

Analytic partial-integrability of a symmetric Hopf-zero degeneracy

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We deal with analytic three-dimensional symmetric systems whose origin is a Hopf-zero singularity. Once it is not completely analytically integrable, we provide criteria on the existence of at least one functionally independent analytic first integral. In the generic case, we characterize the analytic partially integrable systems by using orbitally equivalent normal forms. We also solve the problem through the existence of a class of formal inverse Jacobi multiplier of the system.

Keywords: Hopf-zero bifurcation; First integrals; Inverse Jacobi multiplier

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1. Introduction

We are concerned with analysing the existence of analytic first integrals at the origin of the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ -symmetric analytic systems, i.e. invariant to $(x, y, z) \leftrightarrow (-x, -y, -z)$ and whose origin is a *Hopf-zero singularity*,

$$\begin{aligned} \dot{x} &= -y + \sum_{i,j,k} a_{ijk} x^i y^j z^k, \\ \dot{y} &= x + \sum_{i,j,k} b_{ijk} x^i y^j z^k, \\ \dot{z} &= \sum_{i,j,k} c_{ijk} x^i y^j z^k, \end{aligned} \tag{1.1}$$

with $i, j, k \geq 0$, $i + j + k \geq 3$ and $i + j + k$ odd. Under the conditions of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ -symmetry, performing a change of variables $(x, y, z) = (x + p_3(x, y, z), y + q_3(x, y, z), z + r_3(x, y, z))$ with p_3, q_3, r_3 homogeneous cubic polynomials, we obtain an orbital normal form up to order 3 of the system (1.1)

$$\begin{aligned} \dot{x} &= -y + x(a_1(x^2 + y^2) + a_2z^2) + \text{odd h.o.t.}, \\ \dot{y} &= x + y(a_1(x^2 + y^2) + a_2z^2) + \text{odd h.o.t.}, \\ \dot{z} &= z(b_1(x^2 + y^2) + b_2z^2) + \text{odd h.o.t.}, \end{aligned} \tag{1.2}$$

with

$$\begin{aligned}
 a_1 &= \frac{1}{8}(3b_{030} + a_{120} + b_{210} + 3a_{300}), & a_2 &= \frac{1}{2}(b_{012} + a_{102}), \\
 b_1 &= \frac{1}{2}(c_{021} + c_{201}), & b_2 &= c_{003},
 \end{aligned}
 \tag{1.3}$$

see [1, 12, 13, 15, 17, 23]. This normal form is the simplest orbital normal form up to order 3 of the system (1.1). Therefore, if (a_1, a_2, b_1, b_2) is non-zero, system (1.1) is not orbitally equivalent to $(-y, x, 0)^T$, i.e. it is not linearizable.

Recall that a *first integral at the origin* of a system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, is a scalar function I that is constant on a neighbourhood \mathcal{U} along of any solution of the system and $I(\mathbf{0}) = 0$. If I is a C^1 function, using the chain rule, it means that $F(I) := \nabla I \cdot \mathbf{F} = 0$ on \mathcal{U} , where F denotes the differential operator associated to the system. We say that system (1.1) is *completely analytically integrable* if it admits two functionally independent local analytic first integrals. García [14] has proved that a Hopf-zero singularity is completely analytically integrable if, and only if, it is orbitally equivalent to its linear part $(-y, x, 0)^T$. He also proved that both integrability problem and centre problem (that consists of determining whether there is a neighbourhood of the singularity foliated by period orbits, including a curve of equilibria) are equivalent for system (1.1). So, as a direct consequence of [14], system (1.1) with (a_1, a_2, b_1, b_2) non-zero does not have two functionally independent analytic first integral since it is not linearizable. In other words, the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ -symmetric Hopf-zero singularity we are considering is not completely integrable and our analysis will be focused on detecting the existence of one functionally independent analytic first integral for such a singularity.

This is a difficult problem and there are few known satisfactory methods to solve it. In the present paper, we use the orbital normal form obtained in [1] to establish necessary conditions for the existence of analytic first integrals and formal inverse Jacobi multipliers.

The first result provides an orbital normal form for the systems (1.1) having one, and only one, functionally independent analytic first integral.

THEOREM 1.1. *Consider the analytic system (1.1) with $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$ where (a_1, a_2, b_1, b_2) is given in (1.3). System (1.1) has one, and only one, functionally independent analytic first integral if, and only if, it is orbitally equivalent to the system*

$$\begin{aligned}
 \dot{x} &= -y + x(a_1(x^2 + y^2) + a_2z^2) - y\Psi(x^2 + y^2, z^2), \\
 \dot{y} &= x + y(a_1(x^2 + y^2) + a_2z^2) + x\Psi(x^2 + y^2, z^2), \\
 \dot{z} &= z(b_1(x^2 + y^2) + b_2z^2),
 \end{aligned}
 \tag{1.4}$$

where Ψ is a formal function with $\Psi(0, 0) = 0$ and (a_1, a_2, b_1, b_2) satisfying one of the following two conditions:

- (a) $a_1 = a_2 = 0$ (or $b_1 = b_2 = 0$). Moreover, in this case, an analytic first integral is of the form $x^2 + y^2 + \dots$ (or $z^2 + \dots$).

- (b) there exists a rational m such that $p := b_2(b_1 - a_1)m$, $q := a_1(a_2 - b_2)m$ and $s := (a_1b_2 - a_2b_1)m$ are natural numbers and $\gcd(p, q, s) = 1$. Moreover, in this case, an analytic first integral is of the form $(x^2 + y^2)^p z^{2q} ((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)^s + \dots$.

Theorem 1.1 is proved in § 3.

REMARK 1.2. The assumptions $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$ in theorem 1.1 are necessary. System

$$\begin{aligned} \dot{x} &= -y + x(-2z^2 + (x^2 + y^2)^2), \\ \dot{y} &= x + y(-2z^2 + (x^2 + y^2)^2), \\ \dot{z} &= z(z^2 - 3(x^2 + y^2)^2), \end{aligned} \tag{1.5}$$

satisfies $a_1 = b_1 = 0$ and $a_2 = -2, b_2 = 1$. On the one hand, $(x^2 + y^2)z^2(z^2 - (x^2 + y^2)^2)$ is a polynomial first integral of (1.5). As (a_1, a_2, b_1, b_2) is non-zero, it has one and only one functionally independent analytic first integral. On the other hand, system (1.5) in cylindrical coordinates is

$$\dot{r} = -2rz^2 + r^5, \quad \dot{z} = z^3 - 3r^4z, \quad \dot{\theta} = 1.$$

By [1], system (1.5) is an orbital normal form and it is unique, therefore the terms $r^5\partial_r$ and $-3r^4z\partial_z$ can not be removed. So, it is not orbitally equivalent to system (1.4).

We conclude that if (a_1, a_2, b_1, b_2) is non-zero but $a_1 = b_1 = 0$ or $a_2 = b_2 = 0$, there are systems (1.1) not orbitally equivalent to systems (1.4) with one analytic first integral. Thus, in this case, the analytic partial integrability problem is an open problem.

An *inverse Jacobi multiplier* for a system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$, is a smooth function J which satisfies $F(J) = \operatorname{div}(\mathbf{F})J$ in a neighbourhood of the origin. When J does not vanish in an open set, then the above equality becomes $\operatorname{div}(\frac{\mathbf{F}}{J}) = 0$. For planar systems, the inverse Jacobi multipliers are usually referred as *inverse integrating factors*, see [7].

The inverse Jacobi multiplier is a useful tool in the study of vector fields. So, for example, the existence of a class of inverse Jacobi multipliers (or inverse integrating factors) has been used for the study of the Hopf bifurcation and centre problem, see [9–11, 16, 24], and for the integrability problem in general, see [2, 8, 21].

Here, we solve the analytic partial-integrability problem for systems (1.1) through the existence of an inverse Jacobi multiplier.

THEOREM 1.3. Consider the analytic system (1.1) with $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$ where (a_1, a_2, b_1, b_2) is given in (1.3). System (1.1) has one, and only one, functionally independent analytic first integral if, and only if, it has a formal inverse Jacobi multiplier of the form

$$J = (x^2 + y^2)z((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2) + \dots$$

with (a_1, a_2, b_1, b_2) satisfying one of the following two conditions:

- (a) $a_1 = a_2 = 0$ (or $b_1 = b_2 = 0$),
- (b) $b_2(b_1 - a_1)$, $a_1(a_2 - b_2)$ and $a_1b_2 - a_2b_1$ are rational numbers different from zero with the same sign.

This theorem is proved in § 3.

We note that the degree of the lowest-degree term of the analytic first integral is two or $2(p + q + s)$ while the degree of the lowest-degree of the inverse Jacobi multiplier is five.

2. Computation of necessary conditions of analytic partial integrability of system (1.2)

In this section, we give a result that provides an efficient method for obtaining necessary conditions of existence of one analytic first integral of system (1.2) (orbital normal forms up to order 3 of systems whose origin is a symmetric Hopf-zero singularity). Here, we denote by \mathcal{P}_j the vector space of the homogeneous polynomials of degree j with three variables. First, we present the following result.

PROPOSITION 2.1. *The following statements are satisfied:*

- (i) Consider system (1.2) with $a_1 = a_2 = 0$ and $b_1b_2 \neq 0$. Then, there exists a scalar function $I = x^2 + y^2 + \sum_{k \geq 2} I_{2k}$, with $I_{2k} \in \mathcal{P}_{2k}$ unique module $(x^2 + y^2)^k$, for all k , that verifies

$$F(I) = \sum_{k \geq 2} (\eta_k(x^2 + y^2)^k + \nu_k z^{2k}). \tag{2.1}$$

- (ii) Consider system (1.2) with $b_1 = b_2 = 0$ and $a_1a_2 \neq 0$. Then, there exists a scalar function $I = z^2 + \sum_{k \geq 2} I_{2k}$ with $I_{2k} \in \mathcal{P}_{2k}$ unique module z^{2k} for all k , that verifies

$$F(I) = \sum_{k \geq 2} (\eta_k(x^2 + y^2)^k + \nu_k z^{2k}). \tag{2.2}$$

- (iii) Consider system (1.2) where $b_2(b_1 - a_1)$, $a_1(a_2 - b_2)$ and $a_1b_2 - a_2b_1$ are rational numbers different from zero with the same sign. Let $p := b_2(b_1 - a_1)m$, $q := a_1(a_2 - b_2)m$ and $s := (a_1b_2 - a_2b_1)m$ with m a rational number such that p, q and s are natural numbers and $\gcd(p, q, s) = 1$. Then, there exists a scalar function $I = I_{2M} + \sum_{k > M} I_{2k}$, with $I_{2M} = (x^2 + y^2)^p z^{2q} ((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)^s$, (i.e. $M = p + q + s$) with I_{2k} unique if $k \not\equiv 0 \pmod{M}$, and I_{2Mk} unique module I_{2M}^k , for all k , that verifies

$$F(I) = \sum_{q(k-1) \not\equiv 0 \pmod{M}} \nu_k z^{2k} + \sum_{\substack{q(k-1) \equiv 0 \pmod{M} \\ p(k-1) \not\equiv 0 \pmod{M}}} \eta_k (x^2 + y^2)^k \tag{2.3}$$

$$+ \sum_{k-1 \equiv 0 \pmod{M}} (\nu_k z^{2k} + \eta_k (x^2 + y^2)^k).$$

Proof. Taylor expansion of the associated vector field of system (1.2) is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_3 + \sum_{j \geq 2} \mathbf{F}_{2j+1}$, with $\mathbf{F}_1 = (-y, x, 0)^T$ and $\mathbf{F}_3 = (-y + x(a_1(x^2 + y^2) + a_2z^2), x + y(a_1(x^2 + y^2) + a_2z^2), z(b_1(x^2 + y^2) + b_2z^2))^T$. If $I = I_2 + \sum_{k \geq 2} I_{2k}$ with $I_{2k} \in \mathcal{P}_{2k}$, then $F(I)$ has only even-degree homogeneous terms. The equation $F(I) = 0$ to degree two is satisfied. The $2k$ -degree term of $F(I)$, $k \geq 2$ is

$$F(I)_{2k} = F_1(I_{2k}) + F_3(I_{2k-2}) + R_{2k}$$

where $R_{2k} = \sum_{j=1}^{k-2} F_{2k-2j+1}(I_{2j})$.

The procedure for obtaining the scalar function I will have the following scheme: For each order k , we first compute I_{2k} to get an expression more reduced of R_{2k} . Let us note that the term I_{2k} , in general, is not unique. Later on, in the following step, we will use the no-uniqueness of I_{2k-2} for obtaining the simplest expression of $F(I)_{2k}$.

The above scheme suggests us to consider the following linear operators. The operator

$$\begin{aligned} \ell^{(3)}_{2k} &: \mathcal{P}_{2k} \longrightarrow \mathcal{P}_{2k} \\ &\mu_{2k} \longrightarrow F_1(\mu_{2k}). \end{aligned} \tag{2.4}$$

It is easy to prove that $\text{Ker}(\ell^{(3)}_{2k}) = \text{Span}\{(x^2 + y^2)^k, (x^2 + y^2)^{k-1}z^2, \dots, z^{2k}\}$. Moreover, $\mathcal{P}_{2k} = \text{Range}(\ell^{(3)}_{2k}) \oplus \text{Ker}(\ell^{(3)}_{2k})$. Therefore, for all $k \geq 2$, we can choose $\text{Cor}(\ell^{(3)}_{2k}) = \text{Ker}(\ell^{(3)}_{2k})$, a complementary subspace to $\text{Range}(\ell^{(3)}_{2k})$.

We also consider the linear operator:

$$\begin{aligned} \tilde{\ell}^{(3)}_{2k} &: \text{Ker}(\ell^{(3)}_{2k-2}) \longrightarrow \text{Cor}(\ell^{(3)}_{2k}) \\ &\mu_{2k-2} \longrightarrow F_3(\mu_{2k-2}). \end{aligned} \tag{2.5}$$

The transformed by $\tilde{\ell}^{(3)}_{2k}$ of an element of the basis of $\text{Ker}(\ell^{(3)}_{2k-2})$, $(x^2 + y^2)^{k_0} z^{2j_0}$ with $0 \leq k_0, j_0 \leq k - 1, k_0 + j_0 = k - 1$, is

$$\tilde{\ell}^{(3)}_{2k}((x^2 + y^2)^{k_0} z^{2j_0}) = 2A_{j_0, k_0}(x^2 + y^2)^{k_0+1} z^{2j_0} + 2B_{j_0, k_0}(x^2 + y^2)^{k_0} z^{2j_0+2}, \tag{2.6}$$

where $A_{j_0, k_0} = b_1j_0 + a_1k_0$ and $B_{j_0, k_0} = b_2j_0 + a_2k_0$.

Therefore, the operator $\tilde{\ell}^{(3)}_{2k}$ is well-defined.

We analyse each case:

Consider system (1.2) with $a_1 = a_2 = 0$ and $b_1b_2 \neq 0$. The proof consists on the computation, degree to degree, of the homogeneous terms of $I = I_2 + I_4 + \dots$, with $I_2 = x^2 + y^2$, satisfying (2.1).

The transformed by $\tilde{\ell}^{(3)}_{2k}$ of an element of the basis of $\text{Ker}(\ell^{(3)}_{2k-2})$ is

$$\tilde{\ell}^{(3)}_{2k}((x^2 + y^2)^{k_0} z^{2j_0}) = 2b_1j_0(x^2 + y^2)^{k_0+1} z^{2j_0} + 2b_2j_0(x^2 + y^2)^{k_0} z^{2j_0+2}.$$

Therefore, $\text{Ker}(\tilde{\ell}^{(3)}_{2k}) = \text{Span}\{(x^2 + y^2)^{k-1}\}$ and we can choose $\text{Cor}(\tilde{\ell}^{(3)}_{2k}) = \text{Span}\{(x^2 + y^2)^k, z^{2k}\}$.

We write $I_{2k-2} = I_{2k-2}^a + I_{2k-2}^b$ with I_{2k-2}^b fixed in the previous step and $I_{2k-2}^a \in \text{Ker}(\ell^{(3)}_{2k-2})$. So, for $k \geq 2$,

$$F(I)_{2k} = \ell^{(3)}_{2k}(I_{2k}) + \tilde{\ell}^{(3)}_{2k}(I_{2k-2}^a) + F_3(I_{2k-2}^b) + R_{2k}.$$

We now write $\tilde{R}_{2k} = F_3(I_{2k-2}^b) + R_{2k} = \tilde{R}_{2k}^{(r)} + \tilde{R}_{2k}^{(c)}$ with $\tilde{R}_{2k}^{(r)} \in \text{Range}(\ell^{(3)}_{2k})$ and $\tilde{R}_{2k}^{(c)} \in \text{Cor}(\ell^{(3)}_{2k})$.

Reasoning as in the classical Normal Form Theory, we choose I_{2k} such that $\ell^{(3)}_{2k}(I_{2k}) = -\tilde{R}_{2k}^{(r)}$ and choose I_{2k-2}^a in order to annihilate the part of $\tilde{R}_{2k}^{(c)}$ belonging to the range of the operator $\tilde{\ell}^{(3)}_{2k}$. So, we have that $F(I)_{2k} = \eta_k(x^2 + y^2)^k + \nu_k z^{2k} \in \text{Cor}(\tilde{\ell}^{(3)}_{2k})$, with η_k and ν_k real numbers.

Last on, we note that the solutions of the equation (2.1) are $I_{2k} + \lambda_k(x^2 + y^2)^k$, where I_{2k} is the unique solution chosen following the recursive procedure and λ_k real, that is, I_{2k} is unique module $(x^2 + y^2)^k$.

Consider system (1.2) with $b_1 = b_2 = 0$ and $a_1 a_2 \neq 0$. In this case, we derive, degree to degree, the homogeneous terms of $I = I_2 + I_4 + \dots$, with $I_2 = z^2$, satisfying the equation (2.2).

It has that $\text{Ker}(\tilde{\ell}^{(3)}_{2k}) = \text{Span}\{z^{2(k-1)}\}$ and we can choose $\text{Cor}(\tilde{\ell}^{(3)}_{2k}) = \text{Span}\{(x^2 + y^2)^k, z^{2k}\}$. Reasoning as before, we obtain the result.

For systems (1.2) where $b_2(b_1 - a_1)$, $a_1(a_2 - b_2)$ and $a_1 b_2 - a_2 b_1$ are rational numbers different from zero with the same sign, we compute the homogeneous terms of $I = I_{2M} + \sum_{k>M} I_{2k}$, with $I_{2M} = (x^2 + y^2)^p z^{2q} ((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)^s$, satisfying the equation (2.3). It has that $\tilde{\ell}^{(3)}_{2k}((x^2 + y^2)^{k_0} z^{2j_0})$ is given by (2.6) where

$$A_{j_0, k_0} = b_1 j_0 + a_1 k_0 = \frac{1}{(b_2 - a_2)m} ((p + s)j_0 - qk_0),$$

$$B_{j_0, k_0} = b_2 j_0 + a_2 k_0 = \frac{1}{(a_1 - b_1)m} (-pj_0 + (q + s)k_0),$$

with $0 \leq k_0, j_0 \leq k - 1, k_0 + j_0 = k - 1$. Note that $(p + s)j_0 - qk_0 = Mj_0 - q(k - 1)$, i.e. $\{A_{j_0, k_0}\}, j_0 = 0, \dots, k - 1$, is an arithmetic progression whose difference is $M \neq 0$, and $-pj_0 + (q + s)k_0 = Mk_0 - p(k - 1)$, i.e. $\{B_{j_0, k_0}\}, k_0 = 0, \dots, k - 1$, is an arithmetic progression whose difference is M . Therefore, the numbers A_{j_0, k_0} with $0 \leq k_0, j_0 \leq k - 1, k_0 + j_0 = k - 1$, are different and the numbers B_{j_0, k_0} also are different. Fixed k , this fact allows us to distinguish the following cases:

If $A_{j_0, k_0} \neq 0$ for all $0 \leq k_0, j_0 \leq k - 1$, with $k_0 + j_0 = k - 1$, (i.e. $q(k - 1) \not\equiv 0 \pmod{M}$), it has that $\text{Ker}(\tilde{\ell}^{(3)}_{2k}) = \{0\}$ and we can choose $\text{Cor}(\tilde{\ell}^{(3)}_{2k}) = \text{Span}\{z^{2k}\}$.

If there exists j_1 with $0 < j_1 < k - 1$ such that $A_{j_1, k_1} = 0$, that is $Mj_1 = q(k - 1)$, and $B_{j_0, k_0} \neq 0$ for all $0 \leq k_0, j_0 \leq k - 1$, it has that $\text{Ker}(\tilde{\ell}^{(3)}_{2k}) = \{0\}$ and we can choose $\text{Cor}(\tilde{\ell}^{(3)}_{2k}) = \text{Span}\{(x^2 + y^2)^k\}$.

Otherwise, there exist $(j_1, k_1), (j_2, k_2)$ such that $A_{j_1, k_1} = 0$ and $B_{j_2, k_2} = 0$. We are going to prove that $k - 1$ is a multiple of M . Indeed, we have that $k - 1 = \frac{j_1}{q} M = \frac{k_1}{p+s} M = \frac{j_2}{q+s} M = \frac{k_2}{p} M$, i.e. $\frac{p(k-1)}{M}, = \frac{q(k-1)}{M}$ and $\frac{s(k-1)}{M}$ are natural numbers. Thus, $\frac{(n_1 p + n_2 q + n_3 s)(k-1)}{M}$ is an integer number, for all $n_1, n_2, n_3 \in \mathbb{Z}$.

On the other hand, from Bezout identity, there exist $m_1, m_2, m_3 \in \mathbb{Z}$ such that $m_1p + m_2q + m_3s = 1$ since $\gcd(p, q, s) = 1$. Therefore, $\frac{k-1}{M}$ is a natural number. So, $k - 1$ is a multiple of M .

If we write $k = 1 + \hat{k}M$, it has that $\text{Ker}(\tilde{\ell}_{2k}^{(3)}) = \text{Span}\{I_{2M}^{\hat{k}}\}$. Moreover, $j_1 = \frac{q}{q+s}j_2$. Thus $j_1 < j_2$ and we can choose as a complementary subspace of the range of the operator $\tilde{\ell}_{2k}^{(3)}$ to $\text{Cor}(\tilde{\ell}_{2k}^{(3)}) = \text{Span}\{(x^2 + y^2)^k, z^{2k}\}$. Reasoning as before, we obtain the result. \square

The following result characterizes the analytic integrability of system (1.2). An algorithm for obtaining necessary conditions of existence of a first integral can be derived following the scheme of the proof of proposition 2.1.

THEOREM 2.2. *Consider system (1.2) with (a_1, a_2, b_1, b_2) satisfying one of the following two conditions:*

- (a) $a_1 = a_2 = 0$ and $b_1b_2 \neq 0$ (or $b_1 = b_2 = 0$ and $a_1a_2 \neq 0$).
- (b) $b_2(b_1 - a_1)$, $a_1(a_2 - b_2)$ and $a_1b_2 - a_2b_1$ are rational numbers different from zero with the same sign.

Then, system (1.2) has one, and only one, functionally independent analytic first integral if, and only if, the equations (2.1), (2.2) and (2.3), introduced in proposition 2.1, satisfy $\eta_k = 0$ and $\nu_k = 0$ for all k .

Proof. Consider the function I introduced in proposition 2.1. The sufficient condition is trivial. If $\eta_k = \nu_k = 0$, for all k , then the function I is a formal first integral since $F(I) = 0$. From lemma 3.1, system (1.2) admits an analytic first integral. Moreover, from García [14] it is not completely analytically integrable because it is not linearizable. Therefore, system (1.2) has a unique functionally independent first integral.

Let us prove the necessary condition. If system (1.2) has an analytic first integral then, from theorem 1.1, according to each case, it admits an analytic first integral \tilde{I} of the form $x^2 + y^2 + \dots, z^2 + \dots$ or $(x^2 + y^2)^p z^{2q}((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)^s + \dots$, having only even-degree homogeneous terms. Then, $I_p = \tilde{I} - \sum_{k \geq 2} \beta_k \tilde{I}^k$ is also a formal first integral, i.e. $F(I_p) = 0$. To complete the proof it is enough to choose β_k such that I_p is the unique scalar function given by proposition 2.1. Thus, by the uniqueness of I_p , $\eta_k = \nu_k = 0$, for all k . \square

We can also provide an integrability criterium based on the existence of an inverse Jacobi multiplier of system (1.2). First, we give an auxiliary result.

PROPOSITION 2.3. *The following statements are satisfied:*

- (i) *Consider system (1.2) with $a_1 = a_2 = 0$ and $b_1b_2 \neq 0$. Then, there exists a scalar function*

$$J = J_5 + \sum_{k \geq 3} J_{2k+1}$$

with $J_5 = (x^2 + y^2)z(b_1(x^2 + y^2) + b_2z^2)$ and $J_{2k+1} \in \mathcal{P}_{2k+1}$ unique module $(x^2 + y^2)^{k-1}z(b_1(x^2 + y^2) + b_2z^2)$ for all k , that verifies

$$F(J) - J \operatorname{div}(\mathbf{F}) = \sum_{k \geq 3} (\eta_k z(x^2 + y^2)^k + \nu_k z^{2k+1}). \tag{2.7}$$

(ii) Consider system (1.2) with $b_1 = b_2 = 0$ and $a_1 a_2 \neq 0$. Then, there exists a scalar function

$$J = J_5 + \sum_{k \geq 3} J_{2k+1}$$

with $J_5 = (x^2 + y^2)z(a_1(x^2 + y^2) + a_2z^2)$ and $J_{2k+1} \in \mathcal{P}_{2k+1}$ unique module $(x^2 + y^2)z^{2k-3}(a_1(x^2 + y^2) + a_2z^2)$ for all k , that verifies

$$F(J) - J \operatorname{div}(\mathbf{F}) = \sum_{k \geq 3} (\eta_k z(x^2 + y^2)^k + \nu_k z^{2k+1}). \tag{2.8}$$

(iii) Consider system (1.2) where $b_2(b_1 - a_1)$, $a_1(a_2 - b_2)$ and $a_1 b_2 - a_2 b_1$ are rational numbers different from zero with the same sign. Let $p := b_2(b_1 - a_1)m$, $q := a_1(a_2 - b_2)m$ and $s := (a_1 b_2 - a_2 b_1)m$ where m is a rational such that p, q and s are natural numbers with $\gcd(p, q, s) = 1$. Then, there exists a scalar function

$$J = J_5 + \sum_{k \geq 3} J_{2k+1}$$

with $J_5 = (x^2 + y^2)z((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)$, J_{2k+1} unique if $k \not\equiv 3 \pmod{M}$ being $M = p + q + s$ and J_{2kM+5} unique module $(x^2 + y^2)^{pk+1}z^{2qk+1}((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)^{sk+1}$ for all k , that verifies

$$\begin{aligned} F(J) - J \operatorname{div}(\mathbf{F}) &= \sum_{q(k-3) \not\equiv 0 \pmod{M}} \nu_k z^{2k+1} + \sum_{\substack{q(k-3) \equiv 0 \pmod{M} \\ p(k-3) \not\equiv 0 \pmod{M}}} \eta_k (x^2 + y^2)^k z \\ &+ \sum_{k-3 \equiv 0 \pmod{M}} (\nu_k z^{2k+1} + \eta_k (x^2 + y^2)^k z). \end{aligned} \tag{2.9}$$

Proof. Taylor expansion of the associated vector field of system (1.2) is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_3 + \sum_{k \geq 2} \mathbf{F}_{2k+1}$, with $\mathbf{F}_1 = (-y, x, 0)^T$ and $\mathbf{F}_3 = (-y + x(a_1(x^2 + y^2) + a_2z^2), x + y(a_1(x^2 + y^2) + a_2z^2), z(b_1(x^2 + y^2) + b_2z^2))^T$. Thus $F(J) - J \operatorname{div}(\mathbf{F})$ has only odd-degree homogeneous terms. The term of degree five of $F(J) - J \operatorname{div}(\mathbf{F})$ is zero. The $2k + 1$ -degree term of $F(J) - J \operatorname{div}(\mathbf{F})$, $k \geq 3$ is

$$F_1(J_{2k+1}) + F_3(J_{2k-1}) - J_{2k-1} \operatorname{div}(\mathbf{F}_3) + R_{2k+1} \tag{2.10}$$

where $R_{2k+1} = \sum_{j=1}^{k-2} F_{2k-2j+1}(J_{2j+1}) - J_{2j+1} \operatorname{div}(\mathbf{F}_{2k-2j+1})$.

Applying Euler Theorem for homogeneous function, it has that $F_3(J_{2k-1}) - J_{2k-1} \operatorname{div}(\mathbf{F}_3) = (F_3 - \frac{1}{2k-1} \operatorname{div}(\mathbf{F}_3)(x, y, z)^T)(J_{2k-1})$

The above expression suggests us to consider the following linear operators. The operator

$$\ell^{(3)}_{2k+1} : \mathcal{P}_{2k+1} \longrightarrow \mathcal{P}_{2k+1} \tag{2.11}$$

$$\mu_{2k+1} \longrightarrow F_1(\mu_{2k+1}).$$

It is easy to prove that $\text{Ker}(\ell^{(3)}_{2k+1}) = \text{Span}\{(x^2 + y^2)^k z, (x^2 + y^2)^{k-1} z^3, \dots, z^{2k+1}\}$. Moreover, we can choose $\text{Cor}(\ell^{(3)}_{2k+1}) = \text{Ker}(\ell^{(3)}_{2k+1})$.

We also consider the linear operator:

$$\tilde{\ell}^{(3)}_{2k+1} : \text{Ker}(\ell^{(3)}_{2k-1}) \longrightarrow \text{Cor}(\ell^{(3)}_{2k+1}) \tag{2.12}$$

$$\mu_{2k-1} \longrightarrow (F_3 - \frac{1}{2k-1} \text{div}(\mathbf{F}_3)(x, y, z)^T)(\mu_{2k-1}).$$

The transformed by $\tilde{\ell}^{(3)}_{2k+1}$ of an element of the basis of $\text{Ker}(\ell^{(3)}_{2k-1})$, $(x^2 + y^2)^{k_0} z^{2j_0+1}$ with $0 \leq k_0, j_0 \leq k-1, k_0 + j_0 = k-1$, is

$$\tilde{\ell}^{(3)}_{2k}((x^2 + y^2)^{k_0} z^{2j_0+1}) = 2A_{j_0, k_0}(x^2 + y^2)^{k_0+1} z^{2j_0+1} + 2B_{j_0, k_0}(x^2 + y^2)^{k_0} z^{2j_0+3}, \tag{2.13}$$

where $A_{j_0, k_0} = b_1 j_0 + a_1(k_0 - 2)$ and $B_{j_0, k_0} = b_2(j_0 - 1) + a_2(k_0 - 1)$. Therefore, the operator $\tilde{\ell}^{(3)}_{2k+1}$ is well-defined.

If we write $J_{2k-1} = J_{2k-1}^a + J_{2k-1}^b$ with J_{2k-1}^b chosen in the previous step and $J_{2k-1}^a \in \text{Ker}(\ell^{(3)}_{2k-1})$, the expression (2.10), for $k \geq 3$, becomes

$$\ell^{(3)}_{2k+1}(J_{2k+1}) + \tilde{\ell}^{(3)}_{2k+1}(J_{2k-1}^a) + \tilde{R}_{2k+1},$$

where $\tilde{R}_{2k+1} = F_3(J_{2k-1}^b) - J_{2k-1}^b \text{div}(\mathbf{F}_3) + R_{2k+1}$.

We now write $\tilde{R}_{2k+1} = \tilde{R}_{2k+1}^{(r)} + \tilde{R}_{2k+1}^{(c)}$ with $\tilde{R}_{2k+1}^{(r)} \in \text{Range}(\ell^{(3)}_{2k+1})$ and $\tilde{R}_{2k+1}^{(c)} \in \text{Cor}(\ell^{(3)}_{2k+1})$. We now choose J_{2k+1} such that $\ell^{(3)}_{2k+1}(J_{2k+1}) = -\tilde{R}_{2k+1}^{(r)}$ and choose J_{2k-1}^a in order to annihilate the part of $\tilde{R}_{2k+1}^{(c)}$ belonging to the range of the operator $\tilde{\ell}^{(3)}_{2k+1}$.

Last on, we note that the solution of the equation (2.1) is $J_{2k+1} + J_{2k+1}^b$, for any $J_{2k+1}^b \in \text{Ker}(\tilde{\ell}^{(3)}_{2k+2})$ real and J_{2k+1} is unique module $\text{Ker}(\tilde{\ell}^{(3)}_{2k+1})$.

We finish the proof, obtaining the expression of $\text{Ker}(\tilde{\ell}^{(3)}_{2k+1})$ and $\text{Cor}(\tilde{\ell}^{(3)}_{2k+1})$ for each case:

Consider system (1.2) with $a_1 = a_2 = 0$ and $b_1 b_2 \neq 0$. It has that $A_{j_0, k_0} = b_1 j_0, B_{j_0, k_0} = b_2(j_0 - 1)$. So, $\text{Ker}(\tilde{\ell}^{(3)}_{2k+1}) = \text{Span}\{(x^2 + y^2)^{k-2} z(b_1(x^2 + y^2) + b_2 z^2)\}$ and we can choose $\text{Cor}(\tilde{\ell}^{(3)}_{2k+1}) = \text{Span}\{(x^2 + y^2)^k z, z^{2k+1}\}$.

Consider system (1.2) with $b_1 = b_2 = 0$ and $a_1 a_2 \neq 0$. It has that $A_{j_0, k_0} = a_1(k_0 - 2), B_{j_0, k_0} = a_2(k_0 - 1)$. So, $\text{Ker}(\tilde{\ell}^{(3)}_{2k+1}) = \text{Span}\{(x^2 + y^2) z^{2(k-2)}(a_1(x^2 + y^2) + a_2 z^2)\}$ and we can choose $\text{Cor}(\tilde{\ell}^{(3)}_{2k+1}) = \text{Span}\{(x^2 + y^2)^k z, z^{2k+1}\}$.

For the systems (1.2) with $b_2(b_1 - a_1), a_1(a_2 - b_2)$ and $a_1 b_2 - a_2 b_1$ are rational numbers different from zero with the same sign, it has that $\tilde{\ell}^{(3)}_{2k}((x^2 + y^2)^{k_0} z^{2j_0+1})$ is

given by (2.13) where

$$A_{j_0, k_0} = \frac{1}{(b_2 - a_2)m} ((p + s)j_0 - q(k_0 - 2)),$$

$$B_{j_0, k_0} = \frac{1}{(a_1 - b_1)m} (-p(j_0 - 1) + (q + s)(k_0 - 1)).$$

Note that $(p + s)j_0 - q(k_0 - 2) = Mj_0 - q(k - 3)$ i.e. $\{A_{j_0, k_0}\}$, $j_0 = 0, \dots, k - 1$, is an arithmetic progression whose difference is $M \neq 0$, and $-p(j_0 - 1) + (q + s)(k_0 - 1) = M(k_0 - 1) - p(k - 3)$, i.e. $\{B_{j_0, k_0}\}$, $k_0 = 0, \dots, k - 1$, is an arithmetic progression whose difference is M .

Therefore, the numbers A_{j_0, k_0} with $0 \leq k_0, j_0 \leq k - 1, k_0 + j_0 = k - 1$, are different and the numbers B_{j_0, k_0} also are different. Fixed k , this fact allows us to distinguish the following cases:

If $A_{j_0, k_0} \neq 0$ for all $0 \leq k_0, j_0 \leq k - 1$, with $k_0 + j_0 = k - 1$, then $\text{Ker}(\tilde{\ell}_{2k+1}^{(3)}) = \{0\}$ and we can choose $\text{Cor}(\tilde{\ell}_{2k+1}^{(3)}) = \text{Span}\{z^{2k+1}\}$.

If there exists j_1 with $0 < j_1 < k - 1$ such that $A_{j_1, k_1} = 0$, that is $Mj_1 = q(k - 3)$, and $B_{j_0, k_0} \neq 0$ for all $0 \leq k_0, j_0 \leq k - 1$, it has that $\text{Ker}(\tilde{\ell}_{2k+1}^{(3)}) = \{0\}$ and we can choose $\text{Cor}(\tilde{\ell}_{2k+1}^{(3)}) = \text{Span}\{(x^2 + y^2)^k z\}$.

Otherwise, there exist $(j_1, k_1), (j_2, k_2)$ such that $A_{j_1, k_1} = 0$ and $B_{j_2, k_2} = 0$. Reasoning as in the proof of proposition 2.1, we have that $k - 3$ is a multiple of M .

If we write $k = 3 + \hat{k}M$, it is easy to check that $\text{Ker}(\tilde{\ell}_{2k+1}^{(3)}) = \text{Span}\{J_5 I_{2M}^{\hat{k}}\}$ where $I_{2M} = (x^2 + y^2)^p z^{2q}((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)^s$ and we can choose as a complementary subspace of the range of the operator $\tilde{\ell}_{2k+1}^{(3)}$ to $\text{Cor}(\tilde{\ell}_{2k+1}^{(3)}) = \text{Span}\{(x^2 + y^2)^k z, z^{2k+1}\}$. □

The following result characterizes the analytic integrability of system (1.2). An algorithm for obtaining of necessary conditions of existence of an inverse Jacobi multiplier can be derived following the scheme of the proof of proposition 2.3.

THEOREM 2.4. *Consider system (1.2) with (a_1, a_2, b_1, b_2) satisfying one of the following two conditions:*

- (a) $a_1 = a_2 = 0$ and $b_1 b_2 \neq 0$ (or $b_1 = b_2 = 0$ and $a_1 a_2 \neq 0$).
- (b) $b_2(b_1 - a_1), a_1(a_2 - b_2)$ and $a_1 b_2 - a_2 b_1$ are rational numbers different from zero with the same sign.

Then, system (1.2) has one, and only one, functionally independent analytic first integral if, and only if, the equations (2.7), (2.8) and (2.9), introduced in proposition 2.3, satisfy $\eta_k = 0$ and $\nu_k = 0$ for all k .

Proof. Its proof is similar to the one of theorem 2.2. □

3. Proofs of the main results

In the proof of theorem 1.1, we will use the following result that is a direct consequence of [20, theorem A]. It states that the analysis of the integrability for analytic systems can be reduced to the formal context.

LEMMA 3.1. *Consider the analytic system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ and denote the formal system $\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{F}}(\tilde{\mathbf{x}})$, transformed of \mathbf{F} by the change $\mathbf{x} = \phi(\tilde{\mathbf{x}})$ where ϕ is a formal diffeomorphism. Then, for each \tilde{I} a formal first integral of $\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{F}}(\tilde{\mathbf{x}})$, there exists a formal scalar function \hat{I} with $\hat{I}(0) = 0$, $\hat{I}'(0) = 1$, such that $\hat{I} \circ \tilde{I} \circ \phi^{-1}$ is an analytic first integral of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$.*

Proof of theorem 1.1. As, by hypothesis (a_1, a_2, b_1, b_2) is non-zero, the system (1.1) is not orbitally equivalent to $(-y, x, 0)^T$. From García [14], the system (1.1) is not completely analytically integrable, it has at most one functionally independent analytic first integral.

We prove the sufficient condition. We see that system (1.4), satisfying one of the series of conditions (a) or (b), it has a polynomial first integral. In fact, we distinguish the cases separately:

- (a) System (1.4) with $a_1 = a_2 = 0$, has the first integral $x^2 + y^2$. And system (1.4) with $b_1 = b_2 = 0$, has the first integral z .
- (b) Assume that there is a rational number m such that $p := b_2(b_1 - a_1)m$, $q := a_1(a_2 - b_2)m$ and $s := (a_1b_2 - a_2b_1)m$ are natural numbers. The polynomial $(x^2 + y^2)^p z^{2q} ((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)^s$ is a first integral of system (1.4).

Assume that system (1.1) is orbitally equivalent to system (1.4), i.e. there exists a change of variables ϕ and a reparameterization of the time-variable such that the first one becomes the second one. Let \tilde{I} a polynomial first integral of system (1.4). Undoing the change of variables, system (1.1) is formally integrable and by applying lemma 3.1, there exists a scalar function \hat{I} such that $I = \hat{I} \circ \tilde{I} \circ \phi^{-1}$ with $I(\mathbf{0}) = 0$ is an analytic first integral of the system (1.1).

We see the necessary condition. We assume that the analytic system (1.1) has one functionally independent analytic first integral and (a_1, a_2, b_1, b_2) is non-zero, performing a formal change of variables and a re-parameterization of the time-variable, system (1.1) can be transformed into

$$\begin{aligned} \dot{x} &= -y + xf(x^2 + y^2, z^2), \\ \dot{y} &= x + yf(x^2 + y^2, z^2), \\ \dot{z} &= zg(x^2 + y^2, z^2), \end{aligned} \tag{3.1}$$

where f, g are formal functions with $f(x^2 + y^2, z^2) = a_1(x^2 + y^2) + a_2z^2 + \text{h.o.t.}$ and $g(x^2 + y^2, z^2) = b_1(x^2 + y^2) + b_2z^2 + \text{h.o.t.}$, see [1, 17].

This system is a Poincaré–Dulac normal form, by [22] a formal first integral of system (3.5) is a first integral of its linear part, i.e. $I(x, y, z) = I(x^2 + y^2, z^2)$.

By using cylindrical coordinates, the system is

$$\dot{r} = rf(r^2, z^2), \quad \dot{z} = zg(r^2, z^2), \quad \text{dot}\theta = 1. \tag{3.2}$$

Doing the change $R = r^2$, $Z = z^2$, $\Theta = 2\theta$ and $\tau = 2t$, system (3.2) is transformed into

$$(R', Z', \Theta')^T := \left(\frac{dR}{d\tau}, \frac{dZ}{d\tau}, \frac{d\Theta}{d\tau} \right)^T = (Rf(R, Z), Zg(R, Z), 1)^T \tag{3.3}$$

and if the system (3.3) admits some first integral, then it is of the form $I = I(R, Z)$, i.e. it is invariant under rotations.

Removing the azimuthal component, we obtain the planar system

$$\begin{pmatrix} R' \\ Z' \end{pmatrix} = \begin{pmatrix} R(a_1R + a_2Z) \\ Z(b_1R + b_2Z) \end{pmatrix} + \begin{pmatrix} R\hat{f}(R, Z) \\ Z\hat{g}(R, Z) \end{pmatrix}, \tag{3.4}$$

with \hat{f}, \hat{g} having terms of degree greater than or equals to two in R, Z , that is $R = 0$ and $Z = 0$ invariant curves of the system.

Hence, the analysis of the integrability problem for system (3.3) (or for system (1.2)) is equivalent to the corresponding one for planar system (3.4).

In summary, the system (1.2) with $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$, admits some formal first integral if, and only if, system (3.4) is formally integrable. The integrability problem of these systems is studied in § 4.

From proposition 4.1, if system (3.4) is formally integrable, then one of the following conditions is satisfied:

- (a) $a_1 = a_2 = 0$ (or $b_1 = b_2 = 0$).

We study the first case $a_1 = a_2 = 0$ and $b_1b_2 \neq 0$ (the other case is analogous changing R by Z).

From theorem 4.2, if system (3.4) is formally integrable, then there exist a change of variables and a reparameterization of the time such that system (3.4) is transformed into

$$R' = 0, \quad Z' = Z(b_1R + b_2Z),$$

i.e., the three-dimensional system (3.3) is transformed into a system of the form

$$(R', Z', \Theta')^T := (0, Z(b_1R + b_2Z), 1 + \Psi(R, Z))^T \tag{3.5}$$

where Ψ is a formal function and $\Psi(0, 0) = 0$.

Undoing the change, $(x, y, z, t) = (\sqrt{R}\cos(\Theta/2), \sqrt{R}\sin(\Theta/2), \sqrt{Z}, \tau/2)$, we obtain system (1.4). These changes transform the first integral R of (3.3) into a first integral of system (1.1) which has the expression $I = I(x^2 + y^2, z^2) = x^2 + y^2 + \dots$

- (b) Assume that $p := b_2(b_1 - a_1)m$, $q := a_1(a_2 - b_2)m$ and $s := (a_1b_2 - a_2b_1)m$ are natural numbers and $\text{gcd}(p, q, s) = 1$. By theorem 4.3, if system (3.4) is

formally integrable at the origin then it is orbitally equivalent to

$$R' = R(a_1R + a_2Z), \quad Z' = Z(b_1R + b_2Z),$$

or equivalently, system (1.1) is orbitally equivalent to system (1.4). Moreover, in such a case, it has an analytic first integral of the form $(x^2 + y^2)^p z^{2q}((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)^s + \dots$

□

We give the following result we will use in the proof of theorem 1.3. It states the well-known relationship among inverse Jacobi multipliers of formally orbital equivalent vector fields.

LEMMA 3.2. *Let Φ be a diffeomorphism and η a function on $U \subset \mathbf{C}^n$ such that $\det D\Phi$ has no zero on U and $\eta(\mathbf{0}) \neq 0$. If $J \in \mathbb{C}[[x_1, \dots, x_n]]$ is an inverse Jacobi multiplier of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, then $\eta(\mathbf{y})(\det(D\Phi(\mathbf{y}))^{-1}J(\Phi(\mathbf{y})))$ is an inverse Jacobi multiplier of $\dot{\mathbf{y}} = \Phi_*(\eta\mathbf{F})(\mathbf{y}) := D\Phi(\mathbf{y})^{-1}\eta(\mathbf{y})\mathbf{F}(\Phi(\mathbf{y}))$.*

Proof of theorem 1.3. We assume that $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$. From García [14], the system (1.1) is not completely analytically integrable, it has at most one analytic first integral.

We see the necessary condition. We assume that system (1.1) has one, and only one, functionally independent analytic first integral. From theorem 1.1, it is orbitally equivalent to system (1.4) satisfying one of the series of conditions (a) or (b). It is easy to check that system (1.4) has the polynomial inverse Jacobi multiplier $J_5 = (x^2 + y^2)z((b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2)$. From lemma 3.2, system (1.1) has an inverse Jacobi multiplier of the form $J = J_5 + \dots$

We prove the sufficient condition. We assume that the analytic system (1.1) has an inverse Jacobi multiplier of the form $J = J_5 + \dots$ and (a_1, a_2, b_1, b_2) satisfies one of the series of conditions (a) or (b).

Reasoning as in proof of theorem 1.1 and applying lemma 3.2, system (1.1) has an inverse Jacobi multiplier of the form $J = J_5 + \dots$ if system (3.4) has an inverse integrating factor of the form $\hat{J} = \hat{J}_3 + \dots$ with $\hat{J}_3 = RZ((b_1 - a_1)R + (b_2 - a_2)Z)$.

We notice that $\hat{J}_3 = RZ(b_1R + b_2Z) - RZ(a_1R + a_2Z)$. Thus, applying theorem 4.4, we have the result. □

4. Auxiliary results: analytic integrability of the systems (3.4)

The following result provides necessary conditions of formal integrability of system (3.4).

PROPOSITION 4.1. *Assume that system (3.4) with $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$, has a formal first integral in a neighbourhood of the origin. Then, one of the following conditions is satisfied:*

- (a) $a_1 = a_2 = 0$ (or $b_1 = b_2 = 0$).
- (b) $b_2(b_1 - a_1)$, $a_1(a_2 - b_2)$ and $a_1b_2 - a_2b_1$ are rational numbers different from zero with the same sign.

Proof. Let $I = I_M + \text{h.o.t.}$ a formal first integral of system (3.4) and let $\mathbf{F} = \mathbf{F}_2 + \dots$ be the associated vector field to system (3.4). Equation $F(I) = 0$ for degree $M + 1$ is $F_2(I_M) = 0$, i.e. $\mathbf{F}_2(R, Z) = (R(a_1R + a_2Z), Z(b_1R + b_2Z))^T$ is polynomially integrable and I_M is a first integral of \mathbf{F}_2 . So, if \mathbf{F} is formally integrable then \mathbf{F}_2 is polynomially integrable.

We study the polynomial integrability of \mathbf{F}_2 (necessary condition of formal integrability). We distinguish the following cases separately:

- Assume $a_1 = a_2 = 0$ (or $b_1 = b_2 = 0$). In such a case, $I = R$ (or $I = Z$) is a polynomial first integral of \mathbf{F}_2 .
- Assume $a_1 = b_1 \neq 0$. If $a_2 \neq b_2$, the origin of \mathbf{F}_2 is an isolated singular point and $(R, Z)^T \wedge \mathbf{F}_2$ has multiple factors, thus \mathbf{F}_2 is not polynomially integrable, by [6, theorem 3.1].

Otherwise, $a_1 = b_1 \neq 0$ and $a_2 = b_2 \neq 0$, the vector field \mathbf{F}_2 is of the form $(a_1R + a_2Z)(R, Z)^T$ which is not polynomially integrable.

- Assume $a_2 = b_2 \neq 0$. In this case, the result follows by changing R by Z .
- Assume $a_1 \neq b_1$ and $a_2 \neq b_2$. From [3, proposition 1.7], as the factors of $(R, Z)^T \wedge \mathbf{F}_2$ are R, Z and $(b_1 - a_1)R + (b_2 - a_2)Z$, if it exists a polynomial first integral of \mathbf{F}_2 , then it has the expression $I_M = R^p Z^q ((b_1 - a_1)R + (b_2 - a_2)Z)^s$ with p, q, s natural numbers. By imposing $F_2(I_M) = 0$, we have that

$$a_1(p + s) + b_1q = 0, \quad a_2p + b_2(q + s) = 0.$$

Thus, a_1, a_2, b_1 and b_2 are different from zero. The natural exponents of the invariant curves in the expression of the first integral I_M are of the form

$$p = b_2(b_1 - a_1)m, \quad q = a_1(a_2 - b_2)m, \quad s = (a_1b_2 - a_2b_1)m$$

with m a rational number. So, I_M is polynomial if $b_2(b_1 - a_1), a_1(a_2 - b_2)$ and $a_1b_2 - a_2b_1$ are rational numbers with same sign.

□

We now provide necessary and sufficient conditions of formal integrability of the system (3.4) with $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$.

By proposition 4.1, we only study the system (3.4) satisfying the series of conditions (a) or (b). For the formal integrability of the systems 3.4 case (a), we have the following result.

THEOREM 4.2. *System (3.4) with $a_1 = a_2 = 0$ and $b_1b_2 \neq 0$ (or, $b_1 = b_2 = 0$ and $a_1a_2 \neq 0$) is formally integrable if, and only if, it is formally orbitally equivalent to system $(\dot{R}, \dot{Z})^T = (0, Z(b_1R + b_2Z))^T$ (or, $(\dot{R}, \dot{Z})^T = (R(a_1R + a_2Z), 0)^T$), i.e. it is orbital equivalent to its lowest-degree homogeneous component.*

Proof. We consider the system 3.4 with $a_1 = a_2 = 0$ and $b_1b_2 \neq 0$. The other case is analogous, it is enough to change R by Z .

The sufficient condition is trivial because R is a polynomial first integral of system $(\dot{R}, \dot{Z})^T = (0, Z(b_1R + b_2Z))^T$. Undoing the change of variables system (3.4) is formally integrable.

We see now the necessary condition. Assume that system (3.4) with $a_1 = a_2 = 0$ and $b_1, b_2 \neq 0$, is formally integrable. By theorem A.2, system (3.4) is orbitally equivalent to $\tilde{\mathbf{G}} = \mathbf{G}_2 + \dots$ given in (A.3) which is also formally integrable. We assume that there exists $k_0 := \min\{k \in \mathbb{N}, k \geq 2 : \alpha_k^2 + \beta_k^2 \neq 0\}$. We study two cases separately:

- If $\frac{\alpha_k}{b_1} = \frac{\beta_k}{b_2} = \lambda \neq 0$, for all $k \geq k_0$ then $\tilde{\mathbf{G}} = (b_1R + b_2Z)\bar{\mathbf{G}}$ with $\bar{\mathbf{G}} = (\lambda R^{k_0} + \dots)$, $Z(1 + \lambda R^{k_0-1} + \dots)^T$. The origin of $\bar{\mathbf{G}}$ is an isolated saddle node point (the linear part evaluated at origin only has one eigenvalue zero). From [18, 19], $\bar{\mathbf{G}}$ is not analytically integrable. Therefore, $\tilde{\mathbf{G}}$ is not analytically integrable.
- Otherwise, If $\frac{\alpha_k}{b_1} \neq \frac{\beta_k}{b_2}$ for some $k \geq k_0$ then $b_1R + b_2Z = 0$ is an invariant curve of $\tilde{\mathbf{G}}$ which pass by the origin and whose cofactor is $b_2Z + \sum_{k \geq k_0} (\alpha_k R^k + \beta_k R^{k-1}Z)$. Therefore $b_1R + b_2Z$ is a factor of any analytic first integral at the origin I of $\tilde{\mathbf{G}}$. On the other hand, if $I = I_M + \dots$ is a first integral of $\tilde{\mathbf{G}}$ then I_M is a polynomial first integral of \mathbf{G}_2 first homogeneous component of $\tilde{\mathbf{G}}$. This fact is a contradiction since $I_M = R$.

□

For the case (b), we have a similar result.

THEOREM 4.3. *Consider system (3.4) such that $b_2(b_1 - a_1)$, $a_1(a_2 - b_2)$ and $a_1b_2 - a_2b_1$ are rational numbers different from zero with the same sign. This system is formally integrable if, and only if, it is formally orbital equivalent to system $(\dot{R}, \dot{Z})^T = (R(a_1R + a_2Z), Z(b_1R + b_2Z))^T$, (that is it is orbital equivalent to its lowest-degree homogeneous component).*

Proof. Let m be an integer number such that $p = b_2(b_1 - a_1)m$, $q = a_1(a_2 - b_2)m$ and $s = (a_1b_2 - a_2b_1)m$ are natural numbers with $\gcd(p, q, s) = 1$.

Performing the scaled-change $(R, Z, \tau) \rightarrow (-pa_1R, -qb_2Z, pq\tau)$, system (3.4) turns into

$$(R', Z')^T = (R(-qR + (q + s)Z + \text{h.o.t.}), Z((p + s)R - pZ + \text{h.o.t.}))^T. \tag{4.1}$$

The formal integrability of system (4.1) has been studied in [3]. By [3, theorem 2.11], if system (4.1) is formally integrable at the origin then it is linearizable (orbitally equivalent to its leading homogeneous term). □

The following result characterizes the formal integrability of system (3.4) through the existence of a formal inverse integrating factor of this system.

THEOREM 4.4. *Consider system (3.4) with $a_1^2 + b_1^2 \neq 0$ or $a_2^2 + b_2^2 \neq 0$, satisfying one of the following series of conditions:*

- (a) $a_1 = a_2 = 0$ (or $b_1 = b_2 = 0$).

- (b) $b_2(b_1 - a_1)$, $a_1(a_2 - b_2)$ and $a_1b_2 - a_2b_1$ are rational numbers different from zero with the same sign.

Then, system (3.4) is formally integrable if, and only if, it has a formal inverse integrating factor of the form $V = RZ((b_1 - a_1)R + (b_2 - a_2)Z) + \dots$.

Proof. System (3.4) is of the form $(R', Z')^T = (P_2 + \dots, Q_2 + \dots)^T$, a perturbation of a quadratic system, where $P_2 = R(a_1R + a_2Z)$, $Q_2 = Z(b_1R + b_2Z)$.

The polynomial $xQ_2 - yP_2 = RZ[(b_1 - a_1)R + (b_2 - a_2)Z]$ under the series of condition (a) or (b) has only simple factors on $\mathbb{C}[x, y]$. From Algaba *et al.* [4, theorem 6], system (3.4) has a formal inverse integrating factor $(xQ_2 - yP_2) + \text{h.o.t.}$ if, and only if, it is orbitally equivalent to $(P_2, Q_2)^T$. From theorems 4.2 and 4.3, the result follows. □

5. Applications

We consider the family of differential systems

$$\begin{aligned} \dot{x} &= -y + x(-x^2 - y^2 + 3z^2 + A_{11}(x^2 + y^2)z^2 + A_{02}z^4), \\ \dot{y} &= x + y(-x^2 - y^2 + 3z^2 + a_{11}(x^2 + y^2)z^2 + a_{02}z^4), \\ \dot{z} &= z(3x^2 + 3y^2 - z^2 + b_{11}(x^2 + y^2)z^2 + b_{02}z^4), \end{aligned} \tag{5.1}$$

with $A_{11}, A_{02}, a_{11}, a_{02}, b_{11}, b_{02}$ real numbers. The following result gives the systems of the family with one functionally independent analytic first integral.

THEOREM 5.1. *System (5.1) has one, and only one, independent analytic first integral if, and only if, it satisfies at least one of the following conditions:*

- (i) $A_{02} - a_{02} = A_{11} - a_{11} = b_{11} + 3b_{02} = A_{02} + 2b_{02} + a_{11} = 0$,
- (ii) $A_{02} - a_{02} = A_{11} - a_{11} = b_{11} + a_{11} = A_{02} + 5b_{02} = 0$.

Proof. System (5.1) is a perturbation of system (1.2) with $a_1 = -1, a_2 = 3, b_1 = 3, b_2 = -1$. Therefore, it has at most one functionally independent first integral. We prove the necessary condition. For that, we apply Theorem 2.4. System (5.1) satisfies condition (b) of theorem 2.4 since $b_2(b_1 - a_1) = -4, a_1(a_2 - b_2) = -4$ and $a_1b_2 - a_2b_1 = -8$ are rational numbers different from zero with the same sign.

We impose the existence of a formal function $J = J_5 + \dots$ with $J_5 = (x^2 + y^2)z(x^2 + y^2 - z^2)$ introduced in proposition 2.3 satisfying (2.9) with $p = q = 1, s = 2$ that is $M = 4$. Equation (2.9) to degree 19 is

$$\begin{aligned} F(J) - \text{div}(\mathbf{F})J &= \nu_3z^7 + \nu_4z^9 + \nu_5z^{11} \\ &\quad + \nu_6z^{13} + \nu_7z^{15} + \eta_7z(x^2 + y^2)^7 + \nu_8z^{17} + \nu_9z^{19}. \end{aligned}$$

Following the procedure given in proof of proposition 2.3, we compute the coefficients of $F(J) - \text{div}(\mathbf{F})J$ and the proof consists on the vanishing of its terms:

The coefficient ν_3 is null and $\nu_4 = A_{02} + 10b_{02} + A_{11} + a_{02} + a_{11} + 2b_{11}$. Solving the equation $\nu_4 = 0$, we have $A_{02} = -10b_{02} - A_{11} - a_{02} - a_{11} - 2b_{11}$. For order 11, we have $\nu_5 = (b_{11} + 3b_{02})(2b_{11} + A_{11} + a_{11})$. Imposing $\nu_5 = 0$, we distinguish

two cases: Case 1. $b_{11} = -3b_{02}$. In this case, $\nu_6 = \nu_7 = 0$ and $\nu_7 = (A_{11} + a_{02} + 2b_{02})(A_{11} + 2a_{02} + a_{11} + 4b_{02})$.

Case 1.1. $A_{11} = -a_{02} - 2b_{02}$. To order 17, we have $\nu_8 = (a_{02} + a_{11} + 2b_{02})^2(a_{02} - a_{11} + 4b_{02})$. If $a_{02} = -a_{11} - 2b_{02}$, we arrive to case (i). Otherwise, $a_{02} = a_{11} - 4b_{02}$, we have $\nu_9 = (a_{11} - b_{02})^2$. Therefore $a_{11} = b_{02}$, i.e. case (i).

Case 1.2. $A_{11} = -2a_{02} - a_{11} - 4b_{02}$, we have $\nu_8 = (a_{02} + a_{11} + 2b_{02})^2(a_{02} + 3b_{02})$. If $a_{02} = -a_{11} - 2b_{02}$, we arrive to case (i). Otherwise, $a_{02} = -3b_{02}$, we have $\nu_9 = (a_{11} - b_{02})^2$. Imposing $\nu_9 = 0$, we have $a_{11} = b_{02}$, i.e. case (i).

Case 2. $b_{11} = -\frac{1}{2}(A_{11} + a_{11})$. In this case, $\nu_6 = \nu_7 = 0$ and $\nu_7 = (a_{02} + 5b_{02})(2a_{02} + A_{11} - a_{11} + 10b_{02})$. Imposing the vanishing of ν_7 , we have:

Case 2.1. $a_{02} = -5b_{02}$. To order 17, $\nu_8 = (A_{11} - a_{11})^2(A_{11} + a_{11} + 6b_{02})$. If $A_{11} = a_{11}$, we are in the case (ii). Otherwise, $A_{11} = -a_{11} - 6b_{02}$, we have $\nu_9 = (5b_{02}^2 + 96)(3b_{02} + a_{11})^2$. Thus, $\nu_9 = 0$ if $a_{11} = -3b_{02}$, i.e. case (ii).

Case 2.2. $a_{02} = \frac{1}{2}(-A_{11} + a_{11} - 10b_{02})$. To order 17, we have $\nu_8 = (A_{11} - a_{11})^2(3A_{11} + 3a_{11} - 32b_{02})$. For $A_{11} = a_{11}$, we have the case (ii). Otherwise, $A_{11} = -a_{11} + \frac{32}{3}b_{02}$, we have $\nu_9 = (1152 + 115b_{02}^2)(16b_{02} - 3a_{11})^2$. So, $a_{11} = \frac{16}{3}b_{02}$, i.e. case (ii).

We prove the sufficient condition. It is easy to check that $(x^2 + y^2)z(x^2 + y^2 - z^2)(1 - b_{02}z^2)$ is an inverse Jacobi multiplier of system (5.1) case (i) and $(x^2 + y^2)z(3x^2 + 3y^2 - 3z^2 - a_{11}(x^2 + y^2)z^2 + 3b_{02}z^4)$ is an inverse Jacobi multiplier of system (5.1) case (ii). Thus, both are of the form $J_5 + \dots$. The result follows applying theorem 1.3. □

We consider the family of differential systems

$$\begin{aligned} \dot{x} &= -y + A_{20}(x^2 + y^2)^2x + A_{11}(x^2 + y^2)z^2y, \\ \dot{y} &= x + a_{11}(x^2 + y^2)z^2y + a_{02}z^5, \\ \dot{z} &= z(x^2 + y^2 - z^2 + b_{11}(x^2 + y^2)zx + b_{02}z^4), \end{aligned} \tag{5.2}$$

with $A_{20}, A_{11}, a_{11}, a_{02}, b_{11}, b_{02}$ real numbers. The following result gives the systems of the family with one functionally independent analytic first integral.

THEOREM 5.2. *System (5.2) has one, and only one, independent analytic first integral if, and only if, $A_{20} = A_{11} = a_{11} = a_{02} = 0$.*

Proof. System (5.2) belongs to the family of systems (1.2) with $a_1 = a_2 = 0$ and $b_1 = 1, b_2 = -1$. Therefore, it has at most one functionally independent first integral.

We prove the necessary condition. For that, we apply theorem 2.2. We impose the existence of a formal function $I = (x^2 + y^2) + \dots$ introduced in proposition 2.1 satisfying (2.1). Following the procedure given in proof of proposition 2.1, we compute the coefficients of $F(I)$ and we obtain $\eta = 0, \nu_2 = 0$ and to degree 6, it has that $\eta_3 = A_{20}$ and $\nu_3 = a_{11}$. Thus, we impose that $A_{20} = a_{11} = 0$. In this case, the equation $F(I) = 0$ to order 8 is zero and to order 10 and 12, we have $\eta_5 = 0, \nu_5 = 5a_{02}b_{11}$ and $\eta_6 = 0, \nu_6 = -10a_{02}b_{02}b_{11} - 10a_{02}^2 - \frac{1}{4}A_{11}^2$.

On the one hand, if $a_{02} = 0$, it has that ν_6 is zero if A_{11} is zero. On the other hand, if $b_{11} = 0$, then $\nu_6 = -10a_{02}^2 - \frac{1}{4}A_{11}^2$. So, if $\nu_6 = 0$, we have that $A_{20} = A_{11} = a_{11} = a_{02} = 0$.

The sufficiency is trivial since $x^2 + y^2$ is a first integral of system (5.2) with $A_{20} = A_{11} = a_{11} = a_{02} = 0$. \square

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References

- 1 A. Algaba, E. Freire and E. Gamero. Hypernormal form for the Hopf-zero bifurcation. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **8** (1998), 1857–1887.
- 2 A. Algaba, N. Fuentes, E. Gamero and C. García. On the integrability problem for the Hopf-Zero singularity and its relation with the inverse Jacobi multiplier. *Appl. Math. Comp.* **405** (2021), 126421.
- 3 A. Algaba, C. García and M. Reyes. Analytical integrability of perturbations of quadratic systems. *Mediterr. J. Math.* **18** (2021), 8. doi: 10.1007/s00009-020-01647-8
- 4 A. Algaba, C. García and M. Reyes. Quasi-homogeneous linearization of degenerate vector fields. *J. Math. Anal. Appl.* **483** (2020), 123635.
- 5 A. Algaba, C. García and M. Reyes. Analytical integrability problem for perturbations of cubic Kolmogorov systems. *Chaos, Solitons and Fractals.* **113** (2018), 1–10.
- 6 A. Algaba, C. García and M. Reyes. Integrability of two dimensional quasi-homogeneous polynomial differential systems. *Rocky Mt. J. Math.* **41** (2011), 1–22.
- 7 L. R. Berrone and H. Giacomini. Inverse Jacobi multipliers. *Rend. Circ. Mat. Palermo* **52** (2003), 77–130.
- 8 A. Buică and I. A. García. Inverse Jacobi multipliers and first integrals for nonautonomous differential systems. *Z. Angew. Math. Phys.* **66** (2015), 574–585.
- 9 A. Buică, I. A. García and S. Maza. Multiple Hopf bifurcation in \mathbb{R}^3 and inverse Jacobi multipliers. *J. Differ. Equ.* **256** (2014), 310–325.
- 10 A. Buică, I. A. García and S. Maza. Some remarks on inverse Jacobi multipliers around Hopf singularities. *J. Math. Anal. Appl.* **418** (2014), 1074–1083.
- 11 A. Buică, I. A. García and S. Maza. Existence of inverse Jacobi multipliers around Hopf points in \mathbb{R}^3 : emphasis on the center problem. *J. Differ. Equ.* **252** (2012), 6324–6336.
- 12 G. Chen, D. Wang and J. Yang. Unique orbital normal forms for vector fields of Hopf-zero singularity. *J. Dyn. Differ. Equ.* **17** (2005), 3–20.
- 13 G. Chen, D. Wang and J. Yang. Unique normal forms for Hopf-zero vector fields. *CR Math.* **336** (2003), 345–348.
- 14 I. A. García. Integrable zero-Hopf singularities and three-dimensional centres. *Proc. Roy. Soc. Edinburgh Sect. A* **148** (2018), 327–340.
- 15 M. Gazor and F. Mokhtari. Normal forms of Hopf-zero singularity. *Nonlinearity* **28** (2015), 311–330.
- 16 J. Giné and D. Peralta-Salas. Existence of inverse integrating factors and Lie symmetries for degenerate planar centers. *J. Diff. Equ.* **252** (2012), 344–357.
- 17 J. Guckenheimer and P. J. Holmes. *Nonlinear oscillations, and bifurcations of vector fields, dynamical systems* (Springer-Verlag, 1983).
- 18 W. Li, J. Llibre and X. Zhang. Local first integrals of differential systems and diffeomorphisms. *Z. Angew. Math. Phys.* **54** (2003), 235–255.
- 19 W. Li, J. Llibre and X. Zhang. Planar analytic vector fields with generalized rational first integrals. *Bull. Sci. Math.* **125** (2001), 341–361.

20 J. F. Mattei and R. Moussu. Holonomie et intégrales premières. *Ann. Sci. Ecole Normale Supérieure* **13** (1980), 469–523.

21 S. Weng and X. Zhang. Integrability of vector fields versus inverse Jacobian multipliers and normalizers. *Dis. Cont. Dyn. Sis.* **36** (2016), 6539–6555.

22 S. Yamanaka. Local integrability of Poincaré-Dulac normal forms. *Regul. Chaotic Dyn.* **23** (2018), 933–947.

23 P. Yu and Y. Yuan. The simplest normal form for the singularity of a pure imaginary and a zero eigenvalue. *Dyn. Cont. Dis. Ser. B.* **8** (2000), 219–249.

24 X. Zhang. Inverse Jacobian multipliers and Hopf bifurcation on center manifolds. *J. Differ. Equ.* **256** (2014), 3552–3567.

Appendix A. An orbital normal form for systems (3.4) with $a_1 = a_2 = 0$.

In this section, we give an orbitally equivalent normal form for a general two-dimensional vector field $\mathbf{F} = \mathbf{F}_n + \dots$, where \mathbf{F}_n , the leading term of \mathbf{F} , is a homogeneous polynomial of degree n .

For that, we need to define the following linear operators (in this section, \mathcal{P}_j denotes the vector space of homogeneous polynomials of degree j with two variables):

$$\begin{aligned} \ell_{k-n+1} &: \mathcal{P}_{k-n+1} \longrightarrow \mathcal{P}_k \\ &\mu_{k-n+1} \longrightarrow F_n(\mu_{k-n+1}), \end{aligned} \tag{A.1}$$

(Lie operator of the leading term of \mathbf{F}) and

$$\ell_{n+k}^c : \Delta_{k+1} \longrightarrow \Delta_{n+k} \text{ defined by}$$

$$\ell_{n+k}^c(g) = \text{Proj}_{\Delta_{n+k}} \left(F_n - \frac{1}{n+k} \text{div}(\mathbf{F}_n)(x, y)^T \right) (g), \tag{A.2}$$

where the subspaces Δ_{k+1} and Δ_{n+k} satisfy that $\mathcal{P}_{k+1} = \Delta_{k+1} \oplus h\mathcal{P}_{k-n}$ and $\mathcal{P}_{n+k} = \Delta_{n+k} \oplus h\mathcal{P}_{k-1}$, respectively (such subspaces must be considered as fixed), being $h = \frac{1}{n+1}(x, y)^T \wedge \mathbf{F}_n$ a homogeneous polynomial non-zero of degree $n + 1$.

Next result is [5, theorem A.32]. It provides an expression of an orbital normal form for the vector fields \mathbf{F} with leading term non-conservative.

PROPOSITION A.1. *Let $\mathbf{F} = \mathbf{F}_n + h.o.t.$ with \mathbf{F}_n homogeneous polynomial vector field of degree n . If $\text{Ker}(\ell_{n+k}^c) = \{0\}$ for all $k \in \mathbb{N}$ then \mathbf{F} is orbitally equivalent to*

$$\mathbf{G} = \mathbf{F}_n + \sum_{j \geq n} \mathbf{G}_{j+1}, \text{ with } \mathbf{G}_{j+1} = \begin{pmatrix} -\frac{\partial}{\partial y} g_{j+2} + x\eta_j \\ \frac{\partial}{\partial x} g_{j+2} + y\eta_j \end{pmatrix}$$

where $g_{j+2} \in \text{Cor}(\ell_{j+2}^c)$ (a complementary subspace to $\text{Range}(\ell_{j+2}^c)$) and $\eta_j \in \text{Cor}(\ell_j)$ (a complementary subspace to $\text{Range}(\ell_j)$).

We apply proposition A.1 for obtaining an orbital normal form of system (3.4) with $a_1 = a_2 = 0$ and $b_1, b_2 \neq 0$.

THEOREM A.2. *A formal orbital normal form for the system (3.4) with $a_1 = a_2 = 0$ and $b_1 b_2 \neq 0$ is*

$$\begin{pmatrix} R' \\ Z' \end{pmatrix} = \begin{pmatrix} 0 \\ Z(b_1 R + b_2 Z) \end{pmatrix} + \sum_{k \geq 2} (\alpha_k R^k + \beta_k R^{k-1} Z) \begin{pmatrix} R \\ Z \end{pmatrix}. \tag{A.3}$$

where $\alpha_k, \beta_k \in \mathbb{R}$.

Proof. We first prove that the hypothesis of proposition A.1 are satisfied, i.e. $\text{Ker}(\ell_{k+2}^c) = \{0\}$ for all $k \in \mathbb{N}$.

In this case, $\mathbf{F}_2 = (0, Z(b_1 R + b_2 Z))^T$. Therefore, $h = \frac{1}{3}(b_1 R + b_2 Z)RZ$ and $\text{div}(\mathbf{F}_2) = b_1 R + 2b_2 Z$.

We denote $\mathbf{F}_2^{(k)} = \mathbf{F}_2 - \frac{1}{k+2} \text{div}(\mathbf{F}_2)(R, Z)^T = \frac{1}{k+2} \begin{pmatrix} -R(b_1 R + 2b_2 Z) \\ Z((k+1)b_1 R + kb_2 Z) \end{pmatrix}$. Consider the following bases for departure and arrival spaces of the operator ℓ_{k+2}^c :

$$\Delta_{k+1} = \text{Span}\{R^{k+1}, R^k Z, Z^{k+1}\}, \quad \Delta_{k+2} = \text{Span}\{R^{k+2}, R^{k+1} Z, Z^{k+2}\}.$$

Taking into account that $3R^i Z^j h = b_1 R^{i+2} Z^{j+1} + b_2 R^{i+1} Z^{j+2}$, we have that

$$\begin{aligned} F_2^{(k)}(R^{k+1}) &= -\frac{k+1}{k+2} b_1 R^{k+2} - 2\frac{k+1}{k+2} b_2 R^{k+1} Z, \\ F_2^{(k)}(R^k Z) &= \frac{k+1}{k+2} b_1 R^{k+1} Z - \frac{3k}{k+2} R^{k-1} h, \\ F_2^{(k)}(Z^{k+1}) &= (-1)^{k+1} \frac{(k+1)^2}{k+2} \frac{b_1^{k+1}}{b_2^k} R^{k+1} Z + \frac{k(k+1)}{k+2} b_2 Z^{k+2} + g(R, Z)h \end{aligned}$$

The matrix associated to the operator $\ell_{k+2}^{(c)}$ respect to the basis given is

$$\begin{pmatrix} -\frac{k+1}{k+2} b_1 & 0 & 0 \\ 2\frac{k+1}{k+2} b_2 & \frac{k+1}{k+2} b_1 & A \\ 0 & 0 & \frac{k(k+1)}{k+2} b_2 \end{pmatrix}$$

with $A = (-1)^{k+1} \frac{(k+1)^2}{k+2} \frac{b_1^{k+1}}{b_2^k}$. The determinant of the matrix is not zero, therefore

$\text{Ker}(\ell_{k+2}^{(c)}) = \{0\}$. Moreover, $\text{Cor}(\ell_{k+2}^{(c)}) = \{0\}$ since $\ell_{k+2}^{(c)}$ is full range.

On the other hand, the linear operator $\ell_{k+1}, k \geq 1$, is $\ell_{k+1}(p) = Z(b_1 R + b_2 Z) \frac{\partial}{\partial Z} p$, with $p \in \mathcal{P}_k$. So, the range of ℓ_{k+1} has dimension k and thus the dimension of any complementary subspace of $\text{Range}(\ell_{k+1})$ is 2. To finish the proof of our result, it is enough to check that the subspace $\text{Span}\{R^{k+1}, R^k Z\}$ is a complementary subspace of $\text{Range}(\ell_{k+1})$. □