Canad. Math. Bull. Vol. 30 (1), 1987

COMMENT ON THE NULLENSTELLENSATZ FOR REGULAR RINGS

BY
RON GENTLE

ABSTRACT. An elementary proof of the Nullstellensatz for commutative regular rings is given.

Introduction. The Nullstellensatz for regular rings was established by Saracino and Weispfenning [2] using the model theory techniques of Abraham Robinson. In this paper we will give an 'elementary' proof (using only standard arguments from commutative algebra, no slight intended to the logicians), thereby making the theorem accessible to a wider audience.

Throughout, R will be a commutative regular ring with unity. A ring B is said to be monically closed if every monic polynomial in $B[x]$ has a root in B. For such a ring, every monic polynomial factors into linear terms, but not necessarily uniquely (for example $x^2 - x = (x - e)(x - (1 - e))$ for any idempotent e). If m is a maximal ideal of B then B/m is an algebraically closed field. Pointed brackets $\langle\ \rangle$ will denote 'ideal generated by'.

We will first review the Nullstellensatz for fields. Let k be a field, K an algebraically closed field extending k, I an ideal and f an element of $k[x_1, \ldots, x_n]$. The hypothesis and conclusion are then

(H) Any common zero of I in K^n is a zero of f.

(C) f is in the radical of I, (ie. some power of f is in I).

Now using the fact that $k[x_1, \ldots, x_n]$ is a Jacobson ring (every prime ideal is the intersection of maximal ideals, see [1]) then (C) follows easily from the following statements:

(A) Each maximal ideal of $k[x_1, \ldots, x_n]$ is the contraction of some maximal ideal of $K[x_1, \ldots, x_n]$.

(B) Maximal ideals of $K[x_1, \ldots, x_n]$ are of the form $\langle x_1 - \alpha_1, \ldots, x_n - \alpha_n\rangle = \{h \mid h(\boldsymbol{\alpha}) = 0\}$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in K^n$.

Received by the editors April 11, 1986.
This work was done while the author was a Killam Postdoctoral Fellow at Dalhousie University.
AMS Subject Classification (1980): 13A15.

(B) is known as the Weak Nullstellensatz.

The setting will now be changed to $R \hookrightarrow S$ with R a regular ring and S a regular monically closed ring extending R. (A) generalizes to (A') simply by replacing k and K by R and S respectively (Proposition 8). (B) generalizes to the statement (B$'$) of Proposition 9. However the appropriate generalization of (H) is:

(H$'$) For each maximal ideal m, any common zero of $\bar{I}$ in S/m is a zero of $\bar{f}$.

This means that if $g(\boldsymbol{\alpha}) \in m$ for all g in I then $f(\boldsymbol{\alpha}) \in m$. As for the field case, (H$'$) utilizing (A$'$) and (B$'$) will imply (C). For the hypothesis (H) with k and K replaced by R and S to imply (H$'$), it is sufficient (refer to the proof of Theorem 13) to impose the extra hypothesis:

(Z) the ideal I is finitely generated and $I \cap R = 0$.

As indicated in [2], (H) alone will not imply (C) (or (H$'$)):

(i) let $f = 1$ and $I = \langle e \rangle$ where e is an idempotent $\neq 0, 1$ in R; (H) then holds vacuously but (H$'$) fails for any maximal ideal containing e, and (C) obviously fails.

Now consider $R = \Pi_{i=1}^{\infty} K_i$, where the K_i are fields. Let e_i be the idempotent corresponding to the unity element of K_i and $b_i = 1 - e_i$.

(ii) let $f = 1$ and $I = \langle e_i x, i = 1, 2, \ldots \rangle$

(iii) let $f = x$ and $I = \langle x^2 - b_i x, i = 1, 2, \ldots \rangle$

For these examples $I \cap R = 0$, but I is not finitely generated. In (ii) (H) again holds vacuously and (H$'$) fails for maximal ideals of the form $m_j = \Pi_{i \neq j} K_i$ (the j^{th} co-ordinate is zero). Another common feature of the first two examples is that the ideal generated by the coefficients of elements from I is proper in R. This is not the case in (iii) and also the zero set of I is $\{0\}$ which is non-empty. Any expression $\Sigma_{i=1}^{N} g_i(x)(x^2 - b_i x)$ will have e_j for a zero if $j > N$, and it follows that x is not in the radical. Here (H$'$) fails at any maximal ideal containing all of the e_i, ($\bar{1}$ will be a common zero of $\bar{I}$ but not of $\bar{f} = x$).

I would like to thank Wendy MacCaull for bringing this topic to my attention, and to Pat Stewart for his encouragement and suggestions.

Preliminary Remarks.

1. The prime ideals of a regular ring are maximal (if p is prime then R/p is both regular and a domain, hence it is a field).

2. For any element r of R, let e be its associated idempotent, $\langle r \rangle = \langle e \rangle$. The closed set, in Spec$R$, $V(r) = \{m \mid r \in m\}$ is also open since $V(r) = V(e) = V(1 - e)^c$.

3. Since by 1. above R is obviously a Jacobson ring (every prime ideal is the intersection of maximal ideals), then so is the polynomial ring $R[x_1, \ldots, x_n]$ (see [1] for instance).

https://doi.org/10.4153/CMB-1987-018-9 Published online by Cambridge University Press

LEMMA 4. *Let $\phi : A \hookrightarrow B$ be a ring homomorphism. If the extension m^e, of a maximal ideal m in A, is a proper ideal of B, then m is the contraction of some maximal ideal in B.//*

COROLLARY 5. *Let k be a field and L a field extension of k, then each maximal ideal of* $k[x_1, \ldots, x_n]$ is the contraction of some maximal ideal of $L[x_1, \ldots, x_n]$.

PROOF. Let K be an algebraically closed field extending L. By the Nullstellensatz for fields, any maximal ideal m in $k[x_1, \ldots, x_n]$ will have at least one common zero in K^n. Since these zeroes are preserved under extension, it follows that m^e is proper, so Lemma 4 can be applied.//

COROLLARY 6. *If $R \hookrightarrow S$ is a ring extension, then any maximal ideal of R is the contraction of some maximal ideal of S.*

PROOF. Let m be a maximal ideal of R. If $m^e = S$ then there exists $s_i \in S$ and $m_i \in m$ with $\Sigma_{i=1}^N s_i m_i = 1$. Then $\langle m_1, m_2, \ldots, m_N \rangle = \langle e \rangle$ for some idempotent e in m. However this would yield $0 = (1 - e) \Sigma s_i m_i = 1 - e$, which is impossible. Thus m^e is proper and Lemma 4 can be applied.//

REMARK 7. Suppose M is a maximal ideal of $R[x_1, \ldots, x_n]$ then $m = M \cap R$ is a maximal ideal of R (see Preliminary remark 1). Now $m[x_1, \ldots, x_n] \subseteq M$ and $R[x_1, \ldots, x_n]/m[x_1, \ldots, x_n] \cong R/m[x_1, \ldots, x_n]$, so it follows from the ring epimorphism $R[x_1, \ldots, x_n] \to R/m[x_1, \ldots, x_n]$ that M is the contraction of some maximal ideal in $R/m[x_1, \ldots, x_n]$.

PROPOSITION 8. *Let $R \hookrightarrow S$ be a ring extension, then every maximal ideal in $R[x_1, \ldots, x_n]$ is the contraction of some maximal ideal in $S[x_1, \ldots, x_n]$.*

PROOF. Let M be a maximal ideal of $R[x_1, \ldots, x_n]$. Then by Remark 7 and Corollary 6, there is a maximal ideal m in S such that M is the contraction of a maximal ideal in $R/m^c[x_1, \ldots, x_n]$, where m^c is the contraction of m in R. The result now follows from Corollary 5 and the commutative diagram:

$$\begin{array}{ccc} R[x_1, \ldots, x_n] & \hookrightarrow & S[x_1, \ldots, x_n] \\ \downarrow & & \downarrow \\ R/m^c[x_1, \ldots, x_n] & \hookrightarrow & S/m[x_1, \ldots, x_n] \end{array} //$$

PROPOSITION 9. *(The Weak Nullstellensatz for regular rings). If S is a monically closed regular ring, then the maximal ideals of $S[x_1, \ldots, x_n]$ are of the form $\langle m, x_1 - \alpha_1, x_2 - \alpha_2, \ldots, x_n - \alpha_n \rangle = \{f \mid f(\boldsymbol{\alpha}) \in m\}$ where m is a maximal ideal of S and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S^n$.*

PROOF. By Remark 7, a maximal ideal M will be the contraction of some maximal ideal N in $S/m[x_1, \ldots, x_n]$ where $m = M \cap S$. Since S/m is an algebraically closed field, there is an element $(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)$ in $(S/m)^n$ such that $N = \langle x_1 - \bar{\alpha}_1, \ldots, x_n - \bar{\alpha}_n \rangle$. The desired result now follows readily.//

For the following two technical lemmas, let A be an arbitrary commutative ring with g in $A[x_1, \ldots, x_n]$ regarded as a polynomial function from the A-module A^n to A.

LEMMA 10. *Let* $\{b_i\}$ $i = 1, \ldots, N$ *be a complete set of orthogonal idempotents for* A, *and* $\mathbf{w}_i \in A^n$, *then* $g(\Sigma_{i=1}^N b_i\mathbf{w}_i) = \Sigma_{i=1}^N b_i g(\mathbf{w}_i)$.

PROOF. It suffices to check the case when g is a monomial, and then the result is clear.//

LEMMA 11. *For any idempotent* e *of* A, $g(e\mathbf{w}) = eg(\mathbf{w}) + (1 - e)g(\mathbf{0})$.//

For the remainder of this paper, let $R \hookrightarrow S$ be a ring extension with S regular and monically closed.

PROPOSITION 12. *Let* $\{g_i\}$ $i = 1, 2, \ldots, r$ *be in* $R[x_1, \ldots, x_n]$, *then the following are equivalent*:

(i) $\langle g_1, g_2, \ldots, g_r\rangle \cap R = 0$.

(ii) $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_r\rangle$ *is a proper ideal in* $R/m'[x_1, \ldots, x_n]$ *for any maximal ideal* m' *of* R.

(iii) *For each maximal ideal* m *of* S *there is an* $\boldsymbol{\alpha}$ **in** S^n *with* $g_i(\boldsymbol{\alpha})$ *in* m *for all* i.

(iv) *There is a common zero for* $\{g_i\}$ *in* S^n.

PROOF.

(iv) $\Rightarrow$ (i) clear.

(i) $\Rightarrow$ (ii) Suppose $\Sigma_{i=1}^r \bar{h}_i\bar{g}_i = 1$ in $R/m'[x_1, \ldots, x_n]$, then $\Sigma_{i=1}^r h_i g_i + h = 1$ for some h with coefficients in m'. These coefficients generate an ideal of the form $\langle e\rangle$ for some idempotent e in m'. This yields $\Sigma_{i=1}^r (1 - e)h_i g_i = 1 - e$, hence $\langle g_1, \ldots, g_r\rangle \cap R \neq 0$.

(ii) $\Rightarrow$ (iii) Let m be a maximal ideal of S and m^c its contraction in R. $R/m^c \to S/m$ is an extension of fields with S/m algebraically closed. By assumption $\langle \bar{g}_1, \ldots, \bar{g}_r\rangle$ is a proper ideal in $R/m^c[x_1, \ldots, x_r]$, hence by the weak Nullstellensatz for fields the $\{\bar{g}_i\}$ has a common zero $(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$ in $(S/m)^n$. This means $\overline{g_i(\boldsymbol{\alpha})} = 0$ in S/m, which establishes the implication.

(iii) $\Rightarrow$ (iv) For each $\boldsymbol{\alpha}$ in S^n, consider the open set in Spec S: $O_{\boldsymbol{\alpha}} = \{m \mid g_i(\boldsymbol{\alpha}) \in m$ for all $i\} = \cap_{i=1}^r V(g_i(\boldsymbol{\alpha}))$ (see Preliminary Remark]).

By assumption the $O_{\boldsymbol{\alpha}}$ cover Spec S, and since Spec S is compact there exists a finite subcovering. Hence there exists $\{\boldsymbol{\alpha}_i\}$ $i = 1, \ldots, N$ such that for any given maximal ideal m there is some $\boldsymbol{\alpha}_j$ such that $g_i(\boldsymbol{\alpha}_j) \in m$ for all i. Set $\langle g_1(\boldsymbol{\alpha}_j), g_2(\boldsymbol{\alpha}_j) \ldots g_r(\boldsymbol{\alpha}_j)\rangle = \langle e_j\rangle$ with e_j an idempotent, so that $e_j \in m$. The product $e_1, e_2, \ldots, e_N$ must be zero since it lies in all the maximal ideals of S (the Jacobson radical of a regular ring is zero). Let $b_1 = 1 - e_1$, $b_2 = e_1(1 - e_2)$, $b_3 = e_1e_2(1 - e_3) \ldots b_N = e_1e_2 \ldots e_{N-1}(1 - e_N)$

$= e_1 e_2 \dots e_{N-1}$. For each j, $b_j e_j = 0$ and $\{b_i\}$ form a complete set of orthogonal idempotents for S, hence by Lemma 10, $g_i(\sum_{j=1}^N b_j \boldsymbol{\alpha}_j) = \sum_{j=1}^N b_j g_i(\boldsymbol{\alpha}_j) = 0$. Thus there is a common zero for the g_i in S^n.//

THEOREM 13. *(The Nullstellensatz for regular rings) Let R be a regular ring, S a regular monically closed ring extending R, and $f, g_1, \dots, g_r \in R[x_1, \dots, x_r]$. Suppose further that $\langle g_1, \dots, g_r \rangle \cap R = 0$. Then f vanishes at every common zero of $\{g_i\}$ $i = 1, \dots, r$ in S^n if and only if there is a positive integer J such that f^J is in $\langle g_1, \dots, g_r \rangle$.*

PROOF. The implication one way is trivial. For the other we must show f is in the radical of the ideal $\langle g_1, \dots, g_r \rangle$ in $R[x_1, \dots x_n]$. Since $R[x_1, \dots, x_n]$ is a Jacobson ring (see Preliminary Remark 3) if suffices to show f is in all maximal ideals of $R[x_1, \dots, x_n]$ containing $\{g_i\}$. By Proposition 8 we can assume $R = S$. By Proposition 9 we must show that given a maximal ideal m of S and $\boldsymbol{\alpha}$ in S^n, if $g_i(\boldsymbol{\alpha}) \in m$ for all i, then $f(\boldsymbol{\alpha}) \in m$. Set $G_i(\boldsymbol{x}) = g_i(\boldsymbol{x} + \boldsymbol{\alpha})$ and $F(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{\alpha})$ where $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ is indeterminate. The problem reduces to showing: $G_i(\mathbf{0}) \in m$ for all i implies $F(\mathbf{0}) \in m$. Form the ideal $\langle G_1(\mathbf{0}), \dots, G_r(\mathbf{0}) \rangle = \langle e \rangle$ where e is an idempotent. Let $\boldsymbol{\beta}$ be a common zero of the g_i (which exists by Proposition 12) and hence of f. Then $\boldsymbol{\gamma} = \boldsymbol{\beta} - \boldsymbol{\alpha}$ is a common zero of the G_i and F. By Lemma 11, $G_i(e\boldsymbol{\gamma}) = eG_i(\boldsymbol{\gamma}) + (1 - e)G_i(\mathbf{0}) = 0$ consequently $e\boldsymbol{\gamma}$ is also a zero of F. Again by Lemma 11, $0 = F(e\boldsymbol{\gamma}) = eF(\boldsymbol{\gamma}) + (1 - e)F(\mathbf{0}) = (1 - e)F(\mathbf{0})$, which shows that $F(\mathbf{0})$ is in $\langle e \rangle$ and hence in m.//

REFERENCES

1. N. Bourbaki, *Commutative Algebra*, Addison-Wesley (1972).

2. D. Saracino and V. Weispfenning, *On algebraic curves over commutative regular rings, Model Theory and Algebra: A Memorial Tribute to Abraham Robinson*, Lecture Notes in Mathematics, Vol. 498 (1975) Springer Verlag, pp. 307–383.

DEPARTMENT OF MATHEMATICS, STATISTICS & COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, N.S. B3H 3J5