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CHARACTERIZATIONS OF MODULES

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1. Introduction. In this paper we use the Bourbaki [**2**] conventions for rings and modules. All rings are associative but not necessarily commutative and have a 1; all modules are unital.

Bass [**1**] calls a ring A left perfect if and only if every left A-module has a projective cover, which he shows is equivalent to every flat left A-module being projective. Bass calls a ring A semi-perfect if and only if every finitely generated module has a projective cover and shows that this concept is left-right symmetric.

We will define a ring A to be *quasi-perfect* if and only if every finitely generated flat left A-module is projective.

An exercise [**6**, Exercise 10, p. 136] is given by Lambek to show that every semi-perfect ring is quasi-perfect. However the hint given to solve this exercise seems to be incorrect, since it involves the application of Nakayama's Lemma to a non-finitely generated module. A correct version of the solution will follow from our results.

If $K \subseteq L$ are left A-modules Cohn [**3**] calls K pure in L if and only if $M \otimes K \rightarrow M \otimes L$ is monic for all right A-modules M. Basic properties of purity can be found in [**4**]. A module is pure projective if and only if it is projective relative to the pure exact sequences. We shall use the fact that for flat modules F a submodule K is pure if and only if F/K is flat; in this case K is also flat.

A module will be called a PDS module if and only if every pure submodule is a direct summand. Rings for which all modules are PDS were studied in [**4**].

2. Perfect, quasi-perfect and regular rings. We will need the following facts.

(I) The only small pure submodule of a projective module is 0. For a proof see [**5**].

(II) A module is finitely generated and pure projective if and only if it is finitely presented. For a proof see [**4**].

(III) A module is flat and pure projective if and only if it is projective.

Proof of (III). *Necessity*. Let $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ be a projective resolution of the flat module M. Since M is flat the sequence is pure exact and hence splits since M is pure projective.

Sufficiency. This is obvious.

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THEOREM 2.1. *For any ring A the following conditions are equivalent.* (*All conditions are for the same side.*)

(1) *A is perfect.*

(2) *Every flat module has a projective cover.*

(3) *Every flat module is pure projective.*

(4) *Every flat module is* PDS.

(5) *Every projective module is* PDS.

(6) *Every free module is* PDS.

Proof. (1) $\Rightarrow$ (2). This follows from the definition of perfect.

(2) $\Rightarrow$ (1). Suppose that $0 \to S \to P \to F \to 0$ is a projective cover of the flat module F. Then S is small and pure in P and hence $S = 0$ by (I).

(1) $\Rightarrow$ (3). Every flat is projective, hence pure projective.

(3) $\Rightarrow$ (1). By (III), every flat is projective.

The implications (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(1) $\Rightarrow$ (4). If K is a pure submodule of the flat module F then F/K is flat, hence projective and K is a direct summand.

(6) $\Rightarrow$ (1). If $0 \to K \to L \to F \to 0$ is a free resolution of any flat module F then K is pure in L, hence a direct summand and F is projective.

THEOREM 2.2. *For any ring A the following conditions are equivalent.*

(1)′ *A is quasi-perfect.*

(2)′ *Every finitely generated flat module has a projective cover.*

(3)′ *Every finitely generated flat module is pure projective.*

(3)″ *Every finitely generated flat module is finitely presented.*

(4)′ *Every finitely generated flat module is* PDS.

(5)′ *Every finitely generated projective module is* PDS.

(6)′ *Every finitely generated free module is* PDS.

Proof. This is basically the same as for the preceeding theorem.

COROLLARY. *Every semi-perfect ring is quasi-perfect since in a semi-perfect ring every finitely generated module has a projective cover.*

We now prove a related result.

THEOREM 2.3. *For any ring A the following conditions are equivalent.*

(1) *A is von Neumann regular.*

(2) *Every pure projective is projective.*

(3) *Every finitely presented module is projective.*

(4) *Every pure projective is flat.*

(5) *Every finitely presented module is flat.*

Proof. Clearly we have the implications

$$\begin{array}{ccc} (2) & \Rightarrow & (3) \\ \Downarrow & & \Downarrow \\ (4) & \Rightarrow & (5) \end{array}$$

$(1) \Rightarrow (2)$. Every module is flat. The implication then follows from (III).

$(5) \Rightarrow (1)$. Since the module A/Aa is flat, Aa is a finitely generated pure submodule of the projective module A and hence a direct summand (see [**4**]).

References

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