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Nearby cycles for log smooth families

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Abstract. We calculate l-adic nearby cycles in the étale cohomology for families with log smooth reduction using log étale cohomology. In particular, nearby cycles for log smooth families coincide with tame nearby cycles, as L. Illusie expected, and nearby cycles for semistable families depend only on the first infinitesimal neighborhood of the special fiber.

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0. Introduction

In this paper we calculate l-adic nearby cycles for log smooth families using log étale cohomology.

The point is that, though our concerned families may not be smooth, they start to behave as if they were smooth once equipped with natural log structures. Then our calculation is as easy and transparent as that for usual smooth families.

Our main result (3.4.1) specializes to ((3.4.2))

THEOREM (0.1). *Let $X \to S = \mathrm{Spec}(A)$ be a morphism of schemes with A being a henselian discrete valuation ring. Let s be the closed point of S and $\eta = \mathrm{Spec}\, K$ the generic one. Let $n \geqslant 1$ be an integer invertible on S. For any $q \in \mathbf{Z}$, we have the nearby cycle*

$$\mathrm{R}^q\Psi_\eta \mathbf{Z}/n\mathbf{Z} \ \textit{ on the product topos } \ X_s \times_s S \quad (\text{SGA7 XIII 2.1.1}).$$

Suppose that X has log smooth reduction in the sense explained below. Then the action on $\mathrm{R}^q\Psi_\eta \mathbf{Z}/n\mathbf{Z}$ of the wild inertia group P of K is trivial.

COROLLARY (0.1.1). *In the situation of* (0.1), *we assume further that $X \to S$ is proper. Then the action of P on $\mathrm{H}^q(X_{\overline{\eta}}, \mathbf{Z}/n\mathbf{Z})$ is trivial, where $\overline{\eta}$ is the spectrum of a separable closure of K.*

We explain what is log smooth reduction. For example, a generalized semistable family $\mathrm{Spec}(A[x_1, \ldots, x_d]/(x_1^{m_1} \cdots x_d^{m_d} - \pi))$ with $(m_1, \ldots, m_d, p) = 1$ and

$m_j > 0$ for each j has log smooth reduction, where p is the residual characteristic exponent of A. Further this notion has the advantage of being stable under fiber products and base changes, unlike semistability. Precisely this means that étale locally on X, X is étale over Spec$(A[P]/(\pi - x))$. Here P is a finitely generated, saturated (commutative) monoid, π is a prime element of A, and x is an element of P such that

(i) the order of the torsion part of $P^{\text{gp}}/\langle x\rangle$ is invertible on X; and
(ii) for any $a \in P$, there is an $m \geqslant 1$ and $b \in P$ such that $ab = x^m$ in P. [1]

Here we review some background of this theorem. Let notation and assumptions be the same as in (0.1) except that X does not necessarily have log smooth reduction. We only assume that X is of finite type over S. Even in this general situation, the nearby cycles are known to be an important object, which connect the étale cohomology of the geometric generic fiber with that of the special fiber.

The conclusion of (0.1) has been proved by M. Rapoport and Th. Zink in the following cases (1) and (2) in [RZ] Sections 2, 3:

(1) X has generalized semistable reduction whose multiplicities m_i are all invertible on S.
(2) X is étale locally a product of semistable curves.

In [I], L. Illusie pointed out this is valid also if

(3) X is the product of two semistable families;

and he stated one could expect that the conclusion is always valid whenever X has log smooth reduction ([I] 4.10). Note that X has log smooth reduction if it satisfies one of the above three conditions.

Thus (0.1) gives an affirmative answer to his expectation.

In fact (0.1) and (0.1.1) are easily deduced from the formula (3.2)(i)

$$\mathrm{R}\Phi^{\log}\mathbf{Z}/n\mathbf{Z} = 0$$

on the log vanishing cycle for a log smooth morphism. This formula is the exact analogue of the classical result (cf. SGA7 I 2.4) on the usual vanishing cycle for a smooth morphism. This formula also shows that, under this assumption, $\mathrm{R}\Psi_\eta\mathbf{Z}/n\mathbf{Z}$, as a complex of sheaves with Galois action, depends only on the special fiber endowed with its natural log structure. In the case of semistable reduction, it implies that $\mathrm{R}\Psi_\eta\mathbf{Z}/n\mathbf{Z}$ depends only on $X \otimes_A A/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of A (see (3.3)).

As in the classical case, the above formula is reduced to a case of purity in the log context as follows: Let S, A, n be as in (0.1). Put the canonical log structure M_S defined by the closed point on S (cf. [K1](1.5)(1)).

[1] The condition (ii) is rather a technical one. See (3.4.2) and the paragraph following (0.2).

THEOREM (0.2). *Let* $(X, M_X) \to (S, M_S)$ *be a log smooth morphism of fs log schemes.*

Then the natural homomorphism

$$\mathbf{Z}/n\mathbf{Z} \to \mathrm{R}^{+} j_{*}\mathbf{Z}/n\mathbf{Z}$$

on $X^{\log}_{\text{ét}}$ *is a quasi-isomorphism, where* j *is the open immersion from the maximal open subscheme* X_{triv} *of* X *on which the log structure is trivial (*R^{+} *means the derived functor from* D^{+}*).*

(See [K1] and [N] for the terminology on log schemes and on log étale cohomology.)

To prove this (0.2), we appeal to an unpublished result of K. Fujiwara and K. Kato, the invariance of l-adic log étale cohomology under log blowing-ups ([FK] (2.4)). Admitting this, we reduce (0.2) to the case of generalized semistable reduction $X = A[x_1, \ldots, x_d]/(x_1^{m_1} \cdots x_d^{m_d} - \pi)$. In this case an induction on $m_1 + \cdots + m_d$ works via an argument by T. Saito ([S] Prop. 6′). When $m_1 + \cdots + m_d = 1$, the family is smooth, and we use the smooth base change theorem in the usual étale cohomology. (When an $m_i = 0$, X may no longer have log smooth reduction in the sense explained above. Actually our proof yields the statement (3.4.1) more general than (0.1) because (0.2) treats X with $X_{\text{triv}} \neq X \otimes_S \eta$, that is, X that may contain horizontal log as well. Cf. [N](7.3).)

In Section 1, we treat the case $m_1 + \cdots + m_d = 1$, and in Section 2, we prove (0.2). In Section 3, we prove main results including (0.1), and discuss other corollaries of (0.2), including a formula of SGA7 I 3.3-type. This section could be titled ‘applications of (0.2) to the usual étale cohomology theory’. The last Section 4 could be called ‘applications of (0.2) to log étale cohomology theory’. We prove a relative log Poincaré duality for log smooth families.

Finally we give some comments on two unpublished results that we use in this paper. As was stated above, one of them is [FK] (2.4) which is used in (2.0.3), (2.3)(iii), (2.4) Step 2 and (A.1.1). The other is used in (2.0.2) and (3.6). See (2.0.2) for the detail.

CONVENTIONS. In this paper, a ring (resp. a monoid) means a commutative ring (resp. monoid) having a unit element. A homomorphism of monoids (resp. rings) is required to preserve the unit elements. The log structure defined by the closed point on Spec(A) with A being a discrete valuation ring is called the canonical log structure (cf. [K1] (1.5) (1)).

Terminology and notation in this paper are completely compatible with those in [N] except that we take the abbreviation ‘log’ instead of ‘log.’. Definitions not given here are referred to [K1] and [N]. In particular log structures are always considered on the étale sites of schemes. Here we include a list of Notation for convenience.

Notation

$\twoheadrightarrow$	(bold-headed arrow) strict morphism (cf. [N] (1.4)).
$\overset{\circ}{X}, \overset{\circ}{f}$	the underlying scheme of a log scheme X and the underlying morphism of schemes of that of log schemes f (cf. [N] (1.1.2)).
$X^{\mathrm{cl}}, f^{\mathrm{cl}}$	$X^{\mathrm{cl}} = (\overset{\circ}{X}, \text{trivial log structure})$. $-^{\mathrm{cl}}$ is a functor from (fs log sch) to the category of (fs) log schemes with trivial log structure (cf. [N] (1.1.2)).
$\varepsilon(X), \varepsilon$	the forgetting log morphism $X \to X^{\mathrm{cl}}$ for an fs log scheme X (cf. [N] (1.1.2)).
(fs log sch)	the category of fs log schemes which belong to a fixed universe
$\mathbf{Z}P$	$:= \mathbf{Z}[P]$, semigroup algebra of a monoid P over $\mathbf{Z}$. We regard $\mathbf{Z}$ – as a functor.
$P^{1/n}$	an fs monoid with a homomorphism from P such that $P \to P^{1/n}$ is isomorphic to $P \xrightarrow{n} P$ where P is an fs monoid having no invertible elements (except the unit element) (cf. [N] (2.7)).
X_n	$X \otimes_{\mathbf{Z}P} \mathbf{Z}P^{1/n}$ for $X \twoheadrightarrow \mathrm{Spec}(\mathbf{Z}P)$ in (fs log sch) with $P^{\times} = 1$ and $n \geqslant 1$ an integer invertible on X (cf. [N] (2.7)).
M_X	the log structure of a log scheme X (cf. [N] (1.1.1)).
$(M/\mathcal{O}^*)_X$	$:= M_X/\mathcal{O}_X^*$ for a log scheme X (cf. [N] (1.1.1)).
$X_{\text{ét}}^{\log}$	log étale site of an fs log scheme X (cf. [N] (2.2)).
$\mathcal{S}_X^A$	the category of sheaves of A-modules on $X_{\text{ét}}^{\log}$ for a ring A and an fs log scheme X (cf. [N] (2.3)).
$\mathrm{D}^*(X, A)$	the derived category $\mathrm{D}^*(\mathcal{S}_X^A)$ for $* = +, -, b$ or empty (cf. [N] (2.3)).
(S)	the category of all quasi-compact and quasi-separated fs log schemes over an fs log scheme S and all S-compactifiable morphisms (cf. [N] (5.4)).
I-A-Mod$_{(/X)}$	the category of A-Modules on the usual étale site of a scheme X, on which a commutative profinite group I acts continuously. Here A is a ring (cf. [N] (4.5)).
$\mathcal{M}ap_{\mathrm{c},I}(J, M)$	the sheaves of continuous I-maps from J to M for a homomorphism of commutative profinite groups $I \to J$ and I-A-Module M (cf. [N] (4.7)(i)).
X_{triv}	the maximal open subscheme of an fs log scheme X on which the log structure is trivial.
Λ	$\mathbf{Z}/n\mathbf{Z}$ with n an integer (cf. (3.1.5)).

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1. Log base change by standard affine maps

(1.1). In this section we prove the lemma below. This is the part of log smooth base change theorem that is directly deduced from the usual smooth base change theorem. In general, log smooth base change theorem fails in its naive form ((B.1)), and it seems to be difficult to settle out a suitably restricted statement.

LEMMA. *Let*

$$\begin{array}{ccc} U[\mathbf{N}] & \xrightarrow{\pi'} & X[\mathbf{N}] \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[\pi]{} & X \end{array}$$

be a cartesian diagram in (fs log sch)*where* $X[\mathbf{N}] = X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{Z}[\mathbf{N}])$ *and* f *is the first projection. Here* $\mathrm{Spec}(\mathbf{Z}[\mathbf{N}])$ *is endowed with log structure given by* $\mathbf{N}$. *Let* $F \in \mathcal{S}_U^{\mathbf{Z}}$ *be a log étale sheaf of Abelian groups on* U *that is the inverse image of a sheaf of Abelian groups on* U^{cl} *and that is killed by an integer invertible on* X. *Assume that* $\overset{\circ}{\pi}$ *is quasi-finite and quasi-separated. Then the functorial homomorphism (base change morphism)* $f^*\mathrm{R}^+\pi_*F \to \mathrm{R}^+\pi'_*f'^*F$ *is an isomorphism in* $\mathrm{D}^+(X[\mathbf{N}], \mathbf{Z})$.

Proof. We shall reduce to the case (1.2), where the log str. of X is trivial, by localization on X as follows.

First we may assume that there is a chart $X \longrightarrow \mathrm{Spec}\,(\mathbf{Z}P)$ with P fs (= finitely generated, saturated cf. [N] (1.2)) and having no invertible elements (except the unit element). Here $\longrightarrow$ means that $X \to \mathrm{Spec}(\mathbf{Z}P)$ is strict, that is, the morphism induces an isomorphism of log structures. This reduction is in order to use (1.1.1) later. Take any point y of $X[\mathbf{N}]$ and put $x = f(y)$. Fix a log geometric point ([N] (2.5)) $y(\log) \to X[\mathbf{N}]$, and regard $y(\log) \to X[\mathbf{N}] \to X$ as a log geometric point of X: $x(\log) \to X$. It is enough to show that the natural homomorphism φ: $(\mathrm{R}^q\pi_*F)_{x(\log)} \to (\mathrm{R}^q\pi'_*f'^*F)_{y(\log)}$ is an isomorphism for any q.

On the one hand

$$(\mathrm{R}^q\pi_*F)_{x(\log)} = \varinjlim_{X'} (\mathrm{R}^q(\varepsilon \circ (\pi \times_X X'))_*(F|_{U'}))_{\overline{x'}}$$

where the limit runs over the category of X-morphisms $x(\log) \to X'$ with $X' \in \mathrm{Ob}\, X^{\log}_{\text{ét}}$, $\varepsilon = \varepsilon(X')$ denotes the forgetting morphism (cf. Notation) for X', $U' = U \times_X X'$, and $\overline{x'}$ is the geometric point of X'^{cl} induced by $x(\log) \to X'$. Further for each X', considering the cartesian diagram

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$$\begin{array}{ccc} U'[\mathbf{N}] & \xrightarrow{\pi_1} & X'^{\mathrm{cl}}[\mathbf{N}] \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ U' & \xrightarrow[\varepsilon\circ(\pi\times_X X')]{} & X'^{\mathrm{cl}} \end{array},$$

we have a natural homomorphism

$$\begin{aligned} \varphi_{X'^{\mathrm{cl}}}:\ (\mathrm{R}^q(\varepsilon\circ\pi_{X'})_*(F|_{U'}))_{\overline{x'}} &= (f_1^*\mathrm{R}^q(\varepsilon\circ\pi_{X'})_*(F|_{U'}))_{y'(\log)} \\ &\to (\mathrm{R}^q\pi_{1*}f_1^*(F|_{U'}))_{y'(\log)} \end{aligned}$$

where $\pi_{X'} = \pi\times_X X'$ and $y'(\log) = y(\log) \to X'\times_X X[\mathbf{N}] \to X'^{\mathrm{cl}}[\mathbf{N}]$ defined by $y(\log) = x(\log) \to X'$ and by the fixed $y(\log) \to X[\mathbf{N}]$. (The first = is induced by [N] (2.8)1.) On the other hand we have

$$(\mathrm{R}^q\pi'_*f'^*F)_{y(\log)} = \varinjlim_{Y'} (\mathrm{R}^q(\varepsilon\circ(\pi'\times_{X[\mathbf{N}]}Y'))_*((f'^*F)|_{U\times_X Y'}))_{\overline{y'}}$$

where the limit runs over the category of $X[\mathbf{N}]$-morphisms $y(\log) \to Y'$ with $Y' \in \mathrm{Ob}(X[\mathbf{N}])^{\log}_{\text{ét}}$, and y' is the geometric point of Y'^{cl} induced by $y(\log) \to Y'$.

Thus φ can be viewed as the composition of

$$(\mathrm{R}^q\pi_*F)_{x(\log)} \xrightarrow{\varphi_1} \varinjlim_{X'} (\mathrm{R}^q\pi_{1*}f_1^*(F|_{U'}))_{y'(\log)}$$

and

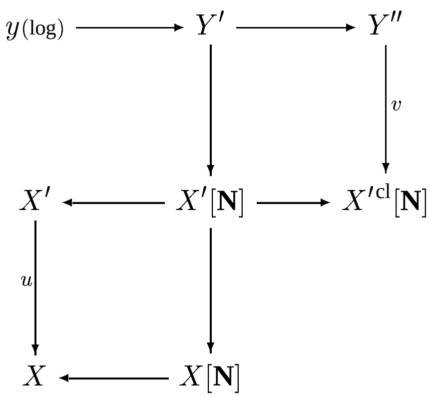
$$\begin{aligned} &\varinjlim_{X'}\varinjlim_{Y''} \mathrm{R}^q(\varepsilon\circ(\pi_1\times_{X'^{\mathrm{cl}}[\mathbf{N}]}Y''))_*(f'^*F|_{U'\times_{X'^{\mathrm{cl}}}Y''})_{\overline{y''}} \\ &\xrightarrow{\varphi_2} \varinjlim_{Y'} \mathrm{R}^q(\varepsilon\circ(\pi'\times_{X[\mathbf{N}]}Y'))_*(f'^*F|_{U\times_X Y'})_{\overline{y'}} \end{aligned}$$

where φ_1 is induced by $\varphi_{X'^{\mathrm{cl}}}$'s, Y'' runs over the category of $X'^{\mathrm{cl}}[\mathbf{N}]$-morphisms $y(\log) \to Y''$ with $Y'' \in \mathrm{Ob}(X'^{\mathrm{cl}}[\mathbf{N}])^{\log}_{\text{ét}}$, and y'' is the geometric point of Y''^{cl} induced by $y(\log) \to Y''$.

We will prove that both φ_1 and φ_2 are isomorphisms. First we treat φ_2. We write $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) for the index category on which the limit of left-hand side (resp. right-hand side) of φ_2 runs. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are essentially the same index categories by the next lemma (1.1.1). Further, since the underlying morphism of $Y' \to Y''$ in (1.1.1) is always an isomorphism, the arguments of limits in both sides are also the same. Thus we see that φ_2 is an isomorphism. To prove that φ_1 is an isomorphism, it is enough to show that each $\varphi_{X'^{\mathrm{cl}}}$ is an isomorphism. But since $(\pi \times_X X')^\circ$ is also quasi-finite and quasi-separated by [N] (1.10), this is implied by the case of the original statement of (1.1) for the map $\pi = \varepsilon \circ (\pi \times_X X')$. Thus we reduce (1.1) to the case where X has the trivial log structure, which will be treated in (1.2). $\square$

LEMMA (1.1.1). *Let* $X \longrightarrow \mathrm{Spec}\,(\mathbf{Z}[P])$ *be a morphism in* (fs log sch) *with* P *being an fs monoid having no invertible elements. Let* $y(\log) \to X[\mathbf{N}]$ *be a log geometric point* ([N](2.5)). *Consider the following two categories:*

$\mathcal{C}_1 :=$ *the category of diagrams*

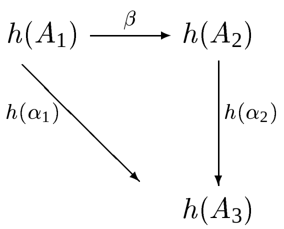


in (fs log sch) *with each square cartesian such that* u *and* v *are log étale and of Kummer type, and that* $y(\log) \to X[\mathbf{N}]$ *coincides with the given one, and*

$\mathcal{C}_2 :=$ *the category of* $X[\mathbf{N}]$*-morphisms* $y(\log) \to Y'$ *with* $Y' \in \mathrm{Ob}\,(X[\mathbf{N}])^{\log}_{\text{ét}}$.

Then the forgetful functor $h\colon \mathcal{C}_1 \to \mathcal{C}_2$ *satisfies the following properties*:

(i) *For any* $B \in \mathrm{Ob}\,\mathcal{C}_2$*, there is an* $A \in \mathrm{Ob}\,\mathcal{C}_1$ *such that* $\mathrm{Hom}_{\mathcal{C}_2}(B, h(A)) \neq \phi$.
(ii) *For any* A_1, $A_2 \in \mathrm{Ob}\,\mathcal{C}_1$ *and any* $\beta\colon h(A_1) \to h(A_2)$ *in* $\mathcal{C}_2$*, there exists a diagram* $A_1 \xrightarrow{\alpha_1} A_3 \xleftarrow{\alpha_2} A_2$ *such that*



commutes in $\mathcal{C}_2$.

Proof. (i) Take a chart of $X[\mathbf{N}] \longrightarrow \mathrm{Spec}\,(\mathbf{Z}[P \oplus \mathbf{N}])$. Let $y(\log) \to Y'$ be in $\mathcal{C}_2$. Then we may assume that there is an integer n invertible on Y' such that Y' has a chart $Y' \longrightarrow \mathrm{Spec}\,(\mathbf{Z}[P^{1/n} \oplus \mathbf{N}^{1/n}])$. Since $X[\mathbf{N}] \to X$ is an open map,

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$Y' \to X[\mathbf{N}]$ factors through $X'[\mathbf{N}] \to X[\mathbf{N}]$ for a certain $X' \to X$ that has a chart $P^{1/n} \leftarrow P$. Putting $Y'' = (\mathring{Y}', (\mathbf{N}^{1/n})^{\mathrm{a}})$ completes the construction of an object of $\mathcal{C}_1$.

(ii) Let $A_1, A_2 \in \mathrm{Ob}\,\mathcal{C}_1$. We may assume that A_1 and A_2 have the common u, Y' and Y'' (but may have different v's), and that the top square of A_i factors through

$$\begin{array}{ccc} Y' & \longrightarrow & Y'' \\ \downarrow & & \downarrow{\scriptstyle v_i} \\ X'[\mathbf{N}^{1/n}] & \xrightarrow{f} & X'^{\mathrm{cl}}[\mathbf{N}^{1/n}] \end{array}$$

$(i = 1, 2)$. Since $\mathring{f}$ is an isomorphism, we have the desired construction. □

CLAIM (1.2). *Lemma* (1.1) *is valid if the log structure of* X *is trivial.*

Proof. This case is reduced to the usual smooth base change theorem as follows. First factor $U \to X$ into $U \to U^{\mathrm{cl}} \to X$ and apply proper base change theorem [N] (5.1) to $U \to U^{\mathrm{cl}}$ (cf. [N] (5.1.1)), then we see that we may assume that the log structure of U is also trivial.

Take any point y of $X[\mathbf{N}]$. It suffices to prove that the homomorphism is bijective for each degree q at y. To prove this, we may suppose by [N] (4.2) that $\mathring{X}$ is strictly local (i.e. the spectrum of a strictly henselian local ring) and that $f(y) =: x$ is its closed point. Let t denote the coordinate of $X[\mathbf{N}]$ on which the log structure lives. On $X[t, t^{-1}]$, the desired bijectivity comes from the usual smooth base change theorem SGA4 XVI 1.2. So we assume that $(M/\mathcal{O}^*)_{X[\mathbf{N}],\overline{y}} = \mathbf{N}$ in the following. We write

$$\begin{array}{ccc} U\{\mathbf{N}\} & \xrightarrow{\pi''} & X\{\mathbf{N}\} \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ U\{t\} & \longrightarrow & X\{t\} \end{array}$$

for the diagram obtained from

$$\begin{array}{ccc} U[\mathbf{N}] & \longrightarrow & X[\mathbf{N}] \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ U[t] & \longrightarrow & X[t] \end{array}$$

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by base change with respect to $\mathcal{O}_{X[t],\overline{y}} \to X[t]$. It suffices to show that $\mathrm{H}^q(U, F) = (\mathrm{R}^q\pi_*F)_{x(\log)} \to (\mathrm{R}^q\pi''_*\varepsilon^*F')_{y(\log)} = \varinjlim_{n\colon \text{invertible on } X\{t\}} \mathrm{H}^q(U\{\mathbf{N}^{1/n}\}, \varepsilon^*F')$ is bijective, where $F' = (U\{t\} \to U)^*F$ and $U\{\mathbf{N}^{1/n}\} := U\{\mathbf{N}\} \otimes_{\mathbf{ZN}} \mathbf{ZN}^{1/n}$ (cf. Notation). But this map factors as

$$\mathrm{H}^q(U, F) \xrightarrow{\alpha} \varinjlim \mathrm{H}^q(U\{t^{1/n}\}, \pi_n^*F') \xrightarrow{\beta} \varinjlim \mathrm{H}^q(U\{\mathbf{N}^{1/n}\}, \varepsilon^*\pi_n^*F'),$$

where π_n is the projection $U\{t^{1/n}\} = U\{t\} \otimes_{\mathbf{Z}[t]} \mathbf{Z}[t^{1/n}] \to U\{t\}$. The bijectivity of β comes from the next Lemma (1.2.1). Thus the problem reduces to a statement in terms of the usual étale cohomology theory, namely that $\mathrm{H}^q(U, F) \to \mathrm{H}^q(U\{t^{1/n}\}, \pi_n^*F')$ is bijective for each n. But the local acyclicity for the smooth morphism $X[t^{1/n}] \to X$ (SGA4 XV 2.1) implies that $X\{t^{1/n}\} \to X$ is acyclic, so that the above homomorphism is bijective for any q (SGA4 XV 1.6 (iii)). This completes the proof of (1.1). □

LEMMA (1.2.1). *Let $\pi\colon U \longrightarrow X$ be a strict morphism of fs log schemes with $\overset{\circ}{X}$ strictly local whose closed point is denoted by x. Let $X \longrightarrow \mathrm{Spec}(\mathbf{Z}P)$ be a chart with P being an fs monoid such that $P \to (M/\mathcal{O}^*)_{X,\overline{x}}$ is bijective. Let $F \in \mathcal{S}_U^{\mathbf{Z}}$ be the inverse image of an $F_0 \in \mathcal{S}_{U^{\mathrm{cl}}}^{\mathbf{Z}}$ such that F_0 is killed by an integer invertible on X. Then the natural map*

$$\varinjlim_n \mathrm{H}^q((U_n^{\mathrm{cl}})_{\text{ét}}, (U_n^{\mathrm{cl}} \to U^{\mathrm{cl}})^*F_0) \to (\mathrm{R}^q\pi_*F)_{x(\log)},$$

where n runs over the set of integers invertible on X, is an isomorphism for all q. (See Notation for $\mathcal{S}_*^{\mathbf{Z}}$, U_n and $(-)^{\mathrm{cl}}$.)

Proof. Since $(\mathrm{R}^q\pi_*F)_{x(\log)} = \varinjlim \mathrm{H}^q((U_n)_{\text{ét}}^{\log}, (U_n \to U)^*F)$, it is enough to show that the map from the usual étale cohomology $\varinjlim \mathrm{H}^q(U_n^{\mathrm{cl}}, (U_n^{\mathrm{cl}} \to U^{\mathrm{cl}})^*F_0)$ to the log étale cohomology $\varinjlim \mathrm{H}^q(U_n, (U_n \to U)^*F)$ is bijective. Let k_n denote the projection $U_n \to U$. We consider the Leray spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(U_n^{\mathrm{cl}}, \mathrm{R}^j\varepsilon_*\varepsilon^*(k_n^{\mathrm{cl}})^*F_0) \Rightarrow \mathrm{H}^{i+j}(U_n, k_n^*F)$$

and its limit

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(U_\infty^{\mathrm{cl}}, \varinjlim (U_\infty^{\mathrm{cl}} \to U_n^{\mathrm{cl}})^*\mathrm{R}^j\varepsilon_*\varepsilon^*(k_n^{\mathrm{cl}})^*F_0) \Rightarrow \varinjlim \mathrm{H}^{i+j}(U_n, k_n^*F),$$

where $U_\infty^{\mathrm{cl}} = \varprojlim U_n^{\mathrm{cl}}$. It suffices to show that the sheaf on U_∞^{cl} is zero for $j > 0$. (As for $j = 0$, $\varepsilon_*\varepsilon^* = \mathrm{id}$.) We investigate transition homomorphisms stalk by stalk as follows.

SUBLEMMA. *Let $U \to \mathrm{Spec}(\mathbf{Z}P)$ be a strict morphism of fs log schemes with P*

being an fs monoid having no invertible elements. Let m be an integer invertible on U, and let k_m denote the projection $U_m \to U$. Let $F \in \mathcal{S}_U^{\mathbf{Z}/m\mathbf{Z}}$ be the inverse image of an $F_0 \in \mathcal{S}_{U^{\mathrm{cl}}}^{\mathbf{Z}/m\mathbf{Z}}$. Then $(k_m^{\mathrm{cl}})^* \mathrm{R}^j \varepsilon_* F \to \mathrm{R}^j \varepsilon_* k_m^* F$ is zero for any $j > 0$.

Once this was proved, applying it with $(U, P, F_0) = (U_n, P^{1/n}, (k_n^{\mathrm{cl}})^* F_0)$ for all n, we get the desired result. The rest is to prove this sublemma. Since ε is proper, [N] (5.1) reduces this to the case where $\overset{\circ}{U}$ is the spectrum of a separably closed field, replacing U by $\overline{u}$ for each $u \in U$. Then $(U_m)^{\circ}_{\mathrm{red}}$ is a disjoint union of $\overset{\circ}{U}$, and [N] (4.7) and (4.7.1) reduce the problem to the one in terms of Galois cohomology: the restriction $\mathrm{H}^j(I, F_0) \overset{\alpha}{\to} \mathrm{H}^j(mI, F_0)$ is zero where $I = \mathrm{Hom}((M/\mathcal{O}^*)^{\mathrm{gp}}_U, \widehat{\mathbf{Z}}'(1))$ and the action of I on F_0 is trivial. But when we identify mI with I by the multiplication by m, this α is identified with the multiplication by m^j, because $H^j = \bigwedge^j H^1$. □

Remark (1.3). We could slightly generalize (1.1), with the aid of [FK] (2.4). See Appendix (A.1).

2. Purity for log smooth families

(2.0). In this section we prove (0.2). See Introduction for the outline.

Though (0.2) is written in terms of log étale cohomology, it can be reformulated in terms of usual étale cohomology ((2.0.2)).

(2.0.1). We start from a general

CONJECTURE (log purity conjecture). *Let X be an fs log scheme with $\overset{\circ}{X}$ locally noetherian. Assume that X is log regular* ([T], *cf.* [K2]) *and let j be the open immersion $X_{\mathrm{triv}} \hookrightarrow X$. Let n be an integer invertible on X. Then for any locally constant sheaf of* $\mathbf{Z}/n\mathbf{Z}$*-modules F on $X^{\log}_{\text{ét}}$, the adjoint homomorphism*

$$F \to \mathrm{R}^+ j_* j^* F$$

is an isomorphism.

PROPOSITION (2.0.2). *In the same notation and assumptions as in* (2.0.1), *assume that X satisfies the conclusion of* (2.0.1). *Then, for any q, we have*

$$\mathrm{R}^q \, \overset{\circ}{j}_* \, \mathbf{Z}/n\mathbf{Z} = \bigwedge^q ((M/\mathcal{O}^*)^{\mathrm{gp}}_X \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z}(-1)),$$

where $\overset{\circ}{j}$ is the open immersion $X_{\mathrm{triv}} \hookrightarrow \overset{\circ}{X}$.

Proof. This comes from a basic result [KN] (2.4) of K. Kato, which we include in Appendix (A.3). □

(2.0.3). In fact the Conjecture (2.0.1), suggested by [FK], is equivalent to the classical purity conjecture of Grothendieck. The outline of the proof of this fact in [FK] (3.6) is as follows: Let D be a divisor with simple normal crossings on a regular locally noetherian scheme Z. Then the case of (2.0.2) for (Z, M_D) is nothing but the classical conjecture for D, where M_D is the log structure defined by D ([K1] (1.5) (1)). Conversely (2.0.1) is reduced to the case where the log structure of X is defined by free monoids $\mathbf{N}^r$ for some r's by using a log blowing up ([K2] (10.4)) and [FK] (2.4). Then we reduce it to the classical conjecture by (A.3). See (2.4) Step 2 in this paper for the similar method.

Remark (2.0.4). Thus the relative purity SGA4 XVI 3.7 implies that if X is log smooth over the spectrum of a field with the trivial log structure, (2.0.1) holds on X. See [N] (7.7.1).

Remark (2.0.5). O. Gabber announced a proof of the classical purity conjecture of Grothendieck in 1994. It implies the validity of (2.0.1) because of (2.0.3). In the following, we prove the case (0.2) of (2.0.1) by a different method.

(2.1.1). While (0.2) is stated for log smooth morphisms, we want to work under slightly weaker conditions on account of induction. So we introduce the weaker condition (W) before starting the proof.

DEFINITION (W). Let A be a discrete valuation ring, and let S be an fs log scheme $\text{Spec}(A)$ with the canonical log structure (cf. Conventions). We say a morphism $f\colon X \to S$ in (fs log sch) is (W) if there exist an étale covering $(X_i \twoheadrightarrow X)$ and a chart (in the sense of [K1] (2.9)) $((P_i)_{X_i} \to M_{X_i}, \mathbf{N}_S \xrightarrow{\varphi_i} M_S, \mathbf{N} \xrightarrow{h_i} P_i)$ of $X_i \to X \to S$ with P_i fs ([N] (1.2)) for each i satisfying the following conditions (i) and (ii).

(i) The homomorphism h_i is injective and $(P_i)_{\text{tor}}$ is trivial.
(ii) The induced (strict) morphism $X_i \to S \times_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[P_i]$ is étale.

By (A.2), if a morphism of fs log schemes $f\colon X \to S$ is log smooth, then f is (W).

EXAMPLE (2.1.2). Let A and S be as in (2.1.1). Let $f\colon X \to S$ be a generalized semistable family $\text{Spec}(A[x_1, \dots, x_d]/(x_1^{m_1} \cdots x_d^{m_d} - \pi)) \to \text{Spec}(A)$ that is the base change of $\text{Spec}(\mathbf{ZN}^d) \xrightarrow{\text{Spec}(\mathbf{Z}h)} \text{Spec}(\mathbf{ZN})$ with respect to a chart $S \xrightarrow{\varphi} \text{Spec}(\mathbf{ZN})$, where $h\colon \mathbf{N} \to \mathbf{N}^d$; $1 \mapsto (m_1, \dots, m_d)$ with $m_1 + \cdots + m_d \geqslant 1$, and π is the prime element $\varphi(1) \in A$. Then f is always (W), and is log smooth if and only if $(m_1, \dots, m_d)$ is invertible on X.

Remark (2.1.3). The condition (W) implies that X is log regular.

(2.2). Now we state a slight generalization of (0.2).

THEOREM. *Let A be a discrete valuation ring, let S be the fs log scheme* $\mathrm{Spec}(A)$ *with the canonical log structure* (*cf. Conventions*). *Let X be an fs log scheme. Assume that there exists a morphism of fs log scheme $f\colon X \to S$ that is* (*W*). (*See* (2.1.1) *above for the definition of* (*W*).) *Then* (2.0.1) *holds on X.*

See Introduction for the outline of the proof of (2.2).

First we treat the case of generalized semistable families:

CLAIM (2.3) $(m_1, \ldots, m_d)_{(A,\pi)}$. *Let $m_1, \ldots, m_d$ be nonnegative integers with $m_1 + \cdots + m_d \geqslant 1$. Then* (2.2) *is valid if* $X = \mathrm{Spec}(A[x_1, \ldots, x_d]/(x_1^{m_1} \cdots x_d^{m_d} - \pi))$ *with log structure defined by* $\mathbf{N}^d \to \mathcal{O}_X$; $e_i \mapsto x_i$ $(1 \leqslant i \leqslant d)$ *where π is a prime element of A and $(e_i)_i$ is the canonical base of* $\mathbf{N}^d$. (*Note that in this case, there exists an f such that $f : X \to S$ is a generalized semistable family and is* (*W*) *as was stated in* (2.1.2).)

Proof. We write $(m_1, \ldots, m_d)_{(A,\pi)}$ for this statement, and $X(m_1, \ldots, m_d)_{(A,\pi)}$ for X in this statement. We will prove $(m_1, \ldots, m_d)_{(A,\pi)}$ by induction on $m_1 + \cdots + m_d$ via an argument in [S] Proposition 6′. First note that the validity of the statement is independent of the order of $m_1, \ldots, m_d$. So it is enough to show the following facts for any (A, π).

(o) $(1, 0, \ldots, 0)_{(A,\pi)}$ is valid by the preliminary Lemma (1.1) as below.

(i) If m is an integer invertible on A, then there is a log étale covering $X(m_1 m, m_2, \ldots, m_d)_{(A,\pi)} \to X(m_1, \ldots, m_d)_{(A,\pi)}$, so $(m_1, \ldots, m_d)_{(A,\pi)} \Rightarrow (m_1 m, m_2, \ldots, m_d)_{(A,\pi)}$.

(ii) If $p>1$ is the residual characteristic of A, then $X(m_1, \ldots, m_d)_{(A[\pi^{1/p}],\pi^{1/p})} = X(m_1 p, \ldots, m_d p)_{(A,\pi)}$, so $(m_1, \ldots, m_d)_{(A[\pi^{1/p}],\pi^{1/p})} \Rightarrow (m_1 p, \ldots, m_d p)_{(A,\pi)}$.

(iii) (key step) $(m_1, \ldots, m_d)_{(A,\pi)}$ and $(m_1+m_2, m_3, \ldots, m_d)_{(A[X]_{(\pi)},X^{-m_1}\pi)} \Rightarrow (m_1, m_1 + m_2, m_3, \ldots, m_d)_{(A,\pi)}$, where $m_1 \geqslant 1$.

To prove (o), we consider a diagram

$$\begin{array}{ccccc} X_{\mathrm{triv}} & \xrightarrow{j_1} & X \otimes_S \eta & \xrightarrow{j_2} & X = X(1, 0, \ldots, 0)_{(A,\pi)} \\ & & \downarrow & & \downarrow f \\ & & \eta & \xrightarrow{j_0} & S \end{array}$$

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in (fs log sch), where η is the generic point of S. By [N] (7.6.5), we have $\mathbf{Z}/n\mathbf{Z} \overset{\cong}{\to} \mathrm{R}^+ j_{1*}\mathbf{Z}/n\mathbf{Z}$. By the preliminary Lemma (1.1), we have $f^*\mathrm{R}^+ j_{0*}\mathbf{Z}/n\mathbf{Z} \overset{\cong}{\to} \mathrm{R}^+ j_{2*}\mathbf{Z}/n\mathbf{Z}$. So it is enough to prove that $\mathbf{Z}/n\mathbf{Z} \overset{\cong}{\to} \mathrm{R}^+ j_{0*}\mathbf{Z}/n\mathbf{Z}$, that is, $(1,0)_{(A,\pi)}$.

To prove this, we may assume that A is strictly henselian. Let q be an integer, and x the closed point of X. We have $(\mathrm{R}^q j_{0*}\mathbf{Z}/n\mathbf{Z})_{x(\log)} = \varinjlim_{\hat{m}} \mathrm{H}^q(K(\pi^{1/m}), \mathbf{Z}/n\mathbf{Z})$ where m runs over the set of integers invertible on A, and K is the fraction field of A. This is equal to $\mathrm{H}^q(\bigcup_m K(\pi^{1/m}), \mathbf{Z}/n\mathbf{Z})$, which is $\mathbf{Z}/n\mathbf{Z}$ for $q = 0$ and is zero for $q > 0$ respectively.

The rest is to prove (iii). To prove (iii), we blow up $X_0 = X(m_1, \ldots, m_d)_{(A,\pi)}$ with center (x_1, x_2), where $(x_i)_i$ is the natural coordinates on X_0 on which the log structure is endowed. There is a natural log structure on the blowing-up X' ([FK] (2.2)) such that X' is covered by $X_1 = X(m_1, m_1 + m_2, m_3, \ldots, m_d)_{(A,\pi)}$ and $X_2 = X(m_1 + m_2, m_2, m_3, \ldots, m_d)_{(A,\pi)}$, and that the projection $X' = X_1 \cup X_2 \xrightarrow{\pi_0} X_0$ is a blowing-up along log structure with center $\langle e_1, e_2\rangle \subset \mathbf{N}^d$ in the sense of [FK] (2.2), where $(e_i)_i$ is the canonical base of $\mathbf{N}^d$. Hence we have $\mathrm{R}\pi_{0*}(\mathbf{Z}/n\mathbf{Z} \overset{\mu}{\to} \mathrm{R}^+ j_*\mathbf{Z}/n\mathbf{Z}$ on X') is isomorphic to $\mathbf{Z}/n\mathbf{Z} \to \mathrm{R}^+ j_*\mathbf{Z}/n\mathbf{Z}$ on X_0 by [FK] (2.4) and the fact that $X'_{\mathrm{triv}} \to (X_0)_{\mathrm{triv}}$ is an isomorphism. (In the course of the proof of (2.2), we use [FK] (2.4) only here and in (2.4) Step 2.) The latter $\mathbf{Z}/n\mathbf{Z} \to \mathrm{R}^+ j_*\mathbf{Z}/n\mathbf{Z}$ is a quasi-isomorphism by $(m_1, \ldots, m_d)_{(A,\pi)}$.

On the other hand, we consider the morphism

$$\widetilde{X_1} = X(m_1 + m_2, m_3, \ldots, m_d)_{(A[X]_{(\pi)}, X^{-m_1}\pi)} \xrightarrow{h} X_1$$

induced by

$$\begin{aligned} X_1 \otimes_{A[x_1]} A[x_1]_{(\pi)} &= A[x_1]_{(\pi)}[x_2, \ldots, x_d]/(x_1^{m_1} x_2^{m_1+m_2} x_3^{m_3} \cdots x_d^{m_d} - \pi) \\ &= A[x_1]_{(\pi)}[x_2, \cdots, x_d]/(x_2^{m_1+m_2} x_3^{m_3} \cdots x_d^{m_d} - x_1^{-m_1}\pi). \end{aligned}$$

Since h is the inverse limit of open immersions, $(m_1 + m_2, m_3, \ldots, m_d)_{(A[X]_{(\pi)}, X^{-m_1}\pi)}$ implies μ: $\mathbf{Z}/n\mathbf{Z} \to \mathrm{R}^+ j_*\mathbf{Z}/n\mathbf{Z}$ on X' is an isomorphism on the image of h by [N] (4.2).

Now we stand at a point to prove that μ is a quasi-isomorphism. We study each fiber of π_0.

Let $T \xrightarrow{g} X_0$ be any strict morphism from the spectrum T of an algebraically closed field k_0. It is enough to show that the inverse image $K^\bullet$ on $T' = T \times_{X_0} X'$ of the mapping cone of μ is acyclic. Consider the cartesian diagrams

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$$\begin{array}{ccccc} \widetilde{T}_1 & \xrightarrow{\widetilde{h}'} & T' & \xrightarrow{\pi'} & T \\ \downarrow & & \downarrow & & \downarrow g \\ \widetilde{X}_1 & \longrightarrow & X' & \xrightarrow{\pi_0} & X_0 \end{array}$$

in (fs log sch). The fiber $\mathring{T}'$ is $\mathbb{P}^1_{k_0}$ or $\text{Spec}(k_0)$. We have already shown: (a) $K^\bullet$ is acyclic on $\widetilde{h}'(\widetilde{T_1})$ and (b) $\text{R}\pi'_* K^\bullet$ is acyclic. When the fiber is $\mathbb{P}^1_{k_0}$, (a) implies $K^\bullet$ is acyclic at the generic point of $\mathbb{P}^1_{k_0}$. Thus $K^\bullet$ lives on only closed points on $\mathring{T}'$, that is, for each q, $\mathcal{H}^q(K^\bullet) = \bigoplus_y \iota_{y*} F^q_y$, where y runs over the set of closed points of $\mathring{T}'$, ι_y is the strict closed immersion $y = \text{Spec}(k_0) \longrightarrow T'$ with the image $\{y\}$, and $F^q_y = \iota^*_y \mathcal{H}^q(K^\bullet)$. On the other hand we have a spectral sequence $\text{E}_2^{p,q} = \text{R}^p \pi'_* \mathcal{H}^q(K^\bullet) \Rightarrow 0$ by (b). Since $\text{R}^p \pi'_*$ preserves $\oplus$, the problem is reduced to the next

CLAIM. *In the situation above,* $(y \xrightarrow{\iota_y} T' \xrightarrow{\pi'} T)_*$ *is faithful and exact, so* $\text{R}^p(\pi' \circ \iota_y)_* = 0$ *for* $p > 0$ *and* $(\pi' \circ \iota_y)_* F^q_y$ *is zero implies that* F^q_y *is zero.*

Proof. Thanks to [N] (4.6) and (4.7), this can be interpreted into the problem on modules with actions of the profinite groups that are determined by the log structures. Since $(M/\mathcal{O}^*)^{\text{gp}}_{T,\bar{t}} \to (M/\mathcal{O}^*)^{\text{gp}}_{T',\bar{y}}$ is surjective (t being the unique point of T), the homomorphism $I \to J$ is injective where $I = \text{Hom}((M/\mathcal{O}^*)^{\text{gp}}_{T',\bar{y}}, \widehat{\mathbf{Z}}'(1))$ and $J = \text{Hom}((M/\mathcal{O}^*)^{\text{gp}}_{T,\bar{t}}, \widehat{\mathbf{Z}}'(1))$. Then the functor $(\pi' \circ \iota_y)_*$, which is isomorphic to the functor $\mathcal{M}ap_{\text{c},I}(J, -)\colon I\text{-}\mathbf{Z}/n\mathbf{Z}\text{-Mod} \to J\text{-}\mathbf{Z}/n\mathbf{Z}\text{-Mod}$ (cf. Notation), is faithful and exact as is explained in [N] (4.7.1). This completes the proof of (2.3). □

(2.4). *Proof of* (2.2). We go through several reduction steps finally to (2.3).

Step 0. We may and will assume that A is strictly henselian by [N] (4.2) in the following.

Step 1. The statement is local on X, so that we may assume by the definition of (W) that X is strict étale over the fiber product Y of a diagram $S \xrightarrow{\pi \leftarrow\!\dashv 1} \text{Spec}(\mathbf{ZN}) \xleftarrow{\text{Spec}(\mathbf{Z}h)} \text{Spec}(\mathbf{Z}P)$, where π is a prime element of A, h is an injective homomorphism of fs monoids, and P_{tor} is trivial.

Step 2. We will show that we can assume that for any $x \in X$, $(M/\mathcal{O}^*)_{X,\bar{x}} \cong \mathbf{N}^{r(x)}$ for some $r(x) \geqslant 0$. First we may assume that X has a chart by Step 1. Now we appeal to [FK] (2.4). By [K2] (10.4), we construct a morphism $g\colon X' \to X$ in (fs log sch) such that:

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(1) For each $x' \in X'$, $(M/\mathcal{O}^*)_{X',\overline{x'}}$ is a free monoid.
(2) g is log étale (and proper), and $X'_{\text{triv}} \to X_{\text{triv}}$ is an isomorphism.
(3) $\mathrm{R}g_*g^* \xleftarrow{\simeq} \mathrm{id}$ on $\mathrm{D}(X, \mathbf{Z}/n\mathbf{Z})$. ([FK] (2.4).)

This construction appears in [FK] (3.6). By the fact that $X'_{\text{triv}} \to X_{\text{triv}}$ is an isomorphism and the fact (3), $\mathrm{R}g_*(\mathbf{Z}/n\mathbf{Z} \to \mathrm{R}^+j_*\mathbf{Z}/n\mathbf{Z}$ on $X')$ is isomorphic to $\mathbf{Z}/n\mathbf{Z} \to \mathrm{R}^+j_*\mathbf{Z}/n\mathbf{Z}$ on X. Further $X' \to S$ is also (W). So (2.2) for X reduces to that for X'. Thus we may and will assume that for any $x \in X$, $(M/\mathcal{O}^*)_{X,\overline{x}} \cong \mathbf{N}^{r(x)}$ for some $r(x) \geqslant 0$ in the following.

Step 3. We will show that in the situation of Step 1, we can assume further that P is free. (Recall that we have already assumed that each stalk of $(M/\mathcal{O}^*)_X$ is free in Step 2.) Take any point x of X. Put $S := \alpha_{\overline{x}}^{-1}(\mathcal{O}^*_{X,\overline{x}})$, a submonoid of P. Then the open subset $X \otimes_{\mathbf{Z}P} \mathbf{Z}[P_S]$ of X contains x where P_S is the submonoid $S^{-1}P$ of P^{gp}. So we can replace P with P_S. But $P_S \cong (P_S)^\times \oplus P_S/(P_S)^\times$, and $(P_S)^\times$ is a free Abelian group because P_{tor} is trivial. Further $P_S/(P_S)^\times \overset{\cong}{\leftarrow} P/S \cong (M/\mathcal{O})_{X,\overline{x}}$ is a free monoid. Hence $P_S \cong \mathbf{N}^r \oplus \mathbf{Z}^{r'}$ for some $r \geqslant 0$ and $r' \geqslant 0$. Since any homomorphism $\mathbf{N} \to \mathbf{N}^r \oplus \mathbf{Z}^{r'}$ factors as $\mathbf{N} \to \mathbf{N}^r \oplus \mathbf{N}^{r'} \xrightarrow{\mathrm{id}\oplus\beta} \mathbf{N}^r \oplus \mathbf{Z}^{r'}$ for some injection β, P can be replaced by a free monoid $\mathbf{N}^{r+r'}$.

Step 4. Finally in the situation of Step 1, we may assume that X itself is $\varprojlim(S \xrightarrow{\pi \leftarrow\!\shortmid 1} \mathrm{Spec}(\mathbf{Z}\mathbf{N}) \xleftarrow{\mathrm{Spec}(\mathbf{Z}h)} \mathrm{Spec}(\mathbf{Z}\mathbf{N}^d))$ for some $d \geqslant 1$, where π is a prime element of A and h is an injection. Thus X is a generalized semistable reduction. This completes the proof of (2.2).

Remark (2.5). K. Kato proved a general log smooth base change theorem for a modified log étale site (not yet published). The same statement as (2.0.1) for the modified sites implies (2.0.1). So another way to prove (0.2) is to reduce by this base change theorem to the case for $X = S$.

3. Nearby cycles

(3.0). In this section, we calculate nearby cycles for log smooth families using (2.2) and an easy limit argument. First we establish the formulation of log nearby cycles and their relationships with classical nearby cycles. Then we deduce results on classical nearby cycles including (0.1) and (0.1.1). The point is that the log smoothness is stable under base changes, unlike semistability; so we can apply (2.2) to each family got by base change.

SITUATION (3.1.1). Let A be a henselian discrete valuation ring. Let $X \to S = \mathrm{Spec}(A)$ be a morphism of fs log schemes, where S has the canonical log structure (cf. Conventions). Let s be the closed point of $\overset{\circ}{S}$ and η the generic one. We fix

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a separable closure K^{sep} of the fraction field K of A, and denote by G, I and P the absolute Galois group, the inertia group and the wild inertia group of K respectively.

NOTATION (3.1.2). In (3.1.1), let L be a finite separable extension of K. Let A^L be the integral closure of A in L, and let S^L be the fs log scheme $\mathrm{Spec}(A^L)$ with the canonical log structure (cf. Conventions). We define the diagram of fs log schemes

$$\begin{array}{ccccc} X_s^L & \longrightarrow & X^L & \longleftarrow & X_\eta^L \\ \downarrow & & \downarrow & & \downarrow \\ s^L & \longrightarrow & S^L & \longleftarrow & \eta^L \end{array} \qquad (1)^L$$

with each square cartesian obtained from

$$\begin{array}{ccccc} X_s & \longrightarrow & X & \longleftarrow & X_\eta \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & S & \xleftarrow{\ j\ } & \eta \end{array} \qquad (1)^K$$

by the base change $S^L \to S$. Note that we have the diagram $((1)^L)^{\mathrm{cl}}$ of fs log schemes with trivial log structures (= schemes) and the forgetting log morphism $(1)^L \to ((1)^L)^{\mathrm{cl}}$. In the following we identify $(1)^L$ and $((1)^L)^{\mathrm{cl}}$ with the associated diagrams of log étale topoi respectively. Now we denote by

$$\begin{array}{ccccc} X_{\overline{s}(\log)} & \xrightarrow{\ \overline{i}(\log)\ } & \overline{X}(\log) & \xleftarrow{\ \overline{j}(\log)\ } & \overline{X}_\eta(\log) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{s}(\log) & \longrightarrow & \overline{S}(\log) & \longleftarrow & \overline{\eta} \end{array} \qquad \overline{(1)}$$

$$\left(\text{resp.} \quad \begin{array}{ccccc} X_s^{\mathrm{tame}} & \xrightarrow{\ i^{\mathrm{tame}}\ } & X^{\mathrm{tame}} & \xleftarrow{\ j^{\mathrm{tame}}\ } & X_\eta^{\mathrm{tame}} \\ \downarrow & & \downarrow & & \downarrow \\ s^{\mathrm{tame}} & \longrightarrow & S^{\mathrm{tame}} & \longleftarrow & \eta^{\mathrm{tame}} \end{array} \qquad (1)^{\mathrm{tame}} \right)$$

the essentially commutative diagram of topoi obtained as the 2-limit (SGA4 VI 8.1.1) $\varprojlim_L ((1)^L)$, where L runs over the set of all separable (resp. tame) extensions of K. These diagrams are uniquely determined up to natural equivalences. Further we consider $\overline{(1)}^{\mathrm{cl}} = \varprojlim_{L:\ \mathrm{sep}} ((1)^L)^{\mathrm{cl}}$, $((1)^{\mathrm{tame}})^{\mathrm{cl}} = \varprojlim_{L:\ \mathrm{tame}} ((1)^L)^{\mathrm{cl}}$ and the diagram of diagrams

$$\begin{array}{ccc} \overline{(1)} & \longrightarrow & \overline{(1)}^{\mathrm{cl}} \\ \downarrow & & \downarrow \\ (1)^{\mathrm{tame}} & \longrightarrow & ((1)^{\mathrm{tame}})^{\mathrm{cl}} \\ \downarrow & & \downarrow \\ (1)^K & \longrightarrow & ((1)^K)^{\mathrm{cl}}. \end{array}$$

On the other hand we define $\overline{(1)^{\mathrm{cl}}}$ and $((1)^{\mathrm{cl}})^{\mathrm{tame}}$ as those obtained from $((1)^K)^{\mathrm{cl}}$ by the base change $\overline{S}_{(\log)}{}^{\mathrm{cl}} \to S^{\mathrm{cl}}$ and $(S^{\mathrm{tame}})^{\mathrm{cl}} \to S^{\mathrm{cl}}$ respectively. Note that horizontal arrows in

$$\begin{array}{ccc} \overline{(1)}^{\mathrm{cl}} & \longrightarrow & \overline{(1)^{\mathrm{cl}}} \\ \downarrow & & \downarrow \\ ((1)^{\mathrm{tame}})^{\mathrm{cl}} & \longrightarrow & ((1)^{\mathrm{cl}})^{\mathrm{tame}} \end{array}$$

are not necessarily equivalent. The group G and G/P 'act' on $X_{\overline{s}(\log)}$ and X_s^{tame} respectively. Precisely speaking, $X_{\overline{s}(\log)}$ (resp. X_s^{tame}) is the unique fiber of a fibered topos over the category with one object associated to G (resp. G/P).

PROPOSITION (3.1.3). *In* (3.1.1), $X_{\overline{s}(\log)} \to X_s^{\mathrm{tame}}$ *is equivalent. So the action of* G *on* $X_{\overline{s}(\log)}$ *factors through* G/P.

Proof. It is enough to show that $X_s^{L_1} \xrightarrow{\varphi} X_s^{L_2}$ induces an equivalence of topoi for any finite extension $L_1 \supset L_2$ such that $L_1 \cap L_2^{\mathrm{tame}} = L_2$. By using exact proper base change theorem for sets-valued sheaves (cf. [N] (5.1)), we may assume that $K = L_2$ and $(\overset{\circ}{X}{}_s^{L_2})$ is the spectrum of a separably closed field. Then $(X_s^{L_1})^{\mathrm{cl}} \to (X_s^{L_2})^{\mathrm{cl}}$ is an equivalence and the cokernel of $(M/\mathcal{O}^*)^{\mathrm{gp}}_{X^{L_1},\overline{x_1}} \hookleftarrow (M/\mathcal{O}^*)^{\mathrm{gp}}_{X^{L_2},\overline{x_2}}$ is killed by a power of the characteristic exponent of $\kappa(x_1)$ where x_i is the unique point of $X_s^{L_i}$ $(i = 1, 2)$. Hence φ induces an equivalence by an interpretation like [N] (4.6) for this case. (Cf. the last part of the proof of Step 1 of [N] (5.6.5).) □

In the followings we identify $X_{\overline{s}(\log)}$ and X_s^{tame}.

DEFINITION (3.1.4). In (3.1.1), we define the category of continuous G (resp. G/P)-sheaves on $X_{\overline{s}(\log)}$ as follows. Let $\mathcal{F}$ be an object (or a sheaf of sets over) $X_{\overline{s}(\log)}$. As in SGA4 XIII 1.1.1, we consider an action of G (resp. G/P) on $\mathcal{F}$ that is compatible with the action of G (resp. G/P) on $X_{\overline{s}(\log)}$. We call the action *continuous* if for any affine $U \in \text{Ob } (X_s)^{\log}_{\text{ét}}$, G (resp. G/P) acts continuously on $\mathcal{F}(U \times_{X_s} X_{\overline{s}(\log)})$ with the discrete topology. Thus we have defined the two categories, which are proven to be topoi by using Giraud's criterion (SGA4 IV 1.2 = [G] Chapitre 0 2.6). Note that the latter topos of continuous G/P-sheaves is equivalent to $(X_s)^{\log\sim}_{\text{ét}}$ (as in SGA7 XIII 1.1.3). Further there is a natural morphism $\begin{pmatrix} \varepsilon' & \text{id} \\ \varepsilon & \varepsilon \end{pmatrix}$ of essentially commutative diagrams of topoi from

$$\begin{array}{ccc} \{\text{continuous } G\text{-sheaf over } X_{\overline{s}(\log)}\} & \xrightarrow{\Gamma} & \eta \\ \Big\downarrow{\scriptstyle \Gamma(P,-)} & & \Big\downarrow{\scriptstyle \text{sp}^{\log}} \\ X_s & \longrightarrow & s \end{array} \quad \text{to} \quad \begin{array}{ccc} X_s^{\text{cl}} \times_{s^{\text{cl}}} \eta & \longrightarrow & \eta \\ \Big\downarrow & & \Big\downarrow \\ X_s^{\text{cl}} & \longrightarrow & s^{\text{cl}}, \end{array}$$

where Γ = (taking the global section), $\Gamma(P, -)$ = (taking P-fixed part), and $\text{sp}^{\log}$ = (taking P-fixed part) under the identifications $X_s = \{\text{continuous } G/P\text{-sheaf over } X_{s(\log)}\}$, $\eta = \{\text{continuous } G\text{-set}\}$, and $s = \{\text{continuous } G/P\text{-set}\}$. For the latter diagram, see SGA7 XIII 1.2. Since the latter diagram is 2-cartesian (see [G] Chapitre VIII 0.5 for the definition), ε' is characterized as the essentially unique morphism that makes $\begin{pmatrix} \varepsilon' & \text{id} \\ \varepsilon & \varepsilon \end{pmatrix}$ a morphism. In fact ε' is induced by the projection $X_{\overline{s}(\log)} \to (X_s^{\text{cl}})^{\text{ur}} = \varprojlim_{L:\,\text{ur}} X_s^{\text{cl}} \times_{s^{\text{cl}}} (s^L)^{\text{cl}}$($L$ runs through the set of all finite unramified extensions of K) under the identification $X_s^{\text{cl}} \times_{s^{\text{cl}}} \eta = \{$continuous G-sheaf over $(X_s^{\text{cl}})^{\text{ur}}\}$.

(3.1.5). In the followings, we denote by $\overline{\varphi}(\log)$ (resp. φ^{tame}) the projections $\overline{(1)} \to (1)^K$ (resp. $(1)^{\text{tame}} \to (1)^K$). In (3.1.1), let n be an integer invertible on S. Put $\Lambda = \mathbf{Z}/n\mathbf{Z}$. We denote by $\text{D}^+(X_{\overline{s}(\log)}, G, \Lambda)$ (resp. $\text{D}^+(X_{\overline{s}(\log)}, G/P, \Lambda)$) the derived category of Λ-Modules on the topos of continuous G (resp. G/P)-sheaves over $X_{\overline{s}(\log)}$. Then we have natural functors

$$\text{R}\Psi\colon \text{D}^+(X_\eta, \Lambda) \to \text{D}^+(X_{\overline{s}(\log)}, G, \Lambda)$$

$$(\text{resp. } \mathrm{R}\Psi_{\mathrm{t}}\colon \mathrm{D}^+(X_\eta,\Lambda) \to \mathrm{D}^+(X_{\bar{s}(\log)}, G/P, \Lambda))$$

such that $\mathfrak{f} \circ \mathrm{R}\Psi = \bar{i}_{(\log)}{}^* \mathrm{R}^+ \bar{j}_{(\log)*} \overline{\varphi}_{(\log)}{}^*$ (resp. $\mathfrak{f} \circ \mathrm{R}\Psi_{\mathrm{t}} = i^{\mathrm{tame}*} \mathrm{R}^+ j^{\mathrm{tame}}_* \varphi^{\mathrm{tame}*}$) where $\mathfrak{f}$ is the forgetful functor to $\mathrm{D}^+(X_{\bar{s}(\log)},\Lambda) = \mathrm{D}^+(X_s^{\mathrm{tame}},\Lambda)$. They are called log nearby cycle (resp. log tame nearby cycle). Note that $\mathrm{R}\Psi_{\mathrm{t}} = \mathrm{R}^+\Gamma(P, \mathrm{R}\Psi -)$. Further we define the log vanishing cycle $\mathrm{R}\Phi^{\log}\colon \mathrm{D}^+(X,\Lambda) \to \mathrm{D}^+(X_{\bar{s}(\log)}, G, \Lambda)$ as the mapping cone of $\bar{i}_{(\log)}{}^* \overline{\varphi}_{(\log)}{}^* \to \mathrm{R}\Psi j^*$.

(3.1.6). The above $\mathrm{R}\Psi$, $\mathrm{R}\Psi_{\mathrm{t}}$ and $\mathrm{R}\Phi^{\log}$ are closely related to the classical ones. Indeed in (3.1.5) we have

$$\begin{aligned} \mathrm{R}\Psi^{\mathrm{cl}}\mathrm{R}^+\varepsilon_* &= \mathrm{R}^+\varepsilon'_*\mathrm{R}\Psi\colon \mathrm{D}^+(X_\eta,\Lambda) \to \mathrm{D}^+(X_s^{\mathrm{cl}} \times_{s^{\mathrm{cl}}} \eta, \Lambda) \\ &= \mathrm{D}^+((X_s^{\mathrm{cl}})^{\mathrm{ur}}, G, \Lambda), \end{aligned}$$

where $\mathrm{R}\Psi^{\mathrm{cl}} := \mathrm{R}\Psi_\eta$ in SGA7 XIII 2.1.1 and $\mathrm{R}^+\varepsilon_*$ is the log forgetting functor: $\mathrm{D}^+(X_\eta,\Lambda) \to \mathrm{D}^+(X_\eta^{\mathrm{cl}},\Lambda)$. Similarly

$$\mathrm{R}\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathrm{R}^+\varepsilon_* = \mathrm{R}^+\varepsilon'_*\mathrm{R}\Psi_{\mathrm{t}}\colon \mathrm{D}^+(X_\eta,\Lambda) \to \mathrm{D}^+((X_s^{\mathrm{cl}})^{\mathrm{ur}}, G/P, \Lambda),$$

where $\mathrm{R}\Psi_{\mathrm{t}}^{\mathrm{cl}}$ is the classical tame nearby cycle functor and $\mathrm{R}^+\varepsilon'\colon \mathrm{D}^+(X_{\bar{s}(\log)}, G/P, \Lambda) \to \mathrm{D}^+((X_s^{\mathrm{cl}})^{\mathrm{ur}}, G/P, \Lambda)$ is induced by the above ε' in (3.1.4) or by the projection $X_{\bar{s}(\log)} \to (X_s^{\mathrm{cl}})^{\mathrm{ur}}$. These formulas are proven by using exact proper base change theorem [N] (5.1). Finally, for $\mathcal{K} \in \mathrm{D}^+(X,\Lambda)$, we have a comparison map $\mathrm{R}\Psi_s\mathcal{K}_s \to \mathrm{R}\Psi^{\mathrm{cl}}\mathrm{R}^+\varepsilon_*\mathcal{K}_\eta$ where $\mathrm{R}\Psi_s := \mathrm{R}^+\varepsilon'_*\overline{\varphi}_{(\log)}{}^*$. Note that the target of this map is in $\mathrm{D}^+((X_s^{\mathrm{cl}})^{\mathrm{ur}}, G, \Lambda)$, whereas the source is in $\mathrm{D}^+((X_s^{\mathrm{cl}})^{\mathrm{ur}}, G/P, \Lambda)$ in virtue of (3.1.3).

THEOREM (3.2). *In* (3.1.5), *assume that* $X \to S$ *is log smooth. Then*

(i) $\mathrm{R}\Phi^{\log}\Lambda = 0$.
(ii) $\Lambda \xrightarrow{\sim} \mathrm{R}\Psi_{\mathrm{t}}\Lambda \xrightarrow{\sim} \mathrm{R}\Psi\Lambda$ *in* $\mathrm{D}^+(X_{\bar{s}(\log)}, G, \Lambda)$.
(iii) $\mathrm{R}\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L} \xrightarrow{\sim} \mathrm{R}\Psi^{\mathrm{cl}}\mathcal{L}$ *where* $\mathcal{L} := \mathrm{R}^+\varepsilon_*\Lambda \in \mathrm{D}^+(X_\eta^{\mathrm{cl}},\Lambda)$.
(iv) $\mathrm{R}^+\varepsilon'_*\Lambda \xrightarrow{\sim} \mathrm{R}\Psi^{\mathrm{cl}}\mathcal{L}$ *in* $\mathrm{D}^+((X_s^{\mathrm{cl}})^{\mathrm{ur}}, G, \Lambda)$.
(v) *The comparison map* (*in* (3.1.6)): $\mathrm{R}\Psi_s\Lambda \to \mathrm{R}\Psi^{\mathrm{cl}}\mathcal{L}$ *is an isomorphism.*

In particular, the action of P *on* $\mathrm{R}^q\Psi^{\mathrm{cl}}\mathcal{L}$ *is trivial for any* q. *Note that if the log structure of* X_η *is trivial, then* $\mathcal{L} \cong \Lambda$.

Proof. By SGA4 VI 8.7.5 and (0.2) for each X^L (L is a finite separable extension of K), we have $\Lambda \xrightarrow{\sim} \mathrm{R}^+\bar{j}_{(\log)*}\Lambda$ and $\Lambda \xrightarrow{\sim} \mathrm{R}^+j^{\mathrm{tame}}_*\Lambda$. Thus (i) and (ii). Actually $\Lambda \xrightarrow{\sim} \mathrm{R}\Psi_{\mathrm{t}}\Lambda$ is deduced from $\Lambda \xrightarrow{\sim} \mathrm{R}\Psi\Lambda$ by applying $\mathrm{R}^+\Gamma(P,-)$ to both sides. Next, applying $\mathrm{R}^+\varepsilon'_*$ to (ii), we have (iii) and (iv) by (3.1.6). Actually (iv) contains (iii). Finally (iv) and (v) are equivalent. □

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Remark (3.2.1). See (3.4) for a reformulation of (3.2) without log terms.

Remark (3.2.2). Even when we weaken the assumption of log smoothness of f to that of (W) (see (2.1.1) for (W)), we still have $\Lambda \xrightarrow{\sim} \mathrm{R}\Psi_{\mathrm{t}}\Lambda$ and $\mathrm{R}^{+}\varepsilon'_{*}\Lambda \xrightarrow{\sim} \mathrm{R}\Psi^{\mathrm{cl}}_{\mathrm{t}}\mathcal{L}$ because (2.2) implies that (2.0.1) still holds on $X \times_S S^L$ for a tame extension L of K.

COROLLARY (3.3). *In* (3.1.5), *suppose that the log structure of* X_η *is trivial. Then* $\mathrm{R}\Psi^{\mathrm{cl}}\Lambda$ *is determined only by* X_s, *that is,* $\mathrm{R}\Psi^{\mathrm{cl}}\Lambda$ *is isomorphic to* $\mathrm{R}\Psi^{\mathrm{cl}}\Lambda$ *for another such* X' *if there is an isomorphism of* s*-log schemes* $X_s \cong X'_s$. *Further, when* X^{cl} *and* X'^{cl} *have semistable reductions,* $\mathrm{R}\Psi^{\mathrm{cl}}\Lambda$*'s are isomorphic if* $X^{\mathrm{cl}} \otimes_A A/\mathfrak{m}^2 \cong X'^{\mathrm{cl}} \otimes_A A/\mathfrak{m}^2$ *where* $\mathfrak{m}$ *is the maximal ideal of* A.

Proof. In (3.2) (v), the left-hand side is determined only by X_s. The last statement comes from the fact that the log structure of X having semistable reduction (in the sense of [K1] (3.7) (2)) depends only on $\mathring{X} \otimes_A A/\mathfrak{m}^2$ (see Appendix (A.4) for the detail). □

Remark (3.3.1). As an application, in a forthcoming paper, we prove that the weight spectral sequence for a proper semistable family degenerates at E_2 regardless of whether the residue field is finite or not.

(3.4). We reformulate (3.2) without log terms for convenience.

VARIANT (3.4.1) (Illusie's expectation). *Let* $U \xrightarrow{j} X \to S = \mathrm{Spec}(A)$ *be a diagram of schemes with* A *being a henselian discrete valuation ring. Let* s *be the closed point of* S *and* η *the generic one. Let* $n \geqslant 1$ *be an integer invertible on* S. *For any* $q \in \mathbf{Z}$, *we have the nearby cycle*

$$\mathrm{R}^q\Psi_\eta\mathcal{L} \text{ on the product topos } X_s \times_s S (\text{SGA7 XIII 2.1.1}),$$

where $\mathcal{L} := \mathrm{R}^{+}j_*\mathbf{Z}/n\mathbf{Z}$. *Assume that étale locally on* X, $U \xrightarrow{j} X$ *is étale over*

$$\mathrm{Spec}(A[P^{\mathrm{gp}}]/(\pi - x)) \hookrightarrow \mathrm{Spec}\,(A[P]/(\pi - x)),$$

where P *is an fs monoid* (= *finitely generated and saturated monoid*), x *is an element of* $P \smallsetminus \{1\}$ *such that the order of the torsion of* $P^{\mathrm{gp}}/\langle x\rangle$ *is invertible on* X, *and* π *is a prime element of* A. *Then the action on* $\mathrm{R}^q\Psi_\eta\mathcal{L}$ *of the wild inertia group of the quotient field of* A *is trivial.*

Proof. Since the statement is local, we may assume that $U \hookrightarrow X$ is étale over $\mathrm{Spec}(A[P^{\mathrm{gp}}]\,/(\pi - x)) \hookrightarrow \mathrm{Spec}(A[P]/(\pi - x))$ where P, x, and π are as above. We endow them with log structures by P. Then (3.4.1) reduces to (3.2) by taking it into account that $\mathrm{Spec}(A[P^{\mathrm{gp}}]/(\pi - x))$ is the maximal open subscheme of $\mathrm{Spec}(A[P]/(\pi - x))$ on which log structure is trivial. Note that $\mathcal{L} = \mathrm{R}^{+}\varepsilon_*\Lambda$ by

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(2.0.1) for X_η ((2.0.4)). □

Remark (3.4.2). In (3.4.1), if further $\mathbf{N} \to P; 1 \mapsto x$ is dominating ([N] (7.3)), that is, the condition (ii) in Introduction is satisfied anywhere, U coincides with the generic fiber X_η of X. (Otherwise U may be strictly contained in X_η.) This case is nothing but (0.1). Further, (0.1.1) is a direct corollary of (0.1), via SGA7 I (2.2.3), (2.7.1) and (2.7.3).

(3.5). Next we prove an SGA7 I 3.3-type formula for the classical tame nearby cycles $\mathrm{R}\Psi_{\mathrm{t}}^{\mathrm{cl}}$ for log smooth families. Again by using (2.2) as in the proof of (3.2), this reduces to an easy problem on the special fiber.

THEOREM (SGA7 I 3.3-type formula for $\mathrm{R}\Psi_{\mathrm{t}}^{\mathrm{cl}}$). *In* (3.1.5), *assume that* $X \xrightarrow{f} S$ *is log smooth. Put* $\mathcal{L} := \mathrm{R}^+\varepsilon_*\Lambda \overset{(2.0.4)}{=} \mathrm{R}^+ j_*^{\mathrm{cl}}\Lambda \in \mathrm{D}^+(X_\eta^{\mathrm{cl}}, \Lambda)$. *Let*

$$(M/\mathcal{O}^*)^{\mathrm{gp}}_{\mathrm{rel}} := \mathrm{cok}(\overset{\circ}{f}{}^*(M/\mathcal{O}^*)^{\mathrm{gp}}_s \xrightarrow{\varphi} (M/\mathcal{O}^*)^{\mathrm{gp}}_{X_s})_{/\mathrm{torsion}} \quad \textit{on } X_s^{\mathrm{cl}}.$$

Then

(i) *For any* $q \geqslant 0$, *there is a natural,* G/P*-isomorphism*

$$\mathrm{R}^q\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L} \cong \mathrm{R}^0\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L} \otimes_{\mathbf{Z}} \bigwedge^q ((M/\mathcal{O}^*)^{\mathrm{gp}}_{\mathrm{rel}} \otimes_{\mathbf{Z}} \Lambda(-1)).$$

(ii) *The stalk of* $\mathrm{R}^0\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L}$ *at* y *in* $(X_s^{\mathrm{cl}})^{\mathrm{ur}}$ *is*

$$(\mathrm{R}^0\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L})_{\overline{y}} \cong \Lambda[E_{\overline{y}}],$$

where

$E_{\overline{y}} := \mathrm{cok}(\mathrm{Hom}(\varphi_{\overline{y}}, \widehat{\mathbf{Z}}'(1)))$; *a finite Abelian group. Here the action of* $I/P = \widehat{\mathbf{Z}}'(1)$ *on the right-hand side is the one through the canonical homomorphism* $\widehat{\mathbf{Z}}'(1) = \mathrm{Hom}((M/\mathcal{O}^*)^{\mathrm{gp}}_{S,\overline{s}}, \widehat{\mathbf{Z}}'(1)) \to E_{\overline{y}}$.

Note that if the log structure of X_η *is trivial, then* $\mathcal{L} \cong \Lambda$.

Remark (3.5.1). We can weaken the assumption of log smoothness of f to that of (W) (see (2.1.1) for (W)) without changing the conclusion of (3.5). We treat this generalization in the proof below.

Remark (3.5.2). The problem of giving a global formula for the sheaf $\mathrm{R}^0\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L}$ should be interesting, though we do not treat it.

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(3.6). *Proof of* (3.5). First note that $\mathrm{R}^{+}\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L} = \mathrm{R}^{+}\varepsilon'_{*}\Lambda$ by (3.2.2). We consider G/P-equivariant homomorphisms

$$\begin{array}{ccc}
\mathrm{R}^{q}\varepsilon'_{*}\Lambda & & \varepsilon'_{*}\Lambda \otimes_{\mathbf{Z}} \bigwedge^{q} ((\varphi^{\mathrm{cl}})^{\mathrm{ur}*}(M/\mathcal{O}^{*})^{\mathrm{gp}}_{\mathrm{rel}} \otimes_{\mathbf{Z}} \Lambda(-1)) \\
\uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \varphi_2} \\
\varepsilon'_{*}\Lambda \otimes_{\mathbf{Z}} \mathrm{R}^{q}\varepsilon_{*}\Lambda & = & \varepsilon'_{*}\Lambda \otimes_{\mathbf{Z}} \bigwedge^{q} ((\varphi^{\mathrm{cl}})^{\mathrm{ur}*}(M/\mathcal{O}^{*})^{\mathrm{gp}}_{X_s} \otimes_{\mathbf{Z}} \Lambda(-1)),
\end{array}$$

where $(\varphi^{\mathrm{cl}})^{\mathrm{ur}}$ is the projection $(X_s^{\mathrm{cl}})^{\mathrm{ur}} \to X_s^{\mathrm{cl}}$, the equality ψ is the isomorphism in (A.3) and φ_1 is induced from the cup product. Since φ_2 is surjective, to prove (i), it is enough to show that there is an isomorphism of modules $(\mathrm{R}^{q}\varepsilon'_{*}\Lambda)_{\overline{y}} \cong (\varepsilon'_{*}\Lambda)_{\overline{y}} \otimes_{\mathbf{Z}} \bigwedge^{q} ((M/\mathcal{O}^{*})^{\mathrm{gp}}_{\mathrm{rel},\overline{y}} \otimes_{\mathbf{Z}} \Lambda(-1))$ which commutes with $\psi_{\overline{y}}$ for any $y \in (X_s^{\mathrm{cl}})^{\mathrm{ur}}$. First $\mathrm{R}^{q}\varepsilon'_{*}\Lambda = \mathrm{R}^{q}\varepsilon_{*}(X_s^{\mathrm{ur}} \to s^{\mathrm{ur}})^{*}((s^{\mathrm{tame}} \to s^{\mathrm{ur}})_{*}\Lambda)$ by exact proper base change theorem [N] (5.1) where X_s^{ur} etc. are defined in the same way as in (3.1.2). Thanks to [N] (4.6) and (4.7), we interpret the problem of log étale sheaves into that on modules with actions of the profinite groups that are determined by the log structures. Then we see that the log étale sheaf $(s^{\mathrm{tame}} \to s^{\mathrm{ur}})_{*}\Lambda$ corresponds via [N] (4.6) to the I_s-Λ-module $\Lambda[I_s] := \mathrm{Map}_{\mathrm{conti}}(I_s, \Lambda)$ on which $I_s := \mathrm{Hom}((M/\mathcal{O}^{*})^{\mathrm{gp}}_{S,\overline{s}}, \widehat{\mathbf{Z}}'(1)) = \widehat{\mathbf{Z}}'(1)$ acts like $z \cdot m(-) \mapsto m(-z)$. Thus $(\mathrm{R}^{q}\varepsilon'_{*}\Lambda)_{\overline{y}} = \mathrm{H}^{q}(I_y, \Lambda[I_s])$ where $I_y := \mathrm{Hom}((M/\mathcal{O}^{*})^{\mathrm{gp}}_{X,\overline{y}}, \widehat{\mathbf{Z}}'(1))$ which acts on $\Lambda[I_s]$ via $\theta = \mathrm{Hom}(\varphi_{\overline{y}}, \widehat{\mathbf{Z}}'(1))\colon I_y \to I_s$.

On the other hand we have an exact sequence

$$0 \to \mathrm{Hom}((M/\mathcal{O}^{*})^{\mathrm{gp}}_{\mathrm{rel},\overline{y}}, \widehat{\mathbf{Z}}'(1)) \to I_y \xrightarrow{\theta} I_s \to E_{\overline{y}} \to 0.$$

So $\Lambda[I_s]$ is decomposed into $\mathrm{Map}(E_{\overline{y}}, \mathrm{Map}_{\mathrm{conti}}(J, \Lambda))$, where $J = \mathrm{Image}(\theta)$. We consider the Hochshild-Serre spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^{p}(J, \mathrm{H}^{q}(\mathrm{Ker}(\theta), \mathrm{Map}_{\mathrm{conti}}(J, \Lambda))) \Rightarrow \mathrm{H}^{p+q}(I_y, \mathrm{Map}_{\mathrm{conti}}(J, \Lambda)).$$

Since $\mathrm{H}^{p}(J, \mathrm{Map}_{\mathrm{conti}}(J, N)) = 0$ for $p > 0$ and N for $p = 0$, we have

$$\begin{aligned}
\mathrm{H}^{q}(I_y, \mathrm{Map}_{\mathrm{conti}}(J, \Lambda)) &= \mathrm{H}^{q}(\mathrm{Ker}(\theta), \Lambda) \\
&= \bigwedge^{q} ((M/\mathcal{O}^{*})^{\mathrm{gp}}_{\mathrm{rel},\overline{y}} \otimes_{\mathbf{Z}} \Lambda(-1)).
\end{aligned}$$

Thus we have the desired isomorphism, which commutes with $\psi_{\overline{y}}$. The above calculation includes $(\mathrm{R}^{0}\varepsilon'_{*}\Lambda)_{\overline{y}} = \mathrm{Map}(E_{\overline{y}}, \Lambda)$ that is (ii). □

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Remark (3.6.1). There is another way to show (3.5) due to L. Illusie, K. Kato, and T. Saito. Consider the different factorization of

$$\mathrm{R}^+\varepsilon'_*\colon \mathrm{D}^+(X_s^{\mathrm{tame}}, G/P, \Lambda) \xrightarrow{\mathrm{R}^+\eta_{2*}} \mathrm{D}^+((X_s^{\mathrm{tame}})^{\mathrm{cl}}, G/P, \Lambda) \xrightarrow{\mathrm{R}^+\eta_{1*}} \mathrm{D}^+((X_s^{\mathrm{cl}})^{\mathrm{ur}}, G/P, \Lambda).$$

Then $\mathrm{R}^+\eta_{1*} = \eta_{1*}$.

On the other hand, since

$$\left(\varinjlim_{L\colon \mathrm{tame}} \overline{\varphi^{L}}^{\mathrm{cl}*}(M/\mathcal{O}^*)^{\mathrm{gp}}_{X_s^L}\right) \otimes \Lambda = (\eta_1^*(M/\mathcal{O}^*)^{\mathrm{gp}}_{\mathrm{rel}}) \otimes_{\mathbf{Z}} \Lambda,$$

where $\overline{\varphi^L}^{\mathrm{cl}}$ is the projection $(X_s^{\mathrm{tame}})^{\mathrm{cl}} \to (X_s^L)^{\mathrm{cl}}$,

$$\mathrm{R}^q\eta_{2*}\Lambda = \bigwedge^q ((\eta_1^*(M/\mathcal{O}^*)^{\mathrm{gp}}_{\mathrm{rel}}) \otimes_{\mathbf{Z}} \Lambda(-1))$$

for any $q \geqslant 0$. Thus we have the desired G/P-equivariant isomorphism by the projection formula.

COROLLARY (3.7) (Cf. SGA7 I 3.4). *In* (3.1.5) *assume that f is log smooth and $\mathring{f}$ is proper. Let l be a prime number invertible in A. Let N be the least integer $\geqslant 1$ that kills the prime-to-p parts of* $\mathrm{cok}((M/\mathcal{O}^*)^{\mathrm{gp}}_{S,\overline{s}} \xrightarrow{\varphi_{\overline{y}}} (M/\mathcal{O}^*)^{\mathrm{gp}}_{X,\overline{y}})_{\mathrm{tor}}$ *for all $y \in X_s$ where p is the residual characteristic exponent of A. Let q be an integer $\geqslant 0$ and let q' be* $\inf\{q+1, \sup\{\mathrm{rank}_{\mathbf{Z}}(M/\mathcal{O}^*)^{\mathrm{gp}}_{X,\overline{y}} \mid y \in X_s\}\}$. *Then for any $T \in I$*

$$(T^N - 1)^{q'} = 0 \quad \text{on } \mathrm{H}^q(X_{\mathrm{triv}} \times_\eta \overline{\eta}, \mathbf{Z}_l).$$

In particular, the action of I on $\mathrm{H}^q(X_{\mathrm{triv}} \times_\eta \overline{\eta}, \mathbf{Z}_l)$ *is quasi-unipotent of échelon q'.*

Remark (3.7.1). Similarly to (3.5.1), we can weaken the log smoothness condition to (W) when we replace $\overline{\eta}$ by η^{tame}. We treat this variant simultaneously below in the proof (3.8).

Remark (3.7.2). Note that X_{triv} is not necessarily proper over K.

(3.8). *Proof of* (3.7). Note that $\mathrm{R}^+\varepsilon_*\Lambda \overset{(2.0.4)}{=} \mathrm{R}^+j_*^{\mathrm{cl}}\Lambda \in \mathrm{D}^+(X_\eta^{\mathrm{cl}}, \Lambda)$. Let m be an integer. We have spectral sequences

$$E_2^{p,q} = \mathrm{H}^p((X_s^{\mathrm{cl}})^{\mathrm{ur}}, \mathrm{R}^q\Psi^{\mathrm{cl}}\mathcal{L}) \Rightarrow \mathrm{H}^{p+q}(X_{\mathrm{triv}} \times_\eta \overline{\eta}, \mathbf{Z}/l^m\mathbf{Z}); I\text{-equivariant}$$

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$$E_2^{p,q} = \mathrm{H}^p((X_s^{\mathrm{cl}})^{\mathrm{ur}}, \mathrm{R}^q\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L})$$
$$\Rightarrow \mathrm{H}^{p+q}(X_{\mathrm{triv}} \times_\eta \eta^{\mathrm{tame}}, \mathbf{Z}/l^m\mathbf{Z});\, I/P\text{-equivariant}$$

(cf. SGA7 I 2.2.3). By (3.2), (3.5) and (3.5.1), we see that T^N acts on $\mathrm{R}^q\Psi^{\mathrm{cl}}\mathcal{L}$ or $\mathrm{R}^q\Psi_{\mathrm{t}}^{\mathrm{cl}}\mathcal{L}$ trivially since the prime-to-p part of $\mathrm{cok}((M/\mathcal{O}^*)^{\mathrm{gp}}_{S,\overline{s}} \xrightarrow{\varphi_{\overline{y}}} (M/\mathcal{O}^*)^{\mathrm{gp}}_{X,\overline{y}})_{\mathrm{tor}}$ is isomorphic to $E_{\overline{y}}$ in (3.5). Thus we get the desired result. □

4. Propositions on log étale cohomology

(4.1). In this section, we prove three propositions based on (2.2). The first one (4.2) says that for a variety over a henselian discrete valuation field with log smooth reduction the l-adic representation is determined by its special fiber endowed with log structure. This had been already pointed out by K. Fujiwara ([F]) for the case of semistable reduction, who applied this fact to the hypersurface case of monodromy-weight conjecture in [F]. The second proposition (4.3) in this section is a case of 'proper log smooth base change theorem': In usual étale cohomology theory for schemes, the statement '$\mathrm{R}^q f_*\mathbf{Z}/n\mathbf{Z}$ is locally constant and constructible for any smooth proper morphism f (n being an integer invertible on the base)' is called proper smooth base change theorem; in fact K. Kato recently proved it is valid when regarded as a statement in log étale cohomology. See (4.3.1). The last one (4.4) is a relative version of log Poincaré duality. Although we worked only on a field in [N], for this time we work over a discrete valuation ring. All proofs of these results are easy applications of (2.2). The second and the third ones are not the final results.

PROPOSITION (4.2). *In* (3.1.5), *assume that f is log smooth and $\mathring{f}$ is proper. Then for any $q \in \mathbf{Z}$,*

$$\mathrm{H}^q(X_s^{\mathrm{tame}}, \mathbf{Z}/n\mathbf{Z}) \cong \mathrm{H}^q(X_\eta^{\mathrm{cl}} \times_\eta \overline{\eta}, \mathcal{L}),$$

where $\mathcal{L} := \mathrm{R}^+ j_*\mathbf{Z}/n\mathbf{Z}$. Note that if the log structure of X_η is trivial, then $\mathcal{L} \cong \mathbf{Z}/n\mathbf{Z}$.

Proof. By (3.2), we have $\Lambda = \mathrm{R}\Psi\Lambda$ on X_s^{tame}. Then $\mathrm{R}^+\Gamma(X_s^{\mathrm{tame}}, \Lambda) = \mathrm{R}^+\Gamma(X_s^{\mathrm{cl}} \times_{s^{\mathrm{cl}}} \eta, \mathrm{R}^+\varepsilon'_*\mathrm{R}\Psi\Lambda) \overset{(3.1.6)}{=} \mathrm{R}^+\Gamma((X_s^{\mathrm{cl}})^{\mathrm{ur}}, \mathrm{R}\Psi^{\mathrm{cl}}\mathcal{L}) = \mathrm{R}^+(X_\eta^{\mathrm{cl}} \times_\eta \overline{\eta}, \mathcal{L})$ with G-action. □

Remark (4.2.1). The left-hand side is described as

$$\varinjlim_{m:\,\text{invertible in } A} \mathrm{H}^q(X_s \otimes_{\mathbf{ZN}} \mathbf{ZN}^{1/m}, \mathbf{Z}/n\mathbf{Z}).$$

The right-hand side is isomorphic to $\mathrm{H}^q(\overline{X}_{\eta(\log)}, \mathbf{Z}/n\mathbf{Z})$ or $\mathrm{H}^q(X_{\mathrm{triv}}^{\mathrm{cl}} \times_\eta \overline{\eta}, \mathbf{Z}/n\mathbf{Z})$. Note that X_{triv} is not necessarily proper over K.

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PROPOSITION (4.3) (a part of proper smooth base change theorem). *Let* $f\colon X \to S$ *be a proper log smooth morphism in* (fs log sch). *Let* n *be an integer invertible on* S. *Assume that*

(i) f *is exact.* (*We review the definition of exactness* [K1] (4.6) *for convenience below.*)

(ii) $\overset{\circ}{S}$ *is noetherian and* M_S *is trivial at each generic point.*

Then $\mathrm{R}^q f_* \mathbf{Z}/n\mathbf{Z}$ *is locally constant and constructible for any* q.

(Review for [K1] (4.6). A homomorphism of integral monoids $h\colon Q \to P$ is said to be *exact* if $Q = (h^{\mathrm{gp}})^{-1}(P)$ in Q^{gp}. A morphism of log schemes with integral log structures $f\colon X \to Y$ is said to be *exact* if the homomorphism $((\overset{\circ}{f})^* M_Y)_{\overline{x}} \to M_{\overline{x}}$ is exact for any $x \in X$.)

Remark (4.3.1). By the similar proof we can replace the condition (ii) by

(ii)′ $\overset{\circ}{S}$ is locally noetherian and $(M/\mathcal{O}^*)_S$ is constant sheaf

without changing the conclusion of (4.3). Further recently K. Kato proved the following general result: In (4.3), we replace both (i) and (ii) by only

(ii)″ $\overset{\circ}{S}$ is locally noetherian.

Let F be a locally constant and constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $X^{\log}_{\text{ét}}$. Then $\mathrm{R}^q f_* F$ is locally constant and constructible for any q. The proof is based on the theory of modified log étale sites mentioned in (2.5) (not yet published). In the following (4.3.3), we prove the case (4.3) by a different method.

Remark (4.3.2). We recall here the definition of constructibility. Let X be an fs log scheme and A a ring. Then $F \in \mathrm{Ob}\, \mathcal{S}_X^A$ is called *constructible* if for any open affine $U \subset \overset{\circ}{X}$, there exists a finite decomposition $(U_i)_{i \in I}$ of U consisting of constructible reduced subschemes such that the inverse image of F to U_i is locally constant whose local values are A-modules of finite presentation, where the log structure of U_i is the restricted one from X. (In (4.3), A is taken to be $\mathbf{Z}/n\mathbf{Z}$.)

(4.3.3). *Proof of* (4.3). Since (i) implies $\overset{\circ}{f}{}^*(M/\mathcal{O}^*)_S \to (M/\mathcal{O}^*)_X$ is injective, the finiteness theorem [N] (5.5.2) implies $\mathrm{R}^q f_* \mathbf{Z}/n\mathbf{Z}$ is a constructible $\mathbf{Z}/n\mathbf{Z}$-Module for any q. Then, by the Lemma (4.3.4) below, it suffices to show that the cospecialization map $(\mathrm{R}^q f_*\, \mathbf{Z}/n\mathbf{Z})_{s(\log)} \to (\mathrm{R}^q f_* \mathbf{Z}/n\mathbf{Z})_{s'(\log)}$ is bijective for any point $s' \in S$ and any specialization s of it. To show this, taking (ii) into account, we may assume that M_S is trivial at s'. We may assume further that $\overset{\circ}{S}$ is the spectrum of a noetherian local domain A with s' being the generic point and s being the closed one. The rest is an induction on $\dim A$.

The case $\dim A = 0$ is trivial. The case $\dim A = 1$ will be treated later. Assume that $\dim A \geqslant 2$. Then we can find a chain of points $s' \rightsquigarrow s_1 \rightsquigarrow s$ ($\rightsquigarrow$ means

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specialization) such that M_S is trivial at s_1 and $s' \neq s_1$ because A is noetherian and the set of the points at which the log structure is trivial is open and non empty, hence (since $\dim A > 1$) cannot consist of s' alone. Thus the induction works.

The rest is the case where $\dim A = 1$. Taking the normalization of A^{sh}, we may assume that A is a strictly henselian discrete valuation ring. Take any chart of S: $P \to M_S$ such that $P \to (M/\mathcal{O}^*)_{S,\bar{s}}$ is bijective ([N] (1.6)). Let π be a prime element of A. Then the modified homomorphism $P \to A \smallsetminus \{0\} \xrightarrow{\mathrm{val}} \mathbf{N} \to A$ is also a chart, where $\mathbf{N} \to A$ is the homomorphism sending 1 to π. Thus we have constructed a morphism $(\mathrm{Spec}(A)$, the canonical log structure$) \to S$ (cf. Convention). In virtue of exact proper base change theorem [N] (5.1) (cf. [N] (5.1.1)), we thus reduce to the case where S has the canonical log structure.

But in this case, we can use (4.2) to get the desired bijection. Note that $\mathrm{H}^q(X^{\mathrm{cl}}_\eta \times_\eta \overline{\eta}, \mathcal{L}) = \mathrm{H}^q(\overline{X}_{\eta(\log)}, \mathbf{Z}/n\mathbf{Z}) = (\mathrm{R}^q f_* \mathbf{Z}/n\mathbf{Z})_{s'(\log)}$. □

LEMMA (4.3.4). *Let X be an fs log scheme, A a ring, F a constructible sheaf of A-modules on X ((4.3.2)), and $x(\log) \to X$ a log geometric point of X* ([N] (2.5)). *Then F is locally constant on a (log étale) neighborhood of x if and only if for any point x' that is a generization of x, the cospecialization map $F_{x(\log)} \to F_{x'(\log)}$ is bijective.*

Proof. This is a log version of SGA4 IX 2.11. The proof is parallel to that of it. See [N] (2.8) 6 for cospecialization maps. □

PROPOSITION (4.4) (log Poincaré duality). *Let A be a discrete valuation ring, let S be the fs log scheme* $\mathrm{Spec}(A)$ *with the canonical log structure (cf. Conventions), let η be the generic point of S, let n be an integer invertible on S, and let*

$$f\colon Y \to X$$

be a vertical ([N] (7.3)), *log smooth, S-compactifiable* ([N] (5.4)) *morphism in* (fs log sch). *Assume that X and Y are connected, and $Y \neq \phi$. Assume further that X is log smooth compactifiable over S. (Note that $f \in \mathrm{Fl}(S)$ (see Notation or* [N] (5.4)), *so that we have the functor* $\mathrm{R}f^!\colon \mathrm{D}^+(X, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{D}^+(Y, \mathbf{Z}/n\mathbf{Z})$ *by* [N] (7.2).)

Then

$$\mathrm{R}f^! \mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}/n\mathbf{Z}(d)[2d],$$

where $d = \dim(Y \otimes_S \eta) - \dim(X \otimes_S \eta)$.

Proof. This is a formal consequence of (2.2) and the log Poincaré duality over a field (in [N]). In fact, writing j_X and j_Y for the strict open immersions $X \otimes_S \eta \hookrightarrow X$ and $Y \otimes_S \eta \hookrightarrow Y$, we have

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$$\begin{aligned} \mathrm{R}f^!\mathbf{Z}/n\mathbf{Z} &\overset{\cong}{\to} \mathrm{R}f^!\mathrm{R}j_{X*}\mathbf{Z}/n\mathbf{Z} \quad ((2.2)) \\ &= \mathrm{R}j_{Y*}\mathrm{R}(f\otimes_S \eta)^!\mathbf{Z}/n\mathbf{Z} \quad (j^* \text{ and } \mathrm{R}f_! \text{ can always interchange}) \\ &= \mathrm{R}j_{Y*}\mathbf{Z}/n\mathbf{Z}(d)[2d] \quad ([\mathrm{N}]\ (7.5)\ (\mathrm{a}),\ \text{verticality assumption}) \\ &\overset{\cong}{\leftarrow} \mathbf{Z}/n\mathbf{Z}(d)[2d] \quad ((2.2)). \qquad \square \end{aligned}$$

Remark (4.4.1). In conjunction with the formal duality [N] (7.2), we have a functorial isomorphism

$$\mathrm{R}^+f_*\mathrm{R}\mathcal{H}om^\bullet(K, \mathbf{Z}/n\mathbf{Z}(d)[2d]) \cong \mathrm{R}\mathcal{H}om^\bullet(\mathrm{R}f_!K, \mathbf{Z}/n\mathbf{Z})$$

for any $K \in \mathrm{Ob}\,\mathrm{D}^-(Y, \mathbf{Z}/n\mathbf{Z})$ as well.

QUESTION (4.4.2). In the case that f is not vertical, is $\mathrm{R}f^!\mathbf{Z}/n\mathbf{Z}$ isomorphic to $j_!\mathbf{Z}/n\mathbf{Z}(d)\ [2d]$? Here j is the strict open immersion $Y_{\mathrm{ver}/f} \hookrightarrow Y$. See [N] (7.3) for notation.

QUESTION (4.4.3). What is a dualizing complex on X or on Y?

Appendix A

(A.1). As was stated in (1.3), here we generalize (1.1) slightly with the aid of a theorem of K. Fujiwara and K. Kato [FK] (2.4).

PROPOSITION (A.1.1). *Let*

$$\begin{array}{ccc} V & \xrightarrow{\pi'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[\pi]{} & X \end{array}$$

be a cartesian diagram in (fs log sch). *Let* $F \in \mathcal{S}_U^{\mathbf{Z}}$ *be a log étale sheaf of Abelian groups on* U *that is the inverse image of a sheaf of Abelian groups on* U^{cl} *and that is killed by an integer invertible on* X. *Assume that* $\mathring{\pi}$ *is quasi-finite and quasi-separated. Assume further that* f *satisfies one of the following two conditions*:

(i) f *is isomorphic to the projection* $X[P] \to X$ *for an fs monoid* P *whose torsion part has an order invertible on* X, *where* $X[P] = X \times_{\mathbf{Z}} \mathbf{Z}P$.

(ii) f *is log smooth and* X *has the trivial log structure.*

Then the functorial homomorphism (*base change morphism*) $f^*\mathrm{R}^+\pi_*F \to \mathrm{R}^+\pi'_*f'^*F$ *is an isomorphism in* $\mathrm{D}^+(Y, \mathbf{Z})$.

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Proof. We reduce (i) to (ii) by localizing X in the same way as in (1.1). So it is enough to prove (ii). As in the beginning of the proof of (1.2), we may assume that U also has the trivial log structure by proper base change theorem [N] (5.1). On the other hand, we blow up Y along the log structure and apply [FK] (2.4) as in (2.4) Step 2, so that we may assume that for any $y \in Y$, $(M/\mathcal{O}^*)_{Y,\overline{y}} \cong \mathbf{N}^{r(y)}$ for some $r(y) \geqslant 0$. Next we take a chart by (A.2) so that we may assume that f factors as $Y \xrightarrow{u} X[P] \to X$ where u is a strict étale morphism and P is a torsionfree fs monoid. Thus we may assume that f is isomorphic to $X[P] \to X$ for a torsionfree fs monoid P. We may assume that $P = \mathbf{N}^r \oplus \mathbf{Z}^{r'}$ for some $r \geqslant 0$ and $r' \geqslant 0$, and further that $r' = 0$ as in (2.4) Step 3. Finally we apply (1.1) r times. □

(A.2). Here we include a proposition used in (2.1.1). This is a slight refinement of [K1] (3.5). The novelty lies in the condition (iii).

PROPOSITION. *Let* $f\colon X \to Y$ *be a log smooth morphism in* (fs log sch). *Assume that we are given a chart* $Y \to \mathrm{Spec}(\mathbf{Z}P)$ *of* Y *with* P *being a torsionfree fs monoid. Then there are a chart covering* $(X \leftarrow X_i \to \mathrm{Spec}(\mathbf{Z}Q_i))_i$ ([N] (1.5)) *and a chart* $(Q_i \to M_{X_i}, P \to M_Y, P \xrightarrow{h_i} Q_i)$ *of* $X_i \to Y$ *such that* $h_i\colon P \to Q_i$ *is an injective homomorphism of fs monoids, satisfying the following* (i)–(iii):

(i) *The order of* $(\mathrm{cok}(h_i^{\mathrm{gp}}))_{\mathrm{tor}}$ *is invertible on* X_i.
(ii) *The induced morphism* $X_i \to Y \times_{\mathbf{Z}P} \mathbf{Z}Q_i$ *is étale.*
(iii) Q_i *is torsionfree.*

Proof.[2] By [K1] (3.5) and (3.6), we may assume that X is strict étale over $Y \times_{\mathbf{Z}P} \mathbf{Z}[Q \oplus R]$ where Q (resp. R) is a torsionfree (resp. torsion) fs monoid and $\mathrm{Spec}(\mathbf{Z}[Q \oplus R]) \to \mathrm{Spec}\,(\mathbf{Z}P)$ is induced by an injection $h\colon P \to Q \oplus R$ such that $n :=$ the order of $(\mathrm{cok}(h^{\mathrm{gp}}))_{\mathrm{tor}}$ is invertible on X. Put $A = \mathbf{Z}_{(n)}[R]$. Then we have two morphisms $\mathrm{Spec}(A[Q]) \to \mathrm{Spec}(\mathbf{Z}P)$ of fs log schemes: one is induced by h via $\mathbf{Z}P \to \mathbf{Z}_{(n)}[Q \oplus R] = A[Q]$; the other is induced by $h' := \mathrm{pr}_1 \circ h\colon P \hookrightarrow Q$ via $\mathbf{Z}P \to \mathbf{Z}_{(n)}[Q] \to A[Q]$.

CLAIM. In the above there is an étale surjective morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ such that induced two morphisms of fs log schemes $\mathrm{Spec}(B[Q]) \to \mathrm{Spec}(A[Q]) \to \mathrm{Spec}(\mathbf{Z}P)$ are isomorphic over the base.

Proof. The difference φ_0 of two $P \to A[Q]$ is contained in $(R \subset)\ (A^\times)_{\mathrm{tor}}$. Take B to be the ring obtained from A by adding $\sqrt[n]{r}$, $r \in R$. Since $(\mathrm{cok}(h'^{\mathrm{gp}}))_{\mathrm{tor}}$ is killed by n, the difference $\varphi_0\colon P \to (A^\times)_{\mathrm{tor}}$ extends to a $\varphi\colon Q \to (B^\times)_{\mathrm{tor}}$. We can make a desired automorphism of $\mathrm{Spec}(B[Q])$ using this φ via $q \mapsto \varphi(q)q$, $q \in Q$. □

[2] Due to a discussion with T. Tsuji and T. Kajiwara.

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By the above claim, we see that $X \times_A B$ is étale over $\varprojlim(Y \longrightarrow \mathrm{Spec}(\mathbf{Z}P) \xleftarrow{\mathrm{Spec}(\mathbf{Z}h')} \mathrm{Spec}(\mathbf{Z}Q) \leftarrow \mathrm{Spec}(B[Q]))$ which is étale over $Y \times_{\mathbf{Z}P} \mathbf{Z}Q$. □

(A.3). Here we include the theorem of K. Kato which was used in (2.0.2) and (3.6).

THEOREM. *Let X be an fs log scheme and F an Abelian étale sheaf on X^{cl} (cf. Notation for X^{cl}) such that*

$$F = \bigcup_{n:\ \text{invertible on } X} \mathrm{Ker}(n\colon F \to F).$$

Then there exists a canonical isomorphism $\mathrm{R}^q\varepsilon_*\varepsilon^*F \overset{\cong}{\leftarrow} F(-q) \otimes_{\mathbf{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathcal{O}_X^*)$ for any q (cf. Notation for $\varepsilon = \varepsilon(X)$). Here $(-q)$ means the Tate twist.

See [KN] (2.4) for the proof. The map is constructed by cup product from the connecting maps $M_X^{\mathrm{gp}}/\mathcal{O}_X^* \to \mathrm{R}^1\varepsilon_*(\mathbf{Z}/n\mathbf{Z})(1)$ of the logarithmic Kummer sequence for various n's.

(A.4). Here we prove the proposition which we used in (3.3).

PROPOSITION (L. Illusie). *Let $(A, \pi A, k)$ be a discrete valuation ring and $S = \mathrm{Spec}(A)$ with the canonical log structure (cf. Conventions). For $i = 1, 2$, let $X_i \to S$ be a morphism of fs log schemes having semistable reductions in the sense of* [K1] (3.7) (2). *Suppose that $\mathring{X}_1 \otimes_A A/(\pi^2)$ and $\mathring{X}_2 \otimes_A A/(\pi^2)$ are isomorphic as $A/(\pi^2)$-schemes. Then the special fibers $X_1 \otimes_A k$ and $X_2 \otimes_A k$ are isomorphic as $S \otimes_A k$-fs log schemes.*

Proof. We reduce to the next local statement.

LEMMA (A.4.1). *In (A.4), we identify $\mathring{X}_1 \otimes_A A/(\pi^2)$ with $\mathring{X}_2 \otimes_A A/(\pi^2)$, and write it as Y. Suppose that there is a strict étale S-morphism $Y \longrightarrow A/(\pi^2)[\mathbf{N}^{n_i}]/(e_1 \cdots e_{r_i} - \pi)$ $(1 \leqslant r_i \leqslant n_i$, e_j's are the canonical base of $\mathbf{N}^{n_i})$ for each $i = 1, 2$. Here the target is regarded as an S-log scheme by $\mathbf{N} \ni \pi \mapsto e_1 \cdots e_{r_i}$. Then*

(i) *Zariski locally on Y, there is a unique M_s-isomorphism $\iota\colon M_{X_1}|_{Y_s} \cong M_{X_2}|_{Y_s}$ (Y_s is the special fiber of Y) satisfying the following three conditions.*

(a) *There is a bijection $\sigma\colon I_1 := \{j | e_j \in \mathbf{N}^{n_1}$ is not invertible on $Y\} \overset{\cong}{\to} \{j | e_j \in \mathbf{N}^{n_2}$ is not invertible on $Y\}$, and for each $j \in I_1$, the subscheme of Y determined by (e_j) is irreducible.*

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(b) *For each* $j \in I_1$, *there is an element* $b_j \in \Gamma(Y, \mathcal{O}_Y^*)$ *such that* $\iota(e_j) = b_j e_{\sigma(j)}$ *in* $\Gamma(Y_s, M_{X_2}|_{Y_s})$ *and* $e_j = b_j e_{\sigma(j)}$ *in* $\Gamma(Y, \mathcal{O}_Y)$.

(c) *If there is another such* (σ', b_j'), *then* $\sigma = \sigma'$ *and* $b_j \equiv b_j' \bmod \pi$ *for each* $j \in I_1$.

(ii) *If further there is an isomorphism* $\iota_0\colon M_{X_1} \to M_{X_2}$ *on* Y, *then the above* ι *is the one restricted from* ι_0.

Proof. We reduce to the next ring-theoretic statement.

LEMMA (A.4.2). *Let* $(A, \pi A, k)$ *be a discrete valuation ring,* $B = A/(\pi^2)[x_1, \ldots, x_n]/(x_1 \cdots x_r - \pi)$ $(1 \leqslant r \leqslant n)$, *and* $\mathfrak{p}$ *a maximal ideal of* B. *Then* $\mathrm{Ann}_{B_{\mathfrak{p}}^{\mathrm{sh}}}(x_i) \subset (\pi)$ *for* $i = 1, \ldots, n$, *where* $B_{\mathfrak{p}}^{\mathrm{sh}}$ *is the strict localization of* B *at* $\mathfrak{p}$. *Further* $\mathrm{Ann}_{B_{\mathfrak{p}}^{\mathrm{sh}}}(\pi) \subset (\pi)$.

Proof. By [M] p. 266 Theorem 83 and Remark 1, we may assume that k is algebraically closed. Then we may assume that $\mathfrak{p} = (x_1, \ldots, x_n, \pi)$ and replace $B_{\mathfrak{p}}^{\mathrm{sh}}$ by $A/(\pi^2)[[x_1, \ldots, x_n]]/(x_1 \cdots x_r - \pi)$. In this ring, $\mathrm{Ann}(x_i) \subset \mathrm{Ann}(\pi) \subset (\pi)$ for $i \leqslant r$ and $\mathrm{Ann}(x_i) = 0$ for $i > r$ respectively. This completes the proof of (A.4). □

Remark (A.4.3). The above proof implies that, in (A.4), it is not necessary that X_i comes from the family over S: For two $S \otimes_A A/(\pi^2)$-fs log schemes X_i $(i = 1, 2)$ which étale locally lift to semistable families over S having the same underlying scheme, the conclusion of (A.4) is satisfied.

Appendix B

(B.1). Here we give the counterexample that we alluded to in (1.1). Let k be a field. Let $h\colon \langle x, y\rangle \to \langle x, z\rangle$ be a homomorphism of fs monoids $\mathbf{N}^2 \to \mathbf{N}^2$; $x \mapsto x$, $y \mapsto xz$. Then the morphism $f\colon Y := \mathrm{Spec}(k[\mathbf{N}^2]) \to \mathrm{Spec}(k[\mathbf{N}^2]) =: X$ induced by h is log étale. We consider the cartesian diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k[x, x^{-1}]) & \xrightarrow{\pi} & X \end{array}$$

in (fs log sch) with π being a strict immersion. Then f' is an isomorphism. Let n be an integer invertible on X. We will see $f^* \mathrm{R}\pi_* \mathbf{Z}/n\mathbf{Z} \neq \mathrm{R}\pi'_* f'^* \mathbf{Z}/n\mathbf{Z}$. First $\mathrm{R}\pi_* \mathbf{Z}/n\mathbf{Z} = (\mathrm{Spec}(k[x]) \longrightarrow X)_* \mathbf{Z}/n\mathbf{Z}$ by [N] (7.6.5). Analogously, $\mathrm{R}\pi'_* \mathbf{Z}/n\mathbf{Z} = (\mathrm{Spec}(k[x]) \longrightarrow Y)_* \mathbf{Z}/n\mathbf{Z}$ whose stalk at $(0, z)$ with $z \neq 0$ is zero. Thus we have got the desired statement.

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