Bull. Aust. Math. Soc. **94** (2016), 1–6
doi:10.1017/S0004972715001501

A UNIQUE REPRESENTATION BI-BASIS FOR THE INTEGERS. II

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(Received 6 September 2015; accepted 25 October 2015; first published online 8 January 2016)

Abstract

For $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$, define $r_A(n)$ and $\delta_A(n)$ by $r_A(n) = \#\{(a_1, a_2) \in A^2 : n = a_1 + a_2, a_1 \leq a_2\}$ and $\delta_A(n) = \#\{(a_1, a_2) \in A^2 : n = a_1 - a_2\}$. We call A a unique representation bi-basis if $r_A(n) = 1$ for all $n \in \mathbb{Z}$ and $\delta_A(n) = 1$ for all $n \in \mathbb{Z}\backslash\{0\}$. In this paper, we prove that there exists a unique representation bi-basis A such that $\limsup_{x\to\infty} A(-x, x)/\sqrt{x} \geq 1/\sqrt{2}$.

2010 *Mathematics subject classification*: primary 11B13; secondary 11B34.

Keywords and phrases: unique representation bi-basis, Sidon set.

1. Introduction

For sets A, B of integers and any integer c, we define the sets

$$A + B = \{a + b : a \in A, b \in B\}, \quad A - B = \{a - b : a \in A, b \in B\}$$

and the translations

$$c + A = \{c + a : a \in A\}, \quad c - A = \{c - a : a \in A\}.$$

For $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$, let

$$r_A(n) = \#\{(a_1, a_2) \in A^2 : n = a_1 + a_2, a_1 \leq a_2\},$$
$$\delta_A(n) = \#\{(a_1, a_2) \in A^2 : n = a_1 - a_2\}.$$

The counting function for the set A is $A(y, x) = \#\{a \in A : y \leq a \leq x\}$.

A set B of integers is called a Sidon set if $r_B(n) \leq 1$ for all $n \in \mathbb{Z}$. A set A of integers is called an additive basis of $\mathbb{Z}$ if $r_A(n) \geq 1$ for all $n \in \mathbb{Z}$, and a unique representation basis if $r_A(n) = 1$ for all $n \in \mathbb{Z}$. A set A of integers is called a *unique representation bi-basis* of $\mathbb{Z}$ if $r_A(n) = 1$ for all $n \in \mathbb{Z}$ and $\delta_A(n) = 1$ for all $n \in \mathbb{Z}\backslash\{0\}$.

In 2003, Nathanson [5] proved that a unique representation basis of $\mathbb{Z}$ can be arbitrarily sparse, but it remains open how dense they can be. Nathanson [6] considered

This work was supported by the National Natural Science Foundation of China (Grant No. 11471017).
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https://doi.org/10.1017/S0004972715001501 Published online by Cambridge University Press

similar problems for asymptotic bases. In 2007, Chen [1] proved that for any $\varepsilon > 0$, there exists a unique representation basis A of $\mathbb{Z}$ such that $A(-x, x) \geq x^{1/2-\varepsilon}$ for infinitely many positive integers x. In 2010, Lee [4] extended this result to the existence of such bases with arbitrary prescribed representation function. In 2011, the present author [7] proved that there exist a real number $c > 0$ and an asymptotic basis A with prescribed representation function such that $A(-x, x) \geq c\sqrt{x}$ for infinitely many positive integers x. In 2013, Cilleruelo and Nathanson [3] proved that the problem of finding a dense set of integers with a prescribed representation function f of order h and $\liminf_{|n|\to\infty} f(n) \geq g$ is 'equivalent' to the classical problem of finding dense $B_h[g]$ sequences of positive integers. In 2014, Xiong and the present author [8] constructed a unique representation bi-basis of $\mathbb{Z}$ whose growth is logarithmic.

In this paper, we obtain the following result.

Theorem 1.1. *There exists a unique representation bi-basis A of $\mathbb{Z}$ such that*

$$\limsup_{x\to\infty} \frac{A(-x, x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}.$$

2. Lemmas

Lemma 2.1 [1, Lemma 1]. *Let A be a nonempty finite set of integers with $r_A(n) \leq 1$ for all $n \in \mathbb{Z}$ and $0 \notin A$. If m is an integer with $r_A(m) = 0$, then there exists a finite set B of integers such that $A \subseteq B$, $r_B(n) \leq 1$ for all $n \in \mathbb{Z}$, $r_B(m) = 1$ and $0 \notin B$.*

Lemma 2.2. *Let A be a nonempty finite set of integers satisfying $r_A(n) \leq 1$ for all $n \in \mathbb{Z}$, $\delta_A(n) \leq 1$ for all $n \in \mathbb{Z}\backslash\{0\}$ and $0 \notin A$. If u and v are integers with $r_A(u) = \delta_A(v) = 0$, then there exists a finite set B of integers such that $A \subseteq B$, $r_B(n) \leq 1$ for all $n \in \mathbb{Z}$, $\delta_B(n) \leq 1$ for all $n \in \mathbb{Z}\backslash\{0\}$, $r_B(u) = \delta_B(v) = 1$ and $0 \notin B$.*

Proof. Since $A \neq \emptyset$, we have $v \neq 0$. Let $b = \max\{|a| : a \in A\}$ and choose positive integers c and d such that

$$c > 4b + 2|u| + |v|, \quad d > 3c + 2|u| + |v|.$$

Put

$$B = A \cup \{u + c, -c, d, v + d\}.$$

Then $0 \notin B$ and

$$B + B = S \cup (A + A) \cup (u + c + A) \cup (-c + A) \cup (d + A) \cup (v + d + A),$$
$$B - B = D \cup (A - A) \cup \pm(u + c - A) \cup \pm(c + A) \cup \pm(d - A) \cup \pm(v + d - A),$$

where

$$S = \{2(d + v), 2d + v, 2d, u + v + c + d, u + c + d, v + d - c, d - c, 2(u + c), u, -2c\},$$
$$D = \{\pm(v + c + d), \pm(c + d), \pm(d - c + v - u), \pm(d - c - u), \pm(u + 2c), \pm v\}.$$

https://doi.org/10.1017/S0004972715001501 Published online by Cambridge University Press

First, we claim that $r_B(n) \le 1$ for all $n \in \mathbb{Z}$ and $r_B(u) = 1$. Observe that

$$A + A \subseteq [-2b, 2b], \quad -c + A \subseteq [-c - b, -3b - 2|u| - |v|),$$
$$u + c + A \subseteq (3b + |u| + |v|, c + b + u],$$
$$d + A \subseteq [d - b, d + b], \quad v + d + A \subseteq [d + v - b, d + b + v].$$

Moreover, $(d + A) \cap (v + d + A) = \varnothing$. In fact, if $(d + A) \cap (v + d + A) \neq \varnothing$, then there are $a, a' \in A$ such that $d + a = v + d + a'$ and thus $v = a - a'$, which contradicts the hypothesis that $\delta_A(v) = 0$. Since $-2c < -c - b$,

$$\min\{2(d + v), 2d + v, 2d\} > \max\{u + v + c + d, u + c + d\},$$
$$\min\{u + v + c + d, u + c + d\} > \max\{d + b, d + b + v\},$$
$$c + b + u < 2(u + c) < v + d - c < \min\{d - b, d + v - b\},$$
$$c + b + u < 2(u + c) < d - c < \min\{d - b, d + v - b\}.$$

Hence, the sets

$$S, A + A, u + c + A, -c + A, v + d + A, d + A$$

are pairwise disjoint. By the hypothesis, if $n \in A + A$, then $r_B(n) = r_A(n) = 1$. Moreover, since

$$u + c + A, -c + A, v + d + A, d + A$$

are translations, if n belongs to one of these four sets, then $r_B(n) = 1$. Consequently, $r_B(n) \le 1$ for all $n \in \mathbb{Z}$ and $r_B(u) = 1$.

Second, we claim that $\delta_B(n) \le 1$ for all $n \in \mathbb{Z}\backslash\{0\}$ and $\delta_B(v) = 1$. In fact, we have $A - A \subseteq [-2b, 2b]$ and

$$u + c - A \subseteq (3b + |u| + |v|, c + b + u], \quad -u - c + A \subseteq [-c - b - u, -3b - |u| - |v|),$$
$$c + A \subseteq (3b + 2|u| + |v|, c + b], \quad -c - A \subseteq [-c - b, -3b - 2|u| - |v|),$$
$$d - A \subseteq [d - b, d + b], \quad -d + A \subseteq [-d - b, -d + b],$$
$$v + d - A \subseteq [d + v - b, d + b + v], \quad -v - d + A \subseteq [-d - b - v, -d + b - v].$$

Since $\delta_A(v) = 0$ and $r_A(u) = 0$,

$$(d - A) \cap (v + d - A) = \varnothing, \quad (-d + A) \cap (-v - d + A) = \varnothing,$$
$$(u + c - A) \cap (c + A) = \varnothing, \quad (-u - c + A) \cap (-c - A) = \varnothing.$$

Moreover,

$$\max\{c + b + u, c + b\} < u + 2c < d - c - u < \min\{d - b, d - b + v\},$$
$$\max\{c + b + u, c + b\} < u + 2c < d - c - u + v < \min\{d - b, d - b + v\},$$
$$\max\{d + b, d + b + v\} < \min\{v + c + d, d + c\}.$$

Hence, the sets

$$A - A, D, \pm(u + c - A), \pm(c + A), \pm(d - A), \pm(v + d - A)$$

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are pairwise disjoint. By the hypothesis, if $n(\neq 0) \in A - A$, then $\delta_B(n) = \delta_A(n) = 1$. Moreover, if $n(\neq 0)$ belongs to one of the sets

$$\pm(u + c - A), \pm(c + A), \pm(d - A), \pm(v + d - A),$$

then $\delta_B(n) = 1$. Consequently, $\delta_B(n) \le 1$ for all $n \in \mathbb{Z}\backslash\{0\}$ and $\delta_B(v) = 1$. □

Lemma 2.3 [2, Lemma 3.1]. *If C_1 and C_2 are Sidon sets such that*

$$(C_i - C_i) \cap (C_j - C_j) = \{0\}, (C_i + C_i) \cap (C_j + C_j) = \emptyset \quad \text{and} \quad (C_i + C_i - C_i) \cap C_j = \emptyset$$

for $i \neq j$, then $C_1 \cup C_2$ is a Sidon set.

Lemma 2.4 [2, Lemma 3.2]. *For each odd prime p, there is a Sidon set B_p such that:*

(i) $B_p \subseteq [1, p^2 - p]$;
(ii) $(B_p - B_p) \cap [-\sqrt{p}, \sqrt{p}] = \{0\}$;
(iii) $|B_p| > p - 2\sqrt{p}$.

3. Proof of Theorem 1.1

We shall use induction to construct an ascending sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite sets of integers such that for any positive integer k:

(i) $r_{A_k}(n) \le 1$ for all $n \in \mathbb{Z}$, $\delta_{A_k}(n) \le 1$ for all $n \in \mathbb{Z}\backslash\{0\}$;
(ii) $r_{A_{2k}}(n) = 1$ for all $n \in \mathbb{Z}$ with $|n| \le k$, $\delta_{A_{2k}}(n) = 1$ for all $n \in \mathbb{Z}\backslash\{0\}$ with $|n| \le k + 2$;
(iii) $0 \notin A_k$.

Let $A_1 = \{-1, 1, 2\}$. Then

$$A_1 + A_1 = \{0, 1, 2, -2, 3, 4\}, \quad A_1 - A_1 = \{0, \pm 1, \pm 2, \pm 3\}.$$

Suppose that we have constructed $A_1, A_2, \ldots, A_{2k-1}$. Let u be an integer with minimum absolute value and $r_{A_{2k-1}}(u) = 0$. Let

$$v = \min\{n > 0 : n \notin A_{2k-1} - A_{2k-1}\}.$$

Then $\delta_{A_{2k-1}}(v) = \delta_{A_{2k-1}}(-v) = 0$.

By Lemma 2.2, there exists a finite set B of integers such that $A_{2k-1} \subseteq B$, $r_B(n) \le 1$ for all $n \in \mathbb{Z}$, $\delta_B(n) \le 1$ for all $n \in \mathbb{Z}\backslash\{0\}$, $r_B(u) = \delta_B(v) = 1$ and $0 \notin B$. If $r_B(-u) = 0$, then by Lemma 2.1 there exists a finite set B' of integers such that $B \subseteq B'$, $r_{B'}(n) \le 1$ for all $n \in \mathbb{Z}$, $r_{B'}(-u) = 1$ and $0 \notin B'$. Now let

$$A_{2k} = \begin{cases} B & \text{if } r_B(-u) \neq 0, \\ B' & \text{if } r_B(-u) = 0. \end{cases}$$

If $k = 1$, then $|u| = 1 = k$ and $v = 4 > k + 2$. If $k > 1$, since $A_{2k-2} \subseteq A_{2k-1}$, we have $r_{A_{2k-2}}(u) = 0$ and $\delta_{A_{2k-2}}(v) = 0$. By the inductive hypothesis and (ii), we have $|u| \ge k$ and $v \ge k + 2$. Thus, A_{2k} satisfies (i), (ii), (iii) and $A_{2k-1} \subseteq A_{2k}$.

Let $x_k = \max\{|a| : a \in A_{2k}\}$ and let p_k denote the least prime greater than $4x_k^2$. By Lemma 2.4, there exists a Sidon set B_{p_k} such that:

(a) $B_{p_k} \subseteq [1, p_k^2 - p_k]$;
(b) $(B_{p_k} - B_{p_k}) \cap [-\sqrt{p_k}, \sqrt{p_k}] = \{0\}$;
(c) $|B_{p_k}| > p_k - 2\sqrt{p_k}$.

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For $k \geq 1$, let

$$A_{2k+1} = A_{2k} \cup (B_{p_k} + p_k^2 + x_k).$$

Then $0 \notin A_{2k+1}$. Now we shall prove that A_{2k+1} is a Sidon set for every $k \geq 1$.

By the construction, A_{2k} and $B_{p_k} + p_k^2 + x_k$ are Sidon sets. We shall apply Lemma 2.3 with $C_1 = A_{2k}$ and $C_2 = B_{p_k} + p_k^2 + x_k$ to show that

$$C_1 \cup C_2 = A_{2k} \cup (B_{p_k} + p_k^2 + x_k)$$

is a Sidon set. Note that

$$C_1 - C_1 \subseteq [-2x_k, 2x_k] \subseteq [-\sqrt{p_k}, \sqrt{p_k}], \quad C_2 - C_2 = B_{p_k} - B_{p_k}.$$

By (b), $(B_{p_k} - B_{p_k}) \cap [-\sqrt{p_k}, \sqrt{p_k}] = \{0\}$. Thus,

$$(C_1 - C_1) \cap (C_2 - C_2) = \{0\}.$$

If $x \in C_2 + C_2$, then $x \geq 2(p_k^2 + x_k + 1) > 2x_k$, but $C_1 + C_1 \subseteq [-2x_k, 2x_k]$. Thus,

$$(C_1 + C_1) \cap (C_2 + C_2) = \varnothing.$$

If $x \in (C_1 + C_1 - C_1)$, then $x \leq 3x_k$, but, if $x \in C_2$, then $x > p_k^2 + x_k > 3x_k$. Thus,

$$(C_1 + C_1 - C_1) \cap C_2 = \varnothing.$$

If $x \in (C_2 + C_2 - C_2)$, then $x \geq 2(p_k^2 + x_k + 1) - (2p_k^2 - p_k + x_k) = p_k + x_k + 2$ and, if $x \in C_1$, then $x \leq x_k$. Thus,

$$(C_2 + C_2 - C_2) \cap C_1 = \varnothing.$$

Hence, $A_{2k+1} = A_{2k} \cup (B_{p_k} + p_k^2 + x_k)$ is a Sidon set.

Let

$$A = \bigcup_{k=1}^{\infty} A_k.$$

By (ii) and $A_{2k-1} \subseteq A_{2k}$, we have $r_A(n) = 1$ for all $n \in \mathbb{Z}$, $\delta_A(n) = 1$ for all $n \in \mathbb{Z}\backslash\{0\}$. That is, A is a unique representation bi-basis of $\mathbb{Z}$. Moreover, by the construction of A, (a), (b) and (c),

$$\begin{aligned}
\limsup_{x\to\infty} \frac{A(-x,x)}{\sqrt{x}} &\geq \limsup_{k\to\infty} \frac{A(1, 2p_k^2 - p_k + x_k)}{\sqrt{2p_k^2 - p_k + x_k}} \\
&\geq \limsup_{k\to\infty} \frac{|B_{p_k}|}{\sqrt{2p_k^2 - p_k + x_k}} \\
&\geq \limsup_{k\to\infty} \frac{p_k - 2\sqrt{p_k}}{\sqrt{2p_k^2 - p_k + \sqrt{p_k}/2}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

This completes the proof of Theorem 1.1.

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https://doi.org/10.1017/S0004972715001501 Published online by Cambridge University Press