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Real Hypersurfaces in Complex Projective Space Whose Structure Jacobi Operator is Lie $\mathbb{D}$-parallel

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Abstract. We prove the non-existence of real hypersurfaces in complex projective space whose structure Jacobi operator is Lie $\mathbb{D}$-parallel and satisfies a further condition.

1 Introduction

We will consider connected real hypersurfaces M in complex projective space $\mathbb{C}P^m$, $m \geq 3$, endowed with the metric of constant holomorphic sectional curvature equal to 4. Let J be the Kaehlerian structure of $\mathbb{C}P^m$ and N a unit normal vector field on M.

The problem of classifying such hypersurfaces is still open, although several partial results have been obtained in works due to Takagi [13] and [14], Okumura [8], Maeda [6], Montiel [7], Kimura [3], among others. In Berndt [1] there is a survey of the most important results along this line.

The fact of a Riemannian manifold being a real hypersurface in $\mathbb{C}P^m$ yields hard restrictions to its intrinsic geometry. For example, it cannot be Einstein, thus its sectional curvature is not constant. It neither can be a locally symmetric space. Therefore some weaker intrinsic conditions have been studied (Ricci-parallelness [4], harmonic curvature [5], and so on).

The Jacobi operator R_X with respect to a unit vector field X is defined as $R_X = R(\cdot, X)X$, where R is the curvature tensor field on M. Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related to Jacobi vector fields, which are solutions of the second order differential equation (the Jacobi equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma} Y) + R(Y, \dot\gamma)\dot\gamma = 0$ along a geodesic γ in M. We call the vector field given by $\xi = -JN$ the structure vector field on M. The corresponding Jacobi operator R_ξ is called the structure Jacobi operator on M. We denote the maximal holomorphic distribution on M by $\mathbb{D}$, given by all vectors orthogonal to ξ at any point of M.

Along the line of characterizing real hypersurfaces of $\mathbb{C}P^m$ in terms of R_ξ it is natural to consider the problem about the parallelism and the invariance, or Lie parallelism. In [9] the non-existence of real hypersurfaces in nonflat complex space forms with parallel structure Jacobi operator is proved. Also in [10] the first author and Santos prove non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, whose structure

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Jacobi operator is Lie-parallel, that is, its Lie derivative in any tangent direction vanishes.

In [12] the present authors and Santos obtain the following result.

Theorem 1.1 *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 3$, such that the structure Jacobi operator R_ξ is invariant under the structure vector field ξ, that is $\mathcal{L}_\xi R_\xi = 0$. Then either M is locally congruent to a tube of radius $\pi/4$ over a complex submanifold in $\mathbb{C}P^m$ or to either a geodesic hypersphere or a tube over a totally geodesic $\mathbb{C}P^k$, $0 < k < m-1$, with radius $r \neq \pi/4$.*

Nevertheless the condition of Lie $\mathbb{D}$-parallelism of the structure Jacobi operator has not been studied until now.

On the other hand, we do not know the classification of real hypersurfaces in $\mathbb{C}P^m$ satisfying

$$AR_\xi = R_\xi A \tag{1.1}$$

where A denotes the shape operator associated to N, although we know that every Hopf real hypersurface (a real hypersurface whose structure vector field is principal) satisfies it.

Thus this paper is devoted to study real hypersurfaces satisfying (1.1) and at the same time

$$\mathcal{L}_X R_\xi = 0 \tag{1.2}$$

for any $X \in \mathbb{D}$. We call such a real hypersurface *a real hypersurface with Lie $\mathbb{D}$-parallel structure Jacobi operator*. We will prove the following theorem.

Theorem A *There does not exist any real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, satisfying (1.1) and (1.2).*

Corollary *There does not exist any Hopf real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, whose structure Jacobi operator is Lie $\mathbb{D}$-parallel.*

2 Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field of M. Let ∇ be the Levi–Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

for vector fields X, Y tangent to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi).$$

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From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$$

and

$$\nabla_X \xi = \phi AX$$

for any vector fields X, Y tangent to M, where A denotes the shape operator of M in $\mathbb{C}P^m$ defined by $AX = -\tilde{\nabla}_X N$ for the connection $\tilde{\nabla}$ of $\mathbb{C}P^m$. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY \end{aligned}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi$$

for any vector fields X, Y and Z tangent to M, where R is the curvature tensor of M.

We will write, in general, $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$. Then M is Hopf (respectively, non-Hopf) if $\beta = 0$ (respectively, $\beta \neq 0$).

We will need the following results.

Theorem 2.1 ([2]) *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 2$, such that $A\phi + \phi A = 0$.*

Theorem 2.2 ([6]) *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$ such that $A\xi = \alpha\xi$. Then α is locally constant and if X is a tangent vector field on M such that $AX = \lambda X$ and X is orthogonal to ξ, then $A\phi X = \frac{\alpha\lambda+2}{2\lambda-\alpha}\phi X$.*

Theorem 2.3 ([11]) *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, whose shape operator satisfies $A\xi = \alpha\xi + U$, $AU = \xi$, $A\phi U = -\frac{1}{\alpha}\phi U$, where U is a unit vector field in $\mathbb{D}$ and α a nonnull function defined on M.*

Theorem 2.4 ([11]) *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$ whose shape operator satisfies $A\xi = \xi + \beta U$, $AU = \beta\xi + (\beta^2 - 1)U$, $A\phi U = -\phi U$ and there exists $Z \in \mathbb{D}_U = \mathrm{Span}\{\xi, U, \phi U\}^\perp$ such that $AZ = -Z$, $A\phi Z = -\phi Z$, where U is a unit vector field in $\mathbb{D}$ and β is a nonnull function defined on M.*

As in [10] we can prove the next theorem.

Theorem 2.5 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, satisfying (1.2). Let E be a subspace of $\mathbb{D}$ that is both ϕ-invariant and A-invariant. Let $G = \{X \in E \mid (\phi A + A\phi)X = 0\}$ and let F be its orthogonal complement in E. Then $AZ = \sigma Z$ for all $Z \in F$, where $1 + \alpha\sigma = 0$. Furthermore, there is a principal basis for G of the form $\{X_i, \phi X_i\}$ with corresponding principal curvatures λ_i and $-\lambda_i$. In particular, F and G are A-invariant.*

3 Proof of Theorem A

We continue writing $A\xi = \alpha\xi + \beta U$ and suppose that M is non-Hopf, that is, $\beta \neq 0$.

Condition (1.2) gives

$$\begin{aligned}(3.1)\qquad & -g(\phi AX, Y)\xi - \eta(Y)\phi AX + g\big(\nabla_X(A\xi), \xi\big)AY + g(A\xi, \phi AX)AY \\ & + \alpha\nabla_X AY - g\big(\nabla_X(AY), \xi\big)A\xi - g(AY, \phi AX)A\xi - \eta(AY)\nabla_X(A\xi) \\ & + \eta(Y)\nabla_\xi X - \alpha\nabla_{AY}X + \eta(AY)\nabla_{A\xi}X - \alpha A\nabla_X Y \\ & + g(\nabla_X Y, A\xi)A\xi - g(\nabla_Y X, \xi)\xi + \alpha A\nabla_Y X - g(\nabla_Y X, A\xi)A\xi = 0\end{aligned}$$

for any $X \in \mathbb{D}$, Y tangent to M.

Taking the scalar product of this equation and ξ we obtain

$$\begin{aligned}(3.2)\qquad & -g(\phi AX, Y) + \eta(AY)g(\phi AX, A\xi) - \alpha g(AY, \phi AX) \\ & + \eta(Y)g(\nabla_\xi X, \xi) - \alpha g(\nabla_{AY}X, \xi) + \eta(AY)g(\nabla_{A\xi}X, \xi) - g(\nabla_Y X, \xi) = 0\end{aligned}$$

for any $X \in \mathbb{D}$, Y tangent to M.

Lemma 3.1 *If M satisfies (1.1), we get $\alpha AU = \alpha\beta\xi + (\beta^2 - 1)U$.*

Proof If we apply (1.1) to ξ, we have $0 = R_\xi(A\xi) = \beta R_\xi(U)$. As we suppose $\beta \neq 0$, we get $R_\xi(U) = U + \alpha AU - \beta A\xi = 0$. From this equality we obtain the result. ■

Thus we have two possibilities:

$$\text{either } \alpha = 0 \text{ and } \beta^2 = 1, \qquad \text{or } \alpha \neq 0 \text{ and } AU = \beta\xi + \tfrac{\beta^2-1}{\alpha}U.$$

Lemma 3.2 *In the above conditions, if M satisfies (1.2), $\alpha \neq 0$.*

Proof Suppose that $\alpha = 0$. Thus $\beta^2 = 1$. So we suppose $\beta = 1$ (if not, we change ξ by $-\xi$). From (3.2) we have

$$\begin{aligned}(3.3)\qquad & -g(\phi AX, Y) + \eta(AY)\eta(A\phi AX) - \eta(Y)g(X, \phi A\xi) \\ & \qquad - \eta(AY)g(X, \phi A^2\xi) + g(X, \phi AY) = 0.\end{aligned}$$

for any $X \in \mathbb{D}$, Y tangent to M.

Take $X = U$, $Y = \phi U$ in (3.3). It yields

$$(3.4)\qquad g(AU, U) + g(A\phi U, \phi U) = 0.$$

Taking $Y \in \mathbb{D}_U$, $X = \phi U$ in (3.3) we have $-g(\phi A\phi U, Y) + g(U, AY) = 0$ for any $Y \in \mathbb{D}_U$. That is, $-\phi A\phi U + AU$ has no component in $\mathbb{D}_U$. As $g(-\phi A\phi U + AU, U) = g(-\phi A\phi U + AU, \phi U) = 0$, where we have used (3.4), and $g(-\phi A\phi U + AU, \xi) = 1$ we get $-\phi A\phi U + AU = \xi$. Applying ϕ to this equality we get

$$(3.5)\qquad A\phi U + \phi AU = 0.$$

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From (1.1),

$$R_\xi(A\phi U) = AR_\xi(\phi U) = A\phi U = -R_\xi(\phi AU) = -\big(\phi AU - g(\phi AU, A\xi)A\xi\big).$$

From (3.5) this yields

$$g(AU, \phi U) = 0.$$

Now suppose that $AU = \xi + \gamma U + \epsilon Z$, for a unit $Z \in \mathbb{D}_U$ and some function ϵ on M.

We have $\phi AU = \gamma\phi U + \epsilon\phi Z$. From (3.5), $A\phi U = -\gamma\phi U - \epsilon\phi Z$. Then $-\phi A\phi U = \gamma U + \epsilon Z$. Thus $-\phi A\phi U + AU = \gamma U + \epsilon Z + \xi + \gamma U + \epsilon Z = \xi$. This yields $2\gamma U + 2\epsilon Z = 0$. Thus $\gamma = \epsilon = 0$. Now we have $AU = \xi$, $A\phi U = 0$.

From the Codazzi equation $(\nabla_U A)\xi - (\nabla_\xi A)U = -\phi U$. This gives us $\nabla_U U + A\nabla_\xi U = 0$. Taking its scalar product with ϕU we get

$$g(\nabla_U U, \phi U) = 0. \tag{3.6}$$

On the other hand, $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$. This gives $-A\nabla_U \phi U + A\nabla_{\phi U} U = -2\xi$. Taking its scalar product with ξ we have

$$g(\nabla_U \phi U, U) = 2. \tag{3.7}$$

From (3.6) and (3.7) we have a contradiction, finishing the proof. ∎

Thus we have $\alpha \neq 0$ and $AU = \beta\xi + \frac{\beta^2-1}{\alpha}U$.

Lemma 3.3 *Let M be a non-Hopf real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, satisfying (1.1) and (1.2). Then $A\phi U = \gamma\phi U$, where either $\gamma = -\frac{1}{\alpha}$ or $\gamma = \frac{1-\beta^2}{\alpha}$.*

Proof We suppose that $A\phi U = \gamma\phi U + \epsilon Z$, where Z is a unit vector field in $\mathbb{D}_U$.

If we take $X = \phi U$, $Y \in \mathbb{D}_U$ in (3.2), we obtain that $\phi A\phi U + \alpha A\phi A\phi U$ has no component in $\mathbb{D}_U$. Taking its scalar product with ξ (respectively, with U and ϕU) we obtain

$$\phi A\phi U + \alpha A\phi A\phi U = -\alpha\beta\gamma\xi - \beta^2\gamma U. \tag{3.8}$$

As $\phi A\phi U = -\gamma U + \epsilon\phi Z$, $A\phi A\phi U = -\gamma\beta\xi - \gamma\frac{\beta^2-1}{\alpha}U + \epsilon A\phi Z$. From (3.8) this yields

$$\epsilon\phi Z + \alpha\epsilon A\phi Z = 0.$$

Suppose $\epsilon \neq 0$. Then $A\phi Z = -\frac{1}{\alpha}\phi Z$.

Taking $X = \phi U$, $Y \in \mathbb{D}_U$ in (3.2) we obtain that $\beta^2 A\phi U + \alpha A^2\phi U$ has no component in $\mathbb{D}_U$. This yields $\beta^2 A\phi U + \alpha A^2\phi U = \frac{1-\beta^2}{\alpha}\phi U$. Taking its scalar product with Z we get $\beta^2\epsilon + \alpha\gamma\epsilon + \alpha\epsilon g(AZ, Z) = 0$. As we suppose $\epsilon \neq 0$ we have

$$\beta^2 + \alpha\gamma + \alpha g(AZ, Z) = 0. \tag{3.9}$$

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Taking $X = \phi Z$, $Y = \phi U$ in (3.2) we obtain $\alpha g(Z, A^2\phi U) = 0$. This yields $\gamma\epsilon + \epsilon g(AZ, Z) = 0$. If $\epsilon \neq 0$ we get

$$\gamma + g(AZ, Z) = 0. \tag{3.10}$$

From (3.9) and (3.10) we have $\beta = 0$, which is impossible. Thus $\epsilon = 0$.

Now, if we take $Y = \phi U$, $X = U$ in (3.2), we obtain $\alpha\gamma^2 + \beta^2\gamma + \frac{\beta^2-1}{\alpha} = 0$. From this equation the result follows. ∎

Now let $Z \in \mathbb{D}_U$ be a unit vector field such that $AZ = \lambda Z$. Let $Y \in \mathbb{D}_U$ such that Y is orthogonal to $\mathrm{Span}\{Z, \phi Z\}$. Applying (3.2) to Z and ϕY we obtain that $(1+\alpha\lambda)g(A\phi Z, \phi Y) + \alpha g(A^2\phi Z, \phi Y) = 0$. Thus $(1+\alpha\lambda)A\phi Z + \alpha A^2\phi Z$ is proportional to ϕZ and we can write $(1+\alpha\lambda)A\phi Z + \alpha A^2\phi Z + \mu\phi Z = 0$ for a certain function μ. But if we take $X = Z$, $Y = \phi Z$ in (3.2) we obtain $(1+\alpha\lambda)g(A\phi Z, \phi Z) + \alpha g(A^2\phi Z, \phi Z) + \lambda = 0$. Then $\mu = \lambda$ and we get

$$(1+\alpha\lambda)A\phi Z + \alpha A^2\phi Z + \lambda\phi Z = 0 \tag{3.11}$$

for any unit principal vector $Z \in \mathbb{D}_U$ such that $AZ = \lambda Z$.

Let ω be one more eigenvalue of A, $\omega \neq \lambda$, and let $W \in \mathbb{D}_U$ be a unit eigenvector associated to ω. We also have

$$(1+\alpha\omega)A\phi W + \alpha A^2\phi W + \omega\phi W = 0. \tag{3.12}$$

If we take the scalar product of (3.11) and W, we obtain

$$\big((1+\alpha\lambda)\omega + \omega^2\alpha + \lambda\big)\, g(\phi Z, W) = 0$$

and taking the scalar product of (3.12) and Z we have

$$\big((1+\alpha\omega)\lambda + \lambda^2\alpha + \omega\big)\, g(\phi Z, W) = 0.$$

From both equations we get

$$(\omega^2 - \lambda^2)g(\phi Z, W) = 0.$$

So we arrive at two possibilities:

(1) For any unit eigenvector W with eigenvalue distinct to λ, $g(\phi Z, W) = 0$. This means $A\phi Z = \lambda\phi Z$.

(2) There exists a unit W as above such that $g(\phi Z, W) \neq 0$. In this case $\omega = -\lambda$.

In the first case, from (3.11) we have $(1+\alpha\lambda)\lambda + \alpha\lambda^2 + \lambda = 0$. This yields $2\lambda(\alpha\lambda+1) = 0$. So either $\lambda = 0$ or $\lambda = -\frac{1}{\alpha}$.

Suppose $\lambda = 0$. The Codazzi equation gives $(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z = -2\xi$. This yields $A[\phi Z, Z] = -2\xi$. Taking its scalar product with ξ, we get

$$g([\phi Z, Z], U) = -\frac{2}{\beta}. \tag{3.13}$$

And the scalar product with U yields

$$\frac{\beta^2 - 1}{\alpha} g([\phi Z, Z], U) = 0. \tag{3.14}$$

From (3.13) $g([\phi Z, Z], U) \neq 0$. Thus from (3.14), $\beta^2 = 1$. We can take $\beta = 1$, replacing ξ by $-\xi$ if necessary. Therefore $AU = \xi$. From Theorem 2.3 this case does not occur if $\gamma = -\frac{1}{\alpha}$. Therefore $A\phi U = 0$. The Codazzi equation yields $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$. This gives $(\phi U)(\alpha)\xi + \nabla_{\phi U} U + A\nabla_\xi \phi U = U$. If we take its scalar product with U we obtain $-1 = 1$, which is impossible. Thus we must suppose $\lambda = -\frac{1}{\alpha}$.

Again $(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z = -2\xi$. Taking its scalar products with ξ and U, we obtain $\frac{2}{\alpha^2\beta} = \frac{2}{\beta}$. Thus $\alpha^2 = 1$ and, probably after a change of ξ by $-\xi$, we can suppose that $\alpha = 1$. From Theorem 2.4 this is not possible if $A\phi U = -\phi U$. Thus we have to suppose $A\phi U = (1 - \beta^2)\phi U$. Let us denote $\delta = \beta^2 - 1$.

From the Codazzi equation $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$. Taking its scalar product with U, we obtain

$$(\phi U)(\beta) + 2\delta g(\nabla_\xi \phi U, U) + \delta - \delta^2 - \beta^2 = 1. \tag{3.15}$$

And the scalar product with ξ gives

$$g(\nabla_\xi \phi U, U) = 3\delta + 1. \tag{3.16}$$

From (3.15) and (3.16), bearing in mind $\delta = \beta^2 - 1$ we get

$$(\phi U)(\beta) = 2 - 2\delta - 5\delta^2. \tag{3.17}$$

Now $(\nabla_U A)\xi - (\nabla_\xi A)U = -\phi U$. This yields

$$U(\beta)U + \beta\nabla_U U + \delta^2\phi U - \xi(\beta)\xi - \xi(\delta)U - \delta\nabla_\xi U + A\nabla_\xi U = 0. \tag{3.18}$$

The scalar product of (3.18) and ξ gives $\xi(\beta) = \xi(\delta) = 0$, and its scalar product with U yields $U(\beta) = 0$. Thus (3.18) becomes

$$\beta\nabla_U U + \delta^2\phi U - \delta\nabla_\xi U + A\nabla_\xi U = 0.$$

Taking its scalar product with ϕU, we get

$$g(\nabla_U U, \phi U) = -\frac{7\delta^2 + 2\delta}{\beta}. \tag{3.19}$$

On the other hand, $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$. Its scalar product with U gives $(\phi U)(\delta) = -2\delta g(\nabla_U \phi U, U) - \delta\beta$. From (3.19) we get

$$2(\phi U)(\beta) = -\frac{2\delta}{\beta^2}(7\delta^2 + 2\delta) - \delta. \tag{3.20}$$

From (3.17) and (3.20) we have $4\delta^3 - 9\delta^2 + \delta + 4 = 0$. Thus δ is constant and β is also constant. From (3.17), $2 - 2\delta - 5\delta^2 = 0$. This means $\delta \neq 0$. From (3.20) we get $2(7\delta^2 + 2\delta) + \beta^2 = 0$. This gives $14\delta^2 + 5\delta + 1 = 0$. As this equation does not have real solutions we have proved the following.

Proposition 3.4 *Case* (1) *does not occur.*

We begin with Case (2). In this case we can write $\phi Z = \omega_1 Z_1 + \omega_2 Z_2$ for unit vector fields Z_1, Z_2 such that $AZ_1 = \lambda Z_1$, $AZ_2 = -\lambda Z_2$, ω_1 and ω_2 as functions on M such that $\omega_1^2 + \omega_2^2 = 1$. Then $A^2\phi Z = \lambda^2(\omega_1 Z_1 + \omega_2 Z_2) = \lambda^2 \phi Z$. From (3.11) we get $(1 + \alpha\lambda)A\phi Z = -\lambda(1 + \alpha\lambda)\phi Z$. So we can conclude that either $A\phi Z = -\lambda\phi Z$ or $\lambda = -\frac{1}{\alpha}$.

As $\mathbb{D}_U$ is holomorphic and A-invariant we can consider $G = \{Z \in \mathbb{D}_U \mid (\phi A + A\phi)Z = 0\}$ and F its orthogonal complement in $\mathbb{D}_U$. By Theorem 2.5, $AX = -\frac{1}{\alpha}X$ for any $X \in F$ and there exists a principal basis of G, $\{Z_i, \phi Z_i\}$ such that $AZ_i = \lambda_i Z_i$, $A\phi Z_i = -\lambda_i \phi Z_i$. Moreover F and G are A-invariant.

Thus we have three possible cases:

Case A: $A = -\frac{1}{\alpha}$ Id on $\mathbb{D}_U$, that means $G = \{0\}$.

Case B: $G = \mathbb{D}_U$.

Case C: $0 < \dim(G) < 2m - 4$. This case only occurs if $m \geq 4$.

For Cases A and C we have the following.

Lemma 3.5 *With the conditions of either Case* A *or Case* C*, we have* $\alpha^2 = 1$.

Proof Take $Z \in \mathbb{D}_U$ such that $AZ = -\frac{1}{\alpha}Z$, $A\phi Z = -\frac{1}{\alpha}\phi Z$. From the Codazzi equation we have $(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z = -2\xi$. Its scalar product with ξ gives

$$g([\phi Z, Z], U) = \frac{2}{\alpha^2 \beta}. \tag{3.21}$$

And the scalar product with U yields

$$g([\phi Z, Z], U) = \frac{2}{\beta}. \tag{3.22}$$

From (3.21) and (3.22) the result follows. ∎

Then, perhaps after changing ξ to $-\xi$, we can suppose $\alpha = 1$. Thus $A\xi = \xi + \beta U$, $AU = \beta\xi + (\beta^2 - 1)U$ and $A\phi U = \gamma\phi U$, where either $\gamma = -1$ or $\gamma = 1 - \beta^2$.

From Theorem 2.4, Cases A and C do not occur if $\gamma = -1$. Suppose now $\gamma = 1 - \beta^2$.

In Case A, for any $X \in \mathbb{D}_U$, $AX = -X$. From the Codazzi equation $(\nabla_X A)U - (\nabla_U A)X = 0$. Its scalar product with U yields $g(\nabla_U U, X) = \frac{2}{\beta}X(\beta)$, and its scalar product with ξ gives $X(\beta) = \beta g(\nabla_U U, X)$. Thus

$$X(\beta) = 0. \tag{3.23}$$

We also have $(\nabla_U A)\xi - (\nabla_\xi A)U = -\phi U$. Its scalar product with ξ implies $\xi(\beta) = 0$ and its scalar product with U gives $U(\beta) = \xi(\beta^2)$. So we get

$$\xi(\beta) = U(\beta) = 0. \tag{3.24}$$

The Codazzi equation also gives $(\nabla_{\phi U}A)\xi - (\nabla_\xi A)\phi U = U$. Its scalar product with ξ yields $g(\nabla_\xi \phi U, U) = 1 - 3(1-\beta^2)$ and from its scalar product with U we get $(\phi U(\beta) - 2 - (\beta^2-1)^2 + 2(\beta^2-1)g(\nabla_\xi \phi U, U) = 0$. Both equations imply

$$(\phi U)(\beta) = 2 - 2\delta - 5\delta^2 \tag{3.25}$$

where we denote $\delta = \beta^2 - 1$. From (3.23), (3.24), and (3.25) we have

$$\operatorname{grad}(\beta) = (2 - 2\delta - 5\delta^2)\phi U = \tau \phi U.$$

As for any X, Y tangent to M we have $g\big(\nabla_X \operatorname{grad}(\beta), Y\big) = g\big(\nabla_Y \operatorname{grad}(\beta), X\big)$, we obtain

$$X(\tau)g(\phi U, Y) + \tau g(\nabla_X \phi U, Y) = Y(\tau)g(\phi U, X) + \tau g(\nabla_Y \phi U, X).$$

If we take $Y = \xi$, this yields $\tau g(\nabla_X \phi U, \xi) = \tau g(\nabla_\xi \phi U, X)$, for any X tangent to M. Taking $X = U$ and bearing in mind that $g(\nabla_\xi \phi U, U) = 1 - 3(1-\beta^2)$, we have $\tau\big(1 - 4(1-\beta^2)\big) = 0$. If $\tau \neq 0$, we obtain $\beta^2 = \frac{3}{4}$, thus β is constant and we have a contradiction. Thus $\tau = 0$ and $\operatorname{grad}(\beta) = 0$. This means

$$5\delta^2 + 2\delta - 2 = 0. \tag{3.26}$$

From the Codazzi equation $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$. If we take its scalar product with U we get

$$2(\beta^2 - 1)g(\nabla_U U, \phi U) = \beta(\beta^2 - 1).$$

If $\beta^2 = 1$, then $\delta = 0$ and (3.26) is not possible. Thus $\beta^2 \neq 1$ and $g(\nabla_U U, \phi U) = \frac{\beta}{2}$. As $g\big((\nabla_U A)\xi - (\nabla_\xi A)U, \phi U\big) = -1$ we get

$$\beta g(\nabla_U U, \phi U) + (\beta^2 - 1)^2 - 2(\beta^2 - 1)g(\nabla_\xi U, \phi U) = 0.$$

Thus

$$\beta g(\nabla_U U, \phi U) = -(\beta^2-1)^2 - 2(\beta^2-1)\big(-1 + 3(\beta^2-1)\big) = -2(\beta^2-1) + 5(\beta^2-1)^2.$$

As $g(\nabla_U U, \phi U) = \frac{\beta}{2}$ we conclude that

$$10\delta^2 - 5\delta - 1 = 0. \tag{3.27}$$

From (3.26) and (3.27), $\delta = \frac{1}{3}$. Now (3.26) and (3.27) are not true and we have proved the following.

Proposition 3.6 *Case* A *does not occur.*

We continue with Case C. Recall that $A\xi = \xi + \beta U$, $AU = \beta\xi + (\beta^2 - 1)U$, $A\phi U = (1 - \beta^2)\phi U$, $F \neq \{0\}$, $G \neq \{0\}$.

Take $Z \in G$ such that $AZ = \lambda Z$. If we take $X = \phi Z$, $Y = Z$ in (3.1) and its scalar product with U we obtain

$$(\lambda + 1)\big(g(\nabla_{\phi Z} Z, U) - g(\nabla_Z \phi Z, U)\big) = 0.$$

Suppose first that $g(\nabla_{\phi Z} Z, U) = g(\nabla_Z \phi Z, U)$. From the Codazzi equation, we have $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z$. If we take the scalar product with ϕZ we get

$$\beta g(\nabla_Z U, \phi Z) - 2\lambda g(\nabla_\xi Z, \phi Z) = -1 - \lambda - \lambda^2. \tag{3.28}$$

As $A\phi Z = -\lambda\phi Z$, if in the above procedure we change Z to ϕZ, we have

$$\beta g(\nabla_{\phi Z} U, Z) + 2\lambda g(\nabla_\xi \phi Z, Z) = 1 - \lambda + \lambda^2. \tag{3.29}$$

From (3.28) and (3.29) we get $2\lambda^2 + 2 = 0$, which is impossible. Thus $g(\nabla_{\phi Z} Z, U) - g(\nabla_Z \phi Z, U) \neq 0$ and $\lambda = -1$.

Taking $X = Z$, $Y = \phi Z$ in (3.1) and its scalar product with U, we get

$$(-\lambda + 1)\big(g(\nabla_Z \phi Z, U) - g(\nabla_{\phi Z} Z, U)\big) = 0.$$

As $g(\nabla_Z \phi Z, U) - g(\nabla_{\phi Z} Z, U) \neq 0$, $\lambda = 1$ and we arrive at a contradiction. Thus the following proposition is proved.

Proposition 3.7 *Case* C *does not occur.*

So, we study Case B. Now $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + \frac{\beta^2 - 1}{\alpha} U$, $A\phi U = \gamma\phi U$, where either $\gamma = -\frac{1}{\alpha}$ or $\gamma = \frac{1 - \beta^2}{\alpha}$, and for any $Z \in \mathbb{D}_U$ such that $AZ = \lambda Z$, $A\phi Z = -\lambda\phi Z$.

A similar proof as in Lemma 4.3 in [10] gives us the following lemma.

Lemma 3.8 *For any* $Z \in \mathbb{D}_U$ *with* $AZ = \lambda Z$ *we get:*

(1) $\alpha\lambda g(\nabla_Z \phi Z, Z) = 0$,
(2) $\alpha\lambda g(\nabla_{\phi Z} Z, \phi Z) = 0$,
(3) $\lambda Z(\alpha) + \alpha Z(\lambda) = 0$,
(4) $(\alpha\lambda + 1)\big(g(\nabla_{\phi Z} Z, U) - g(\nabla_Z \phi Z, U)\big) = 0$.

First suppose that $\lambda = 0$. Then the fourth equation in Lemma 3.5 yields $g(\nabla_{\phi Z} Z, U) - g(\nabla_Z \phi Z, U) = 0$. As $g\big((\nabla_{\phi Z} A)Z - (\nabla_Z A)\phi Z, \xi\big) = 2$ we obtain $\beta\big(g(\nabla_Z \phi Z, U) - g(\nabla_{\phi Z} Z, U)\big) = 2$. This is impossible. Thus $\lambda \neq 0$, and from the two first items in Lemma 3.5, $g(\nabla_{\phi Z} Z, \phi Z) = g(\nabla_Z \phi Z, Z) = 0$. From Lemma 3.5 we have either $\alpha\lambda + 1 = 0$ or $g(\nabla_{\phi Z} Z, U) = g(\nabla_Z \phi Z, U)$. If $\alpha\lambda + 1 = 0$, $\lambda = -\frac{1}{\alpha}$. Then $A\phi Z = \frac{1}{\alpha}\phi Z$, and by the same argument $-\alpha\lambda + 1 = 0$. This is impossible, thus $g(\nabla_{\phi Z} Z, U) = g(\nabla_Z \phi Z, U)$. As in Case C, this gives a contradiction. Therefore we have proved the following proposition.

Proposition 3.9 *Case* B *does not occur.*

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Summing up Propositions 3.2, 3.3 and 3.4, we conclude the following.

Proposition 3.10 *Case* (2) *does not appear.*

So, we have proved that there do not exist non-Hopf real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, satisfying (1.1) and (1.2).

Now we suppose that M is Hopf with $A\xi = \alpha\xi$. Take $X \in \mathbb{D}$ such that $AX = \lambda X$. As above we obtain $\alpha A^2\phi X + (\alpha\lambda + 1)A\phi X + \lambda\phi X = 0$. If $\alpha = 0$, $\lambda = 0$, then by Theorem 2.2 this is impossible. Then $\alpha \neq 0$. As above, bearing in mind Theorem 2.2, the unique possibility is to have $AX = \lambda X$, $A\phi X = -\lambda\phi X$ for any $X \in \mathbb{D}$. This means that $A\phi + \phi A = 0$ and Theorem A follows from Theorem 2.1. ∎

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