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Accumulation set of critical points of the multipliers in the quadratic family

TANYA FIRSOVA† and IGORS GORBOVICKIS‡

† *Kansas State University, Manhattan, KS, USA*
(e-mail: tanyaf@math.ksu.edu)
‡ *Jacobs University, Bremen, Germany*
(e-mail: i.gorbovickis@jacobs-university.de)

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Abstract. A parameter $c_0 \in \mathbb{C}$ in the family of quadratic polynomials $f_c(z) = z^2 + c$ is a *critical point of a period n multiplier* if the map f_{c_0} has a periodic orbit of period n, whose multiplier, viewed as a locally analytic function of c, has a vanishing derivative at $c = c_0$. We study the accumulation set $\mathcal{X}$ of the critical points of the multipliers as $n \to \infty$. This study complements the equidistribution result for the critical points of the multipliers that was previously obtained by the authors. In particular, in the current paper, we prove that the accumulation set $\mathcal{X}$ is bounded, connected, and contains the Mandelbrot set as a proper subset. We also provide a necessary and sufficient condition for a parameter outside of the Mandelbrot set to be contained in the accumulation set $\mathcal{X}$ and show that this condition is satisfied for an open set of parameters. Our condition is similar in flavor to one of the conditions that define the Mandelbrot set. As an application, we get that the function that sends c to the Hausdorff dimension of f_c does not have critical points outside of the accumulation set $\mathcal{X}$.

Key words: quadratic family, the Mandelbrot set, multipliers
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1. *Introduction*

Consider the family of quadratic polynomials

$$f_c(z) = z^2 + c, \quad c \in \mathbb{C}.$$

We say that a parameter $c_0 \in \mathbb{C}$ is a *critical point of a period n multiplier* if the map f_{c_0} has a periodic orbit of period n, whose multiplier, viewed as a locally analytic function of c, has a vanishing derivative at $c = c_0$. The study of critical points of the multipliers is motivated by the problem of understanding the geometry of hyperbolic components of the Mandelbrot set.

As it was observed by Sullivan and Douady and Hubbard [4], the argument of quasiconformal surgery implies that the multipliers of periodic orbits, viewed as analytic functions of the parameter c, are Riemann mappings of the corresponding hyperbolic components of the Mandelbrot set. Existence of analytic extensions of the inverse branches of these Riemann mappings to larger domains can be helpful in estimating the geometry of the hyperbolic components as well as the sizes of some limbs of the Mandelbrot set [8–10] (see also [3]). Critical values of the multipliers are the only obstructions to the existence of these analytic extensions.

It is of special interest to obtain uniform bounds on the shapes of hyperbolic components within renormalization cascades. In particular, this motivates the study of the asymptotic behavior of the critical points of period n multipliers as $n \to \infty$. In [5], the current authors approached this question from the statistical point of view and proved that the critical points of the period n multipliers equidistribute on the boundary of the Mandelbrot set as $n \to \infty$.

More specifically, for each $n \in \mathbb{N}$, let X_n be the set of all parameters $c \in \mathbb{C}$ that are critical points of a period n multiplier (counted with multiplicities). Let $\mathbb{M} \subset \mathbb{C}$ denote the Mandelbrot set and let μ_{bif} be its equilibrium measure (or the bifurcation measure of the quadratic family $\{f_c\}$). Let δ_x denote the δ-measure at $x \in \mathbb{C}$. Then the following theorem is obtained.

THEOREM 1.1. [5] *The sequence of probability measures*

$$\frac{1}{\#X_n} \sum_{x \in X_n} \delta_x$$

converges to the equilibrium measure μ_{bif} *in the weak sense of measures on* $\mathbb{C}$ *as* $n \to \infty$.

At the same time, it was shown in [1] that 0 is a critical point of infinitely many multipliers of different periodic orbits, and hence, since $0 \notin \partial\mathbb{M} = \operatorname{supp}(\mu_{\mathrm{bif}})$, this implies that as the period n grows to infinity, the critical points of period n multipliers accumulate on some set $\mathcal{X} \subset \mathbb{C}$ that is strictly greater than the support of the bifurcation measure μ_{bif}.

The purpose of the current paper is to study this accumulation set $\mathcal{X}$ which can formally be defined as

$$\mathcal{X} := \bigcap_{k=1}^{\infty} \left(\overline{\bigcup_{n=k}^{\infty} X_n} \right).$$

We note that the study of the accumulation set $\mathcal{X}$ complements the statistical approach of Theorem 1.1 in the attempt to understand asymptotic behavior of the critical points of the multipliers.

The first result of this paper is the following.

THEOREM A. *The accumulation set* $\mathcal{X}$ *is bounded, connected, and contains the Mandelbrot set* $\mathbb{M}$*. Furthermore, the set* $\mathcal{X} \setminus \mathbb{M}$ *is nonempty and has a nonempty interior.*

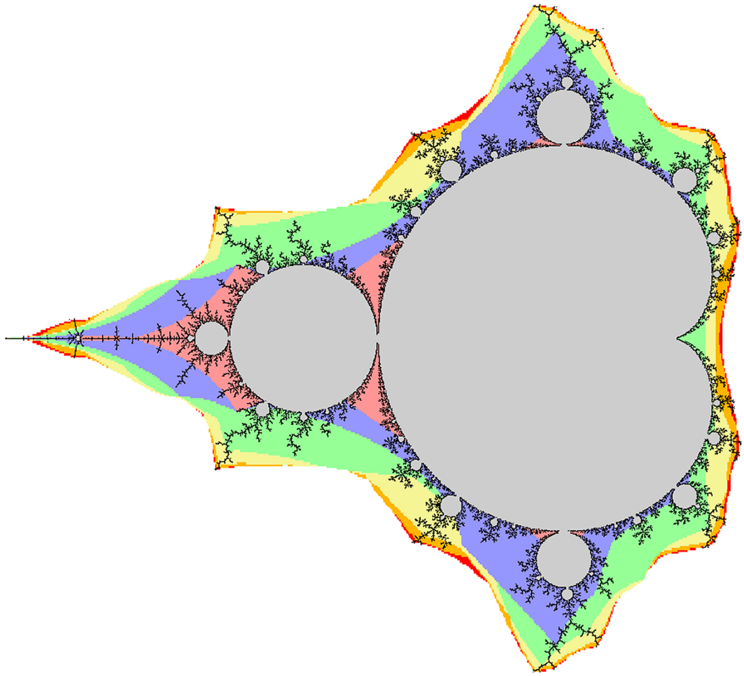

FIGURE 1. The set $\mathcal{X}$ is numerically approximated by the union of the Mandelbrot set and the colored regions. The algorithm for the construction of this picture, as well as the meaning of the colors, are explained in Appendix A.

Figure 1 provides a numerical approximation of the accumulation set $\mathcal{X}$. We also note that the last part of Theorem A complements the following result, previously obtained by the authors in [5].

THEOREM 1.2. [5] *If $c \in \mathbb{C} \setminus \mathbb{M}$ is a critical point of some multiplier, then $c \in \mathcal{X}$. Equivalently, the following identity holds:*

$$\overline{\bigcup_{n=1}^{\infty}(X_n \setminus \mathbb{M})} = \mathcal{X} \setminus \mathbb{M}.$$

We need a few more definitions to state our next result. For a periodic orbit $\mathcal{O}$ of some map f_c, let $|\mathcal{O}|$ stand for its period (that is, the number of distinct points in it).

We recall that a periodic orbit is called *primitive parabolic* if its multiplier is equal to 1. As discussed in [5], for every $c_0 \in \mathbb{C}$ and every periodic orbit $\mathcal{O}$ of f_{c_0} that is not primitive parabolic, the multiplier of this periodic orbit can be viewed as a locally analytic function of the parameter c in the neighborhood of c_0. We denote this function by $\rho_{\mathcal{O}}$. If in addition to that, $\rho_{\mathcal{O}}(c_0) \neq 0$, one can consider a locally analytic function $\nu_{\mathcal{O}}$, defined in a neighborhood of c_0 by the formula

$$\nu_{\mathcal{O}}(c) := \frac{\rho'_{\mathcal{O}}(c)}{|\mathcal{O}|\,\rho_{\mathcal{O}}(c)}. \tag{1}$$

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For each $c \in \mathbb{C}$, let Ω_c denote the set of all repelling periodic orbits of the map f_c. In particular, the locally analytic maps $\nu_{\mathcal{O}}$ are defined for all $\mathcal{O} \in \Omega_c$ in corresponding neighborhoods of the parameter c.

For each $c \in \mathbb{C}$, we consider the set $\mathcal{Y}_c \subset \mathbb{C}$, defined by

$$\mathcal{Y}_c := \overline{\{\nu_{\mathcal{O}}(c) \mid \mathcal{O} \in \Omega_c\}}.$$

Our second result is the following.

THEOREM B. *The following two properties hold.*

(i) *For every parameter $c \in \mathbb{C} \setminus \{-2\}$, the set $\mathcal{Y}_c$ is convex; for $c = -2$, the set $\mathcal{Y}_{-2}$ is the union of a convex set and the point $-\frac{1}{6}$.*

(ii) *For every parameter $c \in \mathbb{C} \setminus \mathbb{M}$, the set $\mathcal{Y}_c$ is bounded. A parameter $c \in \mathbb{C} \setminus \mathbb{M}$ belongs to $\mathcal{X}$ if and only if $0 \in \mathcal{Y}_c$.*

We note that the relation between the sets $\mathcal{Y}_c$ and $\mathcal{X}$, described in part (ii) of Theorem B, resembles the relation between the filled Julia and the Mandelbrot sets, namely that $c \in \mathbb{M}$ if and only if 0 belongs to the filled Julia set K_c of f_c.

As an application of our results and the results of [**6**], we deduce that the Hausdorff dimension function cannot have critical points outside of the accumulation set $\mathcal{X}$. More specifically, let $\delta\colon \mathbb{C} \to \mathbb{R}$ be the function that assigns to each parameter $c \in \mathbb{C}$ the Hausdorff dimension of the Julia set of f_c. It is known that the function δ is real-analytic in each hyperbolic component [**2**] (including the complement of the Mandelbrot set). In [**6**, Theorem 1.3], He and Nie give a necessary condition for a hyperbolic parameter $c \in \mathbb{C}$ to be a critical point of the Hausdorff dimension function δ. Their result is stated for rational maps of degree d, rather than just quadratic polynomials, and the proof is based on ideas from thermodynamical formalism. In the special case of the quadratic family, we can restate their theorem in a concise form as follows.

THEOREM 1.3. [**6**] *If $c \in \mathbb{C}$ is such that f_c is a hyperbolic map and c is a critical point for the map δ, then $0 \in \mathcal{Y}_c$.*

Combining Theorem 1.3 with the main results of this paper, we obtain the following corollary.

COROLLARY 1.4. *The Hausdorff dimension function δ has no critical points in $\mathbb{C} \setminus \mathcal{X}$.*

Proof. According to Theorem A, any parameter $c \in \mathbb{C} \setminus \mathcal{X}$ lies outside of the Mandelbrot set $\mathbb{M}$, and hence the function δ is real-analytic at c. Furthermore, part (ii) of Theorem B implies that $0 \notin \mathcal{Y}_c$, and hence, it follows from Theorem 1.3 that c is not a critical point of the map δ. □

1.1. *Open questions.* Finally, we list some further questions that can be addressed in the study of the geometry of the accumulation set $\mathcal{X}$ and the sets $\mathcal{Y}_c$.

(1) Is the set $\mathcal{X}$ path connected (see Remark 4.9)? Is it simply connected?
(2) Does the boundary of the set $\mathcal{X}$ possess any kind of self-similarity? Is the Hausdorff dimension of $\partial\mathcal{X}$ equal to 1 or is it strictly greater than 1?
(3) For which $c \in \mathbb{C}$ are the sets $\mathcal{Y}_c$ polygonal? How are the points of the finite sets $Y_{c,n} = \{\nu_{\mathcal{O}}(c) \mid \mathcal{O} \in \Omega_c, |\mathcal{O}| = n\}$ distributed inside $\mathcal{Y}_c$ as $n \to \infty$?
(4) What can we say about the geometry of the sets $\mathcal{Y}_c$ when $c \in \partial\mathbb{M}$? Are these sets always unbounded?

2. *On averaging several periodic orbits*

In this section, we state and prove the so called averaging lemma which is the key component of the proofs of Theorems A and B.

LEMMA 2.1. (Averaging lemma) *For any real $\alpha \in [0, 1]$, a complex parameter $c_0 \in \mathbb{C}$ and any two distinct repelling periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of f_{c_0}, such that if $c_0 = -2$, then neither of the orbits $\mathcal{O}_1, \mathcal{O}_2$ is the fixed point $z = 2$, the following holds: there exist a neighborhood U of c_0 and a sequence of distinct repelling periodic orbits $\{\mathcal{O}_j\}_{j=3}^{\infty}$ of f_{c_0}, such that the maps $\nu_{\mathcal{O}_j}$ are defined and analytic in U, for all $j \in \mathbb{N}$, and the sequence of maps $\{\nu_{\mathcal{O}_j}\}_{j=3}^{\infty}$ converges to $\alpha\nu_{\mathcal{O}_1} + (1-\alpha)\nu_{\mathcal{O}_2}$ uniformly in U.*

We need a few preliminary propositions before we can pass to the proof of Lemma 2.1.

For any $c_0 \in \mathbb{C}$ and a periodic orbit $\mathcal{O}$ of f_{c_0} that is non-critical and not primitively parabolic, let $U_{\mathcal{O}} \subset \mathbb{C}$ be a simply connected neighborhood of c_0, such that $\rho_{\mathcal{O}}(c) \neq 0$ for any $c \in U_{\mathcal{O}}$ and let $g_{\mathcal{O}}\colon U_{\mathcal{O}} \to \mathbb{C}$ be the analytic map defined by the relation

$$g_{\mathcal{O}}(c) := (\rho_{\mathcal{O}}(c))^{1/|\mathcal{O}|}, \tag{2}$$

where the branch of the root is chosen so that

$$\arg(g_{\mathcal{O}}(c_0)) \in (-\pi/|\mathcal{O}|, \pi/|\mathcal{O}|].$$

(A particular choice of the branch of the root is not important, but we prefer to make a definite choice.)

For further reference, let us make the following basic observation.

PROPOSITION 2.2. *For any $c_0 \in \mathbb{C}$, a non-critical periodic orbit $\mathcal{O}$ of f_{c_0} and a neighborhood $U_{\mathcal{O}} \subset \mathbb{C}$, satisfying the above conditions, we have*

$$\frac{d}{dc}[\log(g_{\mathcal{O}}(c))] = \nu_{\mathcal{O}}(c),$$

for all $c \in U_{\mathcal{O}}$.

Proof. This follows from a basic computation. □

PROPOSITION 2.3. *Assume $z_0 \in \mathbb{C}$ is a periodic point that belongs to a repelling periodic orbit $\mathcal{O}$ of period n for a map f_{c_0}, where $c_0 \in \mathbb{C}$ is an arbitrary fixed parameter. Let $V \subset \mathbb{C}$ be a simply connected neighborhood of z_0, such that $f_{c_0}^{\circ n}$ is univalent on V and the inclusion $V \Subset f_{c_0}^{\circ n}(V)$ holds. Then there exists a neighborhood $U \subset \mathbb{C}$ of c_0, such that for all $c \in U$, an appropriate branch ϕ_c of the inverse $f_c^{\circ(-n)}$ is defined on V, the inclusion*

$\phi_c(V) \Subset V$ holds, and for any $z \in V$, the analytic functions

$$h_{k,z}(c) := [(f_c^{\circ(nk)})'(\phi_c^{\circ k}(z))]^{1/(nk)}$$

converge to g_O uniformly in $z \in V$ and $c \in U$, for appropriate branches of the roots as $k \to \infty$.

Proof. Since the inverse branch of $f_{c_0}^{\circ(-n)}$, taking V compactly inside itself, is defined on a domain that compactly contains V, it follows that the same holds for $f_c^{\circ(-n)}$, where c is any parameter from a sufficiently small neighborhood U of c_0. For each $c \in U$, let ϕ_c be such an inverse branch of $f_c^{\circ(-n)}$.

According to Denjoy–Wolff theorem applied to the map ϕ_c, we conclude that for any $c \in U$, the map $f_c^{\circ n}$ has a unique repelling fixed point z_c that depends analytically on c and coincides with z_0, when $c = c_0$. This implies that the map g_O is defined for all $c \in U$.

For any $c \in U$ and $z \in V$, consider the sequence of points $z_j = \phi_c^{\circ j}(z)$. Then the chain rule yields

$$h_{k,z}(c) = \bigg(\prod_{j=1}^{k}(f_c^{\circ n})'(z_j)\bigg)^{1/(nk)}.$$

At the same time, we observe that according to the Koebe distortion theorem, after possibly shrinking the neighborhood U, there exists a constant $K > 0$, such that

$$\frac{|(f_c^{\circ n})'(z_c)|}{K} < |(f_c^{\circ n})'(w)| < K|(f_c^{\circ n})'(z_c)| \quad \text{for } c \in U, w \in \phi_c(V).$$

Finally, Denjoy–Wolff theorem implies that for any $c \in U$ and $z \in V$, the sequence of points z_j converges to z_c uniformly in $z \in V$ as $k \to \infty$. Hence, the geometric averages of $(f_c^{\circ n})'(z_1), \ldots, (f_c^{\circ n})'(z_k)$ converge uniformly in $z \in V$, $c \in U$ to $(f_c^{\circ n})'(z_c)$. The latter implies

$$\lim_{k\to\infty} h_{k,z}(c) = ((f_c^{\circ n})'(z_c))^{1/n} = g_O(c),$$

assuming that appropriate branches of the roots are chosen in the definition of $h_{k,z}(c)$. □

PROPOSITION 2.4. *Let $c, z_0 \in \mathbb{C}$ be such that z_0 is a repelling periodic point of f_c. Assume that $(c, z_0) \neq (-2, 2)$. Then there exists a sequence $z_{-1}, z_{-2}, z_{-3}, \ldots \in \mathbb{C}$, such that the following hold simultaneously:*

(i) *the sequence $z_{-1}, z_{-2}, z_{-3}, \ldots$ is dense in the Julia set J_c;*
(ii) *$f(z_{-j}) = z_{1-j}$, for any $j \in \mathbb{N}$;*
(iii) *$z_{-j} \neq 0$, for any $j \in \mathbb{N}$.*

Proof. Existence of a sequence that satisfies properties (i) and (ii) follows immediately from the fact that the set of preimages of any point in the Julia set J_c is dense in J_c. Indeed, from any point z_{-k}, one can land in any arbitrarily small region of J_c by taking an appropriate sequence of preimages of z_{-k}. We can continue this process, making sure that any arbitrarily small region of J_c is eventually visited by our sequence. Furthermore, property (ii) implies that if z_{-k} does not belong to the periodic orbit of z_0, then for every $j \geq k$, the element z_{-j} is different from any other element of the entire sequence $z_0, z_{-1}, z_{-2}, \ldots$, no matter how the sequence of preimages of z_{-k} was chosen.

Property (iii) is equivalent to the property that $z_{-j} \neq c$ for any $j \in \mathbb{N} \cup \{0\}$, since c is the unique point that has only one preimage under the map f_c, and that preimage is 0.

Let $\mathcal{O}$ be the periodic orbit of f_c that contains z_0. First of all, we note that $c \notin \mathcal{O}$. Otherwise, if $c \in \mathcal{O}$, then $0 \in \mathcal{O}$, since 0 is the unique preimage of c, and the orbit $\mathcal{O}$ is super-attracting, which contradicts the assumption of the proposition.

Assume that the sequence, constructed in the first paragraph of the proof, violates property (iii). Let $j \in \mathbb{N}$ be such that $z_{-j} = c$. This number j is unique, since $c \notin \mathcal{O}$, so all further preimages of c must differ from c. If $z_{1-j} \notin \mathcal{O}$, then we can modify z_{-j} by taking it to be equal to another preimage of z_{1-j}. After that, we can construct the remaining 'tail' of the sequence by the same process, as described in the first paragraph. Since $z_{1-j} \notin \mathcal{O}$, no further element of the sequence will ever return to z_{1-j}, and hence, the sequence is guaranteed to avoid the critical value c.

It follows from the construction described in the previous paragraph that the sequence $z_{-1}, z_{-2}, \ldots$ satisfying properties (i)–(iii) can be constructed, if at least one point of the periodic orbit $\mathcal{O}$ has a preimage under f_c that does not belong to $\mathcal{O}$ and is not simultaneously equal to c. This condition is always satisfied, unless z_0 is a fixed point whose two preimages are z_0 and c. The latter happens only when $c = -2$ and $z_0 = 2$. $\square$

Proof of Lemma 2.1. Let n_1 and n_2 be the periods of the periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Let z_1 and z_2 be some periodic points from each of the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$. Since the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are repelling, there exist a simply connected neighborhood U of c_0 and two neighborhoods U_1 and U_2 of z_1 and z_2 respectively, such that for all $c \in U$, the maps $f_c^{\circ n_1}$ and $f_c^{\circ n_2}$ are univalent on U_1 and U_2 respectively, and $f_c^{\circ n_1}(U_1) \setminus U_1$ and $f_c^{\circ n_2}(U_2) \setminus U_2$ are two annuli.

According to Proposition 2.4, there exist $k_1, k_2 \in \mathbb{N}$, $w_1 \in U_2$ and $w_2 \in U_1$, such that

$$f_{c_0}^{\circ k_1}(w_1) = z_1, \quad f_{c_0}^{\circ k_2}(w_2) = z_2,$$
$$(f_{c_0}^{\circ k_1})'(w_1) \neq 0, \quad \text{and} \quad (f_{c_0}^{\circ k_2})'(w_2) \neq 0.$$

Possibly, after shrinking the neighborhood U of c_0, there exist a constant $K > 1$ and the neighborhoods $V_1 \Subset U_2$ and $V_2 \Subset U_1$ of w_1 and w_2 respectively, such that for any $c \in U$ and $j \in \{1, 2\}$, the following hold (see Figure 2).

(a) $f_c^{\circ k_j}$ is univalent on V_j and maps it inside U_j.

(b) The neighborhood $f_c^{\circ k_j}(V_j)$ contains a repelling periodic point of period n_j for the map f_c. (For $c = c_0$, this periodic point is z_j, while for other $c \in U$, it is its perturbation.)

(c) For any $z \in V_j$, we have

$$K^{-1} < |(f_c^{\circ k_j})'(z)| < K. \tag{3}$$

For any $c \in U$, let $\phi_{1,c}$ and $\phi_{2,c}$ be the inverse branches of $f_c^{n_1}$ and $f_c^{n_2}$ respectively, such that $\phi_{j,c}$ takes U_j into itself for $j = 1, 2$. Let $N \in \mathbb{N}$ be a sufficiently large number,

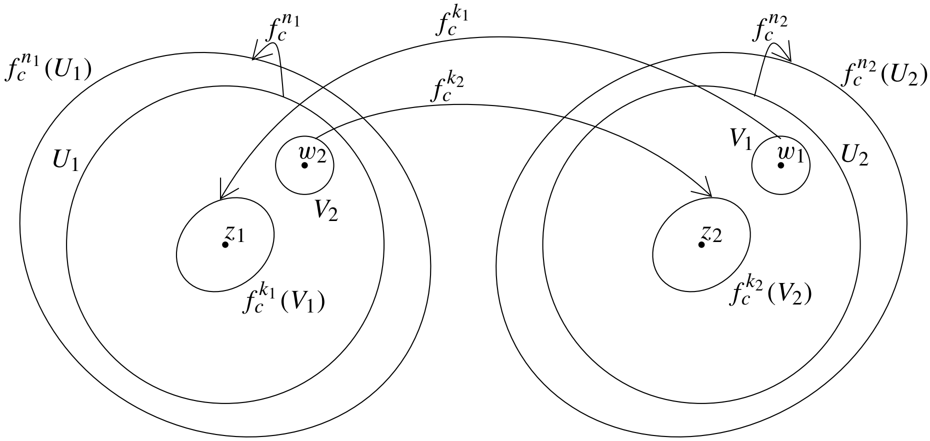

FIGURE 2. Maps and domains from the proof of Lemma 2.1.

such that for any $N_1, N_2 \geq N$ and any $c \in U$, we have

$$\phi_{1,c}^{\circ N_1}(V_2) \Subset f_c^{\circ k_1}(V_1) \quad \text{and} \quad \phi_{2,c}^{\circ N_2}(V_1) \Subset f_c^{\circ k_2}(V_2). \tag{4}$$

The existence of such a number N follows from property (b).

Assume $N_1, N_2 \in \mathbb{N}$ satisfy the condition $N_1, N_2 \geq N$. Then for every $c \in U$, one may consider the following composition of inverse branches of f_c:

$$V_1 \xrightarrow{\phi_{2,c}^{\circ N_2}} f_c^{\circ k_2}(V_2) \xrightarrow{f_c^{\circ(-k_2)}} V_2 \xrightarrow{\phi_{1,c}^{\circ N_1}} f_c^{\circ k_1}(V_1) \xrightarrow{f_c^{\circ(-k_1)}} V_1.$$

Let us denote this composition by $h_c\colon V_1 \to V_1$. By construction, this is a univalent map, and the inclusions (4) imply that $h_c(V_1) \Subset V_1$. Then, according to the Denjoy–Wolff theorem, the map h_c has a unique fixed point z_c in V_1, which is a repelling periodic point of period

$$M = n_1 N_1 + n_2 N_2 + k_1 + k_2$$

for the map f_c. Let $\mathcal{O}_{N_1,N_2}$ denote the periodic orbit of this point when $c = c_0$. Then the map $g_{\mathcal{O}_{N_1,N_2}}$ is defined in U.

Consider the points

$$\begin{aligned} z_c' &= \phi_{2,c}^{\circ N_2}(z_c) \in f_c^{\circ k_2}(V_2), \\ z_c'' &= f_c^{\circ(-k_2)}(z_c') \in V_2, \\ z_c''' &= \phi_{1,c}^{\circ N_1}(z_c'') \in f_c^{\circ k_1}(V_1). \end{aligned}$$

Then, using the chain rule, we get

$$\begin{aligned} g_{\mathcal{O}_{N_1,N_2}}(c) &= [(f_c^{\circ M})'(z_c)]^{1/M} \\ &= [(f_c^{\circ k_1})'(z_c) \cdot (f_c^{\circ(n_1 N_1)})'(z_c''') \cdot (f_c^{\circ k_2})'(z_c'') \cdot (f_c^{\circ(n_2 N_2)})'(z_c')]^{1/M}. \end{aligned}$$

After possibly shrinking the neighborhood U of c_0, we may apply Proposition 2.3 for $V = U_1$ and $V = U_2$. Then, the previous identity can be rewritten in the notation of Proposition 2.3 as

$$g_{O_{N_1,N_2}}(c)=[(f_c^{\circ k_1})'(z_c)]^{1/M}\cdot [h_{N_1,z_c''}(c)]^{n_1N_1/M}\cdot [(f_c^{\circ k_2})'(z_c'')]^{1/M}\cdot [h_{N_2,z_c}(c)]^{n_2N_2/M}. \tag{5}$$

Note that $z_c \in V_1$, $z_c'' \in V_2$. Hence, applying inequality (3) and Proposition 2.3 to the identity (5), we conclude that if $N_1, N_2 \to \infty$ so that

$$\frac{n_1N_1}{n_1N_1+n_2N_2} \to \alpha,$$

then

$$g_{O_{N_1,N_2}}(c) \to s\cdot (g_{O_1}(c))^{\alpha}(g_{O_2}(c))^{1-\alpha}, \tag{6}$$

uniformly in $c \in U$, for appropriate fixed branches of the degree maps $z \mapsto z^{\alpha}$ and $z \mapsto z^{1-\alpha}$, and some constant $s \in \mathbb{C}$, such that $|s| = 1$.

Finally, the proof of Lemma 2.1 can be completed by taking logarithmic derivatives of both sides in equation (6) and applying Proposition 2.2. $\square$

3. *The sets $\mathcal{Y}_c$*

We start this section by giving a proof of Theorem B. We note that our proof of part (ii) of Theorem B, providing the necessary and sufficient condition for $c \in \mathbb{C}\setminus\mathbb{M}$ to be contained in $\mathcal{X}$, seriously depends on the assumption that $c \notin \mathbb{M}$. Furthermore, the condition itself seems to be wrong for some $c \in \partial\mathbb{M}$, (cf. Remark 3.2). Indeed, the case $c \in \mathbb{M}$ appears to be more delicate. In the second part of this section, we provide a sufficient condition for $c \in \mathbb{M}$ to be contained in $\mathcal{X}$. Later, in §4.3, we show that this condition is satisfied for any $c \in \mathbb{M}$.

3.1. *Proof of Theorem B.*

To prove property (ii) of Theorem B, we need the following lemma.

LEMMA 3.1. *For any $c \in \mathbb{C}\setminus\partial\mathbb{M}$, the family of maps $\{\nu_O \mid O \in \Omega_c\}$ is defined and is normal on any simply connected neighborhood $U \subset \mathbb{C}$, such that $c \in U$ and $U \cap \partial\mathbb{M} = \varnothing$. Furthermore, if $c \in \mathbb{C}\setminus\mathbb{M}$, then the identical zero is not a limiting map of the normal family $\{\nu_O \mid O \in \Omega_c\}$.*

Proof. Fix $c \in \mathbb{C}\setminus\partial\mathbb{M}$ and a neighborhood U as in the statement of the lemma. Since $U \cap \partial\mathbb{M} = \varnothing$, all repelling periodic orbits of f_c remain to be repelling after analytic continuation in $c \in U$. This implies that all maps from the family

$$\mathcal{G}_c := \{g_O \mid O \in \Omega_c\}$$

are defined in the neighborhood U and are analytic in it. (We recall that the maps g_O were defined in equation (2) and are appropriate branches of the roots of the multipliers.) Furthermore, for any $\tilde{c} \in U$ and $O \in \Omega_c$, we have $|g_O(\tilde{c})| \le 2\max_{z\in J_{\tilde{c}}}|z| = \mathrm{diam}(J_{\tilde{c}})$, where $J_{\tilde{c}}$ is the Julia set of $f_{\tilde{c}}$. Thus, the family $\mathcal{G}_c$ is locally uniformly bounded and

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hence normal in U. Since all maps from the family $\mathcal{G}_c$ are uniformly bounded away from zero, Proposition 2.2 implies normality of the family $\{\nu_O \mid O \in \Omega_c\}$.

If $c \in \mathbb{C} \setminus \mathbb{M}$, then without loss of generality we may assume that the domain U is simply connected and unbounded. Since for all $\tilde{c} \in \mathbb{C}$ sufficiently close to ∞, the Julia set $J_{\tilde{c}}$ is contained in the annulus centered at zero with inner and outer radii being equal to $\sqrt{|\tilde{c}|} \pm 1$, it follows that for every $\tilde{c} \in U$ sufficiently close to ∞ and for any $O \in \Omega_c$, we have

$$2\sqrt{|\tilde{c}|} - 2 < |g_O(\tilde{c})| < 2\sqrt{|\tilde{c}|} + 2, \tag{7}$$

which implies that none of the limiting maps of the family $\mathcal{G}_c$ is a constant map. Then it follows that the identical zero is not a limiting map of the normal family $\{\nu_O \mid O \in \Omega_c\}$. □

Proof of Theorem B. First, we observe that property (i) of Theorem B is an immediate corollary from the averaging lemma (Lemma 2.1). Indeed, if $c \neq -2$, then convexity of $\mathcal{Y}_c$ is obvious from Lemma 2.1. However, if $c = -2$, then according to the same lemma, the set $\mathcal{Y}_{-2}$ is the union of a convex set and a single point $\nu_{\{2\}}(-2)$, corresponding to the periodic orbit $O = \{2\}$. A direct computation shows that

$$\rho_{\{2\}}(-2) = 4, \quad \rho'_{\{2\}}(-2) = -2/3,$$

and hence $\nu_{\{2\}}(-2) = -1/6$.

We proceed with the proof of part (ii) as follows: for $c \in \mathbb{C} \setminus \mathbb{M}$, let U be a neighborhood of c that satisfies the conditions of Lemma 3.1. First, we observe that according to Lemma 3.1, the family $\{\nu_O \mid O \in \Omega_c\}$, defined on U, is locally uniformly bounded, and hence the set $\mathcal{Y}_c$ is bounded.

Necessary condition for $c \in \mathcal{X}$: If $c \in \mathcal{X}$, then there exists a sequence of points $\{c_k\}_{k=1}^{\infty}$ and a sequence of periodic orbits $\{O_k\}_{k=1}^{\infty} \subset \Omega_c$, such that

$$\lim_{k\to\infty} c_k = c \quad \text{and} \quad \rho'_{O_k}(c_k) = 0 \quad \text{for any } k \in \mathbb{N}.$$

According to Lemma 3.1, after extracting a subsequence, we may assume that the sequence of maps ν_{O_k} converges to some holomorphic map $\nu\colon U \to \mathbb{C}$ uniformly on compact subsets of U. Since for any $k \in \mathbb{N}$ we have $\nu_{O_k}(c_k) = 0$, it follows by continuity that $\nu(c) = 0$. Finally, convergence of the maps ν_{O_k} to ν implies that

$$\lim_{k\to\infty} \nu_{O_k}(c) = \nu(c) = 0,$$

and hence $0 \in \mathcal{Y}_c$.

Sufficient condition for $c \in \mathcal{X}$: However, if $0 \in \mathcal{Y}_c$, then either there exists a periodic orbit $O \in \Omega_c$, such that $\nu_O(c) = 0$, or there exists a sequence of periodic orbits $\{O_k\}_{k=1}^{\infty} \subset \Omega_c$, such that

$$\lim_{k\to\infty} \nu_{O_k}(c) = 0.$$

In the first case, $\rho'_O(c) = 0$, so $c \in \mathcal{X}$ according to Theorem 1.2.

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In the second case, according to Lemma 3.1, after extracting a subsequence, we may assume that the sequence of maps ν_{O_k} converges to some holomorphic map $\nu\colon U \to \mathbb{C}$ uniformly on compact subsets of U. By continuity, we have $\nu(c) = 0$, and, according to Lemma 3.1, $\nu \not\equiv 0$. Then it follows from Rouché's theorem that for any sufficiently large $k \in \mathbb{N}$, there exists $c_k \in U$, such that $\nu_{O_k}(c_k) = 0$ and $\lim_{k\to\infty} c_k = c$. The latter implies that $c \in \mathcal{X}$, and completes the proof of Theorem B. □

Remark 3.2. The above proof of part (ii) of Theorem B fails without the assumption $c \notin \mathbb{M}$. Indeed, if $c \in \partial\mathbb{M}$, then the neighborhood U from Lemma 3.1 does not exist. Furthermore, even though $\partial\mathbb{M} \subset \mathcal{X}$ (since $\partial\mathbb{M}$ is the support of the bifurcation measure μ_{bif}) and $-2 \in \partial\mathbb{M}$, the preliminary computations indicate that the set $\mathcal{Y}_{-2}$ seems to be disjoint from 0. In the case $c \in \mathbb{M} \setminus \partial\mathbb{M}$, the above proof of the sufficient condition for $c \in \mathcal{X}$ fails since the limiting map ν might turn out to be the identical zero.

3.2. *A sufficient condition for $c \in \mathbb{M}$ to be contained in $\mathcal{X}$.* In this subsection, we prove the following sufficient condition for $c \in \mathbb{C} \setminus \{-2\}$ to be contained in $\mathcal{X}$.

LEMMA 3.3. *Let $c \in \mathbb{C} \setminus \{-2\}$ be an arbitrary parameter. If there exist finitely many repelling periodic orbits $O_1, O_2, \ldots, O_k \in \Omega_c$, such that 0 is contained in the convex hull of the points $\nu_{O_1}(c), \ldots, \nu_{O_k}(c)$, then $c \in \mathcal{X}$.*

When $c \in \mathbb{C} \setminus \mathbb{M}$, the sufficient condition, given by Lemma 3.3, is an immediate corollary of Theorem B, but we will use Lemma 3.3 for $c \in \mathbb{M} \setminus \partial\mathbb{M}$.

First, to prove Lemma 3.3, we need the following proposition.

PROPOSITION 3.4. *Let $c \in \mathbb{C}$ be an arbitrary parameter and let $O_1, O_2, \ldots, O_k \in \Omega_c$ be a finite collection of repelling periodic orbits. If $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ are such that $\sum_{j=1}^{k} \alpha_j \neq 0$, then the map*

$$\nu := \sum_{j=1}^{k} \alpha_j \nu_{O_j},$$

defined in a neighborhood of the point c, is not a constant map.

Proof. Since for every $j = 1, \ldots, k$, the multipliers ρ_{O_j} are algebraic (multiple-valued) maps, it follows from equation (1) that the map ν has a single-valued meromorphic extension to any simply connected domain $U \subset \mathbb{C}$ that avoids finitely many branching points of the maps ρ_{O_j}. Note that none of the branching points lie on the real ray $(-\infty, -3)$, since $(-\infty, -3) \cap \mathbb{M} = \varnothing$. Furthermore, since for any parameter $\tilde{c} \in (-\infty, -3)$ the corresponding Julia set $J_{\tilde{c}}$ lies on the real line, it follows that all maps ρ_{O_j} take real values when restricted to the ray $(-\infty, -3)$. Choose the domain U so that it is unbounded and $(-\infty, -3) \subset U$. Then for any $j = 1, \ldots, k$, we have the same asymptotic relation

$$\rho_{O_j}(\tilde{c}) \sim \pm(-4\tilde{c})^{|O_j|/2},$$

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as $\tilde{c} \to -\infty$ within the domain U. A direct computation yields that $\nu_{O_j}(\tilde{c}) \sim 1/(2\tilde{c})$, and hence

$$\nu(\tilde{c}) \sim \frac{\sum_{j=1}^{k} \alpha_j}{2\tilde{c}},$$

as $\tilde{c} \to -\infty$ within the domain U. Since $\sum_{j=1}^{k} \alpha_j \neq 0$, the latter implies that ν is not a constant map. □

Proof of Lemma 3.3. Since the convex hull of the points $\nu_{O_1}(c), \ldots, \nu_{O_k}(c)$ contains zero, it follows that there exist real non-negative constants $\alpha_1, \ldots, \alpha_k$, such that $\sum_{j=1}^{k} \alpha_j = 1$ and the analytic map

$$\nu := \sum_{j=1}^{k} \alpha_j \nu_{O_j},$$

defined in some neighborhood of the point c, satisfies $\nu(c) = 0$.

Since $c \neq -2$, it follows from the averaging lemma (Lemma 2.1) that there exists a sequence of periodic orbits $\{O'_m\}_{m=1}^{\infty} \subset \Omega_c$ and a neighborhood $U \subset \mathbb{C}$ of the point c, such that all maps $\nu_{O'_m}$ are defined and analytic in U and

$$\nu_{O'_m} \to \nu \quad \text{as } m \to \infty, \quad \text{uniformly on } U.$$

According to Proposition 3.4, the map ν is not the identical zero map. Now, since $\nu(c) = 0$, it follows from Rouché's theorem that for any sufficiently large $m \in \mathbb{N}$, the map $\nu_{O'_m}$ has a zero at some point $c_m \in U$, and the points c_m can be chosen so that $\lim_{m\to\infty} c_m = c$. The latter implies that $c \in \mathcal{X}$. □

4. *Proof of Theorem A*

In this section, we complete the proof of Theorem A.

4.1. *The set $\mathcal{X}$ is bounded.*

First, we prove the following.

LEMMA 4.1. *The set $\mathcal{X}$ is bounded.*

Proof. For a fixed parameter $c_0 \in \mathbb{C} \setminus \mathbb{M}$, the Julia set J_{c_0} of the map f_{c_0} is a Cantor set, and all periodic orbits of f_{c_0} are repelling. For any periodic orbit O of f_{c_0}, the locally defined map g_O can be extended by analytic continuation to an analytic map of a double cover of the complement of the Mandelbrot set $\mathbb{M}$ (see [5, §4.1] for details). This means that if

$$\phi_{\mathbb{M}} : \mathbb{C} \setminus \mathbb{M} \to \mathbb{C} \setminus \overline{\mathbb{D}}$$

is a fixed conformal diffeomorphism of $\mathbb{C} \setminus \mathbb{M}$ onto $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $\lambda_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ is a fixed point, such that $\phi_{\mathbb{M}}^{-1}(\lambda_0^2) = c_0$, then the map

$$\lambda \mapsto g_O(\phi_{\mathbb{M}}^{-1}(\lambda^2)),$$

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defined for all λ in a neighborhood of λ_0, extends to a global holomorphic map

$$\gamma_O\colon \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus\overline{\mathbb{D}}.$$

Now assume that the statement of Lemma 4.1 does not hold. Then there exists a sequence of parameters $\{\lambda_n\}_{n\in\mathbb{N}}$ and a corresponding sequence of periodic orbits $\{O_n\}_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty}\lambda_n=\infty \quad \text{and} \quad \gamma'_{O_n}(\lambda_n)=0 \quad \text{for every } n\in\mathbb{N}. \tag{8}$$

Since the family of maps $\{\gamma_O\}$ omits more than three points, which is normal by Montel's theorem (cf. [5, Lemma 4.7]), it follows that after extracting a subsequence, we may assume that the sequence of maps γ_{O_n} converges to a holomorphic map $\gamma\colon \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus\overline{\mathbb{D}}$ uniformly on compact subsets. Since for any $\tilde{c}\in\mathbb{C}$ sufficiently close to ∞ and any $O\in\Omega_{c_0}$ inequality (7) holds, we conclude that γ, as well as each γ_{O_n}, are non-constant maps that have a simple pole at infinity. However, equation (8) implies that γ has at least a double pole at infinity, which provides a contradiction. □

Next, we proceed with proving the remaining statements of Theorem A.

4.2. *The set $\mathcal{X}\setminus\mathbb{M}$.* First we study the set $\mathcal{X}\setminus\mathbb{M}$, that is, the portion of the set $\mathcal{X}$ that is contained in the complement of the Mandelbrot set. We note that even though numerical computations from [1], together with Theorem 1.2, suggest that this set is non-empty, a rigorous computer-free proof of this fact has not been provided so far. We fill this gap by proving the following lemma.

LEMMA 4.2. *The set $\mathcal{X}\setminus\mathbb{M}$ has non-empty interior.*

The idea of the proof of Lemma 4.2 is to show that the sufficient condition from Lemma 3.3 is satisfied for all c in a neighborhood of the parabolic parameter $c_0=-3/4$. The rest of the proof is technical. We will need explicit formulas for the maps ν_O, corresponding to periodic orbits O of periods 1, 2, and 3.

PROPOSITION 4.3. *Let $c_0\in\mathbb{C}$ and a corresponding periodic orbit O of f_{c_0} be such that the map $\nu:=\nu_O$ is defined in a neighborhood of the point $c=c_0$. Then the following holds.*

(i) *If $|O|=1$, then*

$$\nu(c)=\frac{2}{4c-1-\sqrt{1-4c}},$$

where the two branches of the root correspond to the two different periodic orbits of period 1.

(ii) *If $|O|=2$, then*

$$\nu(c)=\frac{1}{2c+2}.$$

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(iii) *If* $|\mathcal{O}| = 3$, *then*

$$\nu(c) = \frac{12c^3 + 37c^2 + 32c + 7 - (c^2 + 6c + 7)\sqrt{-4c - 7}}{6(4c + 7)(c^3 + 2c^2 + c + 1)},$$

where the two branches of the root correspond to the two different periodic orbits of period 3.

Proof. When $|\mathcal{O}| = 1$, that is, $\mathcal{O}$ is a fixed point z, solving the equation $f_c(z) = z$ yields

$$\rho_{c_0,\mathcal{O}}(c) = 2z = 1 + \sqrt{1 - 4c}.$$

Then after a direct computation, we get

$$\nu(c) = \frac{\rho'_{c_0,\mathcal{O}}(c)}{\rho_{c_0,\mathcal{O}}(c)} = \frac{2}{4c - 1 - \sqrt{1 - 4c}}.$$

When $|\mathcal{O}| = 2$, there is only one periodic orbit of period 2. Its multiplier is the free term of the polynomial

$$p(z) = \frac{4(f_c^{\circ 2}(z) - z)}{f_c(z) - z} = 4z^2 + 4z + 4(c + 1).$$

Now, a direct computation yields the formula for $\nu(c)$ in part (ii) of the proposition.

Finally, in the case $|\mathcal{O}| = 3$, there are two periodic orbits of period 3 and according to [**11**], the multiplier $\rho = \rho(c)$ of each of these orbits satisfies the equation

$$c^3 + 2c^2 + (1 - \rho/8)c + (1 - \rho/8)^2 = 0.$$

After solving this equation for ρ, we obtain

$$\rho(c) = 8 + 4c - 4c\sqrt{-4c - 7}.$$

Then a direct computation yields the formula for $\nu(c)$ in part (iii) of the proposition. □

Proof of Lemma 4.2. We consider the maps $\nu_{\mathcal{O}}$ in a neighborhood of the point $c = -3/4$ for periodic orbits $\mathcal{O}$ of periods 1, 2, and 3. The parameter $c = -3/4$ is the point at which the hyperbolic component of period 2 touches the main cardioid of the Mandelbrot set. In particular, all considered functions are defined and analytic in a neighborhood U of that point.

For each $c \in U$, let H_c denote the convex hull of the finite set $\{\nu_{\mathcal{O}}(c) \mid |\mathcal{O}| = 1, 2, 3\}$. It follows from Proposition 4.3 that $\nu_{\mathcal{O}}(-3/4)$ is equal to:

- -1 or $-1/3$, when $|\mathcal{O}| = 1$;
- 2, when $|\mathcal{O}| = 2$;
- $-10/183 \pm (49/183)i$, when $|\mathcal{O}| = 3$;

and hence $H_{-3/4}$ contains 0 in its interior. By continuity, it follows that the convex hull H_c contains 0 for all c in some open complex neighborhood V of the point $-3/4$. Since $c = -3/4$ is a parabolic parameter, it follows that $V \setminus \mathbb{M}$ is a nonempty open set. According to Lemma 3.3, we observe that $V \setminus \mathbb{M} \subset \mathcal{X}$, which completes the proof of Lemma 4.2. □

Next, we prove the following lemma.

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LEMMA 4.4. *For any point $c_0 \in \mathcal{X} \setminus \mathbb{M}$, there exists a continuous path $\gamma\colon [0, 1] \to \mathbb{C}$, such that $\gamma(0) = c_0$, $\gamma([0, 1)) \subset \mathcal{X} \setminus \mathbb{M}$, and $\gamma(1) \in \partial\mathbb{M}$.*

Proof. Step 1: First, we prove this lemma under the assumption that $c_0 \in \mathcal{X} \setminus \mathbb{M}$ is such that there exists a periodic orbit $\mathcal{O}$ of the map f_{c_0}, for which $\nu_{\mathcal{O}}(c_0) = 0$. We fix this periodic orbit $\mathcal{O}$ and let $\mathcal{O}_2$ be the unique periodic orbit of period 2 for the map f_{c_0}. Note that according to Proposition 4.3, we have $\nu_{\mathcal{O}_2} \neq 0$, and hence $\mathcal{O} \neq \mathcal{O}_2$. Then for each $t \in \mathbb{R}$, we consider the map

$$\nu_t := (1 - t)\nu_{\mathcal{O}} + t\nu_{\mathcal{O}_2}, \tag{9}$$

defined in a neighborhood of the point c_0.

Observe that for each $t \in \mathbb{R}$, the map ν_t extends to a multiple valued algebraic map. The set of all poles and branching points of ν_t is contained in a finite set $Q \subset \mathbb{C}$, which is the union of all poles and branching points of the two (global) multiple valued algebraic maps $\nu_{\mathcal{O}}$ and $\nu_{\mathcal{O}_2}$. Since these points are some parabolic parameters and centers of appropriate hyperbolic components, it follows that the set Q is contained in $\mathbb{M}$. Moreover, the set Q is independent of t.

According to Proposition 3.4, for any $t \in \mathbb{R}$, the map ν_t is not identically zero, and hence for any $c \in \mathbb{C} \setminus Q$, $t_0 \in \mathbb{R}$ and a branch $\tilde\nu_{t_0}$ of ν_{t_0}, such that $\tilde\nu_{t_0}(c) = 0$, there exists a continuous (not necessarily unique) local curve $t \mapsto c(t)$, defined for t in a neighborhood of t_0 and satisfying $\tilde\nu_t(c(t)) = 0$ and $c(t_0) = c$, where $\tilde\nu_t$ is a branch of ν_t that is a local perturbation of $\tilde\nu_{t_0}$. (If $\tilde\nu'_{t_0}(c) = 0$, then there can be finitely many such local curves.) We observe that any such curve (that is, the image of the map c) that lies outside of $\mathbb{M}$ is contained in $\mathcal{X}$. Indeed, for each t, the branch $\tilde\nu_t$, satisfying $\tilde\nu_t(c(t)) = 0$, can be represented locally as $\tilde\nu_t = (1 - t)\nu_{\tilde{\mathcal{O}}} + t\nu_{\mathcal{O}_2}$, where $\tilde{\mathcal{O}}$ is a periodic orbit of $f_{c(t)}$, obtained as analytic continuation of the orbit $\mathcal{O}$. Hence, it follows from part (i) of Theorem B that $0 \in \mathcal{Y}_{c(t)}$ and then part (ii) of Theorem B implies that $c(t) \in \mathcal{X}$.

We apply the above observation in the case of $c = c_0$ and $t_0 = 0$ and let $\tilde c\colon [0, s) \to \mathbb{C} \setminus \mathbb{M}$ be a maximal continuous curve satisfying the following conditions:

(i) $s \in (0, +\infty]$;

(ii) $\tilde c(0) = c_0$, and for any $t \in [0, s)$ we have $\nu_t(\tilde c(t)) = 0$, where ν_t is viewed as the analytic continuation of the local map of equation (9) along the curve $\tilde c$ from $\tilde c(0)$ to a neighborhood of $\tilde c(t)$.

(Here, 'maximal' means that there is no other curve $\hat c$ in $\mathbb{C} \setminus \mathbb{M}$, satisfying the same two properties, and defined on a longer interval $[0, \hat s) \supsetneq [0, s)$, so that $\hat c(t) = \tilde c(t)$ for any $t \in [0, s)$.)

First, it follows from Proposition 4.3 that the map $\nu_1 = \nu_{\mathcal{O}_2}$ does not vanish on $\mathbb{C}$, and hence $s \leq 1$. We want to show that the limit

$$\lim_{t \to s^-} \tilde c(t)$$

exists and is finite. The above discussion implies that $\tilde c([0, s)) \subset \mathcal{X}$, and hence according to Lemma 4.1, the set $\tilde c([0, s))$ is bounded, thus, $\tilde c(t)$ accumulates on some bounded set A as $t \to s^-$. Let $c_1 \in A$ be an arbitrary limit point of $\tilde c(t)$ as $t \to s^-$. Then since the branches of ν_t converge to the corresponding branches of ν_s uniformly on compact subsets of $\mathbb{C} \setminus Q$, it follows that either $c_1 \in Q$ or $\tilde{\nu}_s(c_1) = 0$ for some branch $\tilde{\nu}_s$ of ν_s. Since both the set Q and the set of zeros of a (global) algebraic map ν_s are finite sets, it follows that A is a discrete set. Finally, since $\tilde{c}$ is a continuous function, the set A must be connected, and hence consists of a single point. This implies that the limit $\lim_{t\to s^-} \tilde{c}(t)$ exists, and the curve $\tilde{c}$ extends continuously to the closed interval $[0, s_0]$.

Now the proof of Step 1 can be completed by observing that $\tilde{c}(s) \in \partial\mathbb{M}$, because otherwise, if $\tilde{c}(s) \in \mathbb{C} \setminus \mathbb{M}$, then $\tilde{c}(s) \notin Q$ and the curve $\tilde{c}$ can be continued beyond the parameter s; hence, $\tilde{c}$ is not maximal. The required curve γ is obtained from the curve $\tilde{c}|_{[0,s]}$ by an appropriate reparameterization.

Step 2: Now we assume that the point $c_0 \in \mathcal{X} \setminus \mathbb{M}$ is not a critical point of the multiplier of any periodic orbit. Our goal is to show that there is a path that lies in $\mathcal{X} \setminus \mathbb{M}$ and joins c_0 with a critical point of the multiplier of some periodic orbit. Together with Step 1, this will complete the proof of the lemma.

Note that since $c_0 \in \mathcal{X} \setminus \mathbb{M}$, there exists a sequence of periodic orbits $\{\mathcal{O}_j\}_{j\in\mathbb{N}}$ of f_{c_0} and a sequence of parameters $\{c_j\}_{j\in\mathbb{N}} \subset \mathbb{C} \setminus \mathbb{M}$, such that $\lim_{j\to\infty} c_j = c_0$, and for each $j \in \mathbb{N}$, we have $\nu_{\mathcal{O}_j}(c_j) = 0$.

Fix an arbitrary simply connected open domain $V \Subset \mathbb{C} \setminus \mathbb{M}$, such that $c_0 \in V$. Since algebraic extensions of the local maps $\nu_{\mathcal{O}_j}$ do not have branching points outside $\mathbb{M}$, it follows that each of the maps $\nu_{\mathcal{O}_j}$ has an analytic extension to the domain V. For the rest of the proof, we will identify the maps $\nu_{\mathcal{O}_j}$ with these analytic extensions. According to Lemma 3.1, the family of maps $\{\nu_{\mathcal{O}_j}\}_{j\in\mathbb{N}}$ is normal in V, so after extracting a subsequence, we may assume that the sequence of maps $\{\nu_{\mathcal{O}_j}\}_{j\in\mathbb{N}}$ converges to some analytic map $\nu\colon V \to \mathbb{C}$ uniformly on compact subsets of V. By continuity, it follows that $\nu(c_0) = 0$. However, Lemma 3.1 implies that the map ν is not an identical zero, and hence c_0 is an isolated zero of the map ν.

Consider a neighborhood $U \Subset V$, such that $c_0 \in U$ and ν does not vanish on the boundary ∂U. Let $r \in \mathbb{R}$ be defined as

$$r := \min_{z\in\partial U} |\nu(z)| > 0. \tag{10}$$

Then there exists $N \in \mathbb{N}$, such that for any $j \geq N$, we have

$$\max_{z\in\overline{U}} |\nu(z) - \nu_j(z)| < r/2. \tag{11}$$

For each $t \in [0, 1]$, consider the map

$$\nu_t := (1-t)\nu + t\nu_{\mathcal{O}_N},$$

which, by construction, is defined and analytic in V. Similarly to the proof of Step 1, there exist $s \in (0, 1]$ and a maximal continuous curve $\tilde{c}\colon [0, s) \to V$, such that $\tilde{c}(0) = c_0$ and $\nu_t(\tilde{c}(t)) = 0$ for any $t \in [0, s)$. Conditions (10) and (11) imply that for each $t \in [0, 1]$, the map ν_t does not vanish at any point of ∂U, and hence $\tilde{c}([0, s)) \subset U$. In particular, the image $\tilde{c}([0, s))$ is bounded, and the same argument as in Step 1 implies that the map $\tilde{c}$ extends continuously to the parameter $t = s$. Since $\nu_t \to \nu_s$ uniformly in U as $t \to s$ and ν_s does not vanish on ∂U, it follows that $\nu_s(\tilde{c}(s)) = 0$ and $\tilde{c}(s) \in U$. Finally, since

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$U \subset \mathbb{C} \setminus \mathbb{M}$, maximality of the curve $\tilde{c}$ implies that $s = 1$, and thus the curve $\tilde{c}$ connects the point $c_0 = \tilde{c}(0)$ with a point $\tilde{c}(1)$, which is a zero of the function ν_{O_N}.

To complete the proof of the lemma, we will show that the curve $\tilde{c}([0, 1])$ lies in $\mathcal{X} \setminus \mathbb{M}$. Observe that for each $t \in [0, 1]$, the sequence of maps $\nu_{t,j} = (1-t)\nu_{O_j} + t\nu_{O_N}$ converges to ν_t uniformly in $\overline{U}$ as $j \to \infty$. Therefore, for all sufficiently large $j \in \mathbb{N}$, there exists a point $c_{t,j} \in U$, such that

$$\nu_{t,j}(c_{t,j}) = 0 \quad \text{and} \quad c_{t,j} \to \tilde{c}(t) \quad \text{as } j \to \infty.$$

The first condition together with both parts of Theorem B implies that $c_{t,j} \in \mathcal{X} \setminus \mathbb{M}$ for all sufficiently large j. Finally, since the set $\mathcal{X}$ is closed, it follows from the second condition that $\tilde{c}(t) \in \mathcal{X} \setminus \mathbb{M}$, which completes the proof of the lemma. □

4.3. *The set $\mathcal{X} \cap \mathbb{M}$.* Here we turn to the study of the portion of the set $\mathcal{X}$ that is contained in the Mandelbrot set. We show that the whole Mandelbrot set is contained in $\mathcal{X}$.

LEMMA 4.5. *The inclusion $\mathbb{M} \subset \mathcal{X}$ holds.*

Before proving Lemma 4.5, we need several additional results.

For any $c \in \mathbb{C}$ and any $k \in \mathbb{N}$, let Ω_c^k be the set of all periodic orbits of period k for the map f_c. (In particular, Ω_c^k may contain a non-repelling orbit, if it exists.)

LEMMA 4.6. *Let $c_0 \in \mathbb{C}$ be an arbitrary parameter that is neither parabolic nor critically periodic. Then for any $k \in \mathbb{N}$, and the corresponding function $F_k(c) := f_c^{\circ(k-1)}(c)$, the following holds:*

$$\frac{F_k'(c_0)}{kF_k(c_0)} = \sum_{m\in\mathbb{N}, m|k} \; \sum_{O\in\Omega_{c_0}^m} \frac{m}{k}\nu_O(c_0), \tag{12}$$

where the summation goes over all $m \in \mathbb{N}$, such that m divides k, and over all periodic orbits $O \in \Omega_{c_0}^m$.

Proof. For every $k \in \mathbb{N}$, it follows from Vieta's formulas that $F_k(c_0)$ is the product of all fixed points of the map $f_{c_0}^{\circ k}$, counted with multiplicities. Since c_0 is a non-parabolic parameter, all of these fixed points have multiplicity one, and hence we have

$$F_k(c_0) = 2^{-2^k} \prod_{m\in\mathbb{N}, m|k} \; \prod_{O\in\Omega_{c_0}^m} \rho_O(c_0). \tag{13}$$

Since the parameter c_0 is not critically periodic, we have $F_k(c_0) \neq 0$, and for any periodic orbit O of f_{c_0}, the map ν_O is defined and analytic in some fixed neighborhood of the point c_0. This implies that both the left hand side and the right hand side of equation (12) are defined. Finally, the identity (12) can be obtained from equation (13) by a direct computation. □

Next, we prove a slightly refined version of the averaging lemma.

PROPOSITION 4.7. *Under the conditions of Lemma 2.1, if the periods of the periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are relatively prime, then the sequence of repelling periodic orbits $\{\mathcal{O}_j\}_{j=3}^{\infty}$ from Lemma 2.1 can be chosen so that $|\mathcal{O}_j| = j$ for any $j \geq 3$.*

Proof. Here we refer to the proof of the averaging lemma (Lemma 2.1). Define $n_1 := |\mathcal{O}_1|$ and $n_2 := |\mathcal{O}_2|$. It was shown that there exist constants $k_1, k_2 \in \mathbb{N}$ (that depend on c_0, $\mathcal{O}_1$, and $\mathcal{O}_2$), such that the sequence of orbits $\{\mathcal{O}_j\}_{j=3}^{\infty}$ can be chosen to satisfy the following:

$$|\mathcal{O}_j| = n_1 N_{1,j} + n_2 N_{2,j} + k_1 + k_2,$$

for some $N_{1,j}, N_{2,j} \in \mathbb{N}$, where

$$N_{1,j}, N_{2,j} \to \infty \quad \text{and} \quad \frac{n_1 N_{1,j}}{n_1 N_{1,j} + n_2 N_{2,j}} \to \alpha \quad \text{as } j \to \infty. \tag{14}$$

To prove the proposition, it is sufficient to show that for every $\alpha \in [0, 1]$, there exist two sequences $\{N_{1,j}\}_{j=3}^{\infty}$, $\{N_{2,j}\}_{j=3}^{\infty}$ of positive integers that satisfy condition (14) and such that

$$j = n_1 N_{1,j} + n_2 N_{2,j} + k_1 + k_2$$

for all sufficiently large $j \in \mathbb{N}$.

It follows from elementary number theory that for every sufficiently large $j \in \mathbb{N}$, the Diophantine equation

$$j = n_1 N_1 + n_2 N_2 + k_1 + k_2 \tag{15}$$

has a solution $(N_1, N_2) = (K_1, K_2) \in \mathbb{N}^2$ in positive integers. Furthermore, the set of all pairs $(N_1, N_2) \in \mathbb{N}^2$, satisfying condition (15), can be described as

$$\mathcal{N}_j = \{(K_1 - sn_2, K_2 + sn_1) \mid s \in \mathbb{Z}, \quad \text{and} \quad K_1 - sn_2, K_2 + sn_1 > 0\},$$

so the set of all fractions

$$\frac{n_1 N_1}{n_1 N_1 + n_2 N_2} = \frac{n_1 N_1}{j - k_1 - k_2},$$

such that $(N_1, N_2) \in \mathcal{N}_j$, will consist of the real number $n_1 N_1/(j - k_1 - k_2)$ and all other rational numbers from $(0, 1)$ that differ from the first number by an integer multiple of $\theta_j = n_1 n_2/(j - k_1 - k_2)$. Now, since $\theta_j \to 0$ as $j \to \infty$, it follows that for every sufficiently large $j \in \mathbb{N}$, one can choose a pair $(N_{1,j}, N_{2,j}) \in \mathcal{N}_j$ so that condition (14) holds. □

Proof of Lemma 4.5. First, observe that Theorem 1.1 and the fact that $\sup(\mu_{\mathrm{bif}}) = \partial\mathbb{M}$ imply the inclusion $\partial\mathbb{M} \subset \mathcal{X}$. Thus, we only need to show that the interior of $\mathbb{M}$ is contained in $\mathcal{X}$. Let $c_0 \in \mathbb{M}$ be a non-critically periodic interior point of the Mandelbrot set. We note that c_0 belongs to either a hyperbolic or a queer component, in case if the latter ones exist. For each $k \in \mathbb{N}$, consider the map $F_k : \mathbb{C} \to \mathbb{C}$ defined by the formula

$$F_k(c) := f_c^{\circ(k-1)}(c).$$

Since $c_0 \in \mathbb{M}$, the sequence $\{F_k(c_0)\}_{k=1}^{\infty}$ is bounded, and hence there exists a subsequence $\{k_m\}_{m\in\mathbb{N}} \subset \mathbb{N}$, such that the limit $\lim_{m\to\infty} F_{k_m}(c_0)$ exists and is equal to some number

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$w \in \mathbb{C}$. We may assume that $w \neq 0$. Otherwise, if $w = 0$, then take the subsequence $\{k_m + 1\}_{m\in\mathbb{N}}$ instead of the subsequence $\{k_m\}_{m\in\mathbb{N}}$. Since c_0 is an interior point of $\mathbb{M}$, the family of maps $\{F_{k_m}\}_{m\in\mathbb{N}}$ is normal, when restricted to some open neighborhood U of c_0, so after further extracting a subsequence, we may assume that the sequence of functions $\{F_{k_m}\}_{m\in\mathbb{N}}$ converges to some holomorphic function $F\colon U \to \mathbb{C}$ on compact subsets of U.

Let us assume that $c_0 \notin \mathcal{X}$. Then, according to Lemma 3.3, there exists a closed half-plane $H \subset \mathbb{C}$ such that $0 \in \partial H$ and for any repelling periodic orbit $\mathcal{O} \in \Omega_{c_0}$, we have $\nu_{\mathcal{O}}(c_0) \in H$. For any $z \in H$, let $\operatorname{dist}(z, \partial H)$ denote the Euclidean distance from z to the boundary line ∂H of H. Then, under the above assumption, the following holds.

PROPOSITION 4.8. *Assume the set $\mathbb{M} \setminus \mathcal{X}$ is nonempty and $c_0 \in \mathbb{M} \setminus \mathcal{X}$. Let the half-plane H and the sequence $\{k_m\}_{m\in\mathbb{N}}$ be the same as above. Then for any $\varepsilon > 0$, there exists $M = M(\varepsilon) \in \mathbb{N}$ such that for any $m \geq M$ and any periodic orbit $\mathcal{O} \in \Omega_{c_0}$ of period $|\mathcal{O}| = k_m$, the inequality*

$$\operatorname{dist}(\nu_{\mathcal{O}}(c_0), \partial H) < \varepsilon$$

holds. □

Proof. According to Lemma 4.6,

$$\lim_{m\to\infty} \sum_{j\in\mathbb{N}, j|k_m} \sum_{\mathcal{O}\in\Omega_{c_0}^j} \frac{j}{k_m}\nu_{\mathcal{O}}(c_0) = \lim_{m\to\infty} \frac{F'_{k_m}(c_0)}{k_m F_{k_m}(c_0)} = \lim_{m\to\infty} \frac{F'(c_0)}{k_m w} = 0,$$

where F is the limiting map of the sequence of maps $\{F_{k_m}\}_{m\in\mathbb{N}}$, and $w = F(c_0) \neq 0$. Note that for all but possibly one non-repelling orbit $\hat{\mathcal{O}}$ of fixed period $\hat{j}$, the terms in the above summation belong to H. As $k_m \to \infty$, the contribution $(\hat{j}/k_m)\nu_{\hat{\mathcal{O}}}$ of this non-repelling orbit in the summation goes to zero. Then it follows that

$$\lim_{m\to\infty} \operatorname{dist}\Bigg(\sum_{\mathcal{O}\in\Omega_{c_0}^{k_m}} \nu_{\mathcal{O}}(c_0), \quad \partial H\Bigg) = 0,$$

which implies Proposition 4.8. □

Finally, we complete the proof of Lemma 4.5 by observing that under the above assumption where $c_0 \notin \mathcal{X}$, according to Lemma 3.3, the half-plane H can be chosen so that for at least one repelling periodic orbit $\mathcal{O}_1 \in \Omega_{c_0}$, the value $\nu_{\mathcal{O}_1}(c_0)$ lies in the interior of H. Let $\mathcal{O}_2 \in \Omega_{c_0}$ be any other repelling periodic orbit whose period is relatively prime to the period of $\mathcal{O}_1$. Then, according to Lemma 2.1 and Proposition 4.7 with the parameter α fixed at $\alpha = 1/2$, it follows that for each sufficiently large $m \in \mathbb{N}$, there exists a periodic orbit $\mathcal{O} \in \Omega_{c_0}$ of period k_m, such that

$$\operatorname{dist}(\nu_{\mathcal{O}}(c_0), \partial H) > \tfrac{1}{3}\operatorname{dist}(\nu_{\mathcal{O}_1}(c_0), \partial H).$$

The latter contradicts Proposition 4.8, and hence the assumption $c_0 \notin \mathcal{X}$ was false. Since c_0 was an arbitrary non-critically periodic parameter from the interior of $\mathbb{M}$, and critically periodic parameters form a nowhere dense subset of $\mathbb{M}$, this completes the proof of Lemma 4.5. □

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Proof of Theorem A. The proof is a combination of several lemmas. We have $\mathbb{M} \subset \mathcal{X}$ due to Lemma 4.5. The set $\mathcal{X}$ is bounded according to Lemma 4.1. From Lemmas 4.4 and 4.5, it follows that the set $\mathcal{X}$ is connected. Finally, Lemma 4.2 implies that the set $\mathcal{X} \setminus \mathbb{M}$ has nonempty interior. □

Remark 4.9. We note that Lemmas 4.4 and 4.5 would imply path connectedness (instead of just connectedness) of the set $\mathcal{X}$ if the MLC conjecture holds. At the same time, it seems that the MLC conjecture is much stronger than the conjecture that the set $\mathcal{X}$ is path connected. For example, the latter conjecture could be established without the MLC, if one can improve Lemma 4.5 by showing that there is a Jordan domain $U \subset \mathbb{C}$ such that $\mathbb{M} \subset \overline{U} \subset \mathcal{X}$. Figure 1 and the discussion in Appendix A suggest that computer-assisted methods can be used in the attempt to construct such a domain U.

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A. *Appendix. Pictures*

Theorems A and B provide efficient algorithms for constructing numerical approximations of the accumulation set $\mathcal{X}$ and the sets $\mathcal{Y}_c$. For example, to approximate numerically the set $\mathcal{X}$, we first observe that according to Theorem A, the inclusion $\mathbb{M} \subset \mathcal{X}$ holds, so one only has to decide for each point $c \in \mathbb{C} \setminus \mathbb{M}$ whether it belongs to $\mathcal{X}$ or not. The latter can be done by means of Theorem B, which provides an easy to check sufficient condition for $c \in \mathcal{X}$. More specifically, for each $c \in \mathbb{C} \setminus \mathbb{M}$, one should compute the points $\nu_{\mathcal{O}}(c)$, where $\mathcal{O}$ runs over different periodic orbits of the map f_c. If at some point 0 falls into the convex hull of the computed points, then $c \in \mathcal{X}$. The periodic orbits of f_c can in turn be computed by Newton's method (see [7] for a precise algorithm).

Figure 1 is obtained by checking all periodic orbits of periods up to and including eight. The color of a point corresponds to the smallest period, up to which the periodic orbits need to be checked to confirm that $c \in \mathcal{X}$. The dark red strip in Figure 1, corresponding to period 8, is quite thin, so we hope that the picture gives a reasonably good approximation of the accumulation set $\mathcal{X}$ in Hausdorff metric; however, we do not know how to estimate the discrepancy. In particular, it is not clear whether the described algorithm can be used to numerically understand the fine structure of the boundary $\partial\mathcal{X}$.

Part (i) of Theorem B also allows to estimate numerically the sets $\mathcal{Y}_c$. Indeed, for any $c \in \mathbb{C} \setminus \{-2\}$, an approximation of $\mathcal{Y}_c$ can be constructed by taking the convex hull of the points $\nu_{\mathcal{O}}(c)$, where $\mathcal{O}$ runs over different periodic orbits of the map f_c. Figure 3 provides several pictures of the sets $\mathcal{Y}_c$, where the parameter c takes different values on the real line. In particular, Figures 3(a) and 3(f) correspond to the centers of the main cardioid and the hyperbolic component of period 2 respectively, and Figures 3(b) and 3(c) correspond to the parameter c lying slightly to the left and respectively slightly to the right of the cusp

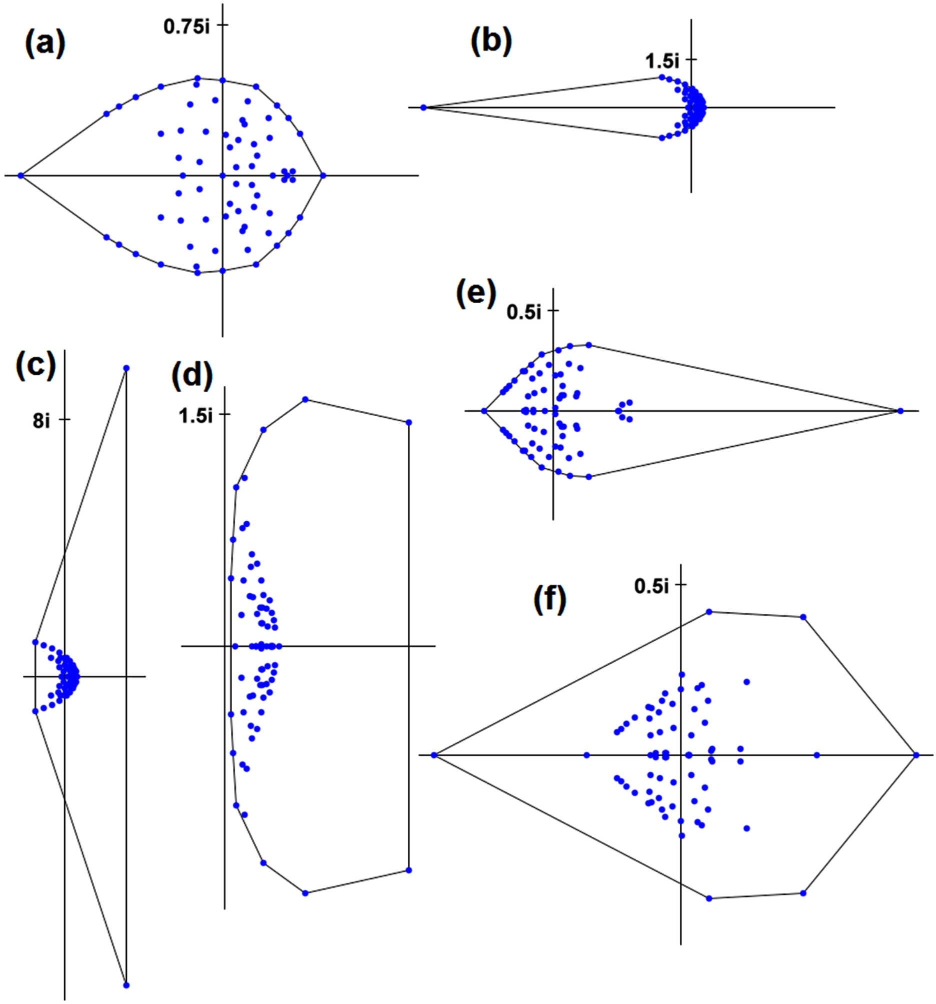

FIGURE 3. Approximations of the sets $\mathcal{Y}_c$ as c changes along the real axis: (a) $c = 0$; (b) $c = 0.24$; (c) $c = 0.26$; (d) $c = 0.42$; (e) $c = -0.71$; (f) $c = -1$.

of the main cardioid. The blue dots are the values of $\nu_{\mathcal{O}}(c)$, for all repelling periodic orbits $\mathcal{O}$ of periods up to and including eight. We do not know how accurate these pictures are, since inclusion of periodic orbits of higher periods can potentially change the convex hulls significantly.

REFERENCES

[1] A. Belova and I. Gorbovickis. Critical points of the multiplier map for the quadratic family. *Exp. Math.* doi:10.1080/10586458.2019.1627682. Published online 21 June 2019.

[2] R. Bowen. Hausdorff dimension of quasicircles. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 11–25.

[3] A. Dezotti. Les langues de Arnold de la famille standard double/Explosion de cycles dans la famille quadratique. *PhD Thesis*, Université Paul Sabatier, Toulouse III, 2011.

[4] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes. Partie II (Publications Mathématiques d'Orsay [Mathematical Publications of Orsay], 85)*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. With the collaboration of P. Lavaurs, T. Lei and P. Sentenac.

[5] T. Firsova and I. Gorbovickis. Equidistribution of critical points of the multipliers in the quadratic family. *Adv. Math.* **380** (2021), 107591.

[6] Y. M. He and H. Nie. A Riemannian metric on polynomial hyperbolic components. *Preprint*, 2020, arXiv:2003.00548.

[7] J. Hubbard, D. Schleicher and S. Sutherland. How to find all roots of complex polynomials by Newton's method. *Invent. Math.* **146**(1) (2001), 1–33.

[8] G. Levin. Multipliers of periodic orbits of quadratic polynomials and the parameter plane. *Israel J. Math.* **170** (2009), 285–315.

[9] G. Levin. Rigidity and non-local connectivity of Julia sets of some quadratic polynomials. *Comm. Math. Phys.* **304**(2) (2011), 295–328.

[10] G. Levin. Addendum to: rigidity and non-local connectivity of Julia sets of some quadratic polynomials. *Comm. Math. Phys.* **325**(3) (2014), 1171–1178.

[11] J. Stephenson. Formulae for cycles in the Mandelbrot set. *Phys. A* **177** (1991), 416–420.