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Examples of Calabi–Yau 3-Folds of $\mathbb{P}^7$ with $\rho = 1$

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Abstract. We give some examples of Calabi–Yau 3-folds with $\rho = 1$ and $\rho = 2$, defined over $\mathbb{Q}$ and constructed as 4-codimensional subvarieties of $\mathbb{P}^7$ via commutative algebra methods. We explain how to deduce their Hodge diamond and top Chern classes from computer based computations over some finite field $\mathbb{F}_p$. Three of our examples (of degree 17 and 20) are new. The two others (degree 15 and 18) are known, and we recover their well-known invariants with our method. These examples are built out of Gulliksen–Negård and Kustin–Miller complexes of locally free sheaves.

Finally, we give two new examples of Calabi–Yau 3-folds of $\mathbb{P}^6$ of degree 14 and 15 (defined over $\mathbb{Q}$). We show that they are not deformation equivalent to Tonoli's examples of the same degree, despite the fact that they have the same invariants $(H^3, c_2 \cdot H, c_3)$ and $\rho = 1$.

1 Introduction

A projective Calabi–Yau 3-fold X is a smooth complex projective 3-dimensional variety with trivial canonical sheaf ($\omega_X \simeq \mathcal{O}_X$) such that $H^1(X, \mathcal{O}_X) = 0$. The current interest in finding examples of projective Calabi–Yau 3-folds comes from mathematical physics. Among Calabi–Yau 3-folds, those with Picard number $\rho = 1$, that is, for which the Picard lattice is generated by a single element, bear special interest. It is indeed believed that they should form only finitely many families. In particular, since Hodge numbers are deformation invariants, there should be a finite number of possible Hodge invariants for these varieties. It is worth pointing out an interesting approach by N.-H. Lee [7, 8] to settling this conjecture, consisting in building families of Calabi–Yau 3-folds with $\rho = 1$ as double covers. A recent up-to-date list of examples of Calabi–Yau 3-folds with $\rho = 1$ can be found in van Straten and van Eckenvordt [14]. Some members of this list were constructed by F. Tonoli [13] as embedded projective Calabi–Yau 3-folds in $\mathbb{P}^6$, using commutative algebra methods.

The original aim of this work was to follow Tonoli's lead and build projective Calabi–Yau 3-folds X (defined over $\mathbb{Q}$) in $\mathbb{P}^7$ with $\rho = 1$, using commutative algebra complexes. With this method we also obtained degenerate examples, *i.e.,* Calabi–Yau 3-folds contained in a hyperplane. Some of these degenerate examples (degree 14 and 15), which can also be realized as Pfaffians in $\mathbb{P}^6$, nevertheless turn out to be interesting, since they give the first examples of nondeformation equivalent Calabi–Yau 3-folds of Picard number one with the same invariants $(H^3, c_2 \cdot H, c_3)$.

For these constructions in codimension 4, we use two complexes of locally free sheaves: the Gulliksen–Negård complex and the Kustin–Miller complex. This last complex had no global version yet, so we first construct a global version of this complex in Section 2.

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Passing from codimension 3 to codimension 4, some new problems arise, such as the determination of the Hodge invariants of the Calabi–Yau 3-folds X we build; their Hodge numbers can, indeed, no longer be computed from the Hilbert polynomial of X. We explain in Section 3 how to deduce these numbers from a single smoothness check computation over $\mathbb{F}_p$. In the last section we present the examples we have found by this method and give their invariants and graded Betti table.

From now on, k will always denote a perfect field, *e.g.*, $\mathbb{Q}$, $\mathbb{C}$, or $\mathbb{F}_p$. We will also denote by $\mathcal{O}$ the structure sheaf of the ambient projective space $\mathbb{P}^N_k$.

2 The Commutative Algebra Complexes in Use

Let $\mathbb{G}_\bullet$ be a complex of locally free sheaves over $\mathbb{P}^N_k$ of length c, such that $\mathbb{G}_c$ is locally free. We will say that $\mathbb{G}_\bullet$ is *quasi self dual* if the dual complex satisfies $\mathbb{G}_\bullet^\vee \simeq \mathbb{G}_\bullet \otimes \mathbb{G}_c^\vee$. If a quasi self dual complex is exact and resolves a codimension c subscheme Z of $\mathbb{P}^N$, Z is locally Gorenstein and subcanonical with $\omega_Z = \mathbb{G}_4^\vee \otimes \mathcal{O}_Z(-N-1)$. Such complexes are thus very useful for constructing *subcanonical varieties* X in $\mathbb{P}^N$, *i.e.*, varieties for which $\omega_X = \mathcal{O}_X(a)$ for some integer $a \in \mathbb{Z}$. For instance, the Pfaffian complex is quasi self dual of length 3 and can be used to construct subcanonical varieties of codimension 3 [10]. Let us recall its construction in case the ambient space is $\mathbb{P}^N_k$.

2.1 Pfaffian Complex over $\mathbb{P}^N$

Given a locally free sheaf $\mathcal{E}$ of odd rank $2s+1 \geq 3$ on $\mathbb{P}^N_k$, a locally free sheaf of rank one $\mathcal{L}$, a non zero section $Y \in H^0(\mathbb{P}^N, \wedge^2\mathcal{E} \otimes \mathcal{L})$ defines a skew symmetric map $\mathcal{E}^\vee \otimes \mathcal{L}^\vee \xrightarrow{\tilde{Y}} \mathcal{E}$. We set $\mathcal{M} = \det(\mathcal{E}) \otimes \mathcal{L}^{\otimes s}$. The Pfaffian complex associated with the data $(\mathcal{E}, \mathcal{L}, Y)$ is then the quasi self dual complex

$$0 \longrightarrow (\mathcal{M}^\vee)^{\otimes 2} \otimes \mathcal{L}^* \xrightarrow{d_1^\vee} \mathcal{E}^\vee \otimes \mathcal{L}^\vee \otimes \mathcal{M}^\vee \xrightarrow{\tilde{Y}} \mathcal{E} \otimes \mathcal{M}^\vee \xrightarrow{d_1 = -\wedge Y^{(s)}} \mathcal{O},$$

where $Y^{(s)}$ is the s-th divided power of Y

$$Y^{(s)} = \frac{1}{s!}\underbrace{Y \wedge \cdots \wedge Y}_{s \text{ times}}$$

The cokernel of d_1 defines a *Pfaffian* subscheme X of $\mathbb{P}^N_k$ by $\mathcal{O}_X = \operatorname{coker}(d_1)$.

Let us recall the following property of Pfaffian subschemes that we shall use later on.

Theorem 2.1 **(Buchbaum–Eisenbud)** *Let X be a Pfaffian subscheme of $\mathbb{P}^N_k$, with $N \geq 4$. Then for each point $x \in X$ we have* $\operatorname{codim}_x(X) \leq 3$. *Moreover, if X is not empty and has codimension 4 in $\mathbb{P}^N_k$, then the associated Pfaffian complex is a resolution of X. In this case, the subscheme X is thus equidimensional, locally Gorenstein, and subcanonical with $\omega_X = \mathcal{O}(2c_1(\mathcal{M}) + c_1(\mathcal{L}) - N - 1)$. Moreover, if $N \geq 5$ and X is smooth, then X is irreducible.*

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The proof follows easily from Buchbaum–Eisenbud results in [1].

To construct Calabi–Yau 3-folds in $\mathbb{P}^7_k$, we will need quasi self dual complexes of locally free sheaves of length 4. Historically, the first known complex of this type is the Gulliksen–Negård complex [2].

2.2 Gulliksen–Negård Complex over $\mathbb{P}^N$

This complex is locally the resolving complex of the locus of submaximal minors of a square matrix. Let us recall in this section the properties of global Gulliksen–Negård complex that we shall use later on. Let $\mathcal{E}$ and $\mathcal{F}$ be two locally free sheaves on $\mathbb{P}^N_k$ of the same rank $e \geq 3$. Choose $\phi \in \operatorname{Hom}(\mathcal{E}, \mathcal{F})$ a morphism of $\mathcal{O}$-modules. Let $\mathcal{L}$ denote the locally free sheaf of rank one $\wedge^e \mathcal{E} \otimes \wedge^e \mathcal{F}^*$. Let s_ϕ denote the composition

$$\wedge^{e-1}\mathcal{E}\otimes\wedge^{e-1}\mathcal{F}^* \simeq \wedge^{e-1}\mathcal{E}\otimes F\otimes\mathcal{O}(-c_1(\mathcal{F})) \xrightarrow{\wedge^{e-1}\phi} \wedge^{e-1}\mathcal{F}\otimes\mathcal{F}\otimes\mathcal{O}(-c_1(\mathcal{F})) \xrightarrow{-\wedge-} \mathcal{O}.$$

The Gulliksen–Negård subscheme $X(\phi)$ of $\mathbb{P}^N_k$ is defined by $\mathcal{O}_{X(\phi)} = \operatorname{coker}(s_\phi)$. In case $X(\phi)$ has codimension 4, the global Gulliksen–Negård complex $\mathbb{F}_\bullet$,

$$0 \to \mathcal{L}^{\otimes 2} \to \mathcal{E} \otimes \mathcal{F}^* \otimes \mathcal{L} \to \wedge^e \mathcal{E} \otimes \wedge_{1,e-1}\mathcal{F}^* \oplus \wedge^e \mathcal{F}^* \otimes \wedge_{1,e-1}\mathcal{E}$$
$$\to \wedge^{e-1}\mathcal{E} \otimes \wedge^{e-1}\mathcal{F}^* \xrightarrow{s_\phi} \mathcal{O},$$

provides a locally free resolution of $X(\phi)$ [6, 12]. The Gulliksen–Negård complex is quasi self dual and satisfies the following properties, which easily follow from [2, théorème 4].

Theorem 2.2 **(Gulliksen–Negård)** *Let $X(\phi)$ be a Gulliksen–Negård subscheme of $\mathbb{P}^N_k$, with $N \geq 5$. Let $X(\phi)$ denote the associated Gulliksen–Negård subscheme; for each point $x \in X(\phi)$, we have $\operatorname{codim}_x(X(\phi)) \leq 4$. Suppose that $X(\phi)$ is not empty and has codimension 4 in $\mathbb{P}^N_k$. Then $\mathbb{F}_\bullet$ is a resolution of $X(\phi)$. Thus, the subscheme $X(\phi)$ is equidimensional, locally Gorenstein, and subcanonical with*

$$\omega_{X(\phi)} = \mathcal{O}(-2(c_1(\mathcal{E}) - c_1(\mathcal{F})) - N - 1).$$

Moreover, if $N \geq 6$ and $X(\phi)$ is smooth, $X(\phi)$ is irreducible.

We will also construct Calabi–Yau 3-folds using the (global) Kustin–Miller complex.

2.3 Kustin-Miller Complex (Local Version)

In order to give a coordinate-free construction of the Kustin–Miller complex, we need to recall how Kustin and Miller constructed their famous complex [4]. In this section, R denotes a commutative ring with unity such that 2 is not a zero divisor. Let τ denote an odd number. Let Y denote a $\tau \times \tau$ alternating matrix (*i.e.*, such that $y_{i,j} = -y_{j,i}$ for all $i, j \in \{1, \ldots, \tau\}$) with coefficients in R. Let us recall the definition and first

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properties of the Pfaffian in the local situation. If F is a free module of rank τ, the choice of a basis $\{e_1, \dots, e_\tau\}$ for F gives an isomorphism $F \simeq R^\tau$. With any $(\tau \times \tau)$ alternating matrix Z we can associate in a unique way the following τ-form:

$$\phi_Y = \sum_{1 \le i < j \le \tau} Z_{i,j} e_i \wedge e_j \in \mathrm{Hom}(R, \wedge^2 F).$$

If τ is even, we set $\tau = 2s$ and let Π denote the set of partitions α of $\{1, \dots, 2s\}$ in ordered pairs $(i_1, j_1), \dots, (i_s, j_s)$ for which $i_t < j_t$ for all $t \in \{1, \dots, s\}$. The Pfaffian of the matrix Z is defined to be

$$Pf(Z) = \begin{cases} \sum_{\alpha \in \Pi} sg(\alpha) Z_{i_1,j_1} \cdots Z_{i_s,j_s} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

where $sg(\alpha)$ denote the sign of the permutation $(i_1, j_1, \dots, i_s, j_s)$ of $\{1, \dots 2s\}$.

Given any multi-index $(i) = (i_1, \dots, i_r) \in \{1, \dots, \tau\}^r$ of length r, the submatrix of Y obtained by removing from Y the rows and columns of index $i_1, \dots, i_r$ is again an alternating matrix; we denote by $Pf_{(i)}(Y)$ the Pfaffian of this matrix. Following Kustin–Miller's sign convention, we can assign to (i) its signed Pfaffian $Y_{(i)}$ as follows. Let us define $\sigma(i)$ to be 0 if (i) has a repeated index and to be the sign of the permutation rearranging $i_1, \dots, i_r$ in ascending order otherwise. We set $|i| = \sum_{i=1}^{r} i_k$.

***Definition 2.3* (Kustin and Miller)** The *signed Pfaffian* $Y_{(i)}$ of Y associated with (i) is

$$Y_{(i)} := \begin{cases} (-1)^{|i|+1} \sigma(i) Pf_{(i)}(Y) & \text{if } r < \tau, \\ (-1)^{|i|+1} \sigma(i) & \text{if } r = \tau, \\ 0 & \text{if } r > \tau. \end{cases} \tag{2.1}$$

The *Pfaffian row* of Y is defined to be $\mathbf{y} := [Y_1, \dots, Y_\tau]$; the ideal $(Y_1, \dots, Y_\tau)$ is the *Pfaffian ideal* of Y.

The (local) *Pfaffian complex* of Y is then

$$0 \longrightarrow R \xrightarrow{\mathbf{y}^\vee} R^\tau \xrightarrow{Y} R^\tau \xrightarrow{\mathbf{y}} R \longrightarrow 0.$$

It resolves the Pfaffian ideal of Y exactly when this ideal is 3-codimensional [1].

The data necessary to build the Kustin–Miller complex are the following:

- $\tau = 2s + 1 \ge 3$, an odd number,
- Y, a $\tau \times \tau$ alternating matrix on R,
- A, a $\tau \times 3$ matrix on R,
- $\mathbf{b}$, a 1×3 row matrix on R,
- u and v, two non-zero scalars of R.

Kustin and Miller also set $X = \begin{pmatrix} A \\ \mathbf{b} \end{pmatrix}$. From this data they define six other matrices $w, \mathbf{z}, Z, S, B$, and T. The scalar w is defined by

$$w = \sum_{1 \le i < j \le \tau} d_{ijk} Y_{ijk},$$

where d_{ijk} is the determinant of the 3×3 submatrices of X obtained by selecting the rows i, j, k in this order. The row matrix $\mathbf{z}$ is defined to be $\mathbf{z} = u\mathbf{b} - \mathbf{y}A$; it is the row Pfaffian of a 3×3 alternating matrix Z. The $(3 \times \tau)$ matrix S is defined to be the matrix with entries

$$s_{l,k} = (-1)^{l+1} \sum_{1 \le i < j \le \tau} Y_{kij} \begin{vmatrix} x_{i,m} & x_{i,n} \\ x_{j,m} & x_{j,n} \end{vmatrix},$$

where $m < n$ and $\{l, m, n\} = \{1, 2, 3\}$. The row matrix $\mathbf{b}$ is the row Pfaffian of some 3×3 matrix B; the $3 \times \tau$ matrix T is then defined by $T = -BA^t$. Kustin and Miller define the differential maps of their length 4 complex by

$$d_1 = \begin{pmatrix} z & v\mathbf{y} - \mathbf{b}S & w - uv \end{pmatrix},$$

$$d_2 = \begin{pmatrix} Z & S & vI_3 & T \\ 0 & uI_\tau & A & Y \\ 0 & \mathbf{y} & \mathbf{b} & 0 \end{pmatrix},$$

$$d_3 = \begin{pmatrix} 0 & I_{\tau+3} \\ I_\tau & 0 \end{pmatrix} d_2^\vee,$$

$$d_4 = d_1^\vee.$$

Theorem 2.4 **(Kustin and Miller [4])** *With the previous notation, the maps d_i are the differentials of a self-dual complex of length 4:*

$$\mathbb{G}_\bullet : 0 \longrightarrow R \xrightarrow{d_4} R^{2(\tau+3)} \xrightarrow{d_2} R^{\tau+4} \xrightarrow{d_1} R \longrightarrow 0.$$

Moreover, this complex is generically exact.

2.4 Global Kustin–Miller Complex on $\mathbb{P}_k^N$

We present in this section a global version of a Kustin–Miller complex; we will give a geometric interpretation of this construction in terms of the Kustin–Miller unprojection in the next section. The construction given here works over any smooth projective variety $\mathbb{P}$. For simplicity, we will assume that $\mathbb{P} = \mathbb{P}_k^N$.

The data required to define a global Kustin–Miller complex are the following:

- an odd number $\tau = 2s + 1 \ge 3$,
- a rank 3 vector bundle $\mathcal{F}$ on $\mathbb{P}_k^N$,
- a vector bundle $\mathcal{E}$ of rank τ on $\mathbb{P}_k^N$,

- two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathbb{P}^N_k$,
- a global section $Y \in H^0(\mathbb{P}^N_k, \wedge^2\mathcal{E} \otimes \mathcal{L}_1)$,
- a morphism A in $\mathrm{Hom}(\mathcal{F}, \mathcal{E})$,
- a morphism b in $\mathrm{Hom}(\mathcal{F}, \mathcal{L}_2)$,
- a non-zero morphism u in $\mathrm{Hom}(\mathcal{L}_2, \wedge^{2s+1}\mathcal{E} \otimes \mathcal{L}_1^{\otimes s})$,
- a non-zero morphism v in $H^0(\mathbb{P}^N_k, \mathcal{L}_2 \otimes \mathcal{L}_1^\vee \otimes \wedge^3(\mathcal{F}^\vee))$,

For convenience we set $\mathcal{M} = \wedge^{2s+1}\mathcal{E} \otimes \mathcal{L}_1^{\otimes s}$, so that $u \in \mathrm{Hom}(\mathcal{L}_2, \mathcal{M})$. We define the (global) morphism w by the composition

$$\wedge^3\mathcal{F} \xrightarrow{\wedge^3 A} \wedge^3\mathcal{E} \xrightarrow{-\wedge Y^{(s-1)}} \wedge^{2s+1}\mathcal{E} \otimes \mathcal{L}_1^{\otimes s-1} = \mathcal{M} \otimes \mathcal{L}_1^\vee.$$

We easily define z to be Kustin–Miller in the local situation by

$$z = u \circ b - y \circ A \in \mathrm{Hom}(\mathcal{F} \otimes \mathcal{L}_1^{\otimes -s}, \wedge^{2s+1}\mathcal{E}).$$

In order to construct the morphism S, we first define the morphism S_0 by the composition

$$\mathcal{F}^\vee \otimes \wedge^3\mathcal{F} \xrightarrow{Sg} \wedge^2\mathcal{F} \xrightarrow{\wedge^2 A} \wedge^2\mathcal{E} \xrightarrow{-\wedge Y^{(s-1)}} \wedge^{2k}\mathcal{E} \otimes \mathcal{L}_1^{\otimes s-1},$$

where Sg is the base change matrix between $\mathcal{F}^\vee \otimes \wedge^3\mathcal{F}$ and $\wedge^2\mathcal{F}$ with their usual bases, *i.e.*,

$$Sg = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The morphism S is then defined by the composition

$$\mathcal{E} \otimes \mathcal{F}^\vee \otimes \wedge^3\mathcal{F} \xrightarrow{\otimes S_0} \mathcal{E} \otimes \wedge^{2s}\mathcal{E} \otimes \mathcal{L}_1^{s-1} \xrightarrow{-\wedge-} \mathcal{M} \otimes \mathcal{L}_1^\vee.$$

Remark 1 Any triple $(\mathcal{G}, \mathcal{L}, y)$, where $\mathcal{G}$ is a rank 3 vector bundle, $\mathcal{L}$ is a line bundle on $\mathbb{P}^N_k$, and y is a morphism from $\mathcal{G}$ to $\wedge^3\mathcal{G} \otimes \mathcal{L}$, gives rise to a Pfaffian complex via the isomorphisms

$$H^0(\mathbb{P}^N_k, \wedge^2\mathcal{G} \otimes \mathcal{L}) \simeq H^0(\mathbb{P}^N_k, \mathcal{G}^\vee \otimes \wedge^3\mathcal{G} \otimes \mathcal{L}) \simeq \mathrm{Hom}(\mathcal{G}, \wedge^3 G \otimes \mathcal{L}).$$

Therefore, we can define the matrix Z (respectively B) to be the skew-symmetric morphism associated with z (respectively b). We set $T = -B \circ A^\vee$. Let us define the following vector bundles:

$$\mathbb{G}_1 = (\mathcal{F} \otimes \mathcal{M}^\vee) \oplus (\mathcal{E} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2^\vee \otimes \wedge^3\mathcal{F} \otimes \mathcal{M}^\vee) \oplus (\mathcal{L}_1 \otimes \wedge^3\mathcal{F} \otimes \mathcal{M}^\vee)$$

$$\mathbb{G}_2 = (\mathcal{E}^\vee \otimes \mathcal{L}_2^\vee \otimes \wedge^3\mathcal{F} \otimes \mathcal{M}^\vee) \oplus (\mathcal{F} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2^\vee \otimes \wedge^3\mathcal{F} \otimes \mathcal{M}^\vee) \oplus (\mathcal{E} \otimes \mathcal{L}_1 \otimes \wedge^3\mathcal{F} \otimes (\mathcal{M}^\vee)^{\otimes 2}) \oplus (\mathcal{F}^\vee \otimes \wedge^3\mathcal{F} \otimes (\mathcal{M}^\vee)^{\otimes 2})$$

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Notice that we have $\mathbb{G}_2^\vee \otimes (\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes (\mathcal{M})^{\otimes 3} \otimes (\wedge^3 \mathcal{F})^{\otimes 2}) \simeq \mathbb{G}_2$; since we wish our complex to be quasi self dual, this forces us to take

$$\mathbb{G}_4 = \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes (\mathcal{M})^{\otimes 3} \otimes (\wedge^3 \mathcal{F})^{\otimes 2} \quad \text{and} \quad \mathbb{G}_3 = \mathbb{G}_1^\vee \otimes \mathbb{G}_4.$$

Let us now define the differentials of our complex of vector bundles:

$$\begin{aligned} d_1 &= \begin{pmatrix} z & -b \circ S + v \circ y & w - v \circ u \end{pmatrix} \in \operatorname{Hom}(\mathbb{G}_1, \mathcal{O}_{\mathbb{P}_k^N}), \\ d_2 &= \begin{pmatrix} T & v & S & Z \\ Y & A & y & 0 \\ 0 & b & y & 0 \end{pmatrix} \in \operatorname{Hom}(\mathbb{G}_2, \mathbb{G}_1), \\ d_4 &= d_1^\vee \otimes \mathbb{G}_4, \\ d_3 &= d_2^\vee \otimes \mathbb{G}_4. \end{aligned}$$

Theorem 2.5 *With the previous notation, the morphisms d_i define a complex of vector bundles*

$$\mathbb{G}_\bullet : 0 \longrightarrow \mathbb{G}_4 \xrightarrow{d_4} \mathbb{G}_3 \xrightarrow{d_3} \mathbb{G}_2 \xrightarrow{d_2} \mathbb{G}_1 \xrightarrow{d_1} \mathcal{O}_{\mathbb{P}_k^N} \longrightarrow 0$$

that is quasi self dual and whose restriction to any trivializing affine open set U of the data is a complex isomorphic to the one constructed by Kustin and Miller. Let X denote the Kustin–Miller subscheme of $\mathbb{P}_k^N$ defined by $\mathcal{O}_X = \operatorname{coker}(d_1)$. Then for each $x \in X$, we have $\operatorname{codim}_x(X) \le 4$. If this subscheme X is non-empty and 4-codimensional, then $\mathbb{G}_\bullet$ resolves $\mathcal{O}_X$ and X is equidimensional, locally Gorenstein, and subcanonical, with

$$\omega_X \simeq (\omega_{\mathbb{P}_k^N} \otimes \mathcal{M}^{\otimes 3} \otimes (\wedge^3(\mathcal{F}^\vee))^{\otimes 2} \otimes \mathcal{L}_2 \otimes \mathcal{L}_1^\vee)|_X,$$

Moreover, if $N \ge 6$ and X is smooth, then the scheme X is irreducible.

Proof To show that the differential maps and modules we have just introduced define a complex whose restriction to any trivializing open set is isomorphic to the Kustin–Miller complex, it is enough to show that in the local setting they give a coordinate free description of the Kustin–Miller complex. So we can assume that we are working with free R-modules, so that we can replace $\mathcal{E}$ by R^{2s+1}, $\mathcal{F}$ by R^3, and the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ by R. Let us denote by $\{e_1, \ldots, e_\tau\}$ the canonical basis of $E := R^{2s+1}$ and by $\{f_1, f_2, f_3\}$ the canonical basis of $F := R^3$. We can now replace the homomorphisms Y, A, b by their matrices (u and v become scalars). We only have to check that our coordinate free constructions of S and w match the Kustin and Miller ones. Let us first work on the construction of S. The morphism $\wedge^2 A$ can be described as

$$f_m \wedge f_n \mapsto \sum_{i<j} \begin{vmatrix} a_{i,m} & a_{i,n} \\ a_{j,m} & a_{j,n} \end{vmatrix} e_i \wedge e_j.$$

Let us denote by $\{f_1^*, f_2^*, f_3^*\}$ the dual basis of $\{f_1, f_2, f_3\}$; the matrix Sg maps $F^\vee$ to $\wedge^2 F$ via $f_l^* \mapsto (-1)^{(l+1)} f_m \wedge f_n$ with $m < n$ and $\{l, m, n\} = \{1, 2, 3\}$. Thus, in order to show that our coordinate free construction of S coincides with

$$(f_l^*; e_k) \mapsto (-1)^{l+1} \sum_{1 \le i < j \le \tau} Y_{kij} \begin{vmatrix} a_{i,m} & a_{i,n} \\ a_{j,m} & a_{j,n} \end{vmatrix},$$

we only have to check that the morphism $\mathrm{id}_E \wedge Y^{(s-1)} \colon \wedge^2 E \otimes E \to \wedge^{2s+1} E \simeq R$ coincides with

$$(e_i \wedge e_j; e_k) \mapsto Y_{kij} e_1 \wedge \cdots \wedge e_{2s+1}. \tag{2.2}$$

Indeed, by skew-symmetry of A we have

$$\begin{vmatrix} a_{i,m} & a_{i,l} \\ a_{j,m} & a_{j,n} \end{vmatrix} = \begin{vmatrix} a_{m,i} & a_{m,j} \\ a_{n,i} & a_{n,j} \end{vmatrix}.$$

In order to prove (2.2), let us first remark that the morphism $- \wedge Y^{(s-1)} \colon R \to \wedge^{2s-2} E$ corresponds to

$$1 \mapsto \frac{1}{(s-1)!} \Big(\sum_{i_1 < j_1} Y_{i_1, j_1} e_{i_1} \wedge e_{j_1} \Big) \wedge \cdots \wedge \Big(\sum_{i_{s-1} < j_{s-1}} Y_{i_{s-1}, j_{s-1}} e_{i_{s-1}} \wedge e_{j_{s-1}} \Big).$$

This can be rewritten as

$$1 \mapsto \sum_{t \ne i} \sum_{\alpha \in \Pi_{i,j,t}} Y_\alpha e_\alpha,$$

where $\Pi_{i,j,t}$ is the set of partitions of $\{1, \ldots, 2s+1\} \setminus \{i, j, t\}$ by pairs

$$(i_1, j_1), \ldots, (i_{s-1}, j_{s-1})$$

for which $i_r < j_r$ for all $r \in \{1, \ldots s-1\}$, Y_α stands for the product $Y_{i_1, j_1} \cdots Y_{i_{s-1}, j_{s-1}}$, and e_α for the wedge product $e_{i_1} \wedge e_{j_1} \wedge \cdots \wedge e_{i_{s-1}} \wedge e_{j_{s-1}}$. Remark now that

$$\sum_{\alpha \in \Pi_{i,j,t}} Y_\alpha e_\alpha = \sum_{\alpha \in \Pi_{i,j,t}} sg(\alpha) Y_\alpha \omega_{i,j,t} = Pf_{i,j,t}(Y) \omega_{i,j,t},$$

where $\omega_{i,j,t}$ is the wedge product of the e_r for $r \ne i, j, t$ written in increasing order of indices. Relation (2.2) can then be deduced from the following observation (using the Kustin–Miller sign convention (2.1)): if $t > j > i$, we have

$$\begin{aligned} Pf_{i,j,t}(Y) e_t \wedge e_i \wedge e_j &= (-1)^{|(i,j,t)|+1} Y_{tij} \sigma(t, i, j) e_t \wedge e_i \wedge e_j \wedge \omega_{t,i,j} \\ &= Y_{tij} e_1 \wedge \cdots \wedge e_{2s+1}. \end{aligned}$$

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Similarly, we can show that our coordinate free construction of w coincides with the Kustin–Miller one. This shows the first part of the theorem.

Let us assume that our global complex resolves a subscheme X of $\mathbb{P}^N_k$, so that $\mathcal{O}_X = \operatorname{coker}(d_1)$. Then localizing at any point $x \in X$, we have $\operatorname{codim}_x(X) \geq 4$. Applying [4, Corollary 2.6], we deduce that $\operatorname{codim}_x(X) = 4$ and that $\mathbb{G}_{\bullet,x}$ is a resolution of $\mathcal{O}_{X,x}$. The subscheme X is thus equidimensional of codimension 4 in $\mathbb{P}^N_k$. Since $\mathbb{G}_\bullet$ resolves X, we deduce that X is locally Gorenstein. Since $\mathbb{G}_\bullet$ is quasi self dual, X is subcanonical with $\omega_X \simeq (\omega_{\mathbb{P}^N_k} \otimes \mathcal{M}^{\otimes 3} \otimes (\wedge^3(\mathcal{F}^\vee))^{\otimes 2} \otimes \mathcal{L}_2 \otimes \mathcal{L}_1^\vee)|_X$. ■

Remark 2 **(Shifts of the data)** Let $\mathcal{N}$ be a line bundle on $\mathbb{P}^N_k$. If we replace $\mathcal{E}$ by $\mathcal{E}' = \mathcal{E} \otimes \mathcal{N}$, $\mathcal{L}_1$ by $\mathcal{L}_1' = \mathcal{L}_1 \otimes (\mathcal{N}^\vee)^{\otimes 2}$, $\mathcal{F}$ by $\mathcal{F}' = \mathcal{F} \otimes \mathcal{N}$, and $\mathcal{L}_2$ by $\mathcal{L}_2' = \mathcal{L}_2 \otimes \mathcal{N}$, keeping the data morphisms Y, A, b, u, v, these new data define the same Kustin–Miller complex.

Remark 3 **(Global Kustin–Miller complex and unprojection)** Assume that the two Pfaffian subschemes of $\mathbb{P}^N$, X_0 and X_1, associated with $(\mathcal{F}, \mathcal{M} \otimes \det(\mathcal{F}^\vee), Z)$ and $(\mathcal{E}, \wedge^2\mathcal{E} \otimes \mathcal{L}_1, Y)$ respectively, are 3-codimensional. Assume also that the section $X_1 \cap (u)$ is of codimension 4. Localizing the data at some fixed point $z \in \mathbb{P}^N$, we can use Kustin and Miller's observation in [5, Example 2.2]. The localized complex $\mathbb{G}_{\bullet,z}$ resolves the section by (v_z) of the unprojection of $\operatorname{spec}(\mathcal{O}_{X_1,z}) \cap (u_z)$ in $\operatorname{spec}(\mathcal{O}_{X_0,z})$ (see [11] for further details on Kustin–Miller unprojection). The global Kustin–Miller scheme X is 4-codimensional, provided that v is generic in $\operatorname{Hom}(\mathbb{P}^N, \mathcal{L}_2 \otimes \mathcal{L}_1^\vee \otimes \wedge^3(\mathcal{F}^\vee)) \neq 0$. Notice that in this case $\omega_X = \omega_{X_1}(\deg(u) + 2\deg(v))$. Let us also point out that the existence of a global version of the Kustin–Miller complex shows that in this very special case, there is a global version of unprojection with complexes, even though, in general, such a process cannot be carried out globally because of the non-vanishing of certain $\operatorname{Ext}^1(*, *)$ groups.

In order to build Calabi–Yau 3-folds of Kustin–Miller type in $\mathbb{P}^7$, we will therefore look for vector bundles $\mathcal{E}$ and $\mathcal{L}_1$, such that the Pfaffian complex associated with $(\mathcal{E}, \wedge^2\mathcal{E} \otimes \mathcal{L}_1, Y)$ is exact for Y generic and such that $\omega_{X_1} = \mathcal{O}_{X_1}(a)$, with $a \in \mathbb{Z}$ such that $a = 2\deg(\mathcal{M}) - 8 + deg(\mathcal{L}_1) \leq -2$. Indeed,

$$w - u \circ v \in \operatorname{Hom}(\mathcal{O}(-\deg(u) - \deg(v)), \mathcal{O})$$

is one of the defining equations of X, so that we need $\deg(u) + \deg(v) \geq 2$, unless X is contained in a hyperplane.

3 Smoothness Check and Computation of Invariants

3.1 Smoothness Check

The Jacobian criterion cannot be used in practice and the Gröbner basis computation exceeds the CPU capacity of most computers. The top Chern class c_3 of a Calabi–Yau 3-fold in $\mathbb{P}^7$ is not a function of the coefficients of its Hilbert polynomial, unlike Calabi–Yau 3-folds in $\mathbb{P}^6$. Therefore, we cannot adapt Tonoli's smoothness criterion [13] to our situation. This type of extremely efficient smoothness test originates in the famous article [3] on computer-based construction of surfaces in projective space. We use instead the following coarse smoothness test, which is, nonetheless, faster than the Jacobian criterion and has the advantage of giving for free the top Chern class of the Calabi–Yau in case the test is positive.

Let $I = (g_1, \dots, g_s)$ denote a generating ideal for X_k in $\mathbb{P}^7_k$; let $S = k[x_0, \dots, x_7]$ denote the polynomial ring of $\mathbb{P}^7_k$. If h_1, h_2, h_3 are polynomials of I, let $I_4(h_1, h_2, h_3)$ denote the ideal generated by the 4×4 minors of the matrices

$$\begin{pmatrix} \frac{\partial h_1}{\partial x_0} & \frac{\partial h_2}{\partial x_0} & \frac{\partial h_3}{\partial x_0} & \frac{\partial h}{\partial x_0} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_3}{\partial x_1} & \frac{\partial h}{\partial x_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_1}{\partial x_7} & \frac{\partial h_2}{\partial x_7} & \frac{\partial h_3}{\partial x_7} & \frac{\partial h}{\partial x_7} \end{pmatrix},$$

where h varies in $\{g_1, \dots, g_s\}$. We denote by $\mathrm{Jac}_3(h_1, h_2, h_3)$ the ideal of 3×3 minors of the Jacobian matrix of ideal (h_1, h_2, h_3). We can now state our very coarse smoothness criterion.

Theorem 3.1 **(Coarse smoothness test)** *Let X_k be a 3-dimensional subscheme of $\mathbb{P}^7_k$ where (k is a perfect field). Let I denote a defining ideal of X_k in $\mathbb{P}^N_k$. Suppose that there exist two triples of polynomials of the same degree e in the ideal I, (f_1, f_2, f_3) and (g_1, g_2, g_3), such that*

(i) *the vanishing loci $V(I_4(f_1, f_2, f_3) + I)$ and $V(\mathrm{Jac}_3(f_1, f_2, f_3) + I)$ are both one dimensional and have the same Hilbert polynomial;*

(ii) *the vanishing loci $V(I_4(g_1, g_2, g_3) + I)$ and $V(\mathrm{Jac}_3(g_1, g_2, g_3) + I)$ are both one dimensional and have the same Hilbert polynomial.*

   (a) *If $\dim(V(Jac_3(f_1, f_2, f_3) + Jac_3(g_1, g_2, g_3) + I) = 0$ and X_k is equidimensional, then X_k has only isolated hypersurface singularities.*

   (b) *If $V(Jac_3(f_1, f_2, f_3) + Jac_3(g_1, g_2, g_3) + I) = \varnothing$, then X_k is smooth.*

The proof of this theorem is obvious; it is only worthwhile to state, because it dramatically improves in practice the efficiency of the Jacobian criterion.

3.2 Determination of c_3

If the coarse smoothness test is positive for the reduction modulo p of a Calabi–Yau 3-fold X defined over $\mathbb{Z}$, we can deduce the value of the top Chern class c_3 of $X_{\mathbb{C}}$ from the results of the computations made over $\mathbb{F}_p$, thanks to the following formula.

Theorem 3.2 **(Computation of c_3)** *Let X be a subscheme of $\mathbb{P}^7$ of Kustin–Miller or Gulliksen–Negård type over $\mathbb{Z}$, such that $X_{\mathbb{F}_p}$ is a Calabi–Yau 3-fold, for some prime number p. Let I be a defining ideal of $X_{\mathbb{F}_p}$ in $\mathbb{P}^7_{\mathbb{F}_p}$. Suppose, moreover, that for a triple of polynomials of I of same degree e we have $\dim(V(\mathrm{Jac}_3(f_1, f_2, f_3) + I)) = 1$. Let p_a denote the arithmetical genus of $C = V(\mathrm{Jac}_3(f_1, f_2, f_3) + I)$. Then the third Chern class of $X_{\mathbb{C}}$ is given by*

$$\frac{c_3}{2} = d(-14e^3 + 84e^2 - 180e + 140) + 3ec_2 \cdot H - 8c_2 \cdot H + 1 - p_a,$$

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where $PH_{X_{\mathbb{F}_p}} := \frac{d}{3!}x^3 - \frac{c_2 \cdot H}{12}x$ *is the Hilbert polynomial of* $X_{\mathbb{F}_p}$ *(or* $X_{\mathbb{C}}$*).*

Proof For $i = 1, 2, 3$, the polynomial f_i induces a section σ_i of $\mathcal{N}^{\vee}_{X_{\mathbb{F}_p}}$. Since $X_{\mathbb{F}_p}$ is smooth by assumption, the curve C coincides with the degeneracy locus D_3 of the section $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$.

Let us first notice that, using Hirzebruch–Riemann–Roch, for any integer h, we find the following.

Lemma 3.3 *Let h be an integer. Then*

$$\chi(\mathcal{N}^{\vee}_{X_{\mathbb{C}}}(h)) = \frac{c_3}{2} + d\Big(\frac{2h^3}{3} - 4h^2 + 4h - \frac{4}{3}\Big) + \frac{2H \cdot c_2}{3}(2h-1),$$

where $\mathcal{N}^{\vee}_{X_{\mathbb{C}}}$ *is the conormal bundle of* $X_{\mathbb{C}}$*,* c_2 *and* c_3 *are the second and third Chern classes of* $X_{\mathbb{C}}$*, and H denotes the class of a hyperplane section of* $X_{\mathbb{C}}$*.*

Proof From the exact sequence $0 \to \mathcal{N}^{\vee}_{X_{\mathbb{C}}} \to \Omega^1_{\mathbb{P}^7_{\mathbb{C}}|X_{\mathbb{C}}} \to \Omega^1_{X_{\mathbb{C}}} \to 0$ we deduce the following expressions:

$$\begin{aligned} c_1(\mathcal{N}^{\vee}_{X_{\mathbb{C}}}) &= c_1(\Omega^1_{\mathbb{P}^7_{\mathbb{C}}|X_{\mathbb{C}}}) = -8H, \\ c_2(\mathcal{N}^{\vee}_{X_{\mathbb{C}}}) &= c_2(\Omega^1_{\mathbb{P}^7_{\mathbb{C}}|X_{\mathbb{C}}}) - c_2 = 28H^2 - c_2, \\ c_3(\mathcal{N}^{\vee}_{X_{\mathbb{C}}}) &= c_3(\Omega^1_{\mathbb{P}^7_{\mathbb{C}}|X_{\mathbb{C}}}) + c_3 - c_1(\mathcal{N}^{\vee}_{X_{\mathbb{C}}}) \cdot c_2 = c_3 - 56d + 8H \cdot c_2. \end{aligned}$$

From these relations, we deduce

$$\begin{aligned} c_1(\mathcal{N}^{\vee}_{X_{\mathbb{C}}} \otimes \mathcal{O}(h)) &= (-8 + 4h)H, \\ c_2(\mathcal{N}^{\vee}_{X_{\mathbb{C}}} \otimes \mathcal{O}(h)) &= (28 + 6h^2 - 24h)H^2 - c_2, \\ c_3(\mathcal{N}^{\vee}_{X_{\mathbb{C}}} \otimes \mathcal{O}(h)) &= d(4h^3 - 24h^2 + 56h - 56) + c_2 \cdot H(8 - 2h) + c_3. \end{aligned}$$

These last relations allow us to compute an expansion of the Chern character $\mathcal{C}h(\mathcal{N}^{\vee}_{X_{\mathbb{C}}} \otimes \mathcal{O}(h))$ up to terms of order ≥ 3:

$$4-(8-4h)H+(4H^2-8hH^2+c_2+2h^2H^2)+d\Big(\frac{2h^3}{3}-4h^2+4h-\frac{4}{3}\Big)+\frac{c_3}{2}+hc_2\cdot H+\cdots.$$

Thus, using $Td(X) = 1 + \frac{c_2}{12}$, Hirzebruch–Riemann–Roch gives

$$\chi(\mathcal{N}^{\vee}_{X_{\mathbb{C}}} \otimes \mathcal{O}(h)) = \frac{c_3}{2} + d\Big(\frac{2h^3}{3} - 4h^2 + 4h - \frac{4}{3}\Big) + \frac{2c_2 \cdot H}{3}(2h-1). \qquad \blacksquare$$

For $h = -3e+8$, we may now find a second expression of $\chi(\mathcal{N}^{\vee}_{X_{\mathbb{C}}} \otimes \mathcal{O}(h))$, in terms of the Hilbert polynomials of $X_{\mathbb{F}_p}$ and $C = V(\mathrm{Jac}_3(f_1, f_2, f_3) + I)$.

Lemma 3.4 *Under the assumption of the theorem, we have, for any integer h,*

$$\chi(\mathcal{N}^{\vee}_{X_{\mathbb{F}_p}} \otimes \mathcal{O}(h)) = \chi\big(\mathcal{N}^{\vee}_{X_{\mathbb{Q}}} \otimes \mathcal{O}(h)\big) = \chi(\mathcal{N}^{\vee}_{X_{\mathbb{C}}} \otimes \mathcal{O}(h)).$$

Proof Let us first show

$$\chi(\mathcal{N}^\vee_{X_{\mathbb{F}_p}} \otimes \mathcal{O}(h)) = \chi(\mathcal{N}^\vee_{X_{\mathbb{Q}}} \otimes \mathcal{O}(h)).$$

As we have already seen, there exist an open affine subset U of spec($\mathbb{Z}$), and a U-scheme $\mathcal{X} \xrightarrow{\phi} U$, such that for all $p \in U$, the fiber $\mathcal{X}_p$ is a smooth Calabi–Yau 3-fold of Gulliksen–Negård or Kustin–Miller type ($\mathcal{X}_0 = X_{\mathbb{Q}}$ and $\mathcal{X}_p = X_{\mathbb{F}_p}$). The coherent sheaf $\mathcal{N}^\vee_U := \widetilde{I_U/I_U^2}$ is flat over U, since $\mathcal{N}^\vee_U$ is locally a locally free sheaf of rank 4. The function $p \to \chi(\mathcal{N}^\vee_{X_p})$ is thus (locally) constant on U (see for instance [9, §5]). Therefore, $\chi(\mathcal{N}^\vee_{X_{\mathbb{Q}}}) = \chi(\mathcal{N}^\vee_{X_{\mathbb{F}_p}})$. The remaining equality follows easily by field extension. ■

Similarly, the Hilbert polynomial of $X_{\mathbb{F}_p}$ coincides with the Hilbert polynomial of $X_{\mathbb{Q}}$, hence also with the Hilbert polynomial of $X_{\mathbb{C}}$. Since $X_{\mathbb{F}_p}$ is smooth,

$$V(\mathrm{Jac}_3(f_1, f_2, f_3) + I_{X_{\mathbb{F}_p}})$$

coincides with the dependency locus on $X_{\mathbb{F}_p}$ of the sections σ_1, σ_2 and σ_3 of $\mathcal{N}^\vee_{X_{\mathbb{F}_p}}$, that is to say, the maximal degeneracy locus $D_3(\phi; X_{\mathbb{F}_p}) := \{x \in X_{\mathbb{F}_p} \mid rg_x(\phi) \leq 2\}$ of the morphism

$$\mathcal{O}_X^{\oplus 3}(-e) \xrightarrow{\phi = (\,\sigma_1\ \sigma_2\ \sigma_3\,)} \mathcal{N}^\vee_{X_{\mathbb{Q}}}.$$

Such a degeneracy locus is resolved by the Eagon–Northcott complex in case $C = D_3(\phi; X_{\mathbb{F}_p})$ has expected codimension 2 in $X_{\mathbb{F}_p}$ ($X_{\mathbb{F}_p}$ is regular at every point, since $\mathbb{F}_p$ is a perfect field and $X_{\mathbb{F}_p}$ is smooth):

$$0 \to \mathcal{O}_X^{\oplus 3}(-4e) \otimes \wedge^4 \mathcal{N}_{X_{\mathbb{F}_p}} \xrightarrow{\phi} \mathcal{N}^\vee_{X_{\mathbb{F}_p}} \otimes \wedge^4 \mathcal{N}_{X_{\mathbb{F}_p}} \otimes \mathcal{O}_X(-3e) \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$

We remark that

$$\wedge^4(\mathcal{N}^\vee_{X_{\mathbb{F}_p}}) \simeq (\wedge^3 \Omega^1_{X_{\mathbb{F}_p}})^{-1} \otimes \wedge^7(\Omega^1_{\mathbb{P}^7}) \simeq \mathcal{O}_{X_{\mathbb{F}_p}} \otimes \mathcal{O}^7_{\mathbb{P}_{\mathbb{F}_p}}(-8).$$

Thus, since $\chi(\mathcal{O}_X) = 0$, from the previous exact sequence we get

$$\chi(\mathcal{N}^\vee_{X_{\mathbb{F}_p}} \otimes \mathcal{O}_X(-3e + 8)) = 3PH_{X_{\mathbb{F}_p}}(-4e + 8) + 1 - p_a.$$

Using the other expression for $\chi(\mathcal{N}^\vee_{X_{\mathbb{F}_p}} \otimes \mathcal{O}_X(-3e+8))$, obtained using the previous lemmas, we find the expression of $c_3/2$ announced in the theorem. ■

3.3 Determination of ρ and of the Hodge Diamond

3.3.1 Invariants of Calabi–Yau 3-folds

Let us recall that the Hodge numbers $h^{i,j}$ of a complex projective variety X are defined by $h^{i,j} := h^i(X, \wedge^j \Omega_X))$ and satisfy Hodge duality: $h^{i,j} = h^{j,i}$. So, the Hodge diamond of a Calabi–Yau 3-fold X has the following shape:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

A Calabi–Yau 3-fold thus has only two Hodge invariants: $h^{1,1}$ and $h^{1,2}$. Using Hirzebruch–Riemann–Roch, we find that

$$\chi(\Omega_X) = -\frac{c_3}{2} = h^{1,2} - h^{1,1}.$$

By Serre duality, we have $h^{1,2} = h^1(\theta_X)$, where $\theta_X = (\Omega_X)^* = \wedge^2\Omega_X$ is the tangent bundle of X; the number $h^{1,2}$ is thus the dimension of the space of first order infinitesimal complex deformations of X. Using the long exact sequence of cohomology associated with the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ and the vanishing $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$, we get an abelian group isomorphism between $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ and $H^2(X, \mathbb{Z})$. Since the rank of $H^2(X, \mathbb{Z})$ is b_2, the second Betti number of X, we get $b_2 = h^{1,1} = \operatorname{rank}(\operatorname{Pic}(X)) =: \rho$. Thus the second Hodge invariant of a Calabi–Yau 3-fold is nothing but the Picard number ρ of X. Therefore, if X is projective, we have $h^{1,1} = \rho \geq 1$, since the Picard lattice contains the rank 1 sublattice generated by H the class of a hyperplane. Finally, the invariants of a Calabi–Yau 3-fold of degree d in $\mathbb{P}^7$ satisfy the following further properties. Using Hirzebruch–Riemann–Roch, we find the following expression for the Hilbert polynomial of X

$$PH_X(t) = \frac{d}{3!}t^3 + \frac{c_2 \cdot H}{12}t.$$

Moreover, if X is linearly normal, we have $c_2 \cdot H = 96 - 2d$. Let us also recall how to relate $\rho = h^{1,1}$ and $h^{1,2}$ to the cohomology of the normal bundle $\mathcal{N}_X$.

Proposition 3.5 *Let X be a Calabi–Yau 3-fold of degree d in $\mathbb{P}^7$. We have*

$$h^{1,1} = h^2(X, \theta_X) = h^1(X, \mathcal{N}_X) - h^2(X, \mathcal{N}_X) + 1,$$
$$h^{1,2} = h^0(X, \mathcal{N}_X) - 8h^0(X, \mathcal{O}_X(1)) + 1.$$

Proof By Kodaira vanishing, we have $h^i(X, \mathcal{O}_X(1)) = 0$ for $i > 0$. Applying this vanishing to the long exact sequence of cohomology of the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X^{\oplus 8}(1) \to \theta_{X|\mathbb{P}^7} \to 0,$$

we get $h^i(X, \theta_{X|\mathbb{P}^7}) = 0$ for $i \geq 3$ or $i = 1$ and $h^2(X, \theta_{X|\mathbb{P}^7}) = h^3(X, \mathcal{O}_X) = 1$. We also get $h^0(X, \theta_{X|\mathbb{P}^7}) = 8h^0(X, \mathcal{O}_X(1)) - 1$. By Serre duality, we get $h^{1,1} = h^2(X, \theta_X)$. Applying $\operatorname{Hom}(-, \mathcal{O}_X)$ to the following exact sequence

$$0 \to \mathcal{N}_X^* \to \Omega_{\mathbb{P}^7} \otimes \mathcal{O}_X \to \Omega_X \to 0,$$

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we get

$$0 \longrightarrow H^0(X, \theta_{X|\mathbb{P}^7}) \longrightarrow H^0(X, \mathcal{N}_X) \longrightarrow H^1(X, \theta_X) \longrightarrow H^1(X, \theta_{X|\mathbb{P}^7}) \longrightarrow \cdots .$$

The formula follows from the previously established vanishing results. ■

Thus, if we know the value of c_3, to get the two Hodge invariants, it is enough to know $h^{1,1} = \rho$ or $h^0(\mathcal{N}_X)$ to get the third invariant. We give in the next section an algebraic criterion that guarantees $\rho = 1$.

3.3.2 Determination of ρ

One can compute ρ directly by computer; but on many examples the computation exceeds the machine capacity. We explain here how to show that $\rho = 1$ on several examples by computing $h^0(X, \mathcal{N}_X)$; it is often a much faster computation provided that we can use the following observation.

Theorem 3.6 *Let k be a perfect field. Let X_k be a smooth 3-dimensional variety in $\mathbb{P}^r_k$. Let I be the saturated ideal of X_k in $\mathbb{P}^r_k$. We denote by S the polynomial ring $k[x_0, \dots, x_r]$ and by R the quotient ring S/I. Let $P \xrightarrow{\phi} L$ be a presentation matrix of $I \otimes_S R$ as R-module. We set $N = \mathrm{Hom}_R(I \otimes R, R)$, so that $\widetilde{N} = \mathcal{N}_X$. Assume the following:*

(i) *The projective dimension of R as S-module is at most r.*
(ii) $\dim_k(P_0^*) = h^0(X, \widetilde{P^*})$ *and* $\dim_k(L_0^*) = h^0(X, \widetilde{L^*}) - \epsilon$ *where* $\epsilon \geq 0$.
(iii) *If $\epsilon \geq 1$, assume, moreover, that*

$$\dim_k(N_0) - 8h^0(X, \mathcal{O}_X(1)) + 1 + \frac{c_3}{2} \leq 1 - \epsilon.$$

Then we have $h^0(X, \mathcal{N}_X) = \dim_k(N_0) + \epsilon$.

Proof By assumption, since N is a locally free R-module, N is the kernel of the transpose of ϕ, so that we have the following sequences of k-vector spaces:

$$0 \longrightarrow H^0(X, \mathcal{N}_X) \longrightarrow H^0(X, \widetilde{L^*}) \xrightarrow{\psi} H^0(X, \widetilde{P^*}),$$

$$0 \longrightarrow N_0 \longrightarrow {L^*}_0 \xrightarrow{\phi^*} {P^*}_0.$$

By assumption on the projective dimension of R as S-module, we have

$$\mathrm{Ext}_S^{r+1}(L, S(-r-1)) = 0,$$

so the exact sequence of comparison with local duality simplifies to

$$0 \longrightarrow L_0^* \longrightarrow H^0(X, \widetilde{L^*}) \longrightarrow \mathrm{Ext}_S^r(L, S(-r-1)) \longrightarrow 0.$$

In particular, the k-vector space L_0^* is a sub-vector space of $H^0(X, \widetilde{L^*})$. Similarly, P_0^* is a sub-vector space of $H^0(X, \widetilde{P^*})$. By assumption, we get $H^0(X, \widetilde{P^*}) = P_0^*$. If $\epsilon = 0$, then clearly $h^0(X, \mathcal{N}_X) = \dim_k(N_0)$. If $\epsilon \geq 1$, then ψ is of the form $(\phi^* \mid c_1 \cdots c_\epsilon)$ for some column matrices $c_1, \dots, c_\epsilon$. Thus, by the dimension formula $\dim_k \ker(\psi) \leq \dim_k \ker(\phi^*) + \epsilon$. Assume that $\ker(\psi) = \ker(\phi^*) \oplus k^\lambda$, for $\lambda \leq \epsilon - 1$; using assumption (iii) we would get $\rho \leq 0$. This gives a contradiction, so that the theorem holds. ■

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4 Examples Found Using Macaulay2

4.1 Examples of Calabi–Yau 3-Folds in $\mathbb{P}^7$

We have gathered here the examples of Calabi–Yau 3-folds of $\mathbb{P}^7$ that we have found and the invariants computed over $\mathbb{F}_p$ by the method explained in Section 2. For all of these examples $\rho = 1$. In this table, KM means of Kustin–Miller type and GN means of Gulliken-Negård type.

H^3	c_3	$c_2 \cdot H$	$h^0(X,H)$	type	ρ	comments
15	-150	66	8	KM	1	$(\mathbb{G}(2,5) \cap \mathbb{P}^7) \cap (3)$
17	-112	62	8	GN	$2^{\dagger}$?	seems to be new
17	-108	62	8	KM	1	seems to be new
18	-162	72	9	KM	$2^{\dagger}$	projection of $\sigma(\mathbb{P}^2 \times \mathbb{P}^2) \cap (3)$
20	-64	56	8	GN	$2^{\dagger}$	seems to be new

†Our computation shows that $h^1(X, \Omega_X) = 2$ over the finite field used for the construction. By semi-continuity, we can only deduce that $1 \leq \rho \leq 2$. G. Kapustka has informed us that he can show that for our examples of degree 18 and 20 the correct Picard number is 2.

4.1.1 Degree 15 of Kustin–Miller Type

Let us choose $\mathcal{E} = \mathcal{O}^7$, $\mathcal{F} = \mathcal{O}^3$, $\mathcal{L}_1 = \mathcal{O}(1)$, and $\mathcal{L}_2 = \mathcal{O}$. Then, choosing random morphisms A, Y, u, v, we get a 4-codimensional subscheme of $\mathbb{P}^7_{\mathbb{F}_{101}}$. Those morphisms are clearly reduction modulo p of morphisms A, Y, u, v defined over $\mathbb{Z}$. The theory explained in Section 3 applies, and we show that the Kustin–Miller subscheme $X_{\mathbb{F}_{101}}$ is a smooth Calabi–Yau 3-fold, reduction modulo p of a smooth Calabi–Yau 3-fold defined over $\mathbb{Q}$ (hence over $\mathbb{C}$). The minimal graded free resolution of this Calabi–Yau $X_{\mathbb{F}_{101}}$ has the following Betti table

0	1	–	–	–	–
1	–	5	5	–	–
2	–	1	–	1	–
3	–	–	5	5	–
4	–	–	–	–	1

The complex Calabi–Yau 3-fold thus obtained has the following Hilbert polynomial

$$PH_X(t) = \frac{5}{2}t^3 + \frac{11}{2}t,$$

thus $c_2 \cdot H = 66$ as expected. Using duality, the numbers $h^1(X, \mathcal{N}_X)$ (resp. $h^2(X, \mathcal{N}_X)$) can be easily computed with Macaulay2 using the command

```
hilbertFunction({0},Ext^5(I/I^2,S^{-8}))
```

Both vanish, so that $\rho = 1$. The coarse smoothness test (Theorems 3.1 and 3.2) over $\mathbb{F}_{101}$ gives $c_3 = -150$, thanks to a Macaulay2 computation.

Remark 4 The simplest idea for constructing a Calabi–Yau 3-fold is to use the adjunction formula. So, for instance, we can build a Calabi–Yau 3-fold by taking a

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generic enough degree 3 section of a Fano 4-fold Y in $\mathbb{P}^7$ such that $K_Y = -3H_Y$. Recall the section Y by two generic hyperplane sections of the Plücker embedding in $\mathbb{P}^9$ of the Grassmanian of lines in $\mathbb{P}^4$, $\mathbb{G}(1,4)$. Taking a generic enough hypersurface section of degree 3 of Y, we thus get a Calabi–Yau 3-fold X of degree 15 in $\mathbb{P}^7$ with the same invariants $\rho = 1$, $c_2 \cdot H = 66$ $c_3 = -150$. (see [14]). This construction leads to a Calabi–Yau 3-fold with the same syzygies as our example of degree 15.

4.1.2 Degree 17 of Gulliksen–Negård Type

Let us choose $\mathcal{E} = \mathcal{O}^3$ and $\mathcal{F} = \mathcal{O}^2 \oplus \mathcal{O}(2)$. Then, taking a random morphism ϕ in $\mathrm{Hom}(\mathcal{E},\mathcal{F})$ over $\mathbb{F}_{101}$, we find a morphism reduction modulo p of some morphism ϕ in $\mathrm{Hom}(\mathcal{E},\mathcal{F})$ defined over $\mathbb{Z}$. The theory explained in Section 2 thus applies and we show that the Gulliksen–Negård subscheme $X_{\mathbb{F}_{101}}$ is a smooth Calabi–Yau 3-fold, reduction modulo p of a smooth Calabi–Yau 3-fold defined over $\mathbb{Q}$ (hence over $\mathbb{C}$). The minimal graded free resolution of this Calabi–Yau $X_{\mathbb{F}_{101}}$ has the following Betti table:

0	1	–	–	–	–
1	–	3	2	–	–
2	–	6	12	6	–
3	–	–	2	3	–
4	–	–	–	–	1

The complex Calabi–Yau 3-fold thus obtained has the Hilbert polynomial

$$PH_X(t) = \frac{17}{6}t^3 + \frac{31}{2}t,$$

thus $c_2 \cdot H = 62$ as expected. Using duality, the numbers $h^1(X,\mathcal{N}_X)$ (resp. $h^2(X,\mathcal{N}_X)$) can be easily computed with Macaulay2 using the command

```
hilbertFunction({0},Ext^5(I/I^2,S^{-8}))
```

We find that $h^1(X,\mathcal{N}_X) = 1$ and $h^2(X,\mathcal{N}_X) = 0$ over $\mathbb{F}_{101}$, so that $h^1(X,\Omega_{X_{\mathbb{F}_{101}}}) = 2$. Thus, by semicontinuity we can only deduce that $\rho \leq 2$ over the complex field. Georg Kaputska has informed us he can prove that $\rho = 2$.

The coarse smoothness test (Theorems 3.1 and 3.2) over $\mathbb{F}_{101}$ gives $c_3 = -112$, thanks to a Macaulay2 computation.

4.1.3 Degree 17 of Kustin–Miller Type

Let us choose $\mathcal{E} = \mathcal{O}^5$, $\mathcal{F} = \mathcal{O}(-1) \oplus \mathcal{O}^2$, $\mathcal{L}_1 = \mathcal{O}(1)$, and $\mathcal{L}_2 = \mathcal{O}(1)$. Then, choosing random morphisms A, Y, u, v, we get a 4-codimensional subscheme of $\mathbb{P}^7_{\mathbb{F}_{101}}$. Those morphisms are clearly reduction modulo p of morphisms A, Y, u, v defined over $\mathbb{Z}$. The theory explained in Section 2 applies, and we show that the Kustin–Miller subscheme $X_{\mathbb{F}_{101}}$ is a smooth Calabi–Yau 3-fold, reduction modulo p of a smooth Calabi–Yau 3-fold defined over $\mathbb{Q}$ (hence over $\mathbb{C}$). The minimal graded free resolution of this

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Calabi–Yau $X_{\mathbb{F}_{101}}$ has the following Betti table:

0	1	–	–	–	–
1	–	3	–	–	–
2	–	4	12	4	–
3	–	–	–	3	–
4	–	–	–	–	1

The complex Calabi–Yau 3-fold thus obtained has the Hilbert polynomial

$$PH_X(t) = \frac{17}{6}t^3 + \frac{31}{2}t,$$

thus $c_2 \cdot H = 62$ as expected. Using duality, the numbers $h^1(X, \mathcal{N}_X)$ (resp. $h^2(X, \mathcal{N}_X)$) can be easily computed with Macaulay2 using the command

```
hilbertFunction({0},Ext^5(I/I^2,S^{-8}))
```

We find that both numbers vanish over $\mathbb{F}_{101}$, so that $\rho = 1$. The coarse smoothness test (Theorems 3.1 and 3.2) over $\mathbb{F}_{101}$ gives $c_3 = -108$, thanks to Macaulay2 computation.

4.1.4 Degree 18 of Kustin–Miller Type

Let us choose $\mathcal{E} = \Omega_{\mathbb{P}^7}(1)$, $\mathcal{F} = \mathcal{O}(-1)^3$, $\mathcal{L}_1 = \mathcal{O}(1)$, and $\mathcal{L}_2 = \mathcal{O}(-1)$. Then, choosing random morphisms A, Y, u, v, we get a 4-codimensional subscheme of $\mathbb{P}^7_{\mathbb{F}_{107}}$. Those morphisms are clearly reduction modulo p of morphisms A, Y, u, v defined over $\mathbb{Z}$. The theory explained in Section 2 applies, and we show that the Kustin–Miller subscheme $X_{\mathbb{F}_{107}}$ is a smooth Calabi–Yau 3-fold, reduction modulo p of a smooth Calabi–Yau 3-fold defined over $\mathbb{Q}$ (hence over $\mathbb{C}$). The minimal graded free resolution of this Calabi–Yau $X_{\mathbb{F}_{107}}$ has the following Betti table:

0	1	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–
2	–	21	55	71	56	28	8	1
3	–	1	8	8	8	1	–	–
4	–	–	–	–	1	–	–	–

The complex Calabi–Yau 3-fold obtained this way has the Hilbert polynomial $PH_X(t) = 3t^3 + 6t$, thus $c_2 \cdot H = 72$. Therefore, X is not linearly normal; it is the projection of some Calabi–Yau 3-fold of $\mathbb{P}^8$. The coarse smoothness test (Theorems 3.1 and 3.2) over $\mathbb{F}_{107}$ gives $c_3 = -162$, thanks to a Macaulay2 computation.

Remark 5 Consider the Segre embedding σ of $\mathbb{P}^2 \times \mathbb{P}^2$ into $\mathbb{P}^8$. Its image $\sigma(\mathbb{P}^2 \times \mathbb{P}^2)$ is a well-known Fano 4-fold of degree 9 such that $K_Y = -3H_Y$. Taking a section of this by a generic enough degree 3 hypersurface in $\mathbb{P}^8$, we get a Calabi–Yau 3-fold of degree 18 such that $\rho = 1$, $c_2 \cdot H = 72$, and $c_3 = -162$. A generic projection to $\mathbb{P}^7$ of such a Calabi–Yau 3-fold has the same syzygies as our example of degree 18. The fastest way to estimate ρ with Macaulay2 is to directly compute ρ for the 3-fold Y in $\mathbb{P}^8$ via the command

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```
HH^1(cotangentSheaf(Y))
```

We find that $h^1(X, \Omega_{X_{\mathbb{F}_{101}}}) = 2$, so again we can only deduce that $\rho \leq 2$.

4.1.5 Degree 20 of Gulliksen–Negård Type

Let us choose $\mathcal{E} = \mathcal{O}^4$ and $\mathcal{F} = \mathcal{O}(1)^4$. Then, taking a random morphism ϕ in $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ over $\mathbb{F}_{101}$, we find a morphism reduction modulo p of some morphism ϕ in $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ defined over $\mathbb{Z}$. The theory explained in Section 2 thus applies, and we show that the Gulliksen–Negård subscheme $X_{\mathbb{F}_{101}}$ is a smooth Calabi–Yau 3-fold, reduction modulo p of a smooth Calabi–Yau 3-fold defined over $\mathbb{Q}$ (hence over $\mathbb{C}$). The minimal graded free resolution of this Calabi–Yau $X_{\mathbb{F}_{101}}$ has the following Betti table:

0	1	–	–	–	–
1	–	–	–	–	–
2	–	16	30	16	–
3	–	–	–	–	–
4	–	–	–	–	1

The complex Calabi–Yau 3-fold thus obtained has the Hilbert polynomial

$$PH_X(t) = \frac{10}{3}t^3 + \frac{16}{3}t,$$

thus $c_2 \cdot H = 56$ as expected. Using the locally free resolution found over $\mathbb{F}_{101}$, the required vanishing in Theorem 3.6 is easy to establish. Thus, we find $\rho = 1$. The coarse smoothness test (Theorems 3.1 and 3.2) over $\mathbb{F}_{101}$ gives $c_3 = -64$, thanks to a Macaulay2 computation.

Remark 6 Thus, we have $h^{1,2} = 33$; it is the dimension of the first order complex infinitesimal deformation of X. It is worth pointing out that this number is not the number of first order infinitesimal deformations of $X_{\mathbb{F}_{101}}$ over $\mathbb{F}_{101}$. This last number is 34, as shown by an easy Macaulay2 calculation.

4.2 Examples of Non Deformation Equivalent Calabi–Yau 3-Folds Sharing the Same Invariants

Our search for good vector bundles $\mathcal{E}$ of (odd) low rank defined over $\mathbb{Z}$, such that the Pfaffian subscheme X_1 associated with the data $(\mathcal{E}, \wedge^2\mathcal{E}(1), Y)$ had expected codimension 3, gave us two vector bundles for which $\deg(u) + \deg(v) = 1$ (see Remark 3). Thus, constructing a Kustin–Miller subscheme out of any of these bundles gives a 3-dimensional subvariety X contained in some hyperplane. It is thus presumably a Calabi–Yau 3-fold of $\mathbb{P}^6$, hence a Pfaffian subscheme of $\mathbb{P}^6$. The two such examples of Calabi–Yau 3-folds of $\mathbb{P}^6$ are new examples of degree 14 and 15. Both examples are not arithmetically Cohen–Macaulay and linearly normal. Since c_3 only depends on d for linearly normal Calabi–Yau 3-folds of $\mathbb{P}^6$, these examples have the same invariants as the examples of degree 14 and 15 that were already known (*cf.* [13]), provided that we can show that $\rho = 1$. Notice that $h^{1,2} = h^0(X, \mathcal{N}_X) - 48$ for every linearly normal

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Calabi–Yau 3-fold of $\mathbb{P}^6$. It is thus enough to show that both examples of degree 14 (respectively 15) $\mathcal{N}_X$ have the same number of global sections. We use the method explained in Theorem 3.6. We shall show here that our new examples in degree 14 and 15 do have Picard number one and are not deformation equivalent to the previously known examples.

4.2.1 Syzygies and Deformations of Calabi–Yau 3-Folds in $\mathbb{P}^6$

Let us recall the meaning of deformation equivalence.

Definition 4.1 Two smooth algebraic varieties over $\mathbb{C}$ are said to be *deformation equivalent* if there exists a smooth morphism $\mathcal{X} \xrightarrow{\phi} T$, of smooth complex algebraic varieties $\mathcal{X}$ with T connected such that the following condition holds: there exist $s, t \in T$ such that the fiber $\mathcal{X}_s$ coincides with X_0 and the fiber $\mathcal{X}_t$ coincides with X_1.

In case of deformation equivalent embedded projective Calabi–Yau 3-folds, we can even assume that ϕ is an embedded deformation.

Proposition 4.2 *Let X_0 and X_1 be two deformation equivalent Calabi–Yau 3-folds of $\mathbb{P}^6$. Any deformation ϕ giving the equivalence is induced by an embedded deformation in $\mathbb{P}^6$. That is to say, we have the following diagram*

$$\begin{array}{ccccc} \mathcal{X} & \subset & T & \times & \mathbb{P}^6 \\ & \overset{\phi}{\searrow} & \swarrow_{\pi_1} & & \downarrow \pi_2 \\ & & T & & \mathbb{P}^6 \end{array}$$

Proof We have the global Zariski–Jacobi sequence that relates deformations and embedded deformations

$$0 \longrightarrow \mathbb{T}^0_X \longrightarrow \mathbb{T}^0_{\mathbb{P}^6}(\mathcal{O}_X) \longrightarrow \mathbb{T}^1_{X|\mathbb{P}^6} \longrightarrow \mathbb{T}^1_X \longrightarrow \mathbb{T}^1_{\mathbb{P}^6}(\mathcal{O}_X) \longrightarrow \mathbb{T}^2_{X|\mathbb{P}^6} \cdots,$$

where $\mathbb{T}^1_{X|\mathbb{P}^6}$ is the space of embedded deformations of X in $\mathbb{P}^6$ and $\mathbb{T}^1_X$ the space of complex deformations of X. We have $\mathbb{T}^1_{\mathbb{P}^6}(\mathcal{O}_X) = H^1(X, \theta_{X|\mathbb{P}^6})$, thus using the fact that X is a Calabi–Yau and the long exact sequence of cohomology associated with

$$0 \to \mathcal{O}_X \to \mathcal{O}^7_X(1) \to \theta_{X|\mathbb{P}^7} \to 0,$$

we get $h^1(X, \theta_{X|\mathbb{P}^7}) = 0$. Therefore, $\mathbb{T}^1_{\mathbb{P}^6}(\mathcal{O}_X) = 0$ for $i = 1$. This shows that $\mathbb{T}^1_{X|\mathbb{P}^6} \twoheadrightarrow \mathbb{T}^1_X$ is surjective.Therefore, any deformation of X is induced by an embedded deformation of X in $\mathbb{P}^6$. ■

Thus, if two Calabi–Yau 3-folds X_1 and X_2 are deformation equivalent by ϕ, their Betti tables cannot be arbitrary. Let us denote by $\mathcal{I}_X$ the ideal sheaf of X in $T \times \mathbb{P}^6$. Take some graded free presentation of $\mathcal{I}_X$,

$$S_2 = \bigoplus_{j=0}^{s_2} \mathcal{O}^{\beta_{2,j}}_{T\times\mathbb{P}^6}(-j-2) \xrightarrow{R} S_1 = \bigoplus_{j=0}^{s_1} \mathcal{O}^{\beta_{1,j}}_{T\times\mathbb{P}^6}(-j-1) \xrightarrow{F} \mathcal{I}_X \longrightarrow 0.$$

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Due to the flatness assumption, we have the following very elementary property used in the examples below.

Proposition 4.3 *Let $s = \min\{j \mid \beta_{1,j} \neq 0\}$. For all $t \in T$, we have $h^0(\mathbb{P}^6, \mathcal{I}_{X_t}(s)) \neq 0$ and $h^0(\mathbb{P}^6, \mathcal{I}_{X_t}(j)) = 0$ for all $j < s$. That is to say, the minimal degree of generators of the saturated ideal defining X_t does not depend on t and is exactly $s + 1$.*

Proof Since by flatness assumption, the graded free presentation of $\mathcal{I}_X$ specializes to a presentation of $\mathcal{I}_{X_t}$, we have $h^0(\mathbb{P}^6, \mathcal{I}_{X_t}(j)) = 0$ for all $j < s$. To show the remaining assertion we only must show that $h^0((\mathbb{P}^6, \mathcal{I}_{X_t}(s)) \neq 0$ for all $t \in T$. Since $\mathcal{I}_{X_t}(s) = (\mathcal{I}_X)_t$ for all $t \in T$, the function $t \mapsto h^0((\mathbb{P}^6, \mathcal{I}_{X_t}(s)) \neq 0$ is upper semicontinuous on T. It is thus enough to show that $h^0((\mathbb{P}^6, \mathcal{I}_{X_t}(1)) \neq 0$ for t generic in T. We can assume for simplicity that T is an affine variety over $\mathbb{C}$, $T = \operatorname{spec}(R)$. Then, by assumption $\beta_{1,s} \neq 0$, there exists a non-zero polynomial F of degree $s + 1$ in $I_X \subset R[x_0, \ldots, x_6]$. We have $F = \sum f_I m_I$, where f_I is some polynomial in R and (m_I) is a basis of $(\mathbb{C}[x_0, \ldots, x_6])_s$. Assume that $h^0((\mathbb{P}^6, \mathcal{I}_{X_t}(s)) = 0$ for t generic in T. For t generic in T, we then have $f_i(t) = 0$ for all $i = 0, \ldots, 6$, so that the vanishing locus of $(f_0, \ldots, f_6)$ in T is exactly T. Thus the ideal $(f_0, \ldots, f_6) = (0)$ and $F = 0$. This is clearly a contradiction. ■

Remark 7 A similar result holds replacing T by $\operatorname{spec}(\mathbb{Z})$, assuming one has a flat family of schemes X over some Zariski open subspace of $\operatorname{spec}(\mathbb{Z})$. Hence, the number $s = \min\{j \mid \beta_{1,j}(X_p) \neq 0\}$ does not depend on p in U.

4.2.2 Degree 14

Take $\mathcal{E} = \Omega_{\mathbb{P}^7} \oplus \mathcal{O}(1)$ and $\mathcal{L}_1 = \mathcal{O}(1)$. Choose a random morphism

$$Y \in \operatorname{Hom}(\mathbb{P}^6, \wedge^2 \mathcal{E} \otimes \mathcal{L}_1)$$

over $\mathbb{F}_{101}$. Then Y is the restriction modulo 101 of a morphism Y in

$$\operatorname{Hom}(\mathbb{P}^6, \wedge^2 \mathcal{E} \otimes \mathcal{L}_1)$$

over $\mathbb{Z}$. The associated Pfaffian subscheme $X_{\mathbb{F}_{101}}$ is a smooth Calabi–Yau 3-fold of degree 14 in $\mathbb{P}^6$(applying Tonoli's smoothness test) and is the restriction modulo 101 of a Calabi–Yau 3-fold defined over $\mathbb{Z}$. The Hilbert polynomial of this Calabi–Yau 3-fold is $\frac{7}{3}t^3 + \frac{14}{3}t$, so that $c_2 \cdot H = 56$, $c_3 = -98$ and $h^{1,2} = 50$ ($\rho = 1$). The minimal graded free resolution of $X_{\mathbb{F}_{101}}$ has the following Betti table:

0	1	–	–	–	–	–	–
1	–	1	–	–	–	–	–
2	–	–	–	–	–	–	–
3	–	14	35	35	21	7	1
4	–	–	–	1	–	–	–

Let us compute ρ. Using the Pfaffian resolution of X, we get $H^i(X, \mathcal{O}_X(m)) = 0$ for all $i > 0$ and $m \geq 2$. Let R denote the quotient ring

$$\frac{k[x_0, \ldots, x_6]}{I_X},$$

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where I_X is the saturated ideal defining X. The Betti table shows that X is 5-regular, so that $h^0(X, \mathcal{O}_X(t)) = \dim_k(R_t)$ for $t \geq 5$. Moreover, a simple Macaulay2 computation gives $h^0(X, \mathcal{O}_X(2)) = \dim_k(R_2) + 1$ and $h^0(X, \mathcal{O}_X(t)) = \dim_k(R_t)$ for $5 \geq t \geq 3$. Using Macaulay2, we find that

$$0 \to N \to R^1(2) \oplus R^{14}4 \xrightarrow{\phi^*} \oplus R^{35}(5).$$

We thus find $\epsilon = 1$. Using Macaulay2 again, we find $\dim_k(N_0) = 97$. Then Theorem 3.6 gives $h^0(X, \mathcal{N}_X) = 98$, so that we find that $h^{1,2} = 50$ and $\rho = 1$.

The known example of degree 14 can be obtained taking $\mathcal{E} = \mathcal{O}^7$ and $\mathcal{L}_1 = \mathcal{O}(1)$ [13]. It has the same invariants as our example. Its Betti table is the following:

0	1	–	–	–
1	–	–	–	–
2	–	7	7	–
3	–	–	–	–
4	–	–	–	1

Clearly for the first example of degree 14 we have $\min\{j \mid \beta_{1,j}(X_{\mathbb{C}}) \neq 0\} = 1$, whereas $\min\{j \mid \beta_{1,j}(X_{\mathbb{C}}) \neq 0\} = 2$ in the second example. Thus, these two examples cannot be deformation equivalent, even though they have the same invariants $(H^3, c_2 \cdot H, c_3, \rho) = (14, 56, -98, 1)$.

4.2.3 Degree 15

Let $\mathcal{E}^2_{0,3}$ denote the first syzygy module, the kernel of the morphism $\mathcal{O}^{10} \xrightarrow{\psi} \mathcal{O}^2(1)$ defined by the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}.$$

This type of syzygy bundle can be thought of as a generalization of $\Omega(1)$. Take $\mathcal{E} = \mathcal{E}^2_{0,3} \oplus \mathcal{O}(1)$ and $\mathcal{L}_1 = \mathcal{O}(1)$. Choose a random morphism $Y \in \mathrm{Hom}(\mathbb{P}^6, \wedge^2 \mathcal{E} \otimes \mathcal{L}_1)$ over $\mathbb{F}_{101}$. Then Y is the restriction modulo 101 of a morphism Y in $\mathrm{Hom}(\mathbb{P}^6, \wedge^2 \mathcal{E} \otimes \mathcal{L}_1)$ over $\mathbb{Z}$. The associated Pfaffian subscheme $X_{\mathbb{F}_{101}}$ is a smooth Calabi–Yau 3-fold of degree 15 in $\mathbb{P}^6$(applying Tonoli's smoothness test) and is the restriction modulo 101 of a Calabi–Yau 3-fold defined over $\mathbb{Z}$. The Hilbert polynomial of this Calabi–Yau 3-fold is $\frac{5}{2}t^3 + \frac{9}{2}t$, so $c_2 \cdot H = 54$, $c_3 = -78$ and $h^{1,2} = 40$ ($\rho = 1$). The minimal graded free resolution of $X_{\mathbb{F}_{101}}$ has the following Betti table:

0	1	–	–	–	–	–	–
1	–	1	–	–	–	–	–
2	–	–	–	–	–	–	–
3	–	4	4	–	–	–	–
4	–	19	70	99	70	26	4

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Let us compute ρ. Using the Pfaffian resolution of X, we get $H^i(X, \mathcal{O}_X(m)) = 0$ for all $i > 0$ and $m \geq 2$. Let R denote the quotient ring $\frac{k[x_0,\dots,x_6]}{I_X}$, where I_X is the saturated ideal defining X. The Betti table shows that X is 5-regular, so that $h^0(X, \mathcal{O}_X(t)) = \dim_k(R_t)$ for $t \geq 5$. Moreover, a simple Macaulay2 computation gives $h^0(X, \mathcal{O}_X(2)) = \dim_k(R_2) + 2$, $h^0(X, \mathcal{O}_X(3)) = \dim_k(R_2) + 4$, and $h^0(X, \mathcal{O}_X(t)) = \dim_k(R_t)$ for $5 \geq t \geq 4$. Using Macaulay2, we find that

$$0 \longrightarrow N \longrightarrow R^1(2) \oplus R^{19}5 \oplus R^4(4) \xrightarrow{\phi^*} \oplus R^{70}(6) \oplus R^4(5).$$

We thus find $\epsilon = 2$. Using again Macaulay2, we find $\dim_k(N_0) = 86$. Then Theorem 3.6 gives $h^0(X, \mathcal{N}_X) = 88$, so that we find that $h^{1,2} = 40$ and $\rho = 1$.

The known example of degree 15 can be obtained taking $\mathcal{E} = \Omega(1) \oplus \mathcal{O}^3$ and $\mathcal{L}_1 = \mathcal{O}(1)$ [13]. It has, of course, the same invariants as our example. Its Betti table is the following:

0	1	–	–	–	–	–	–
1	–	–	–	–	–	–	–
2	–	3	–	–	–	–	–
3	–	11	34	35	21	7	1
4	–	–	–	1	–	–	–

Clearly for the first example of degree 15 we have $\min\{j \mid \beta_{1,j}(X_{\mathbb{C}}) \neq 0\} = 1$, whereas $\min\{j \mid \beta_{1,j}(X_{\mathbb{C}}) \neq 0\} = 2$ in the second example. Thus, these two examples cannot be deformation equivalent, even though they have the same Hodge invariants

Remark 8 Use the family of vector bundles $\mathcal{E}^t_{0,3}$ with $t \geq 1$ defined to be the kernel of the morphism $\mathcal{O}^{4+3t} \xrightarrow{\psi} \mathcal{O}^t(1)$ defined by the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & & & & & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}.$$

The generic Pfaffian subscheme associated with $(\mathcal{E}^t_{0,3} \oplus \mathcal{O}(1), \mathcal{O}(1))$ always seems to be 3-codimensional; it gives a locally Gorenstein subscheme X_t of degree $13 + t$ for which $\omega_{X_t} = \mathcal{O}_{X_t}$. Unfortunately, Tonoli's smoothness test over $\mathbb{F}_{101}$ fails for X_t, for $t \geq 3$.

The libraries of Macaulay2 programs that I have built to construct those Calabi–Yau 3-folds and compute their invariants are available upon request.

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