Can. J. Math., Vol. XXVIII, No. 2, 1976, pp. 301–311

A NEW APPROXIMATION OPERATOR GENERALIZING MEYER-KÖNIG AND ZELLER'S POWER SERIES

BENNY LEVIKSON

0. Introduction. In this paper we introduce a new approximation operator of the Arato-Renyi type and study its properties. Special cases of our operator are the power-series, of W. Meyer-König and K. Zeller (see [**9**]) and the generalized Berenstein power series introduced by A. Jakimovski and D. Leviatan in [**5**] and analyzed by them in [**6**].

We prove that our approximation operator converges uniformly to the approximated function provided this function is continuous. Using Liapounov's central limit theorem we analyze the behavior of the operator near discontinuity points. We use these theorems along with some other probablistic arguments to give new results for the generalized Berenstein power series.

The motivation for using probablistic methods came from the interesting paper by M. Arato and A. Renyi [**1**]. Such methods simplify proofs, give insight to them, and thus enable a better understanding of the approximation mechanism.

In the first section we summarize the main results, while their proofs along with some additional lemmas are given in Section 2.

1. The main results. Let X_n $(n \geqq 1)$ be nonnegative independent random variables with means m_n and variances b_n. We define the following approximation operator

$$(1.1)\qquad Q_n(f,x) \equiv \sum_{k=0}^{\infty} f(C_{n,k}) Q_{n,k}(x)$$

where

$$(1.2)\qquad \begin{cases} Q_{n,0}(x) = \Pr\{X_n > x\} \\ Q_{n,k}(x) = \Pr\{X_n + \ldots + X_{n+k-1} \leqq x < X_n + \ldots + X_{n+k}\}, \quad k \geqq 1 \end{cases}$$

$$(1.3)\qquad C_{n,k} = \sum_{j=n}^{n+k-1} m_j, \quad k \geqq 0.$$

The following conditions

$$(1.4)\qquad m_n > 0 \quad (n \geqq 1)$$

$$(1.5)\qquad m_n \to 0 \quad (n \to \infty)$$

$$(1.6)\qquad \textstyle\sum_{n=1}^{\infty} m_n = +\infty$$

$$(1.7)\qquad q_n = b_n/m_n \to 0 \quad (n \to \infty)$$

Received February 12, 1975 and in revised form, January 8, 1976.

are natural ones to impose in order to ensure that $Q_n(f, x)$ is an approximation operator. Indeed condition (1.5) guarantees that on the set

$$\{X_n + \ldots + X_{n+k-1} \leqq x < X_n + \ldots + X_{n+k}\}$$

x will be "close" to $X_n + \ldots + X_{n+k}$. A condition of type (1.7) assures that X_n will be "close" to its mean m_n and condition (1.6) is needed to ensure that $\sum_{k=0}^{\infty} Q_{n,k}(x) = 1$.

Indeed we have:

THEOREM 1. *Let X_n $(n \geqq 1)$ be independent nonnegative random variables with means m_n and variances b_n. Suppose* (1.4), (1.5), (1.6) *and* (1.7) *are satisfied. Then*

(i) *For every continuous function f on* $[0, \infty]$,

$$\lim_{n\to\infty} Q_n(f, x) = f(x) \text{ uniformly in } [0, a], \quad 0 < a < \infty. \tag{1.8}$$

(ia) *For every continuous function f on $[0, \infty]$ and any numbers $(e_{n,k})$ satisfying $C_{n,k} \leqq e_{n,k} \leqq C_{n,k+1}$ we have*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} f(e_{n,k}) Q_{n,k}(x) = f(x)$$

uniformly in $[0, a], \quad 0 < a < +\infty$.

(ii) *For every bounded function f on $[0, \infty]$ and for any continuity point x_0 $(0 \leqq x_0 < \infty)$ of this function one has*

$$\lim_{n\to\infty} Q_n(f, x_0) = f(x_0). \tag{1.8a}$$

The next theorem analyzes the behavior of $Q_n(f, x)$ at a discontinuity point of f.

THEOREM 2. *Let X_n $(n \geqq 1)$ be as in Theorem 1. For $0 < x_0 < \infty$ and n large enough denote*

$$r(n) \equiv r(n, x_0) = \max\{k: C_{n,k} \leqq x_0\}. \tag{1.10}$$

Suppose (1.4), (1.5), (1.6) *and* (1.7) *are satisfied, and for some $\delta > 0$ we have*

$$\left\{\sum_{j=n}^{n+r(n)} E|X_j - m_j|^{2+\delta}\right\}^{1/(2+\delta)} = o\left\{\left(\sum_{j=n}^{n+r(n)} b_j\right)^{1/2}\right\} \quad (n \to \infty) \tag{1.11}$$

and

$$m_{n+r(n)} = o\left\{\left(\sum_{j=n}^{n+r(n)} b_j\right)^{1/2}\right\} \quad (n \to \infty). \tag{1.11a}$$

Then for any bounded function f on $[0, \infty]$,

$$1/2(l^+ + l^-) \leqq \liminf_{n\to\infty} Q_n(f, x_0) \leqq \limsup_{n\to\infty} Q_n(f, x_0) \leqq 1/2(L^+ + L^-), \tag{1.12}$$

https://doi.org/10.4153/CJM-1976-032-6 Published online by Cambridge University Press

where L^+, L^-, l^+, l^- are respectively, the right and left upper limits and the right and left lower limits of f at x_0.

In particular, if x_0 is a discontinuity point of the first kind we conclude that

$$(1.13)\quad \lim_{n\to\infty} Q_n(f, x_0) = 1/2\{f(x_0+) + f(x_0-)\}.$$

Let us now turn to a special interesting case of these operators, namely the generalized Berenstein power series. To get these operators take the random variables X_n $(n \geqq 1)$ to be exponentially distributed with means $a_n^{-1} > 0$ (respectively), namely their densities are given by

$$(1.14)\quad g_{X_n}(t) \equiv g_n(t) = \begin{cases} a_n \exp(-a_n t) & t \geqq 0 \\ 0 & t < 0 \end{cases} \quad (a_n > 0)$$

In this case $Q_n(f, x)$, which is now denoted by $K_n(f, x)$, takes the form

$$(1.15)\quad K_n(f, x) = \sum_{k=0}^{\infty} f(d_{n,k}) \frac{1}{a_{n+k}} g_{n,k}(x)$$

where

$$(1.16)\quad d_{n,k} = \sum_{j=n}^{n+k-1} a_j^{-1}$$

and

$$(1.17)\quad g_{n,k}(x) = \left(\mathop{*}_{j=n}^{n+k} g_j\right)(x)$$

($*$ stands for the convolution operation). Since in this special case of exponentially distributed independent random variables $C_{n,k} = d_{n,k}$ and, by an argument similar to that of [**1**, pp. 96–97], $Q_{n,k}(x) = a_{n+k}^{-1}\, g_{n,k}(x)$. It is easily shown that

$$(1.17\text{a})\quad g_{n,k}(x) = (-1)^k \left(\prod_{j=n}^{n+k} a_j\right) [a_n, \ldots, a_{n+k}]$$

where $[a_n, \ldots, a_{n+k}]$ denotes the divided difference of the function $\exp(-tx)$ at the points $t = a_n, \ldots, t = a_{n+k}$. We have also (see [**7**, (4.5)], or by differentiating (1.17a))

$$(1.17\text{b})\quad d/dx\, g_{n,k}(x) = a_{n+k}(g_{n,k-1}(x) - g_{n,k}(x)).$$

The generalized Berenstein power series $K_n(f, x)$, $n \geqq 1$, were introduced by A. Jakimovski and D. Leviatan in [**5**]. For $a_n = n$, $n \geqq 1$, $K_n(f, x)$ turns into a variant of the operator defined by W. Meyer-König and K. Zeller in [**9**].

The following theorems describe the approximation properties of $K_n(f, x)$ for continuous, discontinuous and continuously differentiable functions respectively.

To see how Theorem 1 is applied to $K_n(f, x)$ note that conditions (1.4), (1.5), (1.6) take now, respectively, the forms

$$(1.18) \quad a_n > 0 \quad (n \geqq 1),$$

$$(1.19) \quad a_n^{-1} \to 0 \quad (n \to \infty),$$

$$(1.20) \quad \sum_{n=1}^{\infty} a_n^{-1} = +\infty,$$

and condition (1.7) is identical with (1.19). Thus Theorem 1 reduces to

THEOREM 3. *Assume* a_n $(n \geqq 1)$ *satisfy* (1.18), (1.19) *and* (1.20). *Then the conclusions of Theorem* 1 *apply to* $K_n(f, x)$.

To see how Theorem 2 is applied to $K_n(f, x)$, note that condition (1.11) takes now the form

$$(1.21) \quad \left\{ \sum_{j=n}^{n+k(n)} a_j^{-(2+\delta)} \right\}^{1/(2+\delta)} = o\left\{ \left(\sum_{j=n}^{n+k(n)} a_j^{-2} \right)^{1/2} \right\} \quad (n \to \infty)$$

where

$$(1.22) \quad k(n) = k(n, x_0) = \max \{j; d_{n,j} \leqq x_0\}.$$

It is easily seen that (1.21) implies

$$(1.21\text{a}) \quad a_{n+k(n)}^{-1} = o\left\{ \left(\sum_{j=n}^{n+k(n)} a_j^{-2} \right)^{1/2} \right\} \quad (n \to \infty)$$

which is the analogue of (1.11a). Using a modified version of Lemma 2.1 in [**2**] it follows that (1.21) is equivalent to

$$(1.23) \quad \left(\max_{n \leqq j \leqq n+k(n)} a_j^{-2} \right) \Big/ \left(\sum_{j=n}^{n+k(n)} a_j^{-2} \right) \to 0 \quad (n \to \infty).$$

Hence we get

THEOREM 4. *Suppose* (1.18), (1.19), (1.20) *and* (1.23) *are in force, then the conclusions of Theorem* 2 *are valid for* $K_n(f, x)$.

It is easily seen (by comparing with the integral $\int t^\alpha$) that the sequences $a_n = n^\alpha, \alpha \leqq 1$, satisfy (1.23) as well as (1.18), (1.19) and (1.20). Thus (1.13) holds for our operator with these sequences and in particular it holds for the operator of W. Meyer-König and K. Zeller. However one should realize that condition (1.23) is not always satisfied. On the other extreme end we can prove:

THEOREM 5. *Assume that* a_n $(n \geqq 1)$ *is an increasing sequence satisfying* (1.18), (1.19), (1.20) *and*

$$(1.23\text{a}) \quad \left(\max_{n \leqq j \leqq n+k(n)} a_j^{-2} \right) \Big/ \left(\sum_{j=n}^{n+k(n)} a_j^{-2} \right) \to 1 \quad (n \to \infty).$$

Then for any bounded function f on $[0, \infty]$

$$(1.24)\quad e^{-1}l^{+} + (1 - e^{-1})l^{-} \leqq \varliminf_{n\to\infty} K_n(f, x_0) \leqq \varlimsup_{n\to\infty} K_n(f, x_0) \leqq e^{-1}L^{+} + (1 - e^{-1})L^{-}$$

where l^{-}, l^{+}, L^{-}, L^{+} *are as in Theorem* 2.

In particular if x_0 is a discontinuity point of the first kind,

$$(1.25)\quad \lim_{n\to\infty} K_n(f, x_0) = e^{-1}f(x_0+) + (1 - e^{-1})f(x_0-).$$

THEOREM 6. *Suppose* (1.18), (1.19) *and* (1.20) *are in force and* $f \in C^1[0, \infty]$ *then*

$$(1.26)\quad \lim_{n\to\infty} \frac{d}{dx} K_n(f, x) = f'(x) \text{ uniformly in } [0, a], \quad 0 < a < \infty.$$

2. Proof of theorems. The following lemma is basic for this paper.

LEMMA 1. *Let* X_n $(n \geqq 1)$ *be nonnegative independent random variables with means* m_n $(n \geqq 1)$ *and variances* b_n $(n \geqq 1)$. *If* b_n *and* m_n *satisfy* (1.4), (1.5), (1.6), (1.7) *then*

$$(2.1)\quad \sum_{k=0}^{\infty} Q_{n,k}(x) = 1 \quad \text{for} \quad 0 \leqq x < \infty.$$

Proof. By the definition of $Q_{n,k}(x)$ we have

$$\sum_{k=0}^{l} Q_{n,k}(x) = \Pr\{X_n + \ldots + X_{n+l} > x\}$$
$$= \Pr\{X_n + \ldots + X_{n+l} - C_{n,l+1} > x - C_{n,l+1}\}$$
$$\geqq \Pr\{|X_n + \ldots + X_{n+l} - C_{n,l+1}| < C_{n,l+1} - x\}$$

for sufficiently large l,

$$\geqq 1 - \frac{b_n + \ldots + b_{n+l}}{(C_{n,l+1} - x)^2} = 1 - \frac{q_n m_n + \ldots + q_{n+l} m_{n+l}}{(x - C_{n,l+1})^2},$$

by Chebishev's inequality, the independence of the X_j's, and the definition of q_n. Now by (1.7), $q_n \leqq K_1$ for each $n \geqq 1$, hence

$$1 - \frac{b_n + \ldots + b_{n+l}}{(x - C_{n,l+1})^2} \geqq 1 - K_1 \frac{C_{n,l+1}}{(x - C_{n,l+1})^2} \to 1 \quad (l \to \infty).$$

I am indebted to D. Leviatan for pointing out that Lemma 1 requires proof.

Proof of Theorem 1. (i) Given $\epsilon > 0$ there exists $\delta(\epsilon)$ such that $|f(x_1) - f(x_2)|$

https://doi.org/10.4153/CJM-1976-032-6 Published online by Cambridge University Press

$< \epsilon$ for $|x_1 - x_2| < \delta$, $x_1, x_2 \in [0, a]$. Thus we have (by Lemma 1)

$$(2.2) \qquad Q_n(f, x) - f(x) = \sum_1 (n) + \sum_2 (n) + \sum_3 (n)$$

where

$$\sum_j (n) = \sum_{k \in I_j} (f(C_{n,k}) - f(x)) Q_{n,k}(x) \quad j = 1, 2, 3$$

$$I_1 = \{k : |C_{n,k} - x| < \delta\}$$

$$I_2 = \{k : C_{n,k} - x \geqq \delta\}$$

$$I_3 = \{k : C_{n,k} - x \leqq -\delta\}$$

Clearly

$$(2.3) \qquad \sum_1 (n) < \epsilon \quad \text{for all } n.$$

Let $k_1 \equiv k_1(x) = \min \{k\colon\ k \in I_2\}$ $k_2 \equiv k_2(x) = \max \{k\colon\ k \in I_3\}$. As f is bounded, say $|f(x)| \leqq M$,

$$\begin{aligned}\sum_3 (n) &\leqq 2M \sum_{k \in I_3} Q_{n,k}(x) = 2M \sum_{k=0}^{k_2} Q_{n,k}(x) \\ &= 2M \Pr \{x < X_n + \ldots + X_{n+k_2}\} \\ &= 2M \Pr \{x - C_{n,k_2+1} < X_n + \ldots + X_{n+k_2} - C_{n,k_2+1}\} \\ &= 2M \Pr \{x - C_{n,k_2} - m_{n+k_2} < X_n + \ldots + X_{n+k_2} - C_{n,k_2+1}\} \\ &\leqq 2M \Pr \{\delta - m_{n+k_2} \leqq X_n + \ldots + X_{n+k_2} - C_{n+k_2+1}\},\end{aligned}$$

where this inequality follows from the very definition of C_{n,k_2} and the monotonicity of probability.

As $m_j \to 0$ $(j \to \infty)$ we can find N_1 such that $\delta/2 > m_n$ for $n > N_1$ and in particular we have $\delta/2 > m_{n+k_2}$ if $n > N_1$.

So

$$\begin{aligned}\sum_3 (n) &\leqq 2M \Pr \{\delta - m_{n+k_2} < X_n + \ldots + X_{n+k_2} - C_{n,k_2+1}\} \\ &\leqq 2M \Pr \{\delta/2 \leqq X_n + \ldots + X_{n+k_2} - C_{n+k_2+1}\} \\ &\leqq 2M \cdot (4/\delta^2) \cdot \sum_{j=n}^{n+k_2} b_j,\end{aligned}$$

by Chebishev's inequality. As $k_2(x)$ is an increasing function of x, once we show that

$$\sum_{j=n}^{n+k_2(a)} b_j < \epsilon_1$$

it will follow that this sum is less than ϵ_1 for every $k_2(x)$ where $x \in [0, a]$.

As $q_j \to 0$ $(j \to \infty)$ we can choose N_2 such that $q_j < \epsilon\delta^2/(4a)$ for $j > N_2$.

Hence for $n > N_2$,

$$\sum_{i=n}^{n+k_2} b_j = \sum_{j=n}^{n+k_2} q_j m_j < (\epsilon\delta^2/4a) \sum_{j=n}^{n+k_2} m_j = C_{n,k_2+1} \cdot (\epsilon\delta^2/4a)$$
$$\leqq a \cdot (\epsilon\delta^2/4a) = \epsilon\delta^2/4.$$

So

(2.4) $\quad \sum_3 (n) \leqq 2M\epsilon \quad$ for $n > \max(N_1, N_2)$, $x \in [0, a]$.

Similarly

(2.5) $\quad \sum_2 (n) < 2M\epsilon \quad$ for $n > N_3$.

Hence part (i) follows from (2.2), (2.3), (2.4) and (2.5). Part (ii) is proven similarly.

To prove conclusion (ia) observe that given $\epsilon > 0$ we have for any $|x - y| < \delta(\epsilon)$, $|f(x) - f(y)| < \epsilon$ and $0 < C_{n,k+1} - C_{n,k} = m_{n+k} \to 0$ as $n \to \infty$ uniformly in $k = 1, 2, \ldots$. Hence for $n > N(\epsilon)$, $|f(e_{n,k}) - f(C_{n,k})| < \epsilon$ and

$$\left| Q_n(f, x) - \sum_{k=0}^{\infty} f(e_{n,k}) Q_{n,k}(x) \right| \leqq \epsilon \sum_{k=0}^{\infty} Q_{n,k}(x) = \epsilon.$$

Conclusion (ia) follows now from conclusion (i) of our theorem.

LEMMA 2. *Let X_n $(n \geqq 1)$ be independent random variables with means m_n and variances b_n. Suppose that for some x_0, $0 < x_0 < \infty$, and some $\delta > 0$, (1.11) and (1.11a) are satisfied. Then*

(2.6) $$\lim_{n\to\infty} \sum_{k=0}^{r(n)} Q_{n,k}(x_0) = \frac{1}{2}, \textit{ and}$$

(2.7) $$\lim_{n\to\infty} Q_{n,r(n)}(x_0) = 0.$$

Proof. Let

$$B(n, k) = \left\{ \sum_{j=n}^{n+k} b_j \right\}^{1/2}.$$

(2.8) $$\lim_{n\to\infty} \sum_{k=0}^{r(n)} Q_{n,k}(x_0) = \lim_{n\to\infty} \Pr\{x_0 < X_n + \ldots + X_{n+r(n)}\}$$
$$= \lim_{n\to\infty} \Pr\left\{\frac{x_0 - C_{n,r(n)+1}}{B(n, r(n))} < \frac{X_n + \ldots + X_{n+r(n)} - C_{n,r(n)+1}}{B(n, r(n))}\right\}.$$

By the definition of $r(n)$ and condition (1.11a) we have

(2.9) $$0 \leqq \frac{|x_0 - C_{n,r(n)+1}|}{B(n, r(n))} \leqq \frac{m_{n+r(n)}}{B(n, r(n))} \to 0 \quad (n \to \infty)$$

As (1.11) is Liapounov's condition for the triangular array

$$X_{n,k} = \frac{X_k - m_k}{B(n, r(n))}, \quad n \leqq k \leqq n + r(n),$$

we conclude that

$$(2.10)\quad \lim_{n\to\infty} \Pr\left\{\frac{X_n + \ldots + X_{n+r(n)} - C_{n,r(n)+1}}{B(n, r(n))} \leqq z\right\} = \Phi(z)$$

uniformly in $-\infty \leqq z \leqq \infty$.

(compare with Theorem B (ii) on p. 275 in [**8**] and Theorem 4.3.3 in [**10**]). (2.6) follows now from (2.8), (2.9) and (2.10).

Let us prove (2.7). As our triangular array satisfies the Liapounov condition, each $X_{n,k}$ is uniformly asymptotically negligible, so $\sum_{k=n}^{n+r(n)} X_{n,k}$ and $\sum_{k=n}^{n+r(n)-1} X_{n,k}$ have the same continuous limiting distribution. Hence (2.7) follows from

$$\lim_{n\to\infty} Q_{n,r(n)}(x_0)$$

$$= \lim_{n\to\infty} \Pr\{X_n + \ldots + X_{n+r(n)-1} \leqq x_0 < X_n + \ldots + X_{n+r(n)}\}$$

$$= \lim_{n\to\infty} (\Pr\{X_n + \ldots + X_{n+r(n)} > x_0\} - \Pr\{X_n + \ldots + X_{n+r(n)-1} > x_0\}).$$

Proof of Theorem 2. As f is bounded it has limits at x_0, thus given $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that

$$l^+ - \epsilon < f(x) < L^+ + \epsilon \quad \text{for } x_0 < x \leqq x_0 + \delta$$

$$l^- - \epsilon < f(x) < L^- + \epsilon \quad \text{for } x_0 - \delta \leqq x < x_0.$$

$$Q_n(f, x_0) = \left\{\sum_{k\in I_1'} + \sum_{k\in I_2'} + \sum_{k\in I_3'} + \sum_{k\in I_4'}\right\} f(C_{n,k})Q_{n,k}(x_0)$$

$$\equiv \textstyle\sum_1'(n) + \sum_2'(n) + \sum_3'(n) + \sum_4'(n)$$

where

$$I_1' \equiv I_1'(x_0, n) \equiv \{k: |C_{n,k} - x_0| > \delta\}$$

$$I_2' \equiv I_2'(x_0, n) \equiv \{k: x_0 < C_{n,k} \leqq x_0 + \delta\}$$

$$I_3' \equiv I_3'(x_0, n) \equiv \{k: x_0 - \delta \leqq C_{n,k} < x_0\}$$

$$I_4' \equiv I_4'(x_0, n) \equiv \{k: x_0 = C_{n,k}\}.$$

The proof of Theorem 1 yields

$$\textstyle\sum_1'(n) \to 0 \quad (n \to \infty).$$

By the very definition of $\sum_2'(n)$ and the convergence of $\sum_1'(n)$ to zero we get

$$\textstyle\sum_2'(n) \leqq (L^+ + \epsilon) \displaystyle\sum_{k\in I_2'} Q_{n,k}(x_0) = (L^+ + \epsilon)\left(\sum_{\substack{k\\ C_{n,k} > x_0}} Q_{n,k}(x_0) + o(1)\right),$$

$$= (L^+ + \epsilon)\left(\sum_{k=r(n)+1}^{\infty} Q_{n,k}(x_0) + o(1)\right) \to \frac{1}{2}(L^+ + \epsilon), \text{ by Lemma 2.}$$

https://doi.org/10.4153/CJM-1976-032-6 Published online by Cambridge University Press

So

$$\limsup_{n\to\infty} \sum\nolimits_2{}' (n) \leqq \frac{1}{2} L^+.$$

Similarly

$$\liminf_{n\to\infty} \sum\nolimits_2{}' (n) \geqq \frac{1}{2} l^+.$$

As f is bounded we conclude from (2.7) that

$$\lim_{n\to\infty} \sum\nolimits_4{}' (n) = 0.$$

Now one can easily verify that

$$\frac{1}{2} l^- \leqq \liminf_{n\to\infty} \sum\nolimits_3{}' (n) \leqq \limsup_{n\to\infty} \sum\nolimits_3{}' (n) \leqq \frac{1}{2} L^-.$$

Combining together all our estimates yields the required result.

The proofs of Theorems 3 and 4 are indicated in the previous section, so we turn now to the proof of Theorem 5. To this end we need

LEMMA 3. *Assume all the conditions of Theorem 5 are in force. Then*

$$\text{(2.13)} \quad \lim_{n\to\infty} \sum_{k=0}^{k(n)} Q_{n,k}(x_0) = 1 - e^{-1}.$$

Proof. Let

$$A(n, k) = \left\{ \sum_{j=n}^{n+k} a_j^{-2} \right\}^{1/2}.$$

Condition (1.23a) implies that the sum

$$\sum_{k=n+1}^{n+k(n)} \frac{x_k - a_k^{-1}}{A(n, k(n))}$$

is asymptotically negligible with respect to $(X_n - a_n^{-1})(A(n, k(n)))^{-1}$, that is, the limit of

$$(X_n + \ldots + X_{n+k(n)} - d_{n,k(n)+1})(A(n, k(n)))^{-1}$$

is the same as that of $(X_n - a_n^{-1})a_n^{-1}$. So we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=0}^{k(n)} Q_{n,k}(x_0) &= \lim_{n\to\infty} \Pr\{x_0 < X_n + \ldots + X_{n+k(n)}\} \\
&= \lim_{n\to\infty} \Pr\{x_0 - d_{n,k(n)+1} < X_n + \ldots + X_{n+k(n)} - d_{n,k(n)+1}\} \\
&= \lim_{n\to\infty} \Pr\{x_0 - d_{n,k(n)+1} < X_n - a_n^{-1}\} \\
&= \lim_{n\to\infty} \{1 - \exp(-a_n(x_0 - d_{n,k(n)+1} + a_n^{-1}))\} \\
&= 1 - e^{-1} \lim_{n\to\infty} \exp(-a_n(x_0 - d_{n,k(n)+1})) \\
&= 1 - e^{-1},
\end{aligned}$$

since, by the very definition of $k(n)$ and (1.23a).

$$0 < -a_n(x_0 - d_{n,k(n)+1}) < a_n a_{n+k(n)}^{-1} \to 0 \quad (n \to \infty).$$

Proof of Theorem 5. Theorem 5 follows almost immediately from Lemma 3 and from a similar technique to the one we employ in the proof of Theorem 2.

Proof of Theorem 6. The proof of Lemma C in [**7**] (but using now (1.17b) instead of [**7**, (2.3)]) establishes that

$$\begin{aligned}\frac{d}{dx}K_n(f, x) &= \sum_{k=0}^{\infty} f(d_{n,k}) \frac{d}{dx} g_{n,k}(x) \quad (\text{for } 0 \leqq x \leqq +\infty) \\ &= \sum_{k=0}^{\infty} f(d_{n,k})(g_{n,k-1}(x) - g_{n,k}(x)) \quad (\text{by } 1.17b) \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} f(c_{n,k})(g_{n,k-1}(x) - g_{n,k}(x)) \\ &= \lim_{N\to\infty} \left\{ \sum_{k=0}^{N} (f(d_{n,k+1}) - f(d_{n,k}))g_{n,k}(x) \right. \\ &\qquad \left. - f(d_{n,N+1})\, g_{n,N}(x) \right\}.\end{aligned}$$

Observing that the function $g_{n,k}(x)$ is the same as the function $t_n\lambda_{n+k,n}$ in [**3**] ($t_n = a_n$, $t_0 = 0$, $\mu_n = e^{-a_n x}$) we get from [**3**, Satz I] that $\lim_{N\to\infty} g_{n,N}(x) = 0$ for $n = 1, 2, \ldots$ and $0 \leqq x < \infty$. Therefore, for $0 \leqq x < +\infty$,

$$\begin{aligned}\frac{d}{dx}K_n(f, x) &= \sum_{k=0}^{\infty} (f(d_{n,k+1}) - f(d_{n,k}))g_{n,k}(x) \\ &= \sum_{k=0}^{\infty} f'(e_{n,k})a_{n+k}^{-1}g_{n,k}(x) \quad (d_{n,k} < e_{n,k} < d_{n,k+1}) \\ &\to f'(x) \quad \text{uniformly in } 0 \leqq x \leqq a, \text{ for each } a > 0,\end{aligned}$$

by the part of Theorem 2 corresponding to conclusion (ia) of Theorem 1.

Acknowledgements. I am grateful to my teacher, Professor A. Jakimovski, for suggesting this domain and for his encouragement and help through part of this research. I would like also to thank the referee for some helpful comments.

References

1. M. Arato and A. Renyi, *Probablistic proof of a theorem on the approximation of continuous functions by means of generalized Bernstein polynomials*, Acta Math. Acad. Sci. Hungar *8* (1957), 91–97.

2. Z. Ditzian and A. Jakimovski, *Inversion and jump formulae for variation diminishing transforms*, Annali di Mathematica pure et applicate *81*, (1969), 261–318.

3. F. Hausdorff, *Summationsmethoden und momentfolgen II*, Math Zeit. *9* (1921), 280–299.

4. A. Jakimovski, *Generalized Bernstein polynomials for discontinuous and convex functions*, J. d'anal. Math. *23* (1970), 171–183.

5. A. Jakimovski and D. Leviatan, *A property of approximation operators and application to Tauberian constants*, Math. Zeit. *102* (1967), 177–204.

6. ——— *Completeness and approximation operators*, Publ. Ramanujan Inst. (Madras) *1* (1969), 123–129.

7. D. Leviatan, *On approximation operators of the Berenstein type*, J. Approximation Theory *1* (1968), 275–278.

8. M. Loeve, *Probability theory* (D. Van-Nostrand Company, Princeton, N.J. 3rd Edition, 1970).

9. W. Meyer-König and K. Zeller, *Berensteinsche Potenzreichen*, Studia Mathematica *19* (1960), 89–94.

10. A. Renyi, *Foundations of probability* (Holden-Day, Inc. San Francisco, 1970).

Purdue University,
West Lafayette, Indiana