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A Universal Volume Comparison Theorem for Finsler Manifolds and Related Results

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Abstract. In this paper, we establish a universal volume comparison theorem for Finsler manifolds and give the Berger–Kazdan inequality and Santaló's formula in Finsler geometry. Based on these, we derive a Berger–Kazdan type comparison theorem and a Croke type isoperimetric inequality for Finsler manifolds.

1 Introduction

As is well known, the Bishop–Gromov volume comparison theorem [10], the Berger–Kazdan inequality [8], and Santaló's formula [31] play very important roles in global differential geometry and geometric analysis. Some of their applications are Gromov's precompactness theorem [22], the Berger-Kazdan comparison theorem [7] and Croke's isoperimetric inequality [17]. Moreover, with the Berger–Kazdan inequality, the Blaschke conjecture was thereby settled for even dimensions (*cf.* [9, 13]). See [13, 16, 17, 22, 28, 29], *etc.*, for other interesting applications in global Riemannian geometry.

Finsler geometry is just Riemannian geometry without quadratic restriction. A Finsler manifold is a differentiable manifold on which every tangent space is endowed with a Minkowski norm instead of a Euclidean norm. Recently, there has been a revived interest in the study of Finsler manifolds, particularly in their global aspect; see, for example, [3–5, 33, 34, 36].

By [11, Sect. 5.5], there is only one reasonable notion of the volume form for Riemannian manifolds. However, the situation is different in Finsler geometry. The Finsler volume form can be defined in various ways and essentially different results may be obtained, *e.g.*, [1, 2]. Therefore, it is an interesting and important problem to investigate the relations between the volume forms and the geometric properties on a Finsler manifold. A Finsler volume form used frequently is the Busemann–Hausdorff volume form $d\mu_{BH}$ (*cf.* [12]), with respect to which Z. Shen firstly obtained the following Bishop–Gromov type volume comparison theorem in [36].

Theorem 1.1 ([36]) *Let (M, F) be a forward complete Finsler n-manifold satisfying*

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$\mathbf{Ric} \geq (n-1)k$ *and* $\mathbf{S}_{BH} \geq (n-1)h$. *Then the function*

$$\mathcal{P}(r) = \frac{\mu_{BH}(B^+(p,r))}{V_{k,h,n}(r)}$$

is decreasing, and for any $r > 0$,

$$\mu_{BH}(B^+(p,r)) \leq V_{k,h,n}(r), \tag{1.1}$$

where

$$V_{k,h,n}(r) := c_{n-1} \int_0^r \left[e^{-ht} \mathfrak{s}_k(r) \right]^{n-1} dt,$$

c_{n-1} *is the volume of the standard Euclidean unit* $(n-1)$*-sphere,* $\mathfrak{s}_k(t)$ *is the unique solution to* $y'' + ky = 0$ *with* $y(0) = 0$ *and* $y'(0) = 1$*, and* $\mathbf{S}_{BH}$ *is the S-curvature with respect to* $d\mu_{BH}$.

Using the Laplacian comparison theorem in Finsler geometry, Wu and Xin gave a different proof of the above theorem in [39]. But, in general, Theorem 1.1 (more precisely, the inequality (1.1)) is invalid for other volume forms. In [38], there are some comparison theorems that are valid with respect to all volume forms. However, Theorem 1.1 cannot be deduced from the results of [38]. We refer to [19, 23, 32, 34, 36–39] for more results.

The purpose of this paper is to study the influence that the volume measure has on the geometry of a Finsler manifold.

Let (M, F) be a forward complete Finsler n-manifold. Given a point $p \in M$, denote by S_pM the indicatrix at p, namely, $S_pM = \{y \in T_pM : F(p, y) = 1\}$. Let $d\nu_p$ be the Riemannian volume form on S_pM induced by F (see [5]). For each $y \in S_pM$, let $\gamma_y(t)$ denote the unit speed geodesic with $\dot{\gamma}_y(0) = y$. Let i_y denote the cut value of y and let $\mathfrak{i}_p$ (resp. $\mathfrak{i}_M$) denote the injective radius of p (resp. M). Given any volume form $d\mu$ on (M, F), let τ and $\mathbf{S}$ denote the distortion and S-curvature of $d\mu$, respectively. Define

$$\mathcal{V}_{p,k,n}(r) := \left(\int_{S_pM} d\nu_p(y) \right) \cdot \left(\int_0^r e^{-\tau(\dot{\gamma}_y(t))} \mathfrak{s}_k^{n-1}(t) dt \right).$$

Let (r, y) denote the polar coordinate system at p on (M, F) (see Section 3). Set $d\mu = \widehat{\sigma}_p(r, y) dr \wedge d\nu_p(y)$ and $\mathscr{F}(r, y) = e^{\tau(\dot{\gamma}_y(r))} \widehat{\sigma}_p(r, y)$. In fact, $\mathscr{F}(r, y)$ is independent of the choice of $d\mu$. We then shall establish the following theorem.

Theorem 1.2 *Let* $(M, F, d\mu)$ *be a forward complete Finsler n-manifold with a volume form* $d\mu$. *Suppose the Ricci curvature of M is bounded from below by* $(n-1)k$. *Then the function*

$$\mathcal{P}(r) = \frac{\mathscr{F}(r, y)}{\mathfrak{s}_k^{n-1}(r)}$$

is monotonically decreasing in r and converges to 1 *(as* $r \to 0^+$*). Therefore, for any* $r > 0$,

$$\mu\big(B^+(p, r)\big) \leq \mathcal{V}_{p,k,n}(r),$$

with equality for some $r_0 > 0$ *if and only if the flag curvature of* M *satisfies*

$$\mathbf{K}\big(\dot{\gamma}_y(t); \cdot\big) \equiv k, \qquad 0 \le t \le r_0 \le \mathfrak{i}_p,$$

for all $y \in S_pM$.

It is easy to check that the usual Bishop–Gromov volume comparison theorem, Theorem 1.1, and [38, Theorem 5.4] can all be deduced from Theorem 1.2 (see Remark 3.5 and 3.8).

For any $y \in S_pM$, define $y^{\perp} := \{X \in T_pM : g_y(X, y) = 0\}$. Denote by φ_t the geodesic flow of F. Then we have the following theorem.

Theorem 1.3 *Let* (M, F) *be a compact Finsler n-manifold. For each* $y \in SM$ *and* $0 < t \le l \le i_y$, *we have*

$$\int_0^l dr \int_r^l \mathscr{F}\big(t - r, \varphi_r(y)\big)\, dt \ge \frac{\pi c_n}{2c_{n-1}} \Big(\frac{l}{\pi}\Big)^{n+1},$$

with equality if and only if

$$R_{\dot{\gamma}_y(t)}(\,\cdot\,, \dot{\gamma}_y(t))\dot{\gamma}_y(t) = \Big(\frac{\pi}{l}\Big)^2 \mathrm{id}, \quad 0 \le t \le l,$$

where R *is the (Riemannian) curvature tensor acting on* $\dot{\gamma}_y(t)^{\perp}$.

When F becomes Riemannian, $\mathscr{F}(r, y)dr \wedge d\nu_p(y)$ reduces to the Riemannian volume form, and, therefore, Theorem 1.3 becomes the Berger–Kazdan inequality [8].

Another volume form that is used frequently in Finsler geometry is the so-called Holmes–Thompson volume form $d\mu_{HT}$, *cf.* [24]. If F is reversible, then $d\mu_{BH} \ge d\mu_{HT}$, with equality if and only if F is Riemannian (*cf.* [19]). There exist counterexamples to the inequality when F is nonreversible, *e.g.,* [26]. According to [30], the reversibility λ of a Finsler manifold (M, F) is defined by

$$\lambda := \sup_{(x,y)\in SM} F(x, -y), \quad SM = \bigcup_{x\in M} S_xM.$$

Clearly, $\lambda \ge 1$ and $\lambda = 1$ if and only if F is reversible. As an application of Theorem 1.3, we have the following theorem.

Theorem 1.4 *Let* (M, F) *be a compact Finsler n-manifold with reversibility* λ. *Then*

$$\mu_{BH}(M) \ge c_n \Big(\frac{\mathfrak{i}_M}{\lambda\pi}\Big)^n \exp\Big[-\frac{\mathfrak{i}_M}{\lambda}\big(|h_1| + |h_2|\big)\Big], \tag{1.2}$$

where $h_1 := \sup_{y\in SM} \mathbf{S}_{BH}(y)$ *and* $h_2 := \sup_{y\in SM} \mathbf{S}_{HT}(y)$.

In particular, if F *is reversible, then equality holds in* (1.2) *if and only if* (M, F) *is isometric to the standard n-sphere of constant sectional curvature* $(\pi/\mathfrak{i}_M)^2$.

In the Riemannian case, $\mathbf{S}_{BH} = \mathbf{S}_{HT} = 0$ and $\lambda = 1$. Thus, Theorem 1.4 implies the Berger–Kazdan comparison theorem [7].

Let $\Omega \subset M$ be a relatively compact domain in (M, F) with smooth boundary $\partial\Omega$. Denote by $\mathbf{n}$ the unit inward normal vector field along $\partial\Omega$. Thus, $g_{\mathbf{n}}(\mathbf{n}, X) = 0$ for all $X \in T\partial\Omega$ (see [34]). Set $S^+\partial\Omega = \{y \in SM|_{\partial\Omega} : g_{\mathbf{n}}(\mathbf{n}, y) > 0\}$. For $y \in S\Omega \cup S^+\partial\Omega$, define $\widehat{t}(y) := \sup\{T > 0 : \gamma_y(t) \in \Omega,\ \forall t \in (0, T)\}$ and $l(y) := \min\{i_y, \widehat{t}(y)\}$. Let $\pi_1 \colon SM \to M$ be the natural projection and dV_{SM} be the canonical volume form on SM. Given any volume form $d\mu$ on (M, F), the induced volume form on $\partial\Omega$ by $d\mu$ is defined by $dA := i^*(\mathbf{n}\rfloor d\mu)$, where $i \colon \partial\Omega \hookrightarrow M$ is the inclusion map (*cf.* [34]). Then we have the following theorem.

Theorem 1.5 *Let Ω be a relatively compact domain in a reversible Finsler n-manifold $(M, F, d\mu)$, with $\partial\Omega \in C^\infty$. For all integrable function f on $S\Omega$, we have*

$$\int_{\mathcal{V}_\Omega^-} f\,dV_{SM} = \int_{S^+\partial\Omega} e^{\tau(y)} g_{\mathbf{n}}(\mathbf{n}, y)\,d\chi(y) \int_0^{l(y)} f(\varphi_t(y))\,dt,$$

where $\mathcal{V}_\Omega^- := \{y \in S\Omega : \widehat{t}(-y) \leq i_{-y}\}$ and $d\chi(y) = dA(\pi_1(y))\,d\nu_{\pi_1(y)}(y)$.

In the Riemannian case, $e^{\tau(y)} = 1$, $g_{\mathbf{n}} = g$, and, therefore, Theorem 1.5 yields Santaló's formula [31].

Given any point $p \in \Omega$, define

$$U_p := \pi_1^{-1}|_{\mathcal{V}_\Omega^-}(p) \subset S_pM, \quad \omega_p := \frac{1}{c_{n-1}} \int_{U_p} e^{\tau(y)}\,d\nu_p(y), \quad \text{and} \quad \omega := \inf_{p\in\Omega} \omega_p.$$

Theorem 1.3 together with Theorem 1.5 furnishes the following theorem.

Theorem 1.6 *Let Ω be a relatively compact domain with $\partial\Omega \in C^\infty$ in a reversible Finsler n-manifold (M, F). Let $d\mu$ denote either the Busemann–Hausdorff volume form or the Holmes–Thompson volume form. Set*

$$\Xi = \sup_{y\in S\overline{\Omega}} \tau(y),\ \mathcal{M} = \max\Big(\sup_{y\in SM|_{\partial\Omega}} \frac{1}{\sqrt{g_{\mathbf{n}}(y, y)}},\ \sup_{y\in SM|_{\partial\Omega}} \sqrt{g_{\mathbf{n}}(y, y)} \Big).$$

Then

(i)
$$\frac{A(\partial\Omega)}{\mu(\Omega)} \geq \frac{(n-1)c_{n-1}\omega}{c_{n-2}e^{2\Xi}\mathcal{M}^{2n+1}d(\Omega)},$$

where $d(\Omega)$ denotes the diameter of Ω;

(ii)
$$\frac{A(\partial\Omega)}{\mu(\Omega)^{1-1/n}} \geq \frac{c_{n-1}}{\mathcal{M}^{(2n+1)}(c_n/2)^{1-1/n}} \Big(\frac{\omega}{e^{2\Xi}}\Big)^{1+1/n},$$

with equality if and only if $(\overline{\Omega}, F|_{\overline{\Omega}})$ is a hemisphere of a constant sectional curvature sphere.

If F is Riemannian, then $\Xi = 0$ and $\mathcal{M} = 1$. Therefore, Theorem 1.6 gives Croke's isoperimetric inequality [17].

The paper is organized as follows. In Section 2, we give some necessary definitions and properties concerned with Finsler geometry. In Section 3, by investigating the polar coordinate system of a Finsler manifold, we prove Theorem 1.2 and obtain a Bishop–Günther type volume comparison theorem. As applications of Theorem 1.2, we derive a characterization of $\mathbb{R}^n$ in Finsler geometry and give a Finsler version of the Calabi–Yau linear volume growth theorem. In Section 4, by virtue of the study of Jacobi fields on a Finsler manifold, we prove Theorems 1.3 and 1.4. In Section 5, we prove Theorem 1.5 by studying the properties of the distance function from a closed hypersurface in a Finsler manifold. Based on these, Theorem 1.6 will be shown in Section 6.

2 Preliminaries

In this section, we recall some definitions and properties cencerned with Finsler geometry. See [5, 33, 34] for more details.

Let (M, F) be a (connected) Finsler manifold with Finsler metric $F\colon TM \to [0, +\infty)$. Define

$$S_xM := \{y \in T_xM : F(x, y) = 1\} \quad \text{and} \quad SM := \bigcup_{x\in M} S_xM.$$

Let $(x, y) = (x^i, y^i)$ be local coordinates on TM, and let $\pi\colon TM \to M$ and $\pi_1\colon SM \to M$ be the natural projections. Denote by c_{n-1} the volume of the Euclidean unit $(n-1)$-sphere. Define

$$\ell^i := \frac{y^i}{F},\ g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \qquad A_{ijk}(x, y) := \frac{F}{4}\frac{\partial^3 F^2(x, y)}{\partial y^i \partial y^j \partial y^k},$$
$$\gamma^i_{jk} := \frac{1}{2} g^{il}\Big(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\Big), \qquad N^i_j := \big(\gamma^i_{jk}\ell^j - A^i_{jk}\gamma^k_{rs}\ell^r\ell^s\big)\cdot F.$$

The *Chern connection* ∇ is defined on the pulled-back bundle π^*TM, and its forms are characterized by the following structure equations:

(1) torsion freeness: $dx^j \wedge \omega^i_j = 0$;

(2) almost g-compatibility:

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = 2\frac{A_{ijk}}{F}(dy^k + N^k_l dx^l).$$

From this it is easy to obtain $\omega^i_j = \Gamma^i_{jk} dx^k$, and $\Gamma^i_{jk} = \Gamma^i_{kj}$.

The *curvature form* of the Chern connection is defined as

$$\Omega^i_j := d\omega^i_j - \omega^k_j \wedge \omega^i_k =: \frac{1}{2} R^i_{j\,kl} dx^k \wedge dx^l + P^i_{j\,kl} dx^k \wedge \frac{dy^l + N^l_s dx^s}{F}.$$

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Given a non-zero vector $V \in T_xM$, the *flag curvature* $K(y, V)$ on $(x, y) \in TM\backslash 0$ is defined by

$$\mathbf{K}(y, V) := \frac{V^i y^j R_{jikl} y^l V^k}{g_y(y, y) g_y(V, V) - [g_y(y, V)]^2},$$

where $R_{jikl} := g_{is} R^s_{j\,kl}$. And the *Ricci curvature* of y is defined as

$$\mathbf{Ric}(y) := \sum_i K(y, e_i),$$

where $e_1, \dots, e_n$ is a g_y-orthonormal base on $(x, y) \in TM\backslash 0$.

For any $y \neq 0$, let $\gamma_y(t)$ denote a constant speed geodesic with $\dot{\gamma}_y(0) = y$. Given any the volume form $d\mu$ on M, in a local coordinate system (x^i), express $d\mu = \sigma(x) dx^1 \wedge \cdots \wedge dx^n$. For $y \in T_xM\backslash\{0\}$, define the *distortion* of $(M, F, d\mu)$ as

$$\tau(y) := \log \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

And we define the *S-curvature* $\mathbf{S}$ as

$$\mathbf{S}(y) := \frac{d}{dt}\big[\tau\big(\dot{\gamma}_y(t)\big)\big]\,|_{t=0},$$

Two volume forms used frequently are the Busemann–Hausdorff volume form $d\mu_{BH}$ and the Holmes-Thompson volume form $d\mu_{HT}$, respectively. Given a local coordinate system (x^i), $d\mu_{BH} = \sigma_{BH}(x) dx^1 \wedge \cdots \wedge dx^n$ and $d\mu_{HT} = \sigma_{HT}(x) dx^1 \wedge \cdots \wedge dx^n$, where

$$\sigma_{BH}(x) = \frac{\mathrm{Vol}_{\mathbb{R}^n}(\mathbb{B}^n)}{\mathrm{Vol}_{\mathbb{R}^n}\{y \in T_xM : F(x, y) < 1\}},$$

$$\sigma_{HT}(x) = \frac{1}{c_{n-1}} \int_{S_xM} \det g_{ij}(x, y) \Big(\sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n\Big).$$

If F is reversible, then it follows from [5, 19] that $\sigma_{BH}(x) \geq \sigma_{HT}(x)$, with equality if and only if $F(x, \cdot\,)$ is a Euclidean norm.

By Stokes' formula, we have

$$\int_{S_xM} e^{-\tau_{BH}(y)} d\nu_x(y) = \int_{S_xM} e^{\tau_{HT}(y)} d\nu_x(y) = c_{n-1},$$

where

$$d\nu_x(y) := \sqrt{\det g_{ij}(x, y)} \Big(\sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n\Big).$$

Throughout this paper, let τ_{BH}, τ_{HT}, $\mathbf{S}_{BH}$, and $\mathbf{S}_{HT}$ denote the distortions and S-curvatures of the Busemann–Hausdorff volume form and the Holmes–Thompson volume form, respectively.

Given any volume form $d\mu$ on M. The volume form on SM is defined by

$$\begin{aligned} dV_{SM}|_{(x,y)} \\ &= \det g_{ij}(x,y)dx^1 \wedge \cdots \wedge dx^n \wedge \Big(\sum_{j=1}^{n}(-1)^{j-1}y^j dy^1 \wedge \cdots \wedge \widehat{dy^j} \wedge \cdots \wedge dy^n\Big) \\ &= \sqrt{\det g_{ij}(x,y)}dx^1 \wedge \cdots \wedge dx^n \wedge d\nu_x(y) \\ &= e^{\tau(y)}\pi_1^*(d\mu(x)) \wedge d\nu_x(y). \end{aligned}$$

The *spray* $\mathbf{G}(y)$ of F is defined by

$$\mathbf{G}(y) = y^i\frac{\delta}{\delta x^i} = y^i\Big(\frac{\partial}{\partial x^i} - N_i^j\frac{\partial}{\partial y^j}\Big).$$

For any $y \in TM$, let $\varphi_t(y)$ be the integral curve of $\mathbf{G}$ with $\varphi_0(y) = y$. Then φ_t is called the *geodesic flow* of F. Clearly, $\gamma_y(t) = \pi \circ \varphi_t(y)$ and $\dot\gamma_y(t) = \varphi_t(y)$. If M is compact, then $\varphi_t\colon SM \to SM$ is a diffeomorphism (see [33]). P. Dazord proved the following theorem in [18]; see also [34].

Theorem 2.1 ([18]) *The volume form dV_{SM} on SM is invariant with respect to the geodesic flow, that is $\varphi_t^*(dV_{SM}) = dV_{SM}$.*

F is called *reversible* if $F(X) = F(-X)$, for all $X \in TM$. By [30], the *reversibility* λ_F of (M,F) is defined by

$$\lambda_F := \sup_{(x,y)\in TM\setminus 0}\frac{F(x,-y)}{F(x,y)}.$$

Clearly, $\lambda_F \geq 1$ and $\lambda_F = 1$ if and only if F is reversible. According to [20], the *uniformity constant* of (M,F) is defined by

$$\Lambda_F := \sup_{X,Y,Z\in SM}\frac{g_X(Y,Y)}{g_Z(Y,Y)}.$$

Clearly, $\lambda_F \leq \sqrt{\Lambda_F}$ and $\Lambda_F = 1$ if and only if F is Riemannian.

The *Legendre transformation* $\mathcal{L}\colon TM \to T^*M$ is defined by

$$\mathcal{L}(Y) = \begin{cases} 0, & Y = 0, \\ g_Y(Y,\cdot), & Y \neq 0. \end{cases}$$

For any $x \in M$, the Legendre transformation is a smooth diffeomorphism from $T_xM\backslash\{0\}$ onto $T_x^*M\backslash\{0\}$ (see [34]). Given a function $f \in C^1(M)$ and a point $x \in M$, the *gradient* of f at x is defined as

$$\nabla f(x) = \begin{cases} 0, & df|_x = 0, \\ \mathcal{L}^{-1}(df|_x), & df|_x \neq 0. \end{cases}$$

3 Universal Volume Comparison Theorem

In this section, we will investigate the polar coordinate system of a Finsler manifold and prove Theorem 1.2.

Fix a point p of a forward complete Finsler n-manifold (M, F). Let $\{x^i\}$ be local coordinates on some neighborhood of p. Then F induces a Riemannian metric on $T_pM\backslash\{0\}$ by

$$g_p(y) := g(p,y)_{ij} dy^i \otimes dy^j,$$

where $y = y^i \frac{\partial}{\partial x^i}$. Let $\dot{g}_p$ and $d\nu_p$ denote the Riemannian metric and the Riemannian volume form on $S_pM = \{y \in T_pM : F(p,y) = 1\}$ induced by g_p, respectively. Thus,

$$d\nu_p(y) = \sqrt{\det g_{ij}(p,y)} \Big(\sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n \Big).$$

Let $\{\bar{x}^i\}_{i=1}^n$ be (local) polar coordinates on $T_pM\backslash\{0\}$. Namely,

$$\bar{x}^n(v) := \bar{r}(v) := F(v),\ \bar{x}^\alpha(v) := \bar{\theta}^\alpha\Big(\frac{v}{F(v)}\Big),$$

where $\{\bar{\theta}^\alpha\}_{\alpha=1}^{n-1}$ are local coordinates on S_pM. According to [5, p. 412], $g_p = d\bar{r} \otimes d\bar{r} + \bar{r}^2 \dot{g}_p$ and $d\nu_p = \sqrt{\det \dot{g}_p}\, d\bar{\Theta}$, where $d\bar{\Theta} := d\bar{\theta}^1 \wedge \cdots \wedge d\bar{\theta}^{n-1}$.

Given $y \in S_pM$, denote by i_y the cut value of y. Define

$$D_p := \{ty \in T_pM : y \in S_pM, 0 \le t < i_y\},$$

and $\mathcal{D}_p := \exp_p(D_p)$. Thus, $M = \mathcal{D}_p \sqcup \mathrm{Cut}_p$ (see [5, Proposition 8.5.2]). The coordinate system

$$\{x^i\} = \{\bar{x}^i \circ \exp_p^{-1}\} \colon \mathcal{D}_p - \{p\} \longrightarrow \mathbb{R}^n$$

is called the *polar coordinate system* at p. For convenience, let

$$r := \bar{x}^n \circ \exp_p^{-1}, \quad \theta^\alpha := \bar{x}^\alpha \circ \exp_p^{-1}.$$

It is noticeable that the polar coordinates actually describe a diffeomorphism of $(0, +\infty) \times S_pM$ onto $\mathcal{D}_p\backslash\{p\}$, given by $(r, y) \mapsto \exp_p ry$. Hence, we also use (r, y) to denote the polar coordinate system at p. Thus,

$$\frac{\partial}{\partial\theta^\alpha}\Big|_{(r,y)} = (\exp_p)_{*ry}\Big(r\frac{\partial}{\partial\bar{\theta}^\alpha}\Big), \tag{3.1}$$

$$\frac{\partial}{\partial r}\Big|_{(r,y)} = (\exp_p)_{*ry}\, y, \tag{3.2}$$

where y is a point in S_pM whose coordinates are $\{\theta^\alpha\}$.

Given any volume form $d\mu$ on M, in the polar coordinates, express $d\mu|_{(r,y)} = \sigma_p(r,y)dr \wedge d\Theta$, where $d\Theta = d\theta^1 \wedge \cdots \wedge d\theta^{n-1}$. By abuse of notation, we will use $d\nu_p$ to denote $(\exp_p^{-1})^* d\nu_p$. Thus,

$$\begin{aligned} d\mu|_{(r,y)} &= \Big(\frac{\sigma_p(r,y)}{\sqrt{\det \dot{g}_p(y)}}\Big)\, dr \wedge \Big(\sqrt{\det \dot{g}_p(y)}d\Theta\Big) \\ &= \Big(\frac{\sigma_p(r,y)}{\sqrt{\det \dot{g}_p(y)}}\Big)\, dr \wedge d\nu_p(y) =: \widehat{\sigma}_p(r,y) dr \wedge d\nu_p(y). \end{aligned}$$

The following is a key lemma for the proof of Theorem 1.2.

Lemma 3.1 *Let τ be the distortion of $(M, F, d\mu)$. Then*

$$\lim_{r\to 0^+} \frac{\widehat{\sigma}_p(r,y)}{r^{n-1}} = e^{-\tau(y)}.$$

Proof From the above, we have

$$\begin{aligned} \widehat{\sigma}_p(r,y) &= \frac{\sigma_p(r,y)}{\sqrt{\det \dot{g}_p(y)}} \\ &= \frac{\sigma_p(r,y)}{\sqrt{\det g(\exp_p(ry), \frac{\partial}{\partial r})}}\, \frac{\sqrt{\det\Big[g(\exp_p(ry), \frac{\partial}{\partial r})\Big]}}{\sqrt{\det \dot{g}_p(y)}} \\ &= e^{-\tau\big((\exp_p)_{*ry} y\big)}\, \frac{\sqrt{\det\Big[g(\exp_p(ry), \frac{\partial}{\partial r})\Big]}}{\sqrt{\det \dot{g}_p(y)}}, \end{aligned} \tag{3.3}$$

where $\det[g(\exp_p(ry), \frac{\partial}{\partial r})]$ is the determinant of $g_{ij}(\exp_p(ry), \frac{\partial}{\partial r})$ (in the polar coordinates). By the Gauss Lemma (see [5, p. 140]), we have

$$g_{\frac{\partial}{\partial r}}\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) = 1 \quad \text{and} \quad g_{\frac{\partial}{\partial r}}\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^\alpha}\Big) = 0.$$

From (3.1), we obtain

$$\begin{aligned} \sqrt{\det\Big[g\Big(\exp_p(ry), \frac{\partial}{\partial r}\Big)\Big]} &= \sqrt{\det\Big[g_{\frac{\partial}{\partial r}}\Big(\frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \theta^\beta}\Big)\Big]} \\ &= r^{n-1}\sqrt{\det\Big[g_{\frac{\partial}{\partial r}}\Big((\exp_p)_{*ry}\frac{\partial}{\partial \bar{\theta}^\alpha}, (\exp_p)_{*ry}\frac{\partial}{\partial \bar{\theta}^\beta}\Big)\Big]}. \end{aligned} \tag{3.4}$$

Thus, (3.2) together with (3.3) and (3.4) gives

$$\lim_{r\to 0^+} \frac{\widehat{\sigma}_p(r,y)}{r^{n-1}} = e^{-\tau(y)}. \qquad \blacksquare$$

Recall that $\gamma_y(t)$ is a constant speed geodesic with $\dot{\gamma}_y(0) = y$. Hence, $\frac{\partial}{\partial r}|_{(r,y)} = (\exp_p)_{*ry} y = \dot{\gamma}_y(r)$. Throughout this paper, let $\pi/\sqrt{k} := +\infty$ when $k \leq 0$. The first consequence of Lemma 3.1 is the following lemma.

Lemma 3.2 *Let $(M, F, d\mu)$ be a forward complete Finsler n-manifold. If for each $y \in S_pM$, $\mathbf{K}(\dot{\gamma}_y(t); \cdot) \equiv k$, then*

$$\mu(B_p^+(r)) = \mathcal{V}_{p,k,n}(r), \text{ for any } 0 < r \leq \mathfrak{i}_p,$$

where

$$\mathcal{V}_{p,k,n}(r) := \left(\int_{S_pM} d\nu_p(y) \right) \cdot \left(\int_0^r e^{-\tau(\dot{\gamma}_y(t))} \mathfrak{s}_k^{n-1}(t) dt \right),$$

and $\mathfrak{i}_p := \inf_{y \in S_pM} i_y$. *Moreover,*

$$g\left(\frac{\partial}{\partial r}\Big|_{(r,y)} \right) = dr \otimes dr + \mathfrak{s}_k^2(r) \dot{g}_p(y),\ 0 < r < i_y,$$

where (r, y) is the polar coordinate system at p.

Proof Given $y \in S_pM$, set $T = \dot{\gamma}_y(t)$. Let $J(t)$ be a Jacobi field along $\gamma_y(t)$ such that $J(0) = 0$ and $g_T(J, T) = 0$. Since $\mathbf{K}(T; \cdot) = k$, $\nabla_T^T \nabla_T^T J + kJ = 0$. Hence, $i_y \leq \pi/\sqrt{k}$ and $J(t) = \mathfrak{s}_k(t) E(t)$, where $E(t)$ is a parallel field along $\gamma_y(t)$.

Let (r, y) (or (r, θ)) be the polar coordinate system about p. From (3.1), we have

$$\lim_{r \to 0^+} \frac{1}{r} \frac{\partial}{\partial \theta^\alpha}\Big|_{(r,y)} = \frac{\partial}{\partial \overline{\theta}^\alpha}\Big|_y.$$

Hence,

$$\frac{\partial}{\partial \theta^\alpha}\Big|_{(t,y)} = \mathfrak{s}_k(t) E_\alpha(t),$$

where $E_\alpha(t)$ is a parallel field along $\gamma_y(t)$ such that $E_\alpha(0) = \frac{\partial}{\partial \overline{\theta}^\alpha}\big|_y$.

Since $\frac{\partial}{\partial r}|_{(t,y)} = \dot{\gamma}_y(t) = T$,

$$\sqrt{\det\Big[g(\exp_p(ty), \frac{\partial}{\partial r})\Big]} = \mathfrak{s}_k^{n-1}(t) \sqrt{\det g_T(E_\alpha, E_\beta)} = \mathfrak{s}_k^{n-1}(t) \sqrt{\det \dot{g}_p(y)}. \tag{3.5}$$

By the definition of distortion, we deduce

$$\frac{d}{dt} \log \sqrt{\det\Big[g\Big(\exp_p(ty), \frac{\partial}{\partial r}\Big)\Big]} = \frac{d}{dt} \log \widehat{\sigma}_p(t, y) + \frac{d}{dt} \tau(\dot{\gamma}_y(t)). \tag{3.6}$$

Equation (3.5) together with (3.6) yields

$$\frac{d}{dt} \log \widehat{\sigma}_p(t, y) = \frac{d}{dt} \log\big[e^{-\tau(\dot{\gamma}_y(t))} \mathfrak{s}_k^{n-1}(t) \big]. \tag{3.7}$$

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From Lemma 3.1 and (3.7), we have

$$\frac{\widehat{\sigma}_p(t,y)}{e^{-\tau(\dot{\gamma}_y(t))}\mathfrak{s}_k^{n-1}(t)} = \lim_{\varepsilon\to 0^+}\frac{\widehat{\sigma}_p(\varepsilon,y)}{e^{-\tau(y)}\mathfrak{s}_k^{n-1}(\varepsilon)} = 1.$$

Hence, $\mu(B^+(p,r)) = V_{p,k,n}(r)$, for any $0 < r \leq \mathfrak{i}_p$. Since $g_{\frac{\partial}{\partial r}}(\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta^\alpha}) = 0$,

$$g\Big(\frac{\partial}{\partial r}\Big|_{(r,y)}\Big) = dr\otimes dr + \mathfrak{s}_k^2(r)\dot{g}_p(y), \quad 0 < r < i_y. \qquad \blacksquare$$

Corollary 3.3 *Let $(M, F, d\mu)$ be a simply connected reversible complete Finsler n-manifold with $\mathbf{K} \equiv k > 0$. Then*

$$\mu(M) = \mathcal{V}_{p,k,n}\Big(\frac{\pi}{\sqrt{k}}\Big).$$

Proof By [37], for each $p \in M$ and each $y \in S_pM$, $\mathfrak{i}_p = i_y = \frac{\pi}{\sqrt{k}}$. Hence,

$$\mu(M) = \mu\Big(B^+\Big(p, \frac{\pi}{\sqrt{k}}\Big)\Big) = \mathcal{V}_{p,k,n}\Big(\frac{\pi}{\sqrt{k}}\Big). \qquad \blacksquare$$

Let (r, y) denote the polar coordinate system at $p \in M$ and let

$$\mathscr{F}(r,y) := e^{\tau(\dot{\gamma}_y(r))}\widehat{\sigma}_p(r,y) = \sqrt{\frac{\det g(\exp_p(ry), \frac{\partial}{\partial r})}{\det \dot{g}_p(y)}}.$$

Then we have the following theorem.

Theorem 3.4 *Let $(M, F, d\mu)$ be a forward complete Finsler n-manifold. Suppose that*

$$\mathbf{Ric} \geq (n-1)k.$$

Then the function

$$\mathcal{P}(r) = \frac{\mathscr{F}(r,y)}{\mathfrak{s}_k^{n-1}(r)}$$

is monotonically decreasing in r and converges to 1 (as $r \to 0^+$). Therefore,

$$\mu(B^+(p,r)) \leq \mathcal{V}_{p,k,n}(r), \quad \textit{for any } r > 0,$$

with equality for some $r_0 > 0$ if and only if for each $y \in S_pM$,

$$\mathbf{K}(\dot{\gamma}_y(t); \cdot) \equiv k,\ 0 \leq t \leq r_0 \leq \mathfrak{i}_p.$$

In this case,

$$g\Big(\frac{\partial}{\partial r}\Big|_{(r,y)}\Big) = dr\otimes dr + \mathfrak{s}_k^2(r)\dot{g}_p(y), \quad 0 < r < r_0,$$

where (r, y) is the polar coordinate system at p.

Proof By [34, 39], $\triangle r = \frac{\partial}{\partial r}\log\sigma_p(r,y) = \frac{\partial}{\partial r}\log\widehat{\sigma}_p(r,y)$ and $\triangle r + \frac{d}{dr}\tau(\dot{\gamma}_y(r)) \leq \frac{\partial}{\partial r}\log[\mathfrak{s}_k(r)]^{n-1}$, for $0 < r < i_y$. Thus, for any $y \in S_pM$,

$$\frac{\partial}{\partial r}\Big(\frac{\widehat{\sigma}_p(r,y)}{e^{-\tau(\dot{\gamma}_y(r))}\mathfrak{s}_k^{n-1}(r)}\Big) \leq 0, \quad \text{for } 0 < r < i_y. \tag{3.8}$$

Lemma 3.1, together with (3.8), yields $\widehat{\sigma}_p(r,y) \leq e^{-\tau(\dot{\gamma}_y(r))}\mathfrak{s}_k^{n-1}(r)$ for $0 < r < i_y$. Hence, for any $r > 0$,

$$\begin{aligned}\mu(B^+(p,r)) &\leq \int_{S_pM} d\nu_p(y)\int_0^{\min\{r,i_y\}} e^{-\tau(\dot{\gamma}_y(r))}\mathfrak{s}_k^{n-1}(r)dr \\ &\leq \int_{S_pM} d\nu_p(y)\int_0^{r} e^{-\tau(\dot{\gamma}_y(r))}\mathfrak{s}_k^{n-1}(r)dr \\ &= \mathcal{V}_{p,k,n}(r).\end{aligned} \tag{3.9}$$

If we have equality in (3.9) for some $r_0 > 0$, then $r_0 \leq \mathfrak{i}_p$, and equality holds in (3.8). By using Lemma 3.1 again, we have

$$e^{-\tau(\dot{\gamma}_y(r)))}\sqrt{\frac{\det g(\exp_p(ry), \frac{\partial}{\partial r})}{\det \dot{g}_p(y)}} = \widehat{\sigma}_p(r,y) = e^{-\tau(\dot{\gamma}_y(r))}\mathfrak{s}_k^{n-1}(r),$$

for each $y \in S_pM$ and for $0 < r < r_0$. Therefore, for any $y \in S_pM$,

$$\sqrt{\frac{\det g(\exp_p(ry), \frac{\partial}{\partial r})}{\det \dot{g}_p(y)}} = \mathfrak{s}_k(r)^{n-1}, \text{ for } 0 < r < r_0.$$

By the proof of [36, Lemma 4.1], we have $\mathbf{K}(\dot{\gamma}_y(t);\,\cdot\,) \equiv k$ and

$$\frac{\partial}{\partial\theta^\alpha}\Big|_{(t,y)} = \mathfrak{s}_k(t)E_\alpha(t),$$

for $0 < t < r_0$, where $E_\alpha(t)$ is a parallel field along $\gamma_y(t)$ such that $E_\alpha(0) = \frac{\partial}{\partial\theta^\alpha}|_y$. Therefore,

$$g\Big(\frac{\partial}{\partial r}\Big|_{(r,y)}\Big) = dr \otimes dr + \mathfrak{s}_k^2(r)\dot{g}_p(y), \quad 0 < r < r_0.$$

The conclusion then follows from Lemma 3.2. ■

Remark 3.5 From Theorem 3.4 and the standard argument (see [13] or [39]), one can obtain the following relative volume comparison theorems:

(i) Suppose that $\mathbf{Ric} \geq (n-1)k$ and $\mathbf{S} \geq (n-1)h$. Then

$$\mathscr{P}_1(r) = \frac{\mu(B^+(p,r))}{\mathscr{V}_{p,k,h,n}(r)}$$

is monotonically decreasing in r and converges to 1 (as $r \to 0^+$), where

$$\mathscr{V}_{p,k,h,n}(r) = \int_{S_pM} e^{-\tau(y)}d\nu_p(y)\int_0^r\left(e^{-ht}\mathfrak{s}_k(t)\right)^{n-1}dt.$$

(ii) Suppose that $\mathbf{Ric} \geq (n-1)k$ and $a \leq \tau \leq b$. Then

$$\frac{\mu(B^+(p,r))}{\mu(B^+(p,R))} \geq e^{a-b}\frac{\int_0^r \mathfrak{s}_k^{n-1}(t)dt}{\int_0^R \mathfrak{s}_k^{n-1}(t)dt},$$

for any $0 < r \leq R$.

A simple argument yields the following theorem.

Theorem 3.6 *Let $(M, F, d\mu)$ be a forward complete Finsler n-manifold. Suppose that $\mathbf{K} \leq k$. Then the function*

$$\mathcal{P}(r) = \frac{\mathscr{F}(r,y)}{\mathfrak{s}_k^{n-1}(r)}$$

is monotonically increasing in r and converges to 1 (as $r \to 0^+$). Therefore,

$$\mu(B^+(p,r)) \geq \mathcal{V}_{p,k,n}(r), \quad \textit{for any } 0 < r \leq \min\{\mathfrak{i}_p, \pi/\sqrt{k}\},$$

with equality for some $r_0 > 0$ if and only if for each $y \in S_pM$,

$$\mathbf{K}(\dot\gamma_y(t);\,\cdot\,) \equiv k,\ 0 \leq t \leq r_0 \leq \min\{\mathfrak{i}_p, \pi/\sqrt{k}\}.$$

In this case,

$$g\Big(\frac{\partial}{\partial r}\Big|_{(r,y)}\Big) = dr \otimes dr + \mathfrak{s}_k^2(r)\dot{g}_p(y), \quad 0 < r < r_0,$$

where (r, y) is the polar coordinate system at p.

Fix a point p of a forward complete Finsler manifold (M, F). Let (r, y) be the polar coordinate system at p. Define $\widehat{g}_p := g(\frac{\partial}{\partial r}|_{(r,y)})$. Clearly, $\widehat{g}_p$ is a Riemannian metric on $\mathcal{D}_p - \{p\}$. As an application of Theorem 3.4, we have the following characterization of $\mathbb{R}^n$.

Corollary 3.7 *Let $(M, F, d\mu)$ be a forward complete noncompact Finsler n-manifold with $\mathbf{Ric} \geq 0$ and $\mathbf{S} \geq 0$. If for some $p \in M$,*

$$\liminf_{r\to+\infty}\frac{\mu(B^+(p,r))}{r^n} = \frac{1}{n}\int_{S_pM} e^{-\tau(y)}d\nu_p(y),$$

then M is C^1-diffeomorphic to T_pM. In this case, $(M\backslash\{p\}, \widehat{g}_p)$ is isometric to $(T_pM\backslash\{0\}, g_p)$. Moreover, if (M, F) is a Berwald space, then M is C^∞-diffeomorphic to T_pM.

Proof Remark 3.5 guarantees $\mathscr{V}_{p,0,0,n}(r) \geq \mu(B^+(p,r)) \to +\infty$, as $r \to +\infty$. Clearly,

$$\liminf_{r\to+\infty}\frac{r^n}{\mathscr{V}_{p,0,0,n}(r)} = \frac{n}{\int_{S_pM} e^{-\tau(y)}d\nu_p(y)}.$$

By using Remark 3.5 again, we have

$$1 \geq \frac{\mu(B^+(p,r))}{\mathscr{V}_{p,0,0,n}(r)} \geq \liminf_{r\to+\infty} \frac{\mu(B^+(p,r))}{\mathscr{V}_{p,0,0,n}(r)} \geq \liminf_{r\to+\infty} \frac{\mu(B^+(p,r))}{r^n} \cdot \liminf_{r\to+\infty} \frac{r^n}{\mathscr{V}_{p,0,0,n}(r)} \geq 1,$$

which implies that $\mathfrak{i}_p = +\infty$ and $\widehat{g}_p = dr \otimes dr + r^2 \dot{g}_p(y)$. Hence, $\exp_p \colon T_pM \to M$ is a C^1-diffeomorphism and $g_p = \exp_p^*(\widehat{g}_p)$. If F is of Berward type, then exp is C^∞ throughout TM (see [5, p. 127]). This complete the proof. ■

Remark 3.8 If $d\mu$ is the Busemann-Hausdorff volume form, then

$$\frac{1}{n}\int_{S_pM} e^{-\tau(y)} d\nu_p(y) = \frac{c_{n-1}}{n} = \mathrm{Vol}_{\mathbb{R}^n}(\mathbb{B}^n).$$

Clearly, $\widehat{g}_p = g|_{\mathcal{D}_p}$ when F is Riemannian. Therefore, Corollary 3.7 implies [15, Corollary 1.134].

From Theorem 3.4, we also obtain a generalized version of the Calabi-Yau linear volume growth theorem [40]. See [38] for another version.

Corollary 3.9 *Let $(M, F, d\mu)$ be a forward complete noncompact Finsler manifold with* $\mathbf{Ric} \geq 0$, $\mathbf{S} \geq 0$ *and the reversibility $\lambda < +\infty$. For any point $p \in M$, there exists a constant $C > 0$ such that for any $r \geq 1$,*

$$\mu(B^+(p,r)) \geq Cr.$$

Proof By Theorem 3.4 and the standard argument (see [13] or [39]), one can show that

$$\mathcal{Q}(r) := \frac{\mu(B^+(x,r))}{r^n}$$

is monotonically decreasing in r. Fix a point $x \in M$ with $d(x,p) = r \geq \lambda + 1$. Thus,

$$\frac{\mu(B^+(x,r+1)) - \mu(B^+(x,r-\lambda))}{\mu(B^+(x,r-\lambda))} \leq \frac{(r+1)^n - (r-\lambda)^n}{(r-\lambda)^n} \leq \frac{C(n,\lambda)}{r}. \tag{3.10}$$

Note that $B^+(p,1) \subset B^+(x,r+1) - B^+(x,r-\lambda)$ and $B^+(p,\lambda r + r - \lambda) \supset B^+(x,r-\lambda)$. Equation (3.10) then yields

$$\frac{\mu(B^+(p,\lambda r + r - \lambda))}{\mu(B^+(p,1))} \geq \frac{r}{C(n,\lambda)}.$$

Let

$$\widetilde{C} := \inf_{r\in[1,(\lambda+1)^2-\lambda]} \frac{\mu(B^+(p,r))}{r}, \text{ and } C := \min\Big(\frac{\mu(B^+(p,1))}{C(n,\lambda)}, \widetilde{C}\Big).$$

Then we have $\mu(B^+(p,r)) \geq Cr$, for $r \geq 1$. ■

4 Generalized Berger–Kazdan Inequality

Let (M, F) be a forward complete Finsler n-manifold. Given $p \in M$ and $y \in S_pM$, define an inner product $\langle \cdot, \cdot \rangle$ on T_pM by $\langle \cdot, \cdot \rangle := g_y(\cdot, \cdot)$. Let $y^\perp = \{X \in T_pM : \langle y, X \rangle = 0\}$. Thus, $T_pM = \mathbb{R}y \oplus y^\perp$. Denote by $P_{t;y}$ the parallel translation along γ_y from $T_{\gamma_y(0)}M$ to $T_{\gamma_y(t)}M$ (with respect to the Chern connection).

Set $T = \dot{\gamma}_y(t)$. For $0 \leq t < c_y$, let $R_T := R_T(\cdot, T)T$ and

$$\mathcal{R}(t, y) := P_{t;y}^{-1} \circ R_T \circ P_{t;y} \colon y^\perp \to y^\perp.$$

Let $\mathcal{A}(t, y)$ be the solution of the matrix (or linear transformation) ordinary differential equation on $y^\perp$:

$$\begin{cases} \mathcal{A}'' + \mathcal{R}(t, y)\mathcal{A} = 0, \\ \mathcal{A}(0, y) = 0, \\ \mathcal{A}'(0, y) = \mathcal{I}, \end{cases} \tag{4.1}$$

where $\mathcal{A}' = \frac{d}{dt}\mathcal{A}$ and $\mathcal{I}$ is the identity transformation of $y^\perp$.

Given any vector $V(t) \in y^\perp$, it is easy to check that

$$\nabla_T^T\big(P_{t;y}V(t)\big) = P_{t;y}V'(t).$$

Hence, (4.1) is equivalent to

$$\begin{cases} \nabla_T^T\nabla_T^T(P_{t;y}\mathcal{A}X) + R_T(P_{t;y}\mathcal{A}X, T)T = 0, \quad \text{for any } X \in y^\perp, \\ (P_{t;y}\mathcal{A}(t, y)X)|_{t=0} = 0, \\ \nabla_T^T(P_{t;y}\mathcal{A}(t, y)X)|_{t=0} = X. \end{cases}$$

Therefore, $P_{t;y}\mathcal{A}(t, y)X = (\exp_p)_{*ty}tX$ (see [5], p. 131).

Define $\det \mathcal{A}(t, y) := \det \mathcal{A}(t, y)_a^\beta$, where $\mathcal{A}(t, y)e_\alpha = \mathcal{A}(t, y)_\alpha^\beta e_\beta$ and $\{e_\alpha\}$ is any basis of $y^\perp$. It is not hard to see that $\det \mathcal{A}$ is well defined and independent of $\{e_\alpha\}$.

Let (r, y) denote the polar coordinate system at p. From (3.1), we have

$$P_{r;y} \circ \mathcal{A}(r, y)\frac{\partial}{\partial \overline{\theta}^\alpha} = (\exp_p)_{*ry} r\frac{\partial}{\partial \overline{\theta}^\alpha} = \frac{\partial}{\partial \theta^\alpha}.$$

Note that for any $X, Y \in T_pM$,

$$g_{\frac{\partial}{\partial r}}(P_{r;y}X, P_{r;y}Y) = g_y(X, Y).$$

Hence,

$$\begin{aligned} \det\Big[g(\exp_p(ry), \frac{\partial}{\partial r})\Big] &= \det\Big[g_{\frac{\partial}{\partial r}}\Big(P_{r,y} \circ \mathcal{A}(r, y)\frac{\partial}{\partial \overline{\theta}^\alpha}, P_{r,y} \circ \mathcal{A}(r, y)\frac{\partial}{\partial \overline{\theta}^\beta}\Big)\Big] \\ &= \det\Big[g_y\Big(\mathcal{A}(r, y)\frac{\partial}{\partial \overline{\theta}^\alpha}, \mathcal{A}(r, y)\frac{\partial}{\partial \overline{\theta}^\beta}\Big)\Big] \\ &= \det\Big[\mathcal{A}(r, y)_\alpha^\delta \mathcal{A}(r, y)_\beta^\eta g_y\Big(\frac{\partial}{\partial \overline{\theta}^\delta}, \frac{\partial}{\partial \overline{\theta}^\sigma}\Big)\Big] \\ &= \big(\det \mathcal{A}(r, y)\big)^2 \det \dot{g}_p(y), \end{aligned}$$

where $\mathcal{A}(r,y)\frac{\partial}{\partial\bar{\theta}^\alpha} =: \mathcal{A}(r,y)^\beta_\alpha\frac{\partial}{\partial\bar{\theta}^\beta}$. Namely,

$$\sqrt{\frac{\det g(\exp_p(ry),\frac{\partial}{\partial r})}{\det \dot{g}_p(y)}} = \det\mathcal{A}(r,y).$$

Equation (3.4) together with the above equality yields

$$\lim_{r\to 0^+}\frac{\det\mathcal{A}(r,y)}{r^{n-1}} = 1.$$

Denote by $\mathcal{A}^*$ the adjoin of the linear transformation $\mathcal{A}$ on $(y^\perp,\langle\cdot,\cdot\rangle)$; that is, for any $X,Y\in y^\perp$, $\langle\mathcal{A}(X),Y\rangle = \langle X,\mathcal{A}^*(Y)\rangle$. Then $\mathcal{A}^*\mathcal{A}$ is self-adjoint and $\det\mathcal{A}^* = \det\mathcal{A}$. Moreover, we have the following lemma.

Lemma 4.1

(i) $\mathcal{A}'^*\mathcal{A} = \mathcal{A}^*\mathcal{A}'$.

(ii) *For* $0<t<c_y$, $\mathcal{A}'\mathcal{A}^{-1}$ *is self-adjoint.*

Proof (i) Let $T=\dot{\gamma}_y(t)$. For any X, $Y\in y^\perp$, set $J_X = (\exp_p)_{*ty}tX = P_{t;y}\mathcal{A}X$ and $J_Y = (\exp_p)_{*ty}tY = P_{t;y}\mathcal{A}Y$. From the Lagrange identity (see [5, p. 135]), we have

$$\begin{aligned}
0 &= g_T(\nabla^T_T J_X, J_Y) - g_T(J_X,\nabla^T_T J_Y)\\
&= g_T(P_{t;y}\mathcal{A}'X, P_{t;y}\mathcal{A}Y) - g_T(P_{t;y}\mathcal{A}X, P_{t;y}\mathcal{A}'Y)\\
&= g_y(\mathcal{A}'X,\mathcal{A}Y) - g_y(\mathcal{A}X,\mathcal{A}'Y)\\
&= \big\langle X, (\mathcal{A}'^*\mathcal{A}-\mathcal{A}^*\mathcal{A}')(Y)\big\rangle.
\end{aligned}$$

This proves (i).

(ii) From (i), we have $0=\langle X,(\mathcal{A}^{-1})^*[\mathcal{A}'^*\mathcal{A}-\mathcal{A}^*\mathcal{A}']\mathcal{A}^{-1}Y\rangle$, which implies

$$\big\langle X,(\mathcal{A}'\mathcal{A}^{-1})^*Y\big\rangle = \langle X,\mathcal{A}'\mathcal{A}^{-1}Y\rangle. \qquad \blacksquare$$

Let $\mathcal{C}_s(t,y)$ be the solution of the matrix (linear transformation) ordinary equation on $y^\perp$:

$$\begin{cases}\mathcal{C}_s{}'' + \mathcal{R}(t,y)\mathcal{C}_s = 0,\\ \mathcal{C}_s(s,y)=0,\\ \mathcal{C}_s'(s,y) = \mathcal{I}.\end{cases}$$

It is easy to check that $\mathcal{C}_s(t,y) = P^{-1}_{s;y}\circ\mathcal{A}(t-s,\dot{\gamma}_y(s))\circ P_{s;y}$, for $t\ge s$. Hence,

$$\det\mathcal{C}_s(t,y) = \det\mathcal{A}\big(t-s,\dot{\gamma}_y(s)\big),\quad t\ge s.$$

In particular, by Lemma 4.1, one can show the following lemma, whose proof is the same as that of [13, Theorem 5.8, Step 1–2] (*cf.* also [9, Appendix D]).

Lemma 4.2 *Given $y \in S_pM$, for any $0 < s \le t < c_y$,*

$$\mathcal{C}_s(t,y) = \mathcal{A}(t,y)\Big(\int_s^t (\mathcal{A}^*\mathcal{A})^{-1}(r,y)dr\Big)\mathcal{A}^*(s,y).$$

$$(\det \mathcal{C}_s(t,y))^{\frac{1}{n-1}} \ge \big(\det \mathcal{A}(t,y)\big)^{\frac{1}{n-1}}\big(\det \mathcal{A}(s,y)\big)^{\frac{1}{n-1}}\int_s^t \frac{1}{\big(\det \mathcal{A}(r,y)\big)^{\frac{2}{n-1}}}dr,$$

with equality if and only if $\mathcal{A}(t,y) = (\det \mathcal{A}(t,y))^{\frac{1}{n-1}}\mathcal{I}$.

Now we recall the Kazdan inequality (see [9, Appendix E]). Given $\lambda > 0$, let S denote the set of functions

$$S = \Big\{\varphi \in C[0,\pi/\lambda] : \varphi(x) = x^{\alpha}(\pi/\lambda - x)^{\beta}\psi(x) \text{ for some } 0 \le \alpha, \beta, < 2 \\ \text{and some } \psi \in C[0,\pi/\lambda] \text{ with } \psi > 0 \text{ on } [0,\pi/\lambda]\Big\}.$$

Let

$$F(\varphi) := \int_0^{\pi/\lambda} dt \int_t^{\pi/\lambda} dr \int_t^r \frac{\varphi(t)\varphi(r)}{\varphi^2(s)}\rho(r-t)ds,$$

where $\rho \in C[0,\pi/\lambda]$ is a given nonnegative function.

Theorem 4.3 (L. Kazdan) *If $\rho(\pi/\lambda - t) = \rho(t)$, for all $0 \le t \le \pi/\lambda$, then $F(\varphi) \ge F(\sin\circ\lambda)$, where $\sin\circ\lambda(x) := \sin(\lambda x)$.*

Let (r,y) denote the polar coordinate system at $p \in M$. Then

$$\mathscr{F}(r,y) = \sqrt{\frac{\det g(\exp_p(ry), \frac{\partial}{\partial r})}{\det \dot{g}_p(y)}} = \det \mathcal{A}(r,y).$$

From the above, we have the following inequality, which can be interpreted as a generalization of the Berger–Kazdan inequality [8].

Theorem 4.4 *Let (M,F) be a compact Finsler n-manifold. For each $y \in SM$ and $0 \le l \le i_y$, we have*

$$\int_0^l dr \int_r^l \mathscr{F}(t-r, \varphi_r(y))\,dt \ge \frac{\pi c_n}{2c_{n-1}}\Big(\frac{l}{\pi}\Big)^{n+1}, \tag{4.2}$$

with equality if and only if

$$R_{\dot\gamma_y(t)}\big(\,\cdot\,, \dot\gamma_y(t)\big)\,\dot\gamma_y(t) = \Big(\frac{\pi}{l}\Big)^2 \mathrm{id}, \quad \text{for } 0 \le t \le l,$$

where φ_t is the geodesic flow and R is the (Riemannian) curvature tensor acting on $\dot\gamma_y(t)^{\perp}$.

Proof Set $\mathfrak{C} = \frac{\pi}{l}$. Using Hölder's inequality, Lemma 4.2, and Theorem 4.3, we have

$$\begin{aligned}
&\int_0^l dr \int_r^l \mathscr{F}(t-r, \varphi_r(y))\, dt \\
&\quad \geq \frac{\left[\int_0^l dr \int_r^l \mathscr{F}(t-r,\varphi_r(y))^{\frac{1}{n-1}} \sin^{n-2}(\mathfrak{C}(t-r))dt\right]^{n-1}}{\left[\int_0^l dr \int_r^l \sin^{n-1}(\mathfrak{C}(t-r))dt\right]^{n-2}} \\
&\quad \geq \frac{\left[\int_0^l dr \int_r^l dt \int_r^t \frac{\mathscr{F}(t,y)^{\frac{1}{n-1}}\mathscr{F}(r,y)^{\frac{1}{n-1}}}{\mathscr{F}(m,y)^{\frac{2}{n-1}}} \sin^{n-2}(\mathfrak{C}(t-r))dm\right]^{n-1}}{\left[\int_0^l dr \int_r^l \sin^{n-1}(\mathfrak{C}(t-r))dt\right]^{n-2}} \\
&\quad \geq \frac{1}{\mathfrak{C}^{n-1}} \int_0^l dr \int_r^l \sin^{n-1}\big(\mathfrak{C}(t-r)\big)\, dt \\
&\quad = \frac{1}{\mathfrak{C}^{n+1}} \int_0^\pi dr \int_r^\pi \sin^{n-1}(t-r)dt = \frac{\pi c_n}{2c_{n-1}\mathfrak{C}^{n+1}}.
\end{aligned}$$

If we have equality in (4.2), then

$$\begin{cases} \mathscr{F}(t-r,\varphi_r(y)) = \left(\frac{\sin \mathfrak{C}(t-r)}{\mathfrak{C}}\right)^{n-1}, \\ \mathcal{A}(t,y) = (\det \mathcal{A}(t,y))^{\frac{1}{n-1}} \mathfrak{I}, \end{cases}$$

for $0 \leq r \leq t \leq l$. Hence, $\mathcal{A}(t,y) = \frac{\sin(\mathfrak{C}t)}{\mathfrak{C}} \cdot \mathfrak{I}$, $0 \leq t \leq l$. It follows from (4.1) that

$$\mathcal{R}(t,y) = \mathfrak{C}^2 \mathfrak{I}, \qquad \text{for } 0 \leq t \leq l.$$

This completes the proof. ∎

The remainder of this section will be devoted to the proof of Theorem 1.4. First, we recall the following theorem due to C. Kim and J. Yim (see [25, Theorem 4]).

Theorem 4.5 ([25]) *If (M,F) is an n-dimensional reversible Finsler manifold with $\mathbf{K} \equiv 1$ and $\mathbf{S}_{BH} \equiv 0$, then F is a Riemannian metric. In fact, the universal covering of M is isometric to the standard n-sphere of constant sectional curvature one.*

Theorem 4.4 together with Theorems 2.1 and 4.5 yields the following theorem.

Theorem 4.6 *Let (M,F) be a compact Finsler n-manifold with reversibility λ. Then*

$$\mu_{BH}(M) \geq c_n \left(\frac{\mathfrak{i}_M}{\lambda\pi}\right)^n \exp\left[-\frac{\mathfrak{i}_M}{\lambda}\big(|h_1| + |h_2|\big)\right], \tag{4.3}$$

where $h_1 := \sup_{y\in SM} \mathbf{S}_{BH}(y)$, $h_2 := \sup_{y\in SM} \mathbf{S}_{HT}(y)$.

In particular, if F is reversible, then equality holds in (4.3) if and only if (M,F) is isometric to the standard n-sphere of constant sectional curvature $(\pi/\mathfrak{i}_M)^2$.

Proof Note that if we set $\widetilde{F} := \frac{\pi}{\mathrm{i}_M} F$, then the injective radius of $(M, \widetilde{F})$ is equal to π. Hence, without loss of generality, we may assume that $\mathrm{i}_M = \pi$.

Since $\mathrm{i}_M = \pi$ and the reversibility is λ,

$$B^+\Big(\gamma_y(0), \frac{r}{\lambda}\Big) \cap B^+\Big(\gamma_y(\pi), \frac{\pi - r}{\lambda}\Big) = \varnothing, \quad \forall r \in [0, \pi/2].$$

Hence,

$$\mu_{BH}(M) \geq \mu_{BH}\Big(B^+\Big(\gamma_y(0), \frac{r}{\lambda}\Big)\Big) + \mu_{BH}\Big(B^+\Big(\gamma_y(\pi), \frac{\pi - r}{\lambda}\Big)\Big).$$

It is straightforward to compute

$$\begin{aligned}
(4.4)\qquad & \int_{SM} \mu_{BH}\big(B^+(\pi_1(y), r)\big)\, dV_{SM}(y) \\
&= \int_M d\mu_{HT}(x) \int_{S_xM} \mu_{BH}\big(B^+(x, r)\big)\, e^{\tau_{HT}(y)} d\nu_x(y) \\
&= c_{n-1} \int_M \mu_{BH}\big(B^+(x, r)\big)\, d\mu_{HT}(x) \\
&= c_{n-1} \int_M d\mu_{HT}(x) \int_0^r dt \int_{S_xM} \widehat{\sigma}_{BHx}(t, y) d\nu_x(y) \\
&= c_{n-1} \int_M d\mu_{HT}(x) \int_0^r dt \int_{S_xM} \mathscr{F}(t, y) e^{-\tau_{BH}(\varphi_t(y))} d\nu_x(y).
\end{aligned}$$

Since M is compact, the geodesic flow $\varphi_t \colon SM \to SM$ is a diffeomorphism. From Theorem 2.1, we have

$$\begin{aligned}
(4.5)\qquad & \int_{SM} \mu_{BH}\Big(B^+\Big(\gamma_y(\pi), \frac{\pi - r}{\lambda}\Big)\Big)\, dV_{SM}(y) \\
&= \int_{SM} \mu_{BH}\Big(B^+\Big(\pi_1 \circ \varphi_\pi(y), \frac{\pi - r}{\lambda}\Big)\Big)\, dV_{SM}(y) \\
&= \int_{\varphi_\pi^{-1} \circ \varphi_\pi(SM)} \mu_{BH}\Big(B^+\Big(\pi_1 \circ \varphi_\pi(y), \frac{\pi - r}{\lambda}\Big)\Big)\, dV_{SM}(y) \\
&= \int_{\varphi_\pi(SM)} (\varphi_\pi^{-1})^* \Big[\mu_{BH}\Big(B^+\Big(\pi_1 \circ \varphi_\pi(y), \frac{\pi - r}{\lambda}\Big)\Big)\, dV_{SM}(y)\Big] \\
&= \int_{\varphi_\pi(SM)} \mu_{BH}\Big(B^+\Big(\pi_1 \circ \varphi_\pi(y), \frac{\pi - r}{\lambda}\Big)\Big)\, dV_{SM}(\varphi_\pi(y)) \\
&= \int_{SM} \mu_{BH}\Big(B^+\Big(\pi_1(y), \frac{\pi - r}{\lambda}\Big)\Big)\, dV_{SM}(y).
\end{aligned}$$

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Note that $V_{SM} = c_{n-1}\mu_{HT}(M)$. Using (4.4) and (4.5), we deduce that

$$
\begin{aligned}
c_{n-1}\mu_{HT}(M)\mu_{BH}(M) &= \int_{SM} \mu_{BH}(M) dV_{SM}(y) \\
&\geq \int_{SM} \Big[\mu_{BH}\Big(B^{+}\Big(\gamma_y(0), \frac{r}{\lambda} \Big) \Big) + \mu_{BH}\Big(B^{+}\Big(\gamma_y(\pi), \frac{\pi - r}{\lambda} \Big) \Big) \Big] dV_{SM}(y) \\
&= \int_{SM} \Big[\mu_{BH}\Big(B^{+}\Big(\pi_1(y), \frac{r}{\lambda} \Big) \Big) + \mu_{BH}\Big(B^{+}\Big(\pi_1(y), \frac{\pi - r}{\lambda} \Big) \Big) \Big] dV_{SM}(y) \\
&= c_{n-1} \int_M d\mu_{HT}(x) \int_{S_x M} d\nu_x(y) \Big[\int_0^{r/\lambda} + \int_0^{\frac{\pi - r}{\lambda}} D(t, y) dt \Big],
\end{aligned}
$$

where $D(t, y) = \mathscr{F}(t, y) e^{-\tau_{BH}(\varphi_t(y))}$. Therefore,

$$
\begin{aligned}
\text{(4.6)} \quad & \frac{\pi}{2}\mu_{HT}(M)\mu_{BH}(M) \geq \\
& \int_M d\mu_{HT}(x) \int_{S_x M} d\nu_x(y) \int_0^{\pi/2} \Big[\int_0^{r/\lambda} + \int_0^{\frac{\pi - r}{\lambda}} D(t, y) dt \Big] dr.
\end{aligned}
$$

By interchanging the order of integration, we obtain

$$
\begin{aligned}
\int_0^{\pi/2} dr \int_0^{r/\lambda} D(t, y) dt &= \int_0^{\frac{\pi}{2\lambda}} dt \int_{t\lambda}^{\pi/2} D(t, y) dr \\
&= \int_0^{\frac{\pi}{2\lambda}} (\pi/2 - t\lambda) D(t, y) dt
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{\pi/2} dr \int_0^{\frac{\pi - r}{\lambda}} D(t, y) dt &= \int_0^{\frac{\pi}{2\lambda}} dt \int_0^{\pi/2} D(t, y) dr + \int_{\frac{\pi}{2\lambda}}^{\pi/\lambda} dt \int_0^{\pi - \lambda t} D(t, y) dr \\
&= \int_0^{\frac{\pi}{2\lambda}} \frac{\pi}{2} D(t, y) dt + \int_{\frac{\pi}{2\lambda}}^{\pi/\lambda} (\pi - \lambda t) D(t, y) dt.
\end{aligned}
$$

Hence,

$$
\text{(4.7)} \qquad \int_0^{\pi/2} \Big[\int_0^{r/\lambda} + \int_0^{\frac{\pi - r}{\lambda}} D(t, y) dt \Big] dr = \int_0^{\frac{\pi}{\lambda}} (\pi - \lambda t) D(t, y) dt,
$$

By (4.6), (4.7), and Theorem 2.1, we obtain

$$
\begin{aligned}
&\frac{\pi}{2}\mu_{HT}(M)\mu_{BH}(M)\\
&\geq \int_M d\mu_{HT}(x)\int_{S_xM} d\nu_x(y)\int_0^{\frac{\pi}{\lambda}}(\pi-\lambda t)D(t,y)dt\\
&=\lambda\int_M d\mu_{HT}(x)\int_{S_xM} d\nu_x(y)\int_0^{\frac{\pi}{\lambda}} dt\int_0^{\frac{\pi}{\lambda}-t} D(t,y)dr\\
&=\lambda\int_0^{\frac{\pi}{\lambda}} dt\int_0^{\frac{\pi}{\lambda}-t} dr\int_{SM} e^{-\tau_{HT}(y)}D(t,y)dV_{SM}(y)\\
&=\lambda\int_0^{\frac{\pi}{\lambda}} dt\int_0^{\frac{\pi}{\lambda}-t} dr\int_{SM} e^{-\tau_{HT}(\varphi_r(y))}D(t,\varphi_r(y))dV_{SM}(y)\\
&=\lambda\int_0^{\frac{\pi}{\lambda}} dt\int_0^{\frac{\pi}{\lambda}-t} dr\int_{SM} e^{-[\tau_{HT}(\varphi_r(y))+\tau_{BH}(\varphi_{t+r}(y))]}\mathscr{F}(t,\varphi_r(y))dV_{SM}(y).
\end{aligned}
$$

It follows from the definition of S-curvature that

$$
\tau_{BH}\big(\varphi_{t+r}(y)\big)\leq\tau_{BH}(y)+\frac{\pi}{\lambda}|h_1|,\ \tau_{HT}(\varphi_r(y))\leq\tau_{HT}(y)+\frac{\pi}{\lambda}|h_2|. \tag{4.8}
$$

Set $\Lambda=e^{-\frac{\pi}{\lambda}(|h_1|+|h_2|)}$. From the above, we have

$$
\begin{aligned}
&\frac{\pi}{2\lambda}\mu_{HT}(M)\mu_{BH}(M)\\
&\geq\Lambda\int_0^{\frac{\pi}{\lambda}} dt\int_0^{\frac{\pi}{\lambda}-t} dr\int_{SM} e^{-[\tau_{HT}(y)+\tau_{BH}(y)]}\mathscr{F}\big(t,\varphi_r(y)\big)\,dV_{SM}(y)\\
&=\Lambda\int_M d\mu_{HT}(x)\int_{S_xM} e^{-\tau_{BH}(y)}d\nu_x(y)\int_0^{\frac{\pi}{\lambda}} dt\int_0^{\frac{\pi}{\lambda}-t}\mathscr{F}\big(t,\varphi_r(y)\big)\,dr\\
&=\Lambda\int_M d\mu_{HT}(x)\int_{S_xM} e^{-\tau_{BH}(y)}d\nu_x(y)\int_0^{\frac{\pi}{\lambda}} dr\int_0^{\frac{\pi}{\lambda}-r}\mathscr{F}\big(t,\varphi_r(y)\big)\,dt\\
&=\Lambda\int_M d\mu_{HT}(x)\int_{S_xM} e^{-\tau_{BH}(y)}d\nu_x(y)\int_0^{\frac{\pi}{\lambda}} dr\int_r^{\frac{\pi}{\lambda}}\mathscr{F}\big(t-r,\varphi_r(y)\big)\,dt
\end{aligned}
$$

By Theorem 4.4,

$$
\begin{aligned}
\mu_{HT}(M)\mu_{BH}(M)&\geq\frac{c_n\Lambda}{c_{n-1}\lambda^n}\int_M d\mu_{HT}(x)\int_{S_xM} e^{-\tau_{BH}(y)}d\nu_x(y)\\
&=\frac{c_n\Lambda}{\lambda^n}\mu_{HT}(M).
\end{aligned}
$$

Namely, $\mu_{BH}(M)\geq\frac{c_n\Lambda}{\lambda^n}$.

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If F is reversible and we have equality in (4.3), then it follows from (4.8) that

$$\tau_{HT}\big(\varphi_r(y)\big) + \tau_{BH}\big(\varphi_{t+r}(y)\big) = \tau_{HT}(y) + \tau_{BH}(y) + \pi\big(|h_1| + |h_2|\big),$$

for every $y \in SM$, $0 \le t \le \pi$ and $0 \le r \le \pi - t$. Since (M, F) is reversible, $h_1 \ge 0$, $h_2 \ge 0$, and

$$\tau_{HT}(y) + \tau_{BH}(y) + rh_2 + (t + r)h_1 \ge \tau_{HT}\big(\varphi_r(y)\big) + \tau_{BH}\big(\varphi_{t+r}(y)\big),$$

which implies that $h_1 = h_2 = 0$. If there exists $y \in SM$ such that $\mathbf{S}_{HT}(y) < 0$, then $\mathbf{S}_{HT}(-y) = -\mathbf{S}_{HT}(y) > 0$, which is a contradiction. Hence, $\mathbf{S}_{HT} = \mathbf{S}_{BH} = 0$. Theorem 4.4 yields $\mathcal{R}(t, y) = \mathcal{I}$, for $0 \le t \le \pi$. By letting $t \to 0^+$, we have

$$R_y = R_y(\,\cdot\,, y)y = \mathcal{I}\colon y^\perp \to y^\perp,$$

i.e., $\mathbf{K} \equiv 1$. Now, we have shown that (M, F) is a reversible compact Finsler n-manifold with $\mathbf{K} \equiv 1$ and $\mathbf{S}_{BH} \equiv 0$. Theorem 4.5 then implies that F is a Riemannian metric and the universal covering of M is $\mathbb{S}^n$. But since $\mu(M) = c_n$, (M, F) must be isometric to $\mathbb{S}^n$. ∎

When F is Riemannian, $\lambda = 1$, $h_1 = h_2 = 0$, and, therefore, Theorem 4.6 becomes the Berger–Kazdan comparison theorem [7].

5 Generalized Santaló formula

This section is dedicated to the proof of Theorem 1.5. Let (M, F) be a reversible complete Finsler n-manifold and let $\Omega \subset M$ be a relatively compact domain with smooth boundary $\partial\Omega$. Denote by $\mathbf{n}$ the unit inward normal vector field along $\partial\Omega$. Thus, $g_{\mathbf{n}}(\mathbf{n}, X) = 0$ for any $X \in T\partial\Omega$ (see [34]). According to [14], $\mathbf{n}$ always exists. In fact, since $\operatorname{codim}(\partial\Omega) = 1$, by a partition of unity one can construct a nonzero 1-form ω on $\partial\Omega$ such that $i^*(\omega) = 0$, where $i\colon \partial\Omega \hookrightarrow M$ is the inclusion map. Thus, $\mathbf{n} = (\mathcal{L}^{-1}(\omega))/(F(\mathcal{L}^{-1}(\omega)))$ (up to a sign), where $\mathcal{L}$ is the Legendre transformation.

Let $\mathcal{N} = \{k \cdot \mathbf{n}(x) : x \in \partial\Omega,\ k \in \mathbb{R}\}$ denote the normal bundle over $\partial\Omega$. Since F is reversible, $\mathcal{N}$ is an n-dimensional smooth manifold. For convenience, we use (x, k) to denote $k \cdot \mathbf{n}(x) \in \mathcal{N}$. The exponential map Exp of the normal bundle $\mathcal{N}$ is defined by

$$\begin{aligned} \operatorname{Exp}\colon\ \mathcal{N} &\longrightarrow M \\ (x, k) &\longmapsto \exp_x(k\mathbf{n}). \end{aligned}$$

We always identify $\partial\Omega$ with the zero section of $\mathcal{N}$. This implies that for any $x \in \partial\Omega$, we have the inclusion $T_x\partial\Omega \subset T_{(x,0)}\mathcal{N}$. Moreover, from the definition of $\mathcal{N}$, we have

$$T_{(x,0)}\mathcal{N} = T_x\partial\Omega \oplus \mathbb{R}, \qquad T_xM = T_x\partial\Omega \oplus \mathbb{R}\mathbf{n}. \tag{5.1}$$

Lemma 5.1 *The map* $\operatorname{Exp}\colon \mathcal{N} \to M$ *maps a neighborhood of* $\partial\Omega \subset \mathcal{N}$ C^1*-diffeomorphically onto a neighborhood of* $\partial\Omega \subset M$.

Proof Choose a (local) coordinate system $\{x^\alpha\}$ of $\partial\Omega$. Thus, (x^α, k) is a (local) coordinate system of $\mathcal{N}$. Given any point $x \in \partial\Omega$, it is easy to check that

$$(\mathrm{Exp})_{*(x,0)}\Big(\frac{\partial}{\partial x^\alpha}\Big) = \frac{\partial}{\partial x^\alpha}, \ (\mathrm{Exp})_{*(x,0)}\Big(\frac{\partial}{\partial k}\Big) = \mathbf{n}(x).$$

Hence, it follows from (5.1) that $(\mathrm{Exp})_{*(x,0)}\colon T_{(x,0)}\mathcal{N} \to T_xM$ is an isomorphism, for all $x \in \partial\Omega$. The remaining part is the same as the proof of [21, Lemma 2.3] (*cf.* also [27, p. 200]), and we omit it here. ■

Remark 5.2 Lemma 5.1 guarantees that there exists a small positive number $\delta > 0$ such that $\mathrm{Exp}\colon \Omega_\delta \to \mathrm{Exp}(\Omega_\delta)$ is C^1-diffeomorphic, where

$$\Omega_\delta = \Big\{(x,k) \in \mathcal{N} : 0 \le k < \delta\Big\}.$$

It follows from [5, p. 126] that $\mathrm{Exp}\colon \Omega_\delta\backslash\partial\Omega \to \mathrm{Exp}(\Omega_\delta)\backslash\partial\Omega$ is C^∞-diffeomorphic.

Define $\rho\colon \overline{\Omega} \to \mathbb{R}$ by $\rho(x) = d(\partial\Omega, x)$. Let Ω_δ be defined as in Remark 5.2 and let $\mathscr{O} = \mathrm{Exp}(\Omega_\delta)$. From the above, we have the following lemma.

Lemma 5.3

(i) $\rho \in C^\infty(\mathscr{O}\backslash\partial\Omega)$.
(ii) *Given any* $p \in \partial\Omega$, $\lim_{x\to p}(d\rho)|_{\mathscr{O}\backslash\partial\Omega}(x) = g_{\mathbf{n}}(\mathbf{n}, \cdot)$.

Proof For each $q \in \mathscr{O}$, there exists a unique point $(x,k) \in \Omega_\delta$ such that $q = \mathrm{Exp}(x,k) = \exp_x(k\mathbf{n})$. Consider the geodesic $\gamma_{\mathbf{n}}(s) = \exp_x(s\mathbf{n})$, $s \in [0,k]$. By the first variation of arc length, one can check that $\gamma_{\mathbf{n}}(s)$ is the unique minimal unit speed geodesic from $\partial\Omega$ to q. Hence, $F(\mathrm{Exp}^{-1}(q)) = k = \rho(q)$. By Remark 5.2, $\rho \in C^\infty(\mathscr{O}\backslash\partial\Omega)$.

Given any continuous curve $\sigma(t)$, $0 \le t < \epsilon$, with $\sigma(0) = p$ and $\sigma((0,\epsilon)) \subset \mathscr{O}\backslash\partial\Omega$. Let $(x(t), k(t)) := \mathrm{Exp}^{-1}(\sigma(t))$ and $\mathbf{n}(t) := \mathbf{n}(x(t))$. From the above, for each fixed $t \in (0,\epsilon)$, $\gamma_{\mathbf{n}(t)}(s) = \exp_{x(t)}(s\mathbf{n}(t))$, $s \in [0, k(t)]$ is the unique minimal unit speed geodesic from $\partial\Omega$ to $\sigma(t)$. By the proofs of [34, Lemma 3.2.3] and Lemma 5.1, we have

$$\nabla\rho|_{\sigma(t)} = \dot\gamma_{\mathbf{n}(t)}\big(\rho\big(\sigma(t)\big)\big) \neq 0, \quad \forall\, t \in (0,\epsilon). \tag{5.2}$$

Using the triangle inequality, we deduce

$$d\big(p, \pi_1\big(\mathbf{n}(t)\big)\big) \le \rho\big(\sigma(t)\big) + L\big(\sigma(t)\big) \le 2L\big(\sigma(t)\big) \to 0, \quad \text{as } t \to 0^+, \tag{5.3}$$

where $L(\sigma(t))$ is the length of $\sigma([0,t])$. Equation (5.2) together with (5.3) and [5, Exercise 5.3.1(b)] then furnishes $\lim_{t\to 0^+} \nabla\rho|_{\sigma(t)} = \mathbf{n}(p)$, which implies

$$\lim_{x\to p}(d\rho)|_{\mathscr{O}\backslash\partial\Omega}(x) = g_{\mathbf{n}}(\mathbf{n}, \cdot). \qquad ■$$

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Lemma 5.4 *Let $\sigma(t)$, $0 \le t < \epsilon$, be a C^1-curve with $\sigma(0) \in \partial\Omega$ and $\sigma((0,\epsilon)) \subset \Omega$. Then*

$$\frac{d}{dt}\Big|_{t=0^+} \rho \circ \sigma(t) = g_{\mathbf{n}}\big(\mathbf{n}, \dot{\sigma}(0)\big).$$

Hence, $(\nabla\rho)|_{\partial\Omega} = \mathbf{n}$ and $\rho \in C^1(\mathscr{O})$.

Proof Set $\dot{\sigma}(0) = V + g_{\mathbf{n}}(\mathbf{n}, \dot{\sigma}(0))\mathbf{n}$, where $V \in T_{\sigma(0)}\partial\Omega$. Without loss of generality, we may assume that $\sigma((0,\epsilon)) \subset \mathscr{O}$. Thus, by Lemma 5.1, $\operatorname{Exp}^{-1}(\sigma(t)) = (x(t), k(t))$, where $x(t)$ is a C^1-curve in $\partial\Omega$ with $x(0) = \sigma(0)$ and $k(t)$ is a nonnegative C^1-function with $k(0) = 0$. Let (x^α, k) be a (local) coordinate system of $\mathcal{N}$, where $\{x^\alpha\}$ is a (local) coordinate system of Ω. From the proof of Lemma 5.1, we have

$$\begin{aligned} V + g_{\mathbf{n}}\big(\mathbf{n}, \dot{\sigma}(0)\big)\,\mathbf{n} = \dot{\sigma}(0) &= (\operatorname{Exp})_{*(\sigma(0),0)}\Big(\dot{x}^\alpha(0)\frac{\partial}{\partial x^\alpha} + \dot{k}(0)\frac{\partial}{\partial k}\Big) \\ &= \dot{x}^\alpha(0)\frac{\partial}{\partial x^\alpha} + \dot{k}(0)\mathbf{n}\big(\sigma(0)\big). \end{aligned}$$

Hence, $\dot{k}(0) = g_{\mathbf{n}}(\mathbf{n}, \dot{\sigma}(0))$.

By the proof of Lemma 5.3, we have

$$\frac{d}{dt}\Big|_{t=0^+} \rho \circ \sigma(t) = \lim_{t\to 0^+} \frac{\rho(\sigma(t))}{t} = \lim_{t\to 0^+}\frac{k(t)}{t} = \dot{k}(0).$$

Hence, for each $x \in \partial\Omega$, $d\rho|_x = g_{\mathbf{n}}(\mathbf{n}, \cdot\,)$, and therefore $\nabla\rho|_x = \mathcal{L}^{-1}(d\rho|_x) = \mathbf{n}(x)$. Now, it follows from Lemma 5.3 that $\rho \in C^1(\mathscr{O})$. ∎

As an immediate consequence of Lemma 5.4, we have the following corollary.

Corollary 5.5 *Let $\sigma(t)$, $0 \le t \le 1$, be a C^1-curve such that $\sigma([0,1)) \subset \Omega$ and $\sigma(1) \in \partial\Omega$. Then $g_{\mathbf{n}}(\mathbf{n}, \dot{\sigma}(1)) \le 0$.*

Proof Without loss of generality, we assume that $\sigma([0,1)) \subset \mathscr{O}$. From Lemma 5.4, we have

$$0 \ge \lim_{t\to 1^-} \frac{\rho\circ\sigma(t) - \rho\circ\sigma(1)}{t-1} = \frac{d}{dt}\Big|_{t=1^-} \rho\circ\sigma(t) = g_{\mathbf{n}}\big(\mathbf{n}, \dot{\sigma}(1)\big). \qquad ∎$$

Let $S^+\partial\Omega$ be the collection of inward pointing unit vectors along $\partial\Omega$, *i.e.*,

$$\begin{aligned} S^+\partial\Omega :=& \{y \in SM|_{\partial\Omega} : y = V + k\mathbf{n},\ V \in T\partial\Omega,\ k > 0\} \\ =& \{y \in SM|_{\partial\Omega} : g_{\mathbf{n}}(\mathbf{n}, y) > 0\}. \end{aligned}$$

Using an argument similar to that in [6, p. 286], one can show that $S^+\partial\Omega$ is a submanifold of SM.

Define $\mathcal{Z} := \{y \in S\partial\Omega : \exists t > 0 \text{ such that } \gamma_y((0,t)) \subset \Omega\}$. For each $y \in S\Omega \cup S^+\partial\Omega \cup \mathcal{Z}$, we set

$$\widehat{t}(y) := \sup\big\{T > 0 : \gamma_y(t) \in \Omega,\ \forall t \in (0,T)\big\};$$

that is, when $y \in S\Omega$ and $\widehat{t}(y)$ is finite, $\gamma_y(\widehat{t}(y))$ will be the first point on the geodesic to hit the boundary $\partial\Omega$.

Lemma 5.6 *$\widehat{t}(y)$ is low semi-continuous on $S\Omega \cup S^+\partial\Omega$.*

Proof Suppose that $\widehat{t}(y)$ is not low semi-continuous at some point $y_0 \in S\Omega \cup S^+\partial\Omega$.

Case I: $\widehat{t}(y_0) < +\infty$. In this case, there exists $\delta > 0$ such that for any neighborhood $\mathcal{U}$ of y_0, there is $y \in \mathcal{U}$ with $\widehat{t}(y) < \widehat{t}(y_0) - \delta$. Hence, we obtain a sequence $\{y_n\} \subset S\Omega \cup S^+\partial\Omega$ such that $\lim_{n\to\infty} y_n = y_0$, and $0 < \widehat{t}(y_n) < \widehat{t}(y_0) - \delta$, for all n. Set

$$\gamma_n(s) := \exp_{\pi_1(y_n)}(s \cdot \widehat{t}(y_n) y_n), \quad s \in [0,1].$$

By the Arzela–Ascoli theorem (see [11, Theorem 2.5.14]), there exists a uniformly convergent subsequence of $\{\gamma_n\}$. Without loss of generality, we assume $\{\gamma_n\}$ converges uniformly to $\gamma(s) \colon [0,1] \to \overline{\Omega}$. It is easy to check that $\gamma(s)$ is also a geodesic. Note that $\gamma_n(1) \in \partial\Omega$ and $\lim_{n\to\infty} \gamma_n(1) = \gamma(1)$. Since $\partial\Omega$ is compact, $\gamma(1) \in \partial\Omega$. Let $\mathcal{T}$ denote the length of γ, *i.e.*, $\mathcal{T} = L(\gamma)$. Hence,

$$\mathcal{T} = \lim_{n\to\infty} L(\gamma_n) = \lim_{n\to\infty} \widehat{t}(y_n) \le \widehat{t}(y_0) - \delta. \tag{5.4}$$

We claim that $\mathcal{T} > 0$. If not, then $\lim_{n\to\infty} \widehat{t}(y_n) = 0$ and $\gamma(0) = \gamma(1) \in \partial\Omega$. Thus, $y_0 \in S^+\partial\Omega$. However, from Corollary 5.5, we have

$$0 \ge g_{\mathbf{n}}\Big(\mathbf{n}, \frac{\dot\gamma_n(1)}{F(\dot\gamma_n(1))}\Big) = g_{\mathbf{n}}\big(\mathbf{n}, (\exp_{\pi_1(y_n)})_{*\widehat{t}(y_n)\cdot y_n} y_n\big), \text{ for all } n,$$

which implies that $g_{\mathbf{n}}(\mathbf{n}, y_0) \le 0$. This contradicts the definition of $S^+\partial\Omega$.

Since $\mathcal{T} > 0$,

$$\dot\gamma(0) = \lim_{n\to\infty} \dot\gamma_n(0) = \lim_{n\to\infty} \widehat{t}(y_n) y_n = \mathcal{T} y_0 \neq 0.$$

Therefore, $\gamma(\frac{s}{\mathcal{T}}) = \gamma_{y_0}(s)$. In particular, $\gamma_{y_0}(\mathcal{T}) = \gamma(1) \in \partial\Omega$, which implies that $\mathcal{T} \ge \widehat{t}(y_0)$. From (5.4), we get a contradiction.

Case II: $\widehat{t}(y_0) = +\infty$. In this case, there exist a constant $\mathcal{K} > 0$ and a sequence $\{y_n\} \subset S\Omega \cup S^+\partial\Omega$ such that $y_n \to y_0$ and $\widehat{t}(y_n) < \mathcal{K}$, for all n. The rest of the proof is similar to Case I, and we omit it. ■

Since (M, F) is complete, we can define a map

$$\Psi \colon \mathbb{R} \times S^+\partial\Omega \longrightarrow SM$$

by $\Psi(t, y) = \varphi_t(y)$. For each $y \in S\Omega \cup S^+\partial\Omega \cup \mathcal{Z}$, let $l(y) := \min\{i_y, \widehat{t}(y)\}$. Set

$$\begin{aligned} U_\Omega^- &:= \left\{ y \in S\Omega : \widehat{t}(-y) < i_{-y} \right\}, \\ N &:= \left\{ (t, y) : y \in S^+\partial\Omega, t \in (0, l(y)) \right\}, \\ U_{\mathcal{Z}} &:= \left\{ \varphi_t(y) : y \in \mathcal{Z}, t \in (0, l(y)) \right\}. \end{aligned}$$

Then we have the following lemma.

Lemma 5.7 *$\Psi|_N\colon N \to U_\Omega^- \backslash U_{\mathcal{Z}}$ is a one-one map.*

Proof Let $N_{\mathcal{Z}} := \{(t, y) : y \in \mathcal{Z}, t \in (0, l(y))\}$. We extend Ψ to a map

$$\Phi\colon \mathbb{R} \times (S^+\partial\Omega \cup S\partial\Omega) \longrightarrow SM$$

such that $\Phi(t, y) = \varphi_t(y)$. Clearly, $U_{\mathcal{Z}} = \Phi(N_{\mathcal{Z}})$ and $\Phi|_N = \Psi|_N$. We just need to prove that $\Phi|_{N\cup N_{\mathcal{Z}}}\colon N \cup N_{\mathcal{Z}} \to U_\Omega^-$ is a one-one map.

Since $\overline{\Omega}$ is compact, for each $X \in U_\Omega^-$, $\widehat{t}(-X) < i_{-X} < +\infty$. Let $Y := -\dot{\gamma}_{-X}(\widehat{t}(-X))$. Corollary 5.5 implies $Y \in S^+\partial\Omega \cup \mathcal{Z}$. Set $p = \pi_1(X)$ and $q = \pi_1(Y)$, where $\pi_1\colon SM \to M$ is the natural projection. From the definition of U_Ω^-, we have $d(p, q) = \widehat{t}(-X) = L(\gamma_{-X}|_{[0,\widehat{t}(-X)]})$. Since F is reversible, $L(\gamma_Y|_{[0,\widehat{t}(-X)]}) = \widehat{t}(-X) = d(q, p)$. Hence, $i_Y \geq \widehat{t}(-X)$ and γ_Y is the minimal geodesic from q to p with $\dot{\gamma}_Y(\widehat{t}(-X)) = X$.

We now claim that $\widehat{t}(-X) < i_Y$. In fact, if $i_Y = \widehat{t}(-X)$, then p is the cut point of q along γ_Y. Since F is reversible, p is not the first conjugate point of q along γ_Y. By [5, Corollary 8.2.2], there exists an another distinct geodesic ς of the same length $\widehat{t}(-X)$ from q to p. This contradicts $\widehat{t}(-X) < i_{-X}$, since F is reversible.

From the above, we have shown that for each $X \in U_\Omega^-$, there exist $Y \in S^+\partial\Omega \cup \mathcal{Z}$ and $t \in (0, l(Y))$ such that $\varphi_t(Y) = X$. Hence, $\Phi|_{N\cup N_{\mathcal{Z}}}\colon N \cup N_{\mathcal{Z}} \to U_\Omega^-$ is a subjective map. It is not hard to see that $\Phi|_{N\cup N_{\mathcal{Z}}}$ is also an injective map. Therefore, we conclude that $\Phi|_{N\cup N_{\mathcal{Z}}}$ is a one-one map. ∎

Given any volume form $d\mu$ on M, the induced volume form on $\partial\Omega$ by $d\mu$ is defined by $dA := i^*(\mathbf{n}\rfloor d\mu)$, where $i\colon \partial\Omega \hookrightarrow M$ is the inclusion map (see [34, pp. 31–32]). Define a $(2n-1)$-form β on $\mathbb{R} \times S^+\partial\Omega$ by $\beta|_{(t,y)} = dt \wedge dA(\pi_1(y)) \wedge d\nu_{\pi_1(y)}(y)$. Hence, there exists $\eta \in C^\infty(\mathbb{R} \times S^+\partial\Omega)$ such that $(\Psi)^*(dV_{SM}) = \eta \cdot \beta$.

Lemma 5.8 *For any $(t, y) \in \mathbb{R} \times S^+\partial\Omega$, we have $\eta(t, y) = \eta(0, y)$.*

Proof Let ς_t denote the transformation of $\mathbb{R} \times S^+\partial\Omega$ into itself, *i.e.*, $\varsigma_t(s, y) = (s + t, y)$. Clearly, $\varphi_t \circ \Psi = \Psi \circ \varsigma_t$. Using this and Theorem 2.1, we have

$$\begin{aligned}\eta \cdot \beta &= \Psi^*(dV_{SM}) = \Psi^*(\varphi_t^*(dV_{SM})) = (\varphi_t \circ \Psi)^*(dV_{SM}) = (\Psi \circ \varsigma_t)^*(dV_{SM})\\ &= \varsigma_t^*(\Psi^* dV_{SM}) = \varsigma_t^*(\eta \cdot \beta) = \varsigma_t^*(\eta)\varsigma_t^*(\beta).\end{aligned}$$

Since β is invariant under ς_t, it follows that $\eta \cdot \beta = \varsigma_t^*(\eta) \cdot \beta$. Hence, $\eta(0, y) = \eta(t, y)$. ∎

Then we have a generalization of Santaló's formula [31].

Theorem 5.9 *For all integrable function f on $S\Omega$, we have*

$$\int_{\mathcal{V}_\Omega^-} f dV_{SM} = \int_{S^+\partial\Omega} e^{\tau(y)} g_{\mathbf{n}}(\mathbf{n}, y) d\chi(y) \int_0^{l(y)} f\big(\varphi_t(y)\big)\, dt,$$

where $\mathcal{V}_\Omega^- := \{y \in S\Omega : \widehat{t}(-y) \leq i_{-y}\}$ and $d\chi(y) = dA(\pi_1(y)) d\nu_{\pi_1(y)}(y)$.

Proof Given any $y \in S^+\partial\Omega$, identify $T_y(S^+\partial\Omega)$ with its image in $T_{(0,y)}(\mathbb{R} \times S^+\partial\Omega)$. Since $\varphi_0 = \mathrm{id}$, we have $\Psi_{*(0,y)}(X) = X, \forall X \in T_y(S^+\partial\Omega)$. This implies that

$$\Psi^*\big(d\chi(y)\big) \equiv d\chi|_{(0,y)} \pmod{dt}. \tag{5.5}$$

Let $\rho(x) = d(\partial\Omega, x)$. For each $X \in T_y(S^+\partial\Omega)$, there exists a curve $\xi\colon (-\varepsilon, +\varepsilon) \to S^+\partial\Omega$ with $\xi(0) = y$ and $\dot{\xi}(0) = X$. Clearly, $\pi_1(\xi(s)) \subset \partial\Omega$, which implies that $\rho(\pi_1(\xi(s))) = 0$. Hence,

$$\big\langle \pi_{1*}(\Psi_{*(0,y)}X), d\rho\big\rangle = \langle \pi_{1*}X, d\rho\rangle = \frac{d}{ds}\Big|_{s=0}\rho\big(\pi_1\big(\xi(s)\big)\big) = 0.$$

Thus, from Lemma 5.4, we deduce

$$\begin{aligned}\big[\Psi^*\big(\pi_1^*(d\rho)\big)\big]|_{(0,y)} &= \Big\langle \frac{\partial}{\partial t}, \Psi^*\big(\pi_1^*(d\rho)\big)\Big\rangle_{(0,y)} dt \\ &= \Big(\frac{d}{dt}\Big|_{t=0^+}\rho \circ \gamma_y(t)\Big)\, dt = g_{\mathbf{n}}(\mathbf{n}, y)dt.\end{aligned} \tag{5.6}$$

By the co-area formula (see [34, Theorem 3.3.1]), (5.5), and (5.6), we have

$$\begin{aligned}[\eta dt \wedge d\chi]|_{(0,y)} &= \Psi^*\big(dV_{SM}(y)\big) = \Psi^*\big[e^{\tau(y)}\pi_1^*(d\mu)(y) \wedge d\nu_{\pi_1(y)}(y)\big] \\ &= \Psi^*\big[e^{\tau(y)}\pi_1^*(d\rho \wedge dA)(y) \wedge d\nu_{\pi_1(y)}(y)\big] \\ &= \Psi^*\big[e^{\tau(y)}\pi_1^*(d\rho)(y) \wedge d\chi(y)\big] \\ &= \big[e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n}, y)dt \wedge d\chi\big]|_{(0,y)};\end{aligned}$$

that is, $\eta(0, y) = e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n}, y)$. It follows from Lemma 5.8 that

$$\Psi^*\big(dV_{SM}\big(\varphi_t(y)\big)\big) = e^{\tau(y)}g_{\mathbf{n}}(n, y)dt \wedge d\chi, \quad \forall (t, y) \in \mathbb{R} \times S^+\partial\Omega, \tag{5.7}$$

which implies that Ψ is of maximal rank. Hence, from Lemma 5.7, we deduce that $\Psi|_N$ is a diffeomorphism.

Clearly, $U_{\mathscr{Z}} = U_\Omega^- \backslash \Psi(N)$ has measure zero with respect to dV_{SM}. Let $\mathscr{N} := \{y \in S\Omega : \widehat{t}(-y) = i_{-y}\}$. Thus, $\mathcal{V}_\Omega^- = U_\Omega^- \cup \mathscr{N}$. By an argument similar to the proof of Lemma 5.7, one has $\mathscr{N} \subset \{\varphi_{l(y)}y : y \in S^+\partial\Omega \cup \mathcal{Z}, l(y) = i_y\}$, which implies that $\mathscr{N}$ has measure zero with respect to dV_{SM}. By (5.7), we have

$$\begin{aligned}\int_{\mathcal{V}_\Omega^-} f dV_{SM} &= \int_{U_\Omega^-} f dV_{SM} = \int_{\Psi(N)} f dV_{SM} = \int_N \Psi^*(f dV_{SM}) \\ &= \int_{S^+\partial\Omega} e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n}, y)d\chi(y)\int_0^{l(y)} f(\varphi_t(y))dt. \qquad \blacksquare\end{aligned}$$

In the Riemannian case, $e^{\tau(y)} = 1$ and $g_{\mathbf{n}} = g$. Therefore, Theorem 5.9 implies Santaló's formula [13, 31].

6 A Croke Type Isoperimetric Inequality

Let (M, F) be a reversible complete Finsler n-manifold and let $\Omega \subset M$ be a relatively compact domain with smooth boundary $\partial\Omega$. Denote by $\mathbf{n}$ the unit inward normal vector field along $\partial\Omega$. In this section, $d\mu$ is either the Busemann–Hausdorff volume form or the Holmes–Thompson volume form. Let $\Xi := \sup_{y\in S\overline{\Omega}} \tau(y)$, where τ is the distortion of $d\mu$. Given any point $p \in \Omega$, define

$$U_p := \pi_1^{-1}|_{\mathcal{V}_\Omega^-}(p) \subset S_pM, \quad \omega_p := \frac{1}{c_{n-1}} \int_{U_p} e^{\tau(y)} d\nu_p(y), \quad \text{and} \quad \omega := \inf_{p\in\Omega} \omega_p.$$

For each point $p \in \partial\Omega$, define

$$\mathcal{M}_p := \max\Big(\sup_{y\in S_pM} \|y\|^{-1}, \sup_{y\in S_pM} \|y\| \Big), \quad \mathcal{M} = \sup_{p\in\partial\Omega} \mathcal{M}_p,$$

where $\|y\| := \sqrt{g_{\mathbf{n}}(y, y)}$. Since $\overline{\Omega}$ is compact, $1 \leq \mathcal{M} < \infty$. It is not hard to see that

$$\mathcal{M}_p^{-1} F(y) \leq \|y\| \leq \mathcal{M}_p F(y), \quad \forall y \in T_pM, \forall p \in \partial\Omega.$$

Using Stokes' formula, we have the following estimate.

Lemma 6.1 *For each point $p \in \partial\Omega$, set $S_p^+\partial\Omega := \{y \in S_pM : g_{\mathbf{n}}(\mathbf{n}, y) > 0\}$. Then*

$$\int_{S_p^+\partial\Omega} g_{\mathbf{n}}(\mathbf{n}, y) e^{\tau(y)} d\nu_p(y) \leq e^{2\Xi} \frac{c_{n-2}}{n-1} \mathcal{M}_p^{2n+1},$$

with equality if and only if $F(p, \cdot)$ is a Euclidean norm.

Proof Choose a $g_{\mathbf{n}}$-orthnormal basis $\{e_i\}$ of T_pM such that $e_n = \mathbf{n}$. Let $\{y^i\}$ be the corresponding coordinates. Define

$$B_p := \{y \in T_pM : F(y) < 1\}, \quad B_p^+ := \{y \in B_p : y^n > 0\},$$

$$B_{p,r} := \{y \in B_p : y^n = r\}, \quad \mathbb{B}_p(r) := \big\{ y \in T_pM : \|y\| < r \big\} \cong \mathbb{B}^n(r),$$

$$\mathbb{B}_{p,r}(s) := \Big\{ y \in T_pM : y^n = r, \Big\| \sum_{\alpha=1}^{n-1} y^\alpha e_\alpha \Big\| < s \Big\} \cong \mathbb{B}^{n-1}(s).$$

Clearly, $\partial B_p^+ = B_{p,0} \cup S_p^+\partial\Omega$, $\mathbb{B}_p(\mathcal{M}_p^{-1}) \subset B_p$, and $B_{p,r} \subset \mathbb{B}_{p,r}(\sqrt{\mathcal{M}_p^2 - r^2})$. In particular, for each $y \in B_p^+$, $0 < y^n = g_{\mathbf{n}}(\mathbf{n}, y) \leq F(y)F(\mathbf{n}) \leq 1$. Let

$$\varpi := y^n \Big(\sum_i (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n \Big).$$

From Stokes' formula, we have

$$
\begin{aligned}
(6.1)\qquad \int_{S_p^+\partial\Omega} \varpi &= \int_{\partial B_p^+} \varpi = \int_{B_p^+} d\varpi = (n+1)\int_{B_p^+} y^n dy^1 \wedge \cdots \wedge dy^n \\
&= (n+1)\int_0^1 \mathrm{Vol}_{\mathbb{R}^{n-1}}(B_{p,y_n}) y^n dy^n \\
&\le (n+1)\int_0^{\mathcal{M}_p} \mathrm{Vol}_{\mathbb{R}^{n-1}}(B_{p,y_n}) y^n dy^n \\
&\le (n+1)\int_0^{\mathcal{M}_p} \mathrm{Vol}_{\mathbb{R}^{n-1}}\Big(\mathbb{B}_{p,y_n}\Big(\sqrt{\mathcal{M}_p^2-(y^n)^2}\Big)\Big) y^n dy^n \\
&= \frac{c_{n-2}(n+1)}{n-1}\int_0^{\mathcal{M}_p}\Big(\sqrt{\mathcal{M}_p^2-(y^n)^2}\Big)^{n-1} y^n dy^n = \frac{c_{n-2}}{n-1}\mathcal{M}_p^{n+1},
\end{aligned}
$$

with equality if and only if $\mathcal{M}_p = 1$, *i.e.*, $F(p,\cdot)$ is a Euclidean norm.

Let $\{\vartheta^i\}$ be the dual basis of $\{e_i\}$ and let $d\mu(p) = \sigma(p)\vartheta^1 \wedge \cdots \wedge \vartheta^n$. Since F is reversible,

$$
(6.2)\qquad \sigma_{HT}(p) \le \sigma_{BH}(p) = \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}(B_p)} \le \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}(\mathbb{B}_p(\mathcal{M}_p^{-1}))} = \mathcal{M}_p^n.
$$

Using (6.1) and (6.2), we have

$$
\begin{aligned}
&\int_{S_p^+\partial\Omega} g_{\mathbf{n}}(\mathbf{n}, y)e^{\tau(y)} d\nu_p(y) \\
&= \int_{S_p^+\partial\Omega} g_{\mathbf{n}}(\mathbf{n}, y)e^{2\tau(y)}e^{-\tau(y)} d\nu_p(y) \le e^{2\Xi}\int_{S_p^+\partial\Omega} g_{\mathbf{n}}(\mathbf{n}, y)e^{-\tau(y)} d\nu_p(y) \\
&= e^{2\Xi}\sigma(p)\int_{S_p^+\partial\Omega} y^n\Big(\sum_i (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n\Big) \\
&\le e^{2\Xi}\mathcal{M}_p^n \int_{S_p^+\partial\Omega} \varpi \le e^{2\Xi}\frac{c_{n-2}}{n-1}\mathcal{M}_p^{2n+1}. \qquad \blacksquare
\end{aligned}
$$

By Theorem 4.4, Lemma 5.6, Theorem 5.9, and Lemma 6.1, we have the following theorem.

Theorem 6.2 *Let Ω be a relatively compact domain in a reversible Finsler n-manifold (M, F), with $\partial\Omega \in C^\infty$. Let $d\mu$ denote either the Busemann–Hausdorff volume form or the Holmes–Thompson volume form and let ω, $\mathcal{M}$ and Ξ be defined as above.*

(i) *We have*

$$
(6.3)\qquad \frac{A(\partial\Omega)}{\mu(\Omega)} \ge \frac{(n-1)c_{n-1}\omega}{c_{n-2}e^{2\Xi}\mathcal{M}^{2n+1}d(\Omega)},
$$

where $d(\Omega)$ denotes the diameter of Ω. The equality holds in (6.3) if $(\overline{\Omega}, F|_{\overline{\Omega}})$ is a standard hemisphere of a constant sectional curvature sphere.

(ii) *We have*

$$\frac{A(\partial\Omega)}{\mu(\Omega)^{1-1/n}} \geq \frac{c_{n-1}}{\mathcal{M}^{(2n+1)}(c_n/2)^{1-1/n}}\Big(\frac{\omega}{e^{2\Xi}}\Big)^{1+1/n}, \tag{6.4}$$

with equality if and only if $(\overline{\Omega}, F|_{\overline{\Omega}})$ *is a hemisphere of a constant sectional curvature sphere.*

Proof (i) From Theorem 5.9, we have

$$\begin{aligned} c_{n-1}\omega\mu(\Omega) &\leq c_{n-1}\int_\Omega \omega_x d\mu(x) = V_{SM}(\mathcal{V}_\Omega^-) = \int_{S^+\partial\Omega} l(y)e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n},y)d\chi(y) \\ &\leq d(\Omega)\int_{\partial\Omega} dA(x)\int_{S^+_x\partial\Omega} e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n},y)d\nu_x(y) \\ &\leq d(\Omega)A(\partial\Omega)e^{2\Xi}\mathcal{M}^{2n+1}\frac{c_{n-2}}{n-1}. \end{aligned}$$

(ii) Given any point $x \in \Omega$, let (r, y) be the polar coordinate system at x. Recall that $d\mu|_{(r,y)} = \widehat{\sigma}_x(r,y)dr \wedge d\nu_x(y)$, where $\widehat{\sigma}_x(r,y) = e^{-\tau(\varphi_r(y))}\mathscr{F}(r,y)$. Clearly,

$$\mu(\Omega) = \int_{S_xM} d\nu_x(y)\int_0^{l(y)} \widehat{\sigma}_x(r,y)dr.$$

For each $y \in S_x\Omega$, $l(\varphi_t(y)) \geq l(y) - t$. Using Hölder's inequality, Theorems 4.4 and 5.9 and Lemma 6.1, we have

$$\begin{aligned} \mu(\Omega)^2 &= \int_\Omega d\mu(x)\int_{S_xM} d\nu_x(y)\int_0^{l(y)} \widehat{\sigma}_x(r,y)dr \\ &= \int_{S\Omega} dV_{SM}(y)\int_0^{l(y)} e^{-\tau(y)}\widehat{\sigma}_x(r,y)dr \\ &\geq \int_{\mathcal{V}_\Omega^-} dV_{SM}(y)\int_0^{l(y)} e^{-\tau(y)}\widehat{\sigma}_x(r,y)dr \qquad (6.5) \\ &= \int_{S^+\partial\Omega} e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n},y)d\chi(y)\int_0^{l(y)} dt\int_0^{l(\varphi_t(y))} e^{-\tau(\varphi_t(y))}\widehat{\sigma}_x(r,\varphi_t(y))dr \qquad (6.6) \\ &= \int_{S^+\partial\Omega} e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n},y)d\chi(y)\int_0^{l(y)} dt\int_0^{l(\varphi_t(y))} e^{-\tau(\varphi_t(y))-\tau(\varphi_{t+r}(y))}\mathscr{F}(r,\varphi_t(y))dr \\ &\geq \int_{S^+\partial\Omega} e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n},y)d\chi(y)\int_0^{l(y)} dt\int_0^{l(y)-t} e^{-\tau(\varphi_t(y))-\tau(\varphi_{t+r}(y))}\mathscr{F}(r,\varphi_t(y))dr \\ &\geq e^{-2\Xi}\int_{S^+\partial\Omega} e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n},y)d\chi(y)\int_0^{l(y)} dt\int_0^{l(y)-t} \mathscr{F}(r,\varphi_t(y))dr \qquad (6.7) \\ &\geq \frac{c_n}{2e^{2\Xi}c_{n-1}\pi^n}\int_{S^+\partial\Omega} l(y)^{n+1}e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n},y)d\chi(y) \qquad (6.8) \end{aligned}$$

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$$(6.9)\qquad \geq \frac{c_n}{2e^{2\Xi}c_{n-1}\pi^n}\Big(\int_{S^+\partial\Omega} l(y)e^{\tau(y)}g_{\mathbf{n}}(\mathbf{n},y)d\chi(y)\Big)^{n+1}\Big(\int_{S^+\partial\Omega} g_{\mathbf{n}}(\mathbf{n},y)e^{\tau(y)}d\chi(y)\Big)^{-n}$$

$$(6.10)\qquad \geq \frac{c_n}{2e^{2\Xi}c_{n-1}\pi^n}\big(V_{SM}(\mathcal{V}_\Omega^-)\big)^{n+1}\Big(e^{2\Xi}\frac{c_{n-2}}{n-1}\mathcal{M}^{2n+1}A(\partial\Omega)\Big)^{-n}$$

$$= \frac{c_n}{2c_{n-1}e^{2(n+1)\Xi}\mathcal{M}^{n(2n+1)}}\Big(\frac{n-1}{\pi c_{n-2}}\Big)^n\frac{V_{SM}(\mathcal{V}_\Omega^-)^{n+1}}{A(\partial\Omega)^n}$$

$$\geq \frac{c_n\omega^{n+1}}{2e^{2(n+1)\Xi}\mathcal{M}^{n(2n+1)}}\Big(\frac{(n-1)c_{n-1}}{\pi c_{n-2}}\Big)^n\frac{\mu(\Omega)^{n+1}}{A(\partial\Omega)^n}$$

$$= \frac{\omega^{n+1}}{e^{2(n+1)\Xi}\mathcal{M}^{n(2n+1)}}\frac{c_{n-1}^n}{(c_n/2)^{n-1}}\frac{\mu(\Omega)^{n+1}}{A(\partial\Omega)^n},$$

the equality (6.6) is Theorem 5.9; the inequality (6.8) is Theorem 4.4; the inequality (6.9) is Höler's inequality; the inequality (6.10) is Lemma 6.1. Therefore,

$$\frac{A(\partial\Omega)}{\mu(\Omega)^{1-1/n}} \geq \frac{c_{n-1}}{\mathcal{M}^{(2n+1)}(c_n/2)^{1-1/n}}\Big(\frac{\omega}{e^{2\Xi}}\Big)^{1+1/n}.$$

If equality holds in (6.4), then we have equalities in (6.5)–(6.10). From (6.10) and Lemma 6.1, it follows that $F(p,\cdot)$ is a Euclidean norm, for each $p\in\partial\Omega$. Hence, $\mathcal{M}=1$ and $e^{\tau(y)}=1$ for all $y\in S^+\partial\Omega$. Thus, by (6.10), we have

$$(6.11)\qquad \frac{A(\partial\Omega)}{\mu(\Omega)^{1-1/n}} = \frac{c_{n-1}}{(c_n/2)^{1-1/n}e^{2\Xi/n}}\omega^{1+1/n}.$$

Equation (6.5) yields $\mathcal{V}_\Omega^- = S\Omega$, which implies that for each $y\in S\Omega$, $\widehat{t}(-y)\leq i_{-y}$. Since F is reversible, $\widehat{t}(y)\leq i_y$, for all $y\in S\Omega$. From Lemma 5.6, we have

$$\widehat{t}(y_0)\leq \liminf_{y\to y_0}\widehat{t}(y)\leq \liminf_{y\to y_0} i_y = i_{y_0},\quad \forall y_0\in S^+\partial\Omega.$$

Equality in (6.9), the Hölder inequality, implies that $l(y)$ is constant, say, equal to l, on all of $S^+\partial\Omega$. From above, $\widehat{t}(y)=l$, for all $y\in S^+\partial\Omega$. Equality in (6.7) implies that $e^{-\tau(\varphi_t(y))}=e^{-\Xi}$, for all $y\in S^+\partial\Omega$, $t\in(0,l)$. By the proof of Theorem 5.9, we have

$$V_{SM}\big(S\Omega\backslash\Psi(N)\big) = V_{SM}\big(\mathcal{V}_\Omega^-\backslash\Psi(N)\big) = 0.$$

Hence, $e^{-\tau(y)}=e^{-\Xi}$, for all $y\in S\Omega$, which implies that $F|_{\overline{\Omega}}$ is a Riemannian metric and $e^{-\Xi}=1$. Then (6.11) becomes

$$\frac{A(\partial\Omega)}{\mu(\Omega)^{1-1/n}} = \frac{c_{n-1}}{(c_n/2)^{1-1/n}}\omega^{1+1/n}.$$

Equality in (6.8), Theorem 4.4, implies that Ω has constant sectional curvature equal to $(\pi/l)^2$. Thus, for all $y\in S^+\partial\Omega$, $\widehat{t}(y)=l=i_y$, *i.e.*, Ω is a hemisphere. ∎

If F is Riemannian, then $\Xi = 0$ and $\mathcal{M} = 1$. Therefore, the theorem above implies Croke's isoperimetric inequality [17].

Let (M, F) be a compact reversible Finsler n-manifold without boundary. Given $p \in M$, the (open) metric ball of radius r centered at p is denoted by $B(p, r)$, and $S(p, r) := \partial B(p, r)$. Define

$$\omega_{HT}(x) := \frac{1}{c_{n-1}} \int_{U_x} e^{\tau_{HT}(y)} d\nu_x(y), \quad \omega_{HT} := \inf_{x \in B(p,r)} \omega_{HT}(x),$$
$$r_p := \inf\big\{ r : \big(B(p, r), F|_{B(p,r)}\big) \text{ has } \omega_{HT} < 1 \big\}.$$

Clearly, $\omega_{HT} = 1$ if and only if $U_x = S_x M$, $\forall x \in B(p, r)$, which implies that the cut locus of any interior point in $B(p, r)$ lies outside $B(p, r)$. Therefore, $r_p \geq \mathfrak{i}_M/2$. Then we have the following proposition, which implies [17, Proposition 14].

Proposition 6.3 *Let $(M, F, d\mu)$ be a compact reversible Finsler n-manifold without boundary, where $d\mu$ is either the Busemann–Hausdorff volume form or the Holmes–Thompson volume form. For any $p \in M$ and $0 < r \leq r_p$ (or $r \leq \mathfrak{i}_M/2$), we have*

$$\mu\big(B(p, r)\big) \geq \frac{C^n(n, \Lambda)}{n^n} r^n, \quad A\big(S(p, r)\big) \geq \frac{C^n(n, \Lambda)}{n^{n-1}} r^{n-1},$$

where Λ is the uniform constant of (M, F) and $C(n, \Lambda)$ is a constant that depends on n and Λ.

Proof Let (x^i, y^i) be local coordinates on TM. By [38], we have

$$\frac{\max\limits_{y \in S_x M} \det g_{ij}(x, y)}{\min\limits_{y \in S_x M} \det g_{ij}(x, y)} \leq \Lambda^n, \frac{c_{n-1}}{\mathrm{Vol}_{\dot{g}_x}(S_x M)} = \frac{c_{n-1}}{\int_{S_x M} d\nu_x(y)} \leq \Lambda^{n/2},$$

for all $x \in M$. Hence, for each $y \in SM$,

$$e^{\tau_{HT}(y)} = \frac{\sqrt{\det g_{ij}(y)}}{\sigma_{HT}(\pi_1(y))} \leq \Lambda^n, \tag{6.12}$$

where $\pi_1 \colon SM \to M$ is the natural projection.

Since F is reversible, $d\mu_{BH} \geq d\mu_{HT}$ (see Section 2 or [5, 19]). Let $\rho(x) := d(p, x)$. Then ρ is differentiable almost everywhere and $dA := i^*(\nabla\rho \rfloor d\mu)$, where $i \colon S(p, r) \hookrightarrow M$ is the inclusion map (see [34]). Hence, $dA_{BH} \geq dA_{HT}$. Therefore, $\mu(B(p, r)) \geq \mu_{HT}(B(p, r))$ and $A(S(p, r)) \geq A_{HT}(S(p, r))$.

Let $C(n, \Lambda) := \frac{c_{n-1}}{\Lambda^{(6n+5)/2}(c_n/2)^{1-1/n}}$. From Theorem 6.2 and (6.12), we have

$$\frac{\frac{d}{dt}\mu_{HT}(B(p, t))}{\mu_{HT}(B(p, t))^{1-1/n}} = \frac{A_{HT}(S(p, t))}{\mu_{HT}(B(p, t))^{1-1/n}} \geq C(n, \Lambda),$$

which implies that

$$\mu_{HT}\big(B(p, r)\big) \geq \frac{C^n(n, \Lambda)}{n^n} r^n, \quad 0 < r \leq r_p.$$

Thus, we get the first statement. Using Theorem 6.2 again, we have

$$A_{HT}(S(p,r)) \geq C(n,\Lambda)\mu_{HT}(B(p,r))^{1-1/n} \geq \frac{C^n(n,\Lambda)}{n^{n-1}} r^{n-1},\ 0 < r \leq r_p. \qquad \blacksquare$$

By Theorem 3.4, we have the following proposition, which is a Finslerian version of [17, Lemma 3].

Proposition 6.4 *Let $(M, F, d\mu)$ be a compact reversible Finsler n-manifold with*

$$\mathbf{Ric} \geq (n-1)k, \quad \tau \geq \hbar,$$

where $d\mu$ is either the Busemann–Hausdorff volume form or the Holmes–Thompson volume form. If Γ is any $(n-1)$-dimensional compact submanifold of M dividing M into two open submanifolds M_1 and M_2, such that ∂M_1=$\partial M_2 = \Gamma$, then

$$\omega_i \geq \frac{e^{2\hbar}\mu(M_j)}{c_{n-1}\int_0^{d(M)} \mathfrak{s}_k^{n-1}(r)dr}, \quad i \neq j$$

where ω_i is the ω corresponding to M_i, and $d(M)$ is the diameter of M.

In particular, if $\mu(M_1) \leq \mu(M_2)$, then

$$\omega_1 \geq \frac{e^{2\hbar}\mu(M)}{2c_{n-1}\int_0^{d(M)} \mathfrak{s}_k^{n-1}(r)dr}.$$

Proof Given $p \in M_1$, let

$$O_p := \{q \in M : q = \gamma_y(t),\ t \in (0, i_y], y \in U_p\}.$$

Then for any $q \in M_2$, there is a unit speed minimizing geodesic $\gamma_y(t)$ from p to q. Hence, $\gamma_y(t)$ must hit the boundary and therefore $y = \dot{\gamma}_y(0) \in U_p$. This implies that $M_2 \subset O_p$. Thus, by the proof of Theorem 3.4, we obtain

$$\begin{aligned}
\mu(M_2) \leq \mu(O_p) &= \int_{U_p} d\nu_p(y) \int_0^{i_y} \widehat{\sigma}_p(t,y)dt \\
&\leq \int_{U_p} d\nu_p(y) \int_0^{i_y} e^{-\tau(\dot{\gamma}_y(t))}\mathfrak{s}_k^{n-1}(t)dt \\
&= \int_{U_p} e^{\tau(y)} d\nu_p(y) \int_0^{i_y} e^{-[\tau(\dot{\gamma}_y(t))+\tau(y)]}\mathfrak{s}_k^{n-1}(t)dt \\
&\leq c_{n-1}\omega_1(p)e^{-2\hbar}\int_0^{i_y} \mathfrak{s}_k^{n-1}(t)dt. \qquad \blacksquare
\end{aligned}$$

According to [13, 34], the *Cheeger's constant* of a reversible Finsler n-manifold $(M, F, d\mu)$ is defined by

$$I_\infty(M) := \inf_\Gamma \frac{A(\Gamma)}{\min\{\mu(M_1),\ \mu(M_2)\}},$$

where Γ varies over compact $(n-1)$-dimensional submanifolds of M that divide M into two disjoint open submanifolds M_1, M_2 of M. Then we have the following theorem.

Theorem 6.5 *Let $(M, F, d\mu)$ be a compact reversible Finsler manifold, where $d\mu$ is either the Busemann–Hausdorff volume form or the Holmes–Thompson volume form. Then $I_\infty(M) > 0$.*

Proof Given any compact $(n-1)$-dimensional submanifold Γ that divided M into two disjoint open submanifolds. Let $\mathbf{n}$ be the unit inward normal vector field along Γ and let $\mathcal{M}$ be defined as above. Since M is compact, the uniformity constant Λ is finite and $\sqrt{\Lambda} \geq \mathcal{M}$. Combining Theorem 6.2 and Proposition 6.4 completes the proof. ■

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