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RESEARCH ARTICLE

C*-simplicity has no local obstruction

Yuhei Suzuki

Department of Mathematics, Faculty of Science, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido 060-0810, Japan; E-mail: yuhei@math.sci.hokudai.ac.jp.

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Abstract
In 2016, I solved a problem of de la Harpe from 2006: Is there a nondiscrete C*-simple group? However the solution was not fully satisfactory, as the C*-simple groups provided (and their operator algebras) are very close to discrete groups. All previously known examples are of this form. In this article I give yet another construction of nondiscrete C*-simple groups. The statement in the title then follows. This in particular gives the first examples of nonelementary C*-simple groups (in Wesolek's sense).

Contents

1. Introduction

A fundamental motivation of operator algebra theory is to give a framework for understanding locally compact groups. Successful achievements in this direction include Glimm's dichotomy theorem, the Kasparov theory, the Baum–Connes theory, and Popa's deformation/rigidity theory. Also, via the (reduced) crossed product construction, locally compact groups produce interesting examples of concrete operator algebras. On the one hand, for *discrete* groups many deep structural results on these operator algebras have been established. On the other hand, their *nondiscrete* counterparts are not yet on a comparable level.

In this article, we focus on *C*-simplicity* [9], the simplicity of the reduced group C*-algebra, of locally compact groups. For discrete groups, satisfactory characterisations of C*-simplicity were established in the last decade (see e.g., [1, 6]). However, the results do not (at least directly) extend to nondiscrete groups. C*-simplicity of nondiscrete groups is still a mysterious property. A main reason for this difficulty is the lack of interesting examples. Indeed, even the existence of such a group – questioned by de la Harpe [3] – was not known until [13]. Although such groups are now known [11, 13], all the currently known examples are *very close* to discrete groups. (More precisely, they are essentially the projective limit of discrete groups of particularly good form; see [13, Proposition], which is the only previously known result to produce a nondiscrete C*-simple group. In particular, all these groups are *elementary* in Wesolek's sense by [15, Theorem 3.18].)

In this article, we provide a new framework to produce nondiscrete C*-simple groups. Note that C*-simple groups must be *totally disconnected*, by [10, Theorem A]. Thus our attention is naturally restricted to totally disconnected groups. As a result of the new construction, we conclude the statement in the title: every totally disconnected locally compact group realises as an open subgroup of a C*-simple group. In particular we obtain the first examples of C*-simple groups which are *nonelementary* in Wesolek's sense [15]. We believe that our new construction sheds new light on (nondiscrete) C*-simplicity, and that our proof gives a new insight into the analysis of the group and reduced crossed product operator algebras of nondiscrete groups.

2. Preliminaries

Here we fix notations, and prove a basic lemma.

Notations

Throughout the article, let G be a totally disconnected locally compact group. We fix a left Haar measure μ on G. Define $L^2(G) := L^2(G, \mu)$. Let $\lambda\colon G \curvearrowright L^2(G)$ denote the left regular representation. The representation λ integrates to the $*$-representation of the group algebra $C_c(G)$ on $L^2(G)$ which is given by the convolution product. The reduced group C*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$ is the operator norm closure of $C_c(G) \subset \mathbb{B}\big(L^2(G)\big)$. For a compact open subgroup K of G, set

$$p_K := \mu(K)^{-1}\chi_K \in C_c(G) \subset \mathrm{C}^*_{\mathrm{r}}(G).$$

Observe that p_K is the orthogonal projection onto the K-fixed point space $L^2(G)^K$. A theorem of van Dantzig [14] shows that the set of compact open subgroups $K < G$ forms a local basis at the identity element $e \in G$. Hence the net $(p_K)_{K<G}$ forms an approximate unit of $\mathrm{C}^*_{\mathrm{r}}(G)$. For a closed subgroup H of G, we identify $\mathrm{C}^*_{\mathrm{r}}(H)$ with the C*-subalgebra of the multiplier algebra $\mathcal{M}\big(\mathrm{C}^*_{\mathrm{r}}(G)\big)$ in the obvious way (compare [7]). In particular, when H is open, we have $\mathrm{C}^*_{\mathrm{r}}(H) \subset \mathrm{C}^*_{\mathrm{r}}(G)$. When $H < G$ is a closed subgroup normalised by a compact subgroup $K < G$, we equip $\mathrm{C}^*_{\mathrm{r}}(H)$ with the K-action induced from the conjugation action $K \curvearrowright H$.

The symbol $\otimes$ stands for the minimal tensor product of C*-algebras, the Hilbert space tensor product and the tensor product of unitary representations. Denote by $\rtimes_{\mathrm{r}}$ the reduced C*-crossed product. (The underlying actions should be always clear from the context.) For a C*-algebra A equipped with a compact group action $K \curvearrowright A$, denote by A^K the fixed point algebra of the K-action.

On conditional expectations

The following lemma should be well known to experts. For completeness of the article, we include the proof.

Lemma 2.1. *Let $K < G$ be a compact open subgroup. Let $\alpha\colon G \curvearrowright A$ be a C*-dynamical system. Then there is a faithful conditional expectation*

$$E_K\colon p_K(A \rtimes_{\mathrm{r}} G)p_K \to p_K A p_K = A^K p_K$$

satisfying $E_K(p_K a \lambda_s p_K) = \chi_K(s) p_K a p_K$ for all $a \in A, s \in G$.

The analogous statement holds true in the von Neumann algebra setting. Moreover, in this setting, E_K can be chosen to be normal.

Proof. We show only the C*-algebra case. The proof in the von Neumann algebra case is identical to that in the C*-algebra case.

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Take a covariant representation (π, v) of (A, α) on $\mathfrak{H}$ such that π is faithful. (For instance, take a faithful regular covariant representation of (A, α).) We identify $A \rtimes_{\mathrm{r}} G$ with a C*-subalgebra of $\mathbb{B}(\mathfrak{H} \otimes L^2(G))$ via the regular covariant representation associated to π. Then this gives rise to an inclusion $p_K(A \rtimes_{\mathrm{r}} G)p_K \subset \mathbb{B}(\mathfrak{H} \otimes L^2(G)^K)$. We also identify $p_K A p_K$ with the C*-algebra $\pi(A^K) \otimes \mathrm{id}_{\mathbb{C}\chi_K}$ on $\mathfrak{H} \otimes \mathbb{C}\chi_K$ in the obvious way. Let q denote the orthogonal projection from $\mathfrak{H} \otimes L^2(G)^K$ onto $\mathfrak{H} \otimes \mathbb{C}\chi_K$. Define

$$\widetilde{E}\colon \mathbb{B}\Big(\mathfrak{H} \otimes L^2(G)^K\Big) \to \mathbb{B}(\mathfrak{H} \otimes \mathbb{C}\chi_K)$$

by

$$\widetilde{E}(x) := qxq.$$

Then under the foregoing identifications of C*-algebras, we obtain

$$\widetilde{E}(p_K(A \rtimes_{\mathrm{r}} G)p_K) = p_K A p_K.$$

Hence the map $\widetilde{E}$ restricts to a conditional expectation

$$E_K\colon p_K(A \rtimes_{\mathrm{r}} G)p_K \to p_K A p_K.$$

Direct computations show that the map E_K satisfies the required equation. Let $\rho\colon G \curvearrowright L^2(G)$ denote the right regular representation of G. Observe that $(v \otimes \rho)(G)p_K$ commutes with $p_K(A \rtimes_{\mathrm{r}} G)p_K$. As the subset $[(v \otimes \rho)(G)p_K] \cdot (\mathfrak{H} \otimes \mathbb{C}\chi_K)$ spans a dense subspace of $\mathfrak{H} \otimes L^2(G)^K$, the conditional expectation E_K is faithful. $\square$

3. New construction of nondiscrete C*-simple groups

Recall that G is a totally disconnected locally compact group. We will construct an ambient C*-simple group $\mathcal{G}$ of G.

To avoid confusion, we first introduce the following notations. Let Υ_n, $n \in \mathbb{N}$, be pairwise distinct copies of the group

$$\bigoplus_{K<G} \bigoplus_{G/K} \mathbb{Z}_2,$$

where the first direct sum is taken over the set of all compact open subgroups K of G. We equip each Υ_n with the G-action induced from the left translation G-actions on G/K. Let Ξ_n, $n \in \mathbb{N}$, be pairwise distinct copies of the integer group $\mathbb{Z}$. We equip each Ξ_n with the trivial G-action.

Set

$$\Gamma_1 := \Upsilon_1, \qquad \Lambda_1 := \Gamma_1 * \Xi_1,$$

equipped with the obvious G-actions. Assume that Γ_n and Λ_n have been defined. We then define

$$\Gamma_{n+1} := \Lambda_n \times \Upsilon_{n+1}, \qquad \Lambda_{n+1} := \Gamma_{n+1} * \Xi_{n+1},$$

equipped with the obvious G-actions. As a result, we obtain the increasing sequence

$$\Gamma_1 < \Lambda_1 < \Gamma_2 < \Lambda_2 < \cdots$$

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of discrete groups. Define Λ to be the inductive limit of this sequence. As the inclusions are G-equivariant, we have a natural G-action α on Λ. Now set

$$\mathcal{G} := \Lambda \rtimes_\alpha G.$$

Clearly $\mathcal{G}$ contains an open subgroup isomorphic to G. Define $\mathcal{G}_n := \Lambda_n \rtimes G < \mathcal{G}$ for $n \in \mathbb{N}$.

For an open compact subgroup $K < G$ $(< \mathcal{G})$, the following observation on $p_K \mathrm{C}^*_\mathrm{r}(\mathcal{G}) p_K$ is useful. Note first that the $*$-subalgebra $p_K C_c(\mathcal{G}) p_K$ is dense in $p_K \mathrm{C}^*_\mathrm{r}(\mathcal{G}) p_K$. Since K is open in $\mathcal{G}$, the characteristic functions χ_S, $S \in K\backslash\mathcal{G}/K$, form a basis of $p_K C_c(\mathcal{G}) p_K$. Therefore one can approximate a given element $x \in p_K \mathrm{C}^*_\mathrm{r}(\mathcal{G}) p_K$ arbitrarily well by an element of the form

$$\sum_{s \in F} p_K x_s \lambda_s p_K,$$

where $n \in \mathbb{N}$, $F = \{e, s_1, \ldots, s_l\}$ is a finite subset in G having the pairwise disjoint K-double cosets, $x_s \in \mathrm{C}^*_\mathrm{r}(\Lambda_n)$ for all $s \in F$ and $x_e \in \mathrm{C}^*_\mathrm{r}(\Lambda_n)^K$.

Note that by [15, Theorem 3.18], the class of elementary totally disconnected locally compact groups is closed under taking open subgroups and group extensions. Therefore the group $\mathcal{G}$ is elementary if and only if the original group G is elementary. Typical examples of nonelementary totally disconnected locally compact groups include $\mathrm{PSL}_d(\mathbb{Q}_p)$ and $\mathrm{Aut}(T_d)$, where p is a prime number, $d \in \{3, 4, \ldots\}$ and T_d is a d-regular tree [15, Proposition 6.3].

The following theorem is the main result of this article:

Theorem 3.1. *The locally compact group $\mathcal{G}$ is C^*-simple.*

Proof. Let I be a nonzero (closed two-sided) ideal of $\mathrm{C}^*_\mathrm{r}(\mathcal{G}) = \mathrm{C}^*_\mathrm{r}(\Lambda) \rtimes_{\mathrm{r},\alpha} G$. Take a nonzero positive element $x \in I$. Let $K < G$ be a compact open subgroup satisfying $a := p_K x p_K \neq 0$. We will show that $p_K \in I$.

Let

$$E_K \colon p_K \, \mathrm{C}^*_\mathrm{r}(\mathcal{G}) p_K \to \mathrm{C}^*_\mathrm{r}(\Lambda)^K p_K$$

be the faithful conditional expectation provided in Lemma 2.1. Since a is positive and nonzero, so is $E_K(a)$. By rescaling a if necessary, we may further assume that

$$\|E_K(a)\| = 1.$$

Choose an $n \in \mathbb{N}$ and $a_0 \in p_K C_c(\mathcal{G}_n) p_K$ satisfying

$$\|a - a_0\| < 1/2, \qquad \|E_K(a_0)\| = 1.$$

Write

$$a_0 = \sum_{s \in F} p_K x_s \lambda_s p_K, \quad x_s \in \mathrm{C}^*_\mathrm{r}(\Lambda_n),\ x_e \in \mathrm{C}^*_\mathrm{r}(\Lambda_n)^K,$$

where $F = \{e, s_1, \ldots, s_l\}$ is a finite subset of G having the pairwise distinct K-double cosets. Note that $\|x_e\| = \|E_K(a_0)\| = 1$.

Let $\Upsilon_{n+1,K} < \Upsilon_{n+1}$ be the Kth direct summand of Υ_{n+1}. Let $\mathrm{C}^*_\mathrm{r}(\Upsilon_{n+1,K}) \cong C(\{0,1\}^{G/K})$ be the obvious G-equivariant $*$-isomorphism. Define

$$U := \left\{ (\epsilon_{gK})_{gK \in G/K} \in \{0,1\}^{G/K} : \epsilon_K = 0,\ \epsilon_{gK} = 1 \text{ for } gK \subset \bigsqcup_{i=1}^{l} K s_i K \right\}.$$

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Observe that U is a K-invariant (nonempty) clopen subset of $\{0,1\}^{G/K}$. Moreover, for each i we have $\alpha_{s_i}(U) \cap U = \emptyset$. We regard $p := \chi_U$ as an element of $\mathrm{C}^*_{\mathrm{r}}(\Lambda)$. Then $p\lambda_{s_i}p = 0$ for $i = 1, \ldots, l$. The projection p is nonzero and commutes with p_K and x_s, $s \in F$. Thus

$$pa_0p = px_ep_K.$$

Since x_e and p sit in the first and second tensor product components of $\mathrm{C}^*_{\mathrm{r}}(\Gamma_{n+1}) = \mathrm{C}^*_{\mathrm{r}}(\Lambda_n) \otimes \mathrm{C}^*_{\mathrm{r}}(\Upsilon_{n+1})$, respectively, we have

$$\|px_e\| = \|p\|\|x_e\| = 1.$$

Let B be the C^*-subalgebra of $\mathrm{C}^*_{\mathrm{r}}(\Lambda)^K$ generated by $\mathrm{C}^*_{\mathrm{r}}(\Gamma_{n+1})^K$ and $\mathrm{C}^*_{\mathrm{r}}(\Xi_{n+1})$. By [5, Theorem 2], B is simple. Note that $px_e \in B$. Therefore, by [16, Lemma 2.3], one has a sequence $b_1, \ldots, b_r \in B$ satisfying

$$\left\| \sum_{i=1}^{r} b_i {b_i}^* \right\| \leq 2, \qquad \sum_{i=1}^{r} b_i p x_e b_i^* = 1_B.$$

This implies

$$\sum_{i=1}^{r} b_i p a_0 p b_i^* = \sum_{i=1}^{r} b_i p x_e b_i^* p_K = p_K.$$

Since $\left\|\sum_{i=1}^{r} b_i p(a - a_0)pb_i^*\right\| \leq 2\|a - a_0\| < 1$, we have $\|p_K + I\|_{\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})/I} < 1$. As p_K is a projection, this yields $p_K \in I$. Since $K < G$ can be chosen arbitrarily small, we conclude $I = \mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$. □

We keep the settings $G, \mathcal{G}$ and so on until the end of this article.

4. Uniqueness of KMS weight on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$

By modifying the proof of Theorem 3.1, we also obtain the uniqueness of KMS weight on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ with respect to the modular flow. From now on, we freely use the basic facts on the Plancherel weight observed in [10, Section 2.6].

For a locally compact group H, let $\Delta_H \colon H \to \mathbb{R}_{>0}$ denote the modular function of H. Define $H_0 := \ker(\Delta_H) < H$. Note that for totally disconnected H, it is not hard to see that H_0 is open in H and that $\Delta_H(H) \subset \mathbb{Q}$. Observe that for our G and $\mathcal{G}$,

$$\Delta_G(G) = \Delta_{\mathcal{G}}(\mathcal{G}), \qquad \mathcal{G}_0 = \Lambda \rtimes G_0.$$

Let φ denote the Plancherel weight on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$. Let σ^φ be the modular flow on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$:

$$\left[\sigma_t^\varphi(f)\right](s) := \Delta_{\mathcal{G}}(s)^{\mathrm{i}t} f(s) \quad \text{for } f \in C_c(\mathcal{G}),\ s \in \mathcal{G},\ t \in \mathbb{R}.$$

Throughout the paper, a weight on a C^*-algebra is always assumed to be *densely defined*, *lower semicontinuous* and nonzero (i.e., *proper*) without being stated. (See [8] or [10, Section 2.6] for the definitions.)

For a weight ψ on a C^*-algebra A, as in [8, Definition 1.1], denote by $\mathcal{M}_\psi$ the linear span of $\psi^{-1}([0,\infty))$. Note that $\mathcal{M}_\psi$ is a hereditary $*$-subalgebra of A. In addition, when ψ is tracial, $\mathcal{M}_\psi$ is a norm dense ideal of A, and hence it contains all projections in A. We call the $*$-subalgebra

$$\left\{ a \in A : \begin{array}{l} \mathcal{M}_\psi a \cup a\mathcal{M}_\psi \subset \mathcal{M}_\psi, \\ \psi(ax) = \psi(xa) \text{ for all } x \in \mathcal{M}_\psi \end{array} \right\} \subset A$$

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the *centraliser* of ψ. We set

$$C_{cc}(\mathcal{G}) := \bigcup_{K<G} p_K C_c(\mathcal{G}) p_K.$$

Here the union is taken over all compact open subgroups $K < G$. Note that $C_{cc}(\mathcal{G})$ is a $*$-subalgebra of $C_c(\mathcal{G})$. Observe that for any σ^φ-KMS weight ψ on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$, as every p_K is a projection fixed by σ^φ, we have $C_{cc}(\mathcal{G}) \subset \mathcal{M}_\psi$. Indeed, as p_K is a projection, one has an analytic element $a \in \mathcal{M}_\psi$ with $p_K \leq p_K a^* a p_K$. Then, by the KMS condition, we have

$$\psi(p_K) \leq \psi(p_K a^* a p_K) = \psi\left(\sigma^\varphi_{\mathrm{i}/2}(a) p_K \sigma^\varphi_{\mathrm{i}/2}(a^*)\right) \leq \psi\left(\sigma^\varphi_{\mathrm{i}/2}(a) \sigma^\varphi_{\mathrm{i}/2}(a^*)\right) = \psi(a^* a).$$

Theorem 4.1. *Up to scalar multiple, the Plancherel weight φ is the only σ^φ-KMS weight on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$. When $\mathcal{G}$ is nonunimodular, there is no tracial weight on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$.*

Proof. We consider the following claim:

Claim. *Let ψ be a weight on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ whose centraliser contains $C_{cc}(\mathcal{G}_0)$ and satisfies $C_{cc}(\mathcal{G}) \subset \mathcal{M}_\psi$. Then for any compact open subgroup $K < G$ and any $s \in \mathcal{G} \setminus K$, we have*

$$\psi(\lambda_s p_K) = 0.$$

Note that by the foregoing observations, any σ^φ-KMS weights and tracial weights on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ satisfy the assumption of the claim.

We first prove the theorem under the assumption that the claim holds true. In the case that ψ is a σ^φ-KMS weight, we will show that ψ is a scalar multiple of φ. Take any two compact open subgroups $K_1, K_2 < G$. Define $K := K_1 \cap K_2$ and take $s_1, \ldots, s_l, t_1, \ldots, t_r \in G$ satisfying $K_1 = \bigsqcup_{i=1}^l s_i K, K_2 = \bigsqcup_{i=1}^r t_i K$. Then

$$p_{K_1} = \frac{1}{l} \sum_{i=1}^{l} \lambda_{s_i} p_K, \qquad p_{K_2} = \frac{1}{r} \sum_{i=1}^{r} \lambda_{t_i} p_K.$$

Since $r\mu(K_1) = l\mu(K_2)$, the hypothesis implies

$$C := \psi(p_{K_1})\mu(K_1) = \psi(p_{K_2})\mu(K_2).$$

By [10, Lemma 2.23] (see also [8]), we obtain $\psi = C\varphi$. Next consider the case that $\mathcal{G}$ is nonunimodular and that ψ is a tracial weight. In this case, the equality in the claim implies that the weight ψ vanishes on $C_{cc}(\mathcal{G})$. Since the projections p_K, where $K < G$ are compact open subgroups, form an approximate unit of $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$, it follows from the tracial condition and lower semicontinuity of ψ that $\psi = 0$. This proves the statement of the theorem. Hence it suffices to show the claim.

We now prove the claim. Let $\psi, K < G, s \in \mathcal{G} \setminus K$ be as in the claim. Write $s = gu, g \in \Lambda, u \in G$. To show the claimed equation $\psi(\lambda_s p_K) = 0$, we first recall from the proof of Theorem 3.1 that when $u \notin K$, one has a nonzero projection $p \in \mathrm{C}^*_{\mathrm{r}}(\Gamma_1)^K$ satisfying $p\lambda_u p = 0$. In fact, p is taken from the group algebra $\mathbb{C}[\Gamma_1]$. When $u \in K$, define $p := \lambda_e \in \mathbb{C}[\Gamma_1]$. Choose $n \in \mathbb{N}$ satisfying $g \in \Lambda_n$. Consider the subgroup $\Sigma := \Lambda_n * \Xi_{n+1} * \Xi_{n+2} < \Lambda$. Denote by τ the canonical tracial state on $\mathrm{C}^*_{\mathrm{r}}(\Sigma)$. As observed in [12, Lemma 3.8], thanks to [4, Lemma 5], one can proceed with the *Powers averaging argument* [9] for Σ by using only elements in $\Xi_{n+1} * \Xi_{n+2}$. This implies that for any $\varepsilon > 0$, one has $t_1, \ldots, t_r \in \Xi_{n+1} * \Xi_{n+2}$ satisfying

$$\left\| \frac{1}{r} \sum_{i=1}^{r} \lambda_{t_i} x \lambda_{t_i}^* - \tau(x) \right\| < \varepsilon \quad \text{for } x = p, \lambda_g.$$

Then, as $\Xi_{n+1} * \Xi_{n+2}$ commutes with G, we have

$$\begin{aligned}\left\|\frac{1}{r^2}\sum_{i=1}^r\sum_{j=1}^r \lambda_{t_i}p\lambda_{t_i}^*\lambda_{t_j}p_K\lambda_s p_K\lambda_{t_j}^*\lambda_{t_i}p\lambda_{t_i}^*\right\| &= \left\|\frac{1}{r}\sum_{i=1}^r\left[\lambda_{t_i}p\lambda_{t_i}^*p_K\left(\frac{1}{r}\sum_{j=1}^r\lambda_{t_j}\lambda_g\lambda_{t_j}^*\right)\lambda_u p_K\lambda_{t_i}p\lambda_{t_i}^*\right]\right\| \\ &< \varepsilon + \frac{\tau(\lambda_g)}{r}\left\|\sum_{i=1}^r \lambda_{t_i}p_K p\lambda_u p p_K\lambda_{t_i}^*\right\| \\ &= \varepsilon.\end{aligned}$$

Here the last equation holds true because the condition $u \in K$ implies $g \neq e$. Since p is a K-invariant projection and $\Sigma, K \subset \mathcal{G}_0$, the previous inequality yields

$$\begin{aligned}\left|\psi\left(\lambda_s p_K\sum_{i=1}^r\sum_{j=1}^r\lambda_{t_j}^*\lambda_{t_i}p\lambda_{t_i}^*\lambda_{t_j}\right)\right| &= \left|\psi\left(\sum_{i=1}^r\sum_{j=1}^r\lambda_s p_K\left(\lambda_{t_j}^*\lambda_{t_i}p\lambda_{t_i}^*\right)\left(\lambda_{t_i}p\lambda_{t_i}^*\lambda_{t_j}p_K\right)\right)\right| \\ &= \left|\psi\left(\sum_{i=1}^r\sum_{j=1}^r\lambda_{t_i}p\lambda_{t_i}^*\lambda_{t_j}p_K\lambda_s p_K\lambda_{t_j}^*\lambda_{t_i}p\lambda_{t_i}^*\right)\right| \\ &\leq r^2\psi(p_K)\varepsilon.\end{aligned}$$

This yields

$$\begin{aligned}|\tau(p)\psi(\lambda_s p_K)| &\leq \left|\psi\left(p_K\lambda_s p_K\left(\tau(p) - \frac{1}{r^2}\sum_{i=1}^r\sum_{j=1}^r\lambda_{t_j}^*\lambda_{t_i}p\lambda_{t_i}^*\lambda_{t_j}\right)\right)\right| + \psi(p_K)\varepsilon \\ &\leq 2\psi(p_K)\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily small (independent on p), we conclude

$$\psi(\lambda_s p_K) = 0.$$

□

5. On factoriality and types of group von Neumann algebras $L(\mathcal{G})$

In this section we observe the factoriality of the group von Neumann algebra $L(\mathcal{G})$. We then determine its Murray–von Neumann–Connes type. For the definition of Connes' S-invariant, we refer the reader to [2, Section III].

Theorem 5.1. *The von Neumann algebra $L(\mathcal{G})$ is a nonamenable factor of type*

$$\begin{cases} II_1 & \text{when } G \text{ is discrete,} \\ II_\infty & \text{when } G \text{ is nondiscrete and unimodular,} \\ III & \text{otherwise,}\end{cases}$$

whose Connes' S-invariant is the closure of $\Delta_G(G)$ in $\mathbb{R}_{\geq 0}$.

Proof. We first show that $L(\mathcal{G})$ and $L(\mathcal{G}_0)$ are factors. By [10, Proposition 2.25], the centraliser of the Plancherel weight on $L(\mathcal{G})$ is equal to $L(\mathcal{G}_0)$. Hence it suffices to show the factoriality of $L(\mathcal{G}_0)$. Let φ be the Plancherel weight on $L(\mathcal{G}_0)$. Note that φ is a faithful normal semifinite tracial weight on $L(\mathcal{G}_0)$. Hence if $L(\mathcal{G}_0)$ is not a factor, then one has a normal semifinite tracial weight ψ which is dominated by

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φ but is not a scalar multiple of φ. This contradicts the uniqueness of the tracial weight on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_0)$ (up to scalar multiple), which follows from the proof of Theorem 4.1.

We next show the nonamenability of $L(\mathcal{G})$. Take any compact open subgroup $K < G$. Then by Lemma 2.1, the corner $p_K L(\mathcal{G}) p_K$ of $L(\mathcal{G})$ admits a conditional expectation

$$E_K \colon p_K L(\mathcal{G}) p_K \to L(\Lambda)^K p_K.$$

Since $L(\Lambda)^K p_K$ is nonamenable, so is $L(\mathcal{G})$.

Finally we determine the Murray–von Neumann–Connes type of $L(\mathcal{G})$. When G is discrete, it is clear from Theorem 4.1 that $L(\mathcal{G})$ is of type II_1. (Alternatively, in the discrete-group case, the statement follows from the fact that $\mathcal{G}$ is a nonamenable ICC discrete group.) When G is nondiscrete and unimodular, observe that the Plancherel weight on $L(\mathcal{G})$ is tracial and unbounded. Since $L(\mathcal{G})$ is nonamenable, it must be of type II_∞. The nonunimodular case and the last statement follow from Connes' theorem [2] (see [10, Theorem 2.27]). $\square$

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