna theory and complex differential equations* (Walter de Gruyter, Berlin, 1993).
9. G. VALIRON, *Lectures on the general theory of integral functions* (translated by E. F. Collingwood) (Chelsea, New York, 1949).
10. H. WITTICH, Über das Anwachsen der Lösungen linearer Differentialgleichungen, *Math. Ann.* **124** (1952), 277–288.
11. H. WITTICH, *Neuere Untersuchungen über eindeutige analytische Functionen*, 2nd edn (Springer, Berlin, 1968).