LIMITING DISTRIBUTION OF THE PRESENT VALUE OF A PORTFOLIO

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Abstract

An approximation of the distribution of the present value of the benefits of a portfolio of temporary insurance contracts is suggested for the case where the size of the portfolio tends to infinity. The model used is the one presented in Parker (1922b) and involves random interest rates and future lifetimes. Some justifications of the approximation are given. Illustrations for limiting portfolios of temporary insurance contracts are presented for an assumed Ornstein-Uhlenbeck process for the force of interest.

Keywords

Force of interest; Ornstein-Uhlenbeck process; Portfolio of policies; Present value function; Limiting distribution.

1. Introduction

When considering random interest rates in actuarial functions, a question of particular interest is the distribution of the present value of a portfolio of policies. Studying such distributions could be very useful in areas such as pricing, valuation, solvency analysis and reinsurance.

Some references which considered stochastic interest rates in actuarial functions are Boyle (1976), Wilkie (1976), Waters (1978), Panjer and Bellhouse (1980), Devolder (1986), Giacotto (1986), Dhaene (1989), Dufresne (1988), Beekman and Fuelling (1990), Parker (1992b).

Recently, Dufresne (1990) derived the distribution of a perpetuity for i.i.d. interest rates. Frees (1990) recursively expressed by an integral equation the distribution of a block of n-year annuities for i.i.d. interest rates.

This paper, taken for the most part from the author's Ph.D. thesis (Parker (1992a)), presents an approximation of the limiting distribution, as the number of policies tend to infinity, of the average present value of the benefits for a specific type of portfolio of insurance contracts. Although, theoretically, the approach may be used for any stochastic process for the interest rates, it is more convenient for Gaussian processes. The approximation is justified by two correlation coefficients which happen to be relatively high mainly because of the definition of the present value function. Some illustrations of the distribution function of the present value of portfolios using the Ornstein-Uhlenbeck process are presented. Finally, the moments of some approximate distributions are compared with the corresponding exact moments.

2. A PORTFOLIO

Consider a portfolio of temporary insurance contracts, each with sum insured 1, issued to c lives insured aged x. Let $Z(c)$ be the random present value of the benefits of the portfolio.

PARKER (1922b) used a definition of $Z(c)$ involving a summation over the c contracts of the portfolio. That is

$$Z(c) = \sum_{i=1}^{c} Z_i, \tag{2.1}$$

where Z_i is the present value of the benefit for the ith life insured of the portfolio. This definition is convenient for calculating the moments of $Z(c)$ because it is possible to simplify the expressions for these moments under the assumption that the future lifetimes of the c policyholders are mutually independent.

Another definition which is equivalent appears to be more appropriate for studying the limiting distribution of the random variable $Z(c)$.

Instead of summing over the c policies, one could consider summing the present value of the benefits in a given year over the n policy-years of the contract. Algebraically, we have

$$Z(c) = \sum_{i=0}^{n-1} c_i \cdot e^{-y(i+1)}, \tag{2.2}$$

where

$$y(i+1) = \int_0^{i+1} \delta_s \, ds, \tag{2.3}$$

d_s is the force of interest at time s and c_i, $i = 0, 1, \ldots, n-1$ is the random variable denoting the number of policies where the death benefit is actually paid at time $i+1$. We let c_n be the number of lives insured surviving to the end of the term, n. Note that the sum of the c_i's from i equal 0 to n is c, the total number of policies in the portfolio. Thus,

$$\sum_{i=0}^{n} c_i = c. \tag{2.4}$$

When studying $Z(c)$, we will assume that the future lifetimes of the lives insured are mutually independent and independent of the forces of interest $\{\delta_s\}_{s \geq 0}$. In this case, the $\{c_i\}_{i=1}^{n}$ is multinominal. We will also assume that the discounting of all the benefits for the policies in the portfolios is done with the same Gaussian forces of interest.

In the next section, we consider limiting portfolios, i.e. portfolios where the number of contracts tends to infinity.

3. LIMITING DISTRIBUTION

Using (2.2), one could intuitively derive that the average cost per policy (defined as $Z(c)/c$) as the number of such policies tends to infinity would simply be a weighted average of the present value functions from year 1 to year n. The weights being the expected proportion of contracts payable in each year, i.e. ${}_{i|}q_x$. The probabilistic version of this intuition is presented in Theorem 1.

Theorem 1: As c tends to infinity, the average cost per policy for a portfolio of n-year temporary insurance contracts tends in distribution to: (see also proposition 5 of FREES (1990))

$$\zeta_n = \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)}. \tag{3.1}$$

Proof: This result is true if

$$Z(c)/c - \zeta_n = \sum_{i=0}^{n-1} (c_i/c - {}_{i|}q_x) \cdot e^{-y(i+1)} \tag{3.2}$$

tends in probability to 0.

We use the well-known result that if X tends in probability to 0 and Y has finite mean and variance, then $X \cdot Y$ tends in probability to 0 (see, for example, CHUNG (1974, p. 92)).

Here, c_i is binomial $(c, {}_{i|}q_x)$ so, $(c_i/c - {}_{i|}q_x)$ tends in probability to 0 for each i. And as $e^{-y(i+1)}$ is log-normally distributed with finite mean and variance, it follows that

$$\sum_{i=0}^{n-1} (c_i/c - {}_{i|}q_x) \cdot e^{-y(i+1)}$$

tends in probability to 0. □

Now, one could theoretically obtain the density function of ζ_n by integrating the joint density function of the $y(i)$'s over the appropriate domain. The expression would look like the following:

$$f_{\rho\zeta_n}(z) = \int_{y_n} \cdots \int_{y_2} \int_{y_1} f_{\underline{Y}}(y_1, y_2, \ldots, y_n)\, dy_1\, dy_2 \ldots dy_n \ , \tag{3.3}$$

where $\underline{Y} = (y(1), y(2), \ldots, y(n))$ and is multivariate normal.

But this approach is not possible from a practical point of view as it is almost impossible to evaluate (3.3) even for n as small as 5. In the next section, however, we derive a recursive equation from which one can approximate the distribution of ζ_n.

4. APPROXIMATION

Since ζ_n is a summation over the policy-years, it is easy to break it down into the sum of ζ_{n-1} and a term for the nth policy year. The recursive equation for ζ_n is then given by:

$$\zeta_n = \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} = \sum_{i=0}^{n-2} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_{n-1|}q_x \cdot e^{-y(n)}$$

$$\zeta_n = \zeta_{n-1} + {}_{n-1|}q_x \cdot e^{-y(n)}. \tag{4.1}$$

Let z_i be a possible realization of z_i and y_j be a possible realization of $y(j)$.

Let the function $g_n(z_n, y_n)$, a somewhat unusual function based on the distribution of ζ_n and the density function of $y(n)$, be defined as:

$$g_n(z_n, y_n) = P(\zeta_n \leq z_n) \cdot f_{y(n)}(y_n | \zeta_n \leq z_n), \tag{4.2}$$

or equivalently,

$$g_n(z_n, y_n) = f_{y(n)}(y_n) \cdot P(\zeta_n \leq z_n | y(n) = y_n). \tag{4.3}$$

From this last definition, it follows immediately that the distribution function of ζ_n is given by:

$$F_{\zeta_n}(z_n) = \int_{-\infty}^{\infty} g_n(z_n, y_n) \cdot dy_n, \tag{4.4}$$

where the function $g_n(z_n, y_n)$ may be calculated with a high degree of accuracy from the following recursive equation:

$$g_n(z_n, y_n) \cong \int_{-\infty}^{\infty} f_{y(n)}(y_n | y(n-1) = y_{n-1}) \times \times g_{n-1}(z_n - {}_{n-1|}q_x \cdot e^{-y_n}, y_{n-1})\, dy_{n-1} \tag{4.5}$$

with the starting value:

$$g_1(z_1, y_1) = \begin{cases} \phi\left(\dfrac{y_1 - E[y(1)]}{V[y(1)]^{.5}}\right) & \text{if} \quad z_1 \geq q_x \cdot e^{-y_1} \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

We use the notation $\phi(\cdot)$ to denote the probability density function of a zero mean and unit variance normal random variable. Note also that given that $y(n-1)$ equal y_{n-1}, $y(n)$ is normally distributed with mean

$$E[y(n) | y(n-1) = y_{n-1}] = E[y(n)] + \frac{\operatorname{cov}(y(n), y(n-1))}{V[y(n)]} \cdot \{y_{n-1} - E[y(n-1)]\} \tag{4.7}$$

and variance

$$(4.8) \qquad V[y(n)|y(n-1)=y_{n-1}] = V[y(n)] - \frac{\operatorname{cov}^2(y(n), y(n-1))}{V[y(n-1)]}$$

(see, for example, MORRISON (1990, p. 92)).

To derive (4.5), we start by noting that from (4.1), we have that:

$$(4.9) \qquad P(\zeta_n \leq z_n | y(n) = y_n) = P(\zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n} | y(n) = y_n).$$

Now using (4.2), (4.3) and (4.9), we have

$$(4.10) \qquad g_n(z_n, y_n) = P(\zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n}) \times \times f_{y(n)}(y_n | \zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n}).$$

The conditional probability density function of $y(n)$ in (4.10) may be written as: (MELSA and SAGE (1973, p. 98))

$$(4.11) \qquad f_{y(n)}(y_n | \zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n}) = \int_{-\infty}^{\infty} f_{y(n)}(y_n | y(n-1) = y_{n-1}, \zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n}) \times \times f_{y(n-1)}(y_{n-1} | \zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n})\, dy_{n-1}.$$

Equation (4.3) implies that

$$(4.12) \qquad f_{y(n-1)}(y_{n-1} | \zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n}) = \frac{g_{n-1}(z_n - {}_{n-1|}q_x \cdot e^{-y_n}, y_{n-1})}{P(\zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n})}.$$

If we now make the following approximation (see the next section for some justifications)

$$(4.13) \qquad f_{y(n)}(y_n | y(n-1) = y_{n-1}, \zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n}) \cong \cong f_{y(n)}(y_n | y(n-1) = y_{n-1}),$$

then equation (4.11) becomes

$$(4.14) \qquad f_{y(n)}(y_n | \zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n}) \cong \int_{-\infty}^{\infty} f_{y(n)}(y_n | y(n-1) = y_{n-1}) \times \times \frac{g_{n-1}(z_n - {}_{n-1|}q_x \cdot e^{-y_n}, y_{n-1})}{P(\zeta_{n-1} \leq z_n - {}_{n-1|}q_x \cdot e^{-y_n})}\, dy_{n-1}.$$

Finally substituting this last expression (4.14) into (4.10), we obtain (4.5).

To obtain the starting value (4.6), we simply have to note that:

$$(4.15) \qquad \zeta_1 = q_x \cdot e^{-y(1)}$$

and that

(4.16)
$$g_1(z_1, y_1) = P(\zeta_1 \le z_1 | y(1) = y_1) \cdot f_{y(1)}(y_1)$$
$$= P(\zeta_1 \le z_1 | y(1) = y_1) \cdot \phi\left(\frac{y_1 - E[y(1)]}{V[y(1)]^{.5}}\right).$$

Then, since

(4.17)
$$\zeta_1 = q_x \cdot e^{-y_1} \quad \text{if} \quad y(1) = y_1,$$

we have that

(4.18)
$$P(\zeta_1 \le z_1 | y(1) = y_1) = \begin{cases} 1 & \text{if} \quad z_1 \ge q_x \cdot e^{-y_1} \\ 0 & \text{otherwise} \end{cases}.$$

Finally, by combining (4.18) and (4.16), we obtain (4.6). This completes the derivation of (4.5) and (4.6).

Before doing numerical evaluations of approximation (4.5), it is important to study in greater details and to justify the approximation (4.13) involved here. This is done in the next section.

5. JUSTIFICATIONS

Looking at the steps leading to (4.5), we note that the result is not exact due only to approximation (4.13) made in order to obtain a recursive equation involving only known quantities. This approximation may be justified theoretically by looking at two particular correlation coefficients, one of which validates the approximation for large values of n and the other for small values of n.

5.1 Correlation between $y(n)$ and $y(n-1)$

From the subject of multivariate analysis, we know that the approximation (4.13) will be acceptable if $y(n)$ and $y(n-1)$ are highly correlated (see, for example, MARDIA, KENT and BIBBY (1979, Section 6.5)). This is true since if they are highly correlated, knowing $y(n-1)$ would explain much of $y(n)$. Now if this is the case, introducing any other variable, correlated or not with $y(n)$, in the regression model to further explain $y(n)$ cannot improve the situation much.

Looking back at the definition of $y(n)$ (see (2.3)) it is clear that $y(n-1)$ and $y(n)$ must be highly correlated. Their correlation coefficient will be given by: (ROSS (1988, p. 280))

(5.1)
$$\varrho(y(n), y(n-1)) = \frac{\text{cov}(y(n), y(n-1))}{\{V[y(n)] \cdot V[y(n-1)]\}^{1/2}}.$$

Note that if the force of interest is modeled by a White Noise process, i.e.

(5.2)
$$\delta_t \sim N(\Delta, \sigma_w^2),$$

where it is understood that its integral, $y(t)$, is a Wiener process, it can be shown that, the expected value of $y(t)$ is

$$E[y(t)] = \Delta \cdot t \tag{5.3}$$

and its autocovariance function is

$$\operatorname{cov}(y(s), y(t)) = \sigma_w^2 \cdot \min(s, t). \tag{5.4}$$

If the force of interest is modeled by the following Ornstein-Uhlenbeck process:

$$d\delta_t = -\alpha(\delta_t - \delta)\,dt + \sigma \cdot dW_t, \tag{5.5}$$

with initial value δ_0, then $y(t)$ has an expected value of

$$E[y(t)] = \delta \cdot t + (\delta_0 - \delta) \cdot \left(\frac{1 - e^{-\alpha t}}{\alpha}\right) \tag{5.6}$$

and its autocovariance function is

$$\operatorname{cov}(y(s), y(t)) = \frac{\sigma^2}{\alpha^2} \min(s, t) + + \frac{\sigma^2}{2\alpha^3}[-2 + 2e^{-\alpha s} + 2e^{-\alpha t} - e^{-\alpha|t-s|} - e^{-\alpha(t+s)}] \tag{5.7}$$

(see, PARKER (1922b, equations 38 and 39)).

The correlation coefficients between $y(n)$ and $y(n-1)$ for different values of n, when the force of interest is modeled by a White Noise (see (5.2)) and when it is modeled by an Ornstein-Uhlenbeck process (see (5.5)) with parameter $\alpha = .1$, .2 or .5 are presented in Table 1.

TABLE 1

CORRELATION COEFFICIENT BETWEEN $y(n)$ AND $y(n-1)$
FORCE OF INTEREST AS WHITE NOISE AND ORNSTEIN-UHLENBECK PROCESSES

n	White Noise	Ornstein-Uhlenbeck		
		$\alpha = .1$	$\alpha = .2$	$\alpha = .5$
2	.7071	.8773	.8707	.8516
3	.8165	.9474	.9423	.9270
4	.8660	.9701	.9659	.9535
5	.8944	.9804	.9769	.9664
6	.9129	.9860	.9829	.9739
7	.9258	.9894	.9867	.9788
8	.9354	.9916	.9891	.9821
9	.9428	.9931	.9909	.9846
10	.9487	.9942	.9922	.9865
20	.9747	.9980	.9969	.9940
40	.9874	.9992	.9987	.9972
60	.9916	.9995	.9991	.9981

Results for the White Noise process are presented here because this process involves i.i.d. forces of interest, therefore, leading to the lowest correlation coefficients. Results for the Ornstein-Uhlenbeck process are presented because it is the process used for illustration purposes in the next section.

Note that the correlation coefficient between $y(n)$ and $y(n-1)$ is not influenced by the parameter σ_w of the White Noise process. For the Ornstein-Uhlenbeck process, the parameter δ_0, δ and σ have no incidence on the correlation coefficients.

Table 1 clearly shows that $y(n)$ and $y(n-1)$ are very highly correlated, especially for large values of n. Therefore, approximation (4.13) made to obtain the recursive equation (4.5) should be acceptable.

Another correlation coefficient could also justify approximation (4.13), independently of the one discussed here. This is the subject of the next section.

5.2. Correlation between $e^{-y(n)}$ and ζ_n

Again from the subject of multivariate analysis, we know that the approximation (4.13) would also be acceptable if $y(n-1)$ and ζ_{n-1} contained about the same useful information to explain $y(n)$ (see, for exemple, MARDIA, KENT and BIBBY (1979, Section 6.5)). This may be investigated by studying the correlation coefficients between $e^{-y(n-1)}$ and ζ_{n-1}.

If $e^{-y(n)}$ and ζ_n are highly correlated, the approximation would be reasonable. The correlation coefficient between these two random variables is: (ROSS (1988, p. 280))

$$\varrho(e^{-y(n)}, \zeta_n) = \frac{\operatorname{cov}(e^{-y(n)}, \zeta_n)}{\{V[e^{-y(n)}] \cdot V[\zeta_n]\}^{1/2}}. \tag{5.8}$$

Using (3.1), we obtain

$$\varrho(e^{-y(n)}, \zeta_n) = \frac{\sum_{i=0}^{n-1} {}_{i|}q_x \cdot \operatorname{cov}(e^{-y(n)}, e^{-y(i+1)})}{\left\{V[e^{-y(n)}] \cdot \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot \operatorname{cov}(e^{-y(i+1)}, e^{-y(j+1)})\right\}^{.5}}, \tag{5.9}$$

where $\operatorname{cov}(e^{-y(i)}, e^{-y(j)})$ is given by

$$\operatorname{cov}(e^{-y(i)}, e^{-y(j)}) = E[e^{-y(i)} \cdot e^{-y(j)}] - E[e^{-y(i)}] \cdot E[e^{-y(j)}]. \tag{5.10}$$

Note that if the force of interest is Gaussian, the expected values involved in (5.10) are simply the expected values of lognormal variables (see PARKER (1992b, Section 6)).

The correlation coefficients between $e^{-y(n)}$ and ζ_n, for different values of n, when the force of interest is modeled by a White Noise or an Ornstein-Uhlenbeck process with particular parameters are presented in the following table. The mortality rates used are the male ultimate rates of the CA 1980-82 mortality table (COWARD (1988, pp. 227-231)).

TABLE 2

CORRELATION COEFFICIENT BETWEEN $e^{-y(n)}$ AND ζ_n
FORCE OF INTEREST AS WHITE NOISE AND ORNSTEIN-UHLENBECK PROCESSES

n	White Noise $\Delta = .06$, $\sigma_w = .01$ $x = 30$	Ornstein-Uhlenbeck $\delta = .06$, $\delta_0 = .1$, $\alpha = .1$		
		$\sigma = .01$ $x = 30$	$\sigma = .02$ $x = 30$	$\sigma = .01$ $x = 50$
1	1.0000	1.0000	1.0000	1.0000
2	.9447	.9899	.9899	.9912
3	.9199	.9824	.9824	.9849
4	.9064	.9770	.9770	.9802
5	.8980	.9728	.9727	.9765
6	.8925	.9693	.9692	.9735
7	.8890	.9665	.9663	.9708
8	.8868	.9642	.9638	.9684
9	.8856	.9622	.9617	.9662
10	.8851	.9605	.9599	.9641
20	.8969	.9535	.9518	.9455
40	.8999	.9368	.9321	.8693
60	.8486	.8730	.8494	—

Note that $\varrho(e^{-y(1)}, \zeta_1)$ is 1. This implies that approximation (4.13) is exact for $n = 2$. The correlation coefficients of Table 2 suggest that the approximation should be good, especially for small values of n.

Combining the two conclusions drawn from the results presented in Table 1 and Table 2, we note that the approximation should be acceptable for all values of n.

Now that approximation (4.5) appears to be justified, we may use it to find the distribution of ζ_n. Equations (4.4) and (4.5) may be computed by numerical integration or by some discretization method. Although some methods are certainly more accurate than others, it is not our intention in this paper to discuss or compare the possible methods. In the next section, we present some results obtained by an arbitrarily chosen discretization of (4.5).

6. ILLUSTRATIONS

Figure 1 illustrates the cumulative distribution function of ζ_n, $n = 5, 10, 15, 20$ and 25, the limiting average cost per policy for temporary insurance contracts issued at age 30 and with the force of interest modeled by a Ornstein-Uhlenbeck process with parameters $\delta = .06$, $\delta_0 = .1$, $\alpha = .1$ and $\sigma = .01$. The mortality rates are again the male ultimate rates of the CA 1980-82.

The range of possible values for ζ_5 is much shorter than the one for ζ_{25}. This is due to the fact that with a limiting portfolio, there is no fluctuation due to mortality, and therefore, all the possible variations in the random variable ζ_n are caused by the force of interest. When there are only five years of fluctuating force of interest involved, it is clear that the results will be less spread than when there are 25 years of fluctuating force of interest. Finally, it should be obvious why ζ_{25} takes larger values than ζ_5.

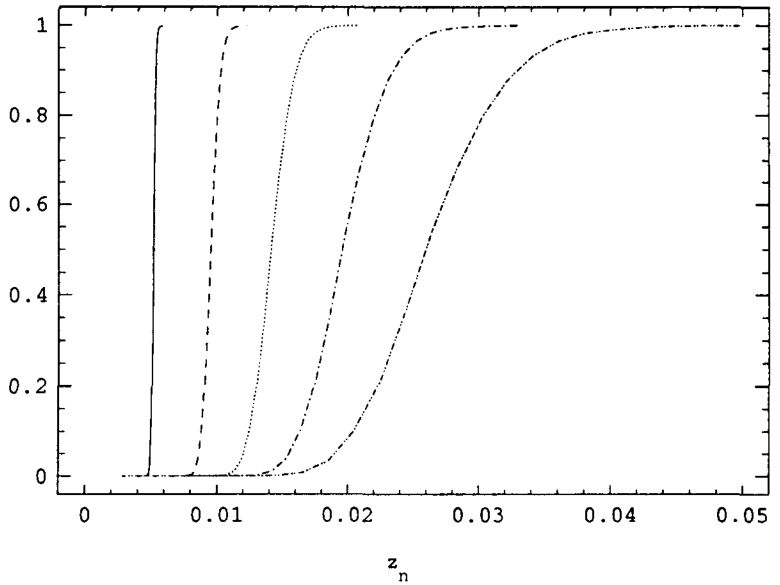

FIGURE 1. Cumulative distribution function of ζ_n
Temporary insurance policies issued at age 30, Ornstein-Uhlenbeck $\delta = .06$ $\delta_0 = .1$ $\alpha = .1$ $\sigma = .01$.

——— 5 years
— — 10 years
· · · 15 years
— · — 20 years
— · · 25 years

There is no doubt that the distribution of ζ_n provides very useful information in solvency problems. One may also be interested in using such information for pricing or valuation of a portfolio of insurance policies. In this regard, the relevant information is contained in the right tail of the distribution of ζ_n.

Table 3 contains some numerical values of the right tail of the distributions of ζ_5 and ζ_{25} illustrated in Figure 1.

From Table 3, we know, for example, that a company charging a single premium of .005602 to each life insured of a very large portfolio of 5-year temporary contracts will meet its future liabilities with a probability of about .995.

TABLE 3

RIGHT TAIL OF THE APPROXIMATE DISTRIBUTION OF ζ_n, 5 AND 25 YEARS TEMPORARY INSURANCE ISSUED AT AGE 30, ORNSTEIN-UHLENBECK $\delta = .06$ $\delta_0 = .1$ $\alpha = .1$ $\sigma = .01$

5 years temporary		25 years temporary	
z_5	$F_{\zeta_5}(z_5)$	z_{25}	$F_{\zeta_{25}}(z_{25})$
.005381	.940609	.036135	.966095
.005436	.972183	.038092	.982494
.005547	.992830	.040048	.989498
.005602	.995229	.042004	.994551
.005823	.997927	.049827	.999505

7. VALIDATIONS

A validation of the results described above has been done by comparing the exact first three moments of ζ_n with its estimated first three moments from the approximate distribution.

A discretization of the variable ζ_n has been used to estimate the moments of the approximate distribution. Algebraically, the mth moment of ζ_n about the origin has been approximated by the following equation:

$$\hat{E}[\zeta_n^m] \cong \sum_{i=0}^{h} \left(\frac{z_n[i] + z_n[i+1]}{2}\right)^m \cdot (F_{\zeta_n}(z_n[i+1]) - F_{\zeta_n}(z_n[i])), \tag{7.1}$$

where $z_n[i]$, $i = 1, 2, \ldots, h$ is the ith ordered value of ζ_n at which F_{ζ_n} was evaluated. For the illustrations presented above, h was chosen to be 25. To deal with the extremities of the distributions the following values were arbitrarily defined as:

$$z_n[0] = z_n[1] - \left(\frac{z_n[2] - z_n[1]}{2}\right) \tag{7.2}$$

$$z_n[h+1] = z_n[h] + \left(\frac{z_n[h] - z_n[h-1]}{2}\right) \tag{7.3}$$

$$F_{\zeta_n}(z_n[0]) = 0 \tag{7.4}$$

$$F_{\zeta_n}(z_n[h+1]) = 1. \tag{7.5}$$

The exact moments of ζ_n about the origin may be obtained by using the definition of ζ_n given by (3.1). Its mth moment about the origin is then given by

$$E[\zeta_n^m] = E\left[\left(\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)}\right)^m\right]. \tag{7.6}$$

Now, with m equal 1, the first moment is

$$E[\zeta_n] = \sum_{i=0}^{n-1} E[{}_{i|}q_x \cdot e^{-y(i+1)}]. \tag{7.7}$$

With m equal 2, the second moment is

$$E[\zeta_n^2] = E\left[\left(\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)}\right) \cdot \left(\sum_{j=0}^{n-1} {}_{j|}q_x \cdot e^{-y(j+1)}\right)\right] \tag{7.8}$$

$$= E\left[\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot e^{-y(i+1)-y(j+1)}\right] \tag{7.9}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot E[e^{-y(i+1)-y(j+1)}]. \tag{7.10}$$

With m equal 3, the third moment is

$$(7.11) \qquad E[\zeta_n^3] = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot {}_{k|}q_x \cdot E[e^{-y(i+1)-y(j+1)-y(k+1)}].$$

Note that the moments of ζ_n are exactly the limiting moments of the average cost per policy studied in PARKER (1992b).

Table 4 presents, for different terms of temporary insurance contracts issued at age 30, the exact moments of ζ_n, $E[\zeta_n^m]$, and the difference between the exact and the estimated moments (given by (7.1)), i.e. $E[\zeta_n^m] - \hat{E}[\zeta_n^m]$, for m equal 1, 2 and 3. The force of interest is modeled by an Ornstein-Uhlenbeck process with parameters $\delta = .06$, $\delta_0 = .1$, $\alpha = .1$ and $\sigma = .01$.

TABLE 4

COMPARISON OF EXACT AND APPROXIMATE MOMENTS OF ζ_n, n-YEAR TEMPORARY INSURANCE ISSUED AT AGE 30, ORNSTEIN-UHLENBECK $\delta = .06$ $\delta_0 = .1$ $\alpha = .1$ $\sigma = .01$

n	$E[\zeta_n^m]$			$E[\zeta_n^m] - \hat{E}[\zeta_n^m]$		
	$m = 1$ ($\times 10$)	$m = 2$ ($\times 100$)	$m = 3$ ($\times 1000$)	$m = 1$ ($\times 10$)	$m = 2$ ($\times 100$)	$m = 3$ ($\times 1000$)
1	.01197	.00014	.00000	.00000	.00000	.00000
2	.02284	.00052	.00001	.00000	.00000	.00000
3	.03291	.00108	.00004	.00000	.00000	.00000
4	.04246	.00180	.00008	−.00001	.00000	.00000
5	.05160	.00266	.00014	−.00003	.00000	.00000
10	.09517	.00909	.00087	−.00017	−.00004	−.00001
15	.14163	.02023	.00292	−.00031	−.00011	−.00003
20	.19731	.03964	.00811	−.00041	−.00024	−.00009
25	.26356	.07167	.02013	−.00054	−.00053	−.00030

Note that, in order to present more significant digits, the first moment has been multiplied by 10, the second moment multiplied by 100 and the third moment multiplied by 1000.

From Table 4, we note that the exact and approximate first three moments of ζ_n agree to at least four, five and six decimal places respectively (for $n \leq 25$). This is excellent, especially if one considers that many approximations were involved before obtaining the estimated moments of ζ_n, $\hat{E}[\zeta_n]$.

Let the relative error for the mth moment of ζ_n be:

$$(7.12) \qquad \frac{|E[\zeta_n^m] - \hat{E}[\zeta_n^m]|}{E[\zeta_n^m]}.$$

Then, for any term, n, the relative error on the expected value of ζ_n is about .2% or less. For its second moment, it is about .7% or less. And for its third moment, it is about 1.5% or less.

The results for other parameters of the Ornstein-Uhlenbeck process and for other ages at issue, not illustrated here, were all excellent. The maximum relative error observed, generally for the third moment, being about 3%. Although for the illustrations presented here, the error is always negative, for other situations it may be positive or even alternate over different ranges of values of the term, n. In all cases, however, the relative error is small.

From the justifications made in Section 5 and from the validations presented here, it appears that the approximation (4.13) suggested to obtain the resursive equation (4.5) has to be highly acceptable.

8. CONCLUSION

The results of this paper provides a way of approximating the distribution of limiting portfolios that is valid for any process for the force of interest as long as the conditional density function of $y(n)$ given $y(n-1)$ is known and expression (5.10) can be evaluated. As indicated earlier, choosing a Gaussian process simplify things considerably.

Although equation (4.5) might not be acceptable for any random variables, the very nature of the problem under consideration here, i.e. the present value of future benefits, has some particular properties which imply that the approximation is good. The worse possible case for Gaussian interest rates is when they are independent, i.e. White Noise process. Even in this case, the correlation resulting between consecutive present value functions is fairly high.

There is no doubt that knowing the distribution of the average cost per policy is useful for pricing, valuation, solvency and reinsurance. The approximation suggested in this paper is certainly accurate enough for most situations one may encounter, it is more justifiable and less subjective than the testing of a limited number of scenarios and it avoids the extremely lengthy simulations required to obtain reasonable information about the tail of the distribution.

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