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ON THE RANGE OF THE Y-TRANSFORM

VU KIM TUAN

The ranges of the Y-integral transform in some spaces of functions are described.

1. INTRODUCTION

The Y-transform Y_ν is defined by [**8, 6**]

$$f(x) = (Y_\nu g)(x) = \int_0^\infty \sqrt{xy}Y_\nu(xy)g(y)\,dy, \quad x \in R_+ = (0,\infty), \tag{1}$$

if the integral converges in some sense (absolutely, improper, mean convergence), where $Y_\nu(x)$ is the Bessel function of the second kind [**1**]. The Y-transform Y_ν has been considered in $\mathcal{L}_{\mu,p}$ in [**3, 6, 7**]. In particular, it follows that in $L_2(R_+) = \mathcal{L}_{1/2,2}$ the Y-transform Y_ν is bounded if $|\mathcal{R}e\,\nu| < 1$, and if, moreover, $0 < |\mathcal{R}e\,\nu| < 1$, then the range of the Y-transform Y_ν is $L_2(R_+)$:

$$\|Y_\nu g\|_{L_2(R_+)} \leqslant C\,\|g\|_{L_2(R_+)}, \qquad |\mathcal{R}e\,\nu| < 1, \tag{2}$$

$$\|g\|_{L_2(R_+)} \leqslant C\,\|Y_\nu g\|_{L_2(R_+)}, \quad 0 < |\mathcal{R}e\,\nu| < 1, \tag{3}$$

where C is an independent constant, (but different in distinct inequalities). The **H**-transform $\mathbf{H}_\nu$ [**8, 6**] denoted by

$$g(x) = (\mathbf{H}_\nu f)(x) = \int_0^\infty \sqrt{xy}\mathbf{H}_\nu(xy)f(y)\,dy, \quad x \in R_+, \tag{4}$$

is the inverse of Y-transform Y_ν in $L_2(R_+)$ if $-1 < \mathcal{R}e\ \nu < 0$. If $0 < \mathcal{R}e\ \nu < 1$ the inverse formula (4) should be replaced by formula (51) or, equivalently, (52). Here $\mathbf{H}_\nu(x)$ is the Struve function [**1**]. The Y- and **H**-transforms are of importance in many singular axially symmetric potential problems [**6**]. In this work we describe precisely the range of the Y-transform in some spaces of functions. The range of the Y-transform of functions with compact supports (analogous to the Paley-Wiener theorem for the Fourier transform [**5**]) is also considered. It is worth remarking that our Paley-Wiener

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theorem (Theorem 2) is different from the classical ones describing Fourier transform of compactly supported functions in terms of entire functions of exponential type [5]. (For the Hankel transform of compactly supported functions see [4].) The theorem stated here involves the spectral radius [12] of some differential operator obtained from the Bessel differential equation and having the kernel of the Y-transform as "eigenfunctions", (similar ideas have been applied in [2, 11] to the Fourier transform). Nevertheless, its proof is straightforward, without referring to spectral theory. Since the **H**-transform $\mathbf{H}_\nu$ is the inverse of the Y-transform Y_ν in all spaces we considered in this paper, corresponding theorems on the range of the **H**-transform can be easily derived.

2. Y-Transform of Polynomial Decreasing Functions

We describe the range of the Y-transform on the space of functions $g(y)$ square integrable together with $y^n g(y)$, $n = 1, 2, \ldots$ (polynomial decreasing functions):

Theorem 1. *A function $f(x)$ is the Y-transform Y_ν, $0 < |\mathcal{R}e\,\nu| < 1/2$, of a function $g(y)$, square integrable together with $y^n g(y)$, $n = 1, 2, \ldots$, if and only if*

(i) *$f(x)$ is infinitely differentiable on R_+;*

(ii) *$\left(d^2/dx^2 + (1/x^2)\left((1/4) - \nu^2\right)\right)^n f(x)$, $n = 0, 1, \ldots$, belongs to $L_2(R_+)$;*

(iii) *$\left(d^2/dx^2 + (1/x^2)\left((1/4) - \nu^2\right)\right)^n f(x)$, $n = 0, 1, \ldots$, tends to 0 as x tends both to 0 and to infinity;*

(iv) *$x(d/dx)\left(d^2/dx^2 + (1/x^2)\left((1/4) - \nu^2\right)\right)^n f(x)$, $n = 0, 1, \ldots$, is bounded at 0;*

(v) *$(d/dx)\left(d^2/dx^2 + (1/x^2)\left((1/4) - \nu^2\right)\right)^n f(x)$, $n = 0, 1, \ldots$, tends to 0 as x tends to infinity;*

(vi) *The improper integrals*

$$\int_{\to 0}^{\to\infty} x^{\nu - 1/2} \frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)^n f(x)\,dx$$

exist and vanish for all $n = 1, 2, \ldots$, as well as for $n = 0$ if $-1/2 < \mathcal{R}e\,\nu < 0$.

Proof: (a) Let $y^n g(y)$ belong to $L_2(R_+)$ for all $n = 0, 1, 2, \ldots$, then $y^n g(y)$ belongs to $L_1(R_+)$ for all $n = 0, 1, 2, \ldots$. Let $f(x)$ be the Y-transform Y_ν, $0 < |\mathcal{R}e\,\nu| < 1/2$, of $g(y)$ (the Y-transform Y_ν of $g(y)$ with other values of ν also appears in the proof, but it is not denoted by $f(x)$).

(a-i) We have [1]

$$\frac{d^n}{dx^n} Y_\nu(x) = 2^{-n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} Y_{\nu - n + 2j}(x). \tag{5}$$

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Therefore,

$$\frac{\partial^n}{\partial x^n}(\sqrt{xy}Y_\nu(xy)) = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{n+j-k}2^{-k}(-1/2)_{n-k}\binom{n}{k}\binom{k}{j} \tag{6}$$
$$x^{1/2+k-n}y^{1/2+k}Y_{\nu-k+2j}(xy),$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol [1]. The Bessel function of the second kind $Y_\nu(y)$ has the asymptotics [1]

(7)
$$Y_\nu(y) = \begin{cases} \sqrt{\frac{2}{\pi y}}\left[\sin\left(y - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \frac{4\nu^2-1}{8y}\cos\left(y - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] + O\left(y^{-5/2}\right), & y \to \infty \\ O\left(y^{-|\mathcal{R}e\,\nu|}\right), & y \to 0. \end{cases}$$

Consequently, $\frac{\partial^n}{\partial x^n}[\sqrt{xy}Y_\nu(xy)]$, $|\mathcal{R}e\,\nu| < 1$, as a function of y has the asymptotics $O\left(y^{1/2-|\mathcal{R}e\,\nu|}\right)$ in the neighbourhood of 0 and $O(y^n)$ at infinity. Hence, $\frac{\partial^n}{\partial x^n}[\sqrt{xy}Y_\nu(xy)]g(y)$, $|\mathcal{R}e\,\nu| < 1$, as a function of y belongs to $L_1(R_+)$ for all $n = 0, 1, 2, \ldots$, and therefore, $f(x)$ is infinitely differentiable on R_+.

(a-ii) Since $Y_\nu(x)$ satisfies the Bessel differential equation [1]

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0, \tag{8}$$

the function $\sqrt{x}Y_\nu(x)$ is a solution of the equation

$$x^2u'' + \left(x^2 + \frac{1}{4} - \nu^2\right)u = 0. \tag{9}$$

Therefore, we have

$$\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n (\sqrt{xy}Y_\nu(xy)) = \left(-y^2\right)^n\sqrt{xy}Y_\nu(xy). \tag{10}$$

Consequently,

(11)
$$\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n f(x) = (-1)^n\int_0^\infty \sqrt{xy}Y_\nu(xy)y^{2n}g(y)\,dy, \quad |\mathcal{R}e\,\nu| < 1/2.$$

By using inequality (2) for the Y-transform (11) of $y^{2n}g(y) \in L_2(R_+)$, we obtain that $[d^2/dx^2 + (1/x^2)((1/4) - \nu^2)]^n f(x)$, $|\mathcal{R}e\,\nu| < 1/2$, $n = 0, 1, \ldots$, belongs to $L_2(R_+)$.

(a-iii) From (7) we see that the function $\sqrt{xy}Y_\nu(xy)$, $|\mathcal{R}e\ \nu| < 1/2$, has the asymptotics $x^{1/2-|\mathcal{R}e\ \nu|}$ as x tends to 0, and is uniformly bounded on R_+. Because $y^{2n}g(y) \in L_1(R_+)$, by applying the dominated convergence theorem [12] we have

$$\lim_{x\to 0}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x) = (-1)^n\int_0^\infty \lim_{x\to 0}[\sqrt{xy}Y_\nu(xy)]y^{2n}g(y)\,dy = 0, \tag{12}$$
$$|\mathcal{R}e\ \nu| < 1/2.$$

Since $\sqrt{xy}Y_\nu(xy)$, $|\mathcal{R}e\ \nu| < 3/2$, is uniformly bounded for $x, y \in [1,\infty)$ and $y^n g(y) \in L_1(R_+)$, for every $\varepsilon > 0$ and for every n, $n = 0, 1, \dots$, one can choose b large enough so that

$$\left|\int_b^\infty \sqrt{xy}Y_\nu(xy)y^n g(y)\,dy\right| < \varepsilon, \quad |\mathcal{R}e\ \nu| < 3/2, \tag{13}$$

uniformly with respect to $x \in [1,\infty)$. On the other hand, from (7) one can conclude that the integral

$$\int_{ax}^{bx} \sqrt{y}Y_\nu(y)\,dy, \quad |\mathcal{R}e\ \nu| < 1/2, \tag{14}$$

is uniformly bounded for all non-negative a, b and x. Hence,

$$\int_a^b \sqrt{xy}Y_\nu(xy)\,dy = \frac{1}{x}\int_{ax}^{bx}\sqrt{y}Y_\nu(y)\,dy, \quad |\mathcal{R}e\ \nu| < 1/2, \tag{15}$$

tends to 0 uniformly in a, b for $0 \leqslant a < b < \infty$ as x tends to infinity. Consequently, applying the generalised Riemann-Lebesgue lemma [8] we get

$$\lim_{x\to\infty}\int_0^b \sqrt{xy}Y_\nu(xy)y^{2n}g(y)\,dy = 0, \quad 0 < b < \infty,\ |\mathcal{R}e\ \nu| < 1/2. \tag{16}$$

Because ε can be taken arbitrarily small, from (13) and (16) we obtain

$$\lim_{x\to\infty}\int_0^\infty \sqrt{xy}Y_\nu(xy)y^{2n}g(y)\,dy = 0, \quad |\mathcal{R}e\ \nu| < 1/2. \tag{17}$$

Hence,

$$\lim_{x\to\infty}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x) = 0, \quad n = 0, 1, \dots,\ |\mathcal{R}e\ \nu| < 1/2. \tag{18}$$

(a-iv) Since [1]

$$2\frac{d}{dx}\left(\sqrt{x}Y_\nu(x)\right) = \sqrt{x}Y_{\nu-1}(x) - \sqrt{x}Y_{\nu+1}(x) + \frac{1}{\sqrt{x}}Y_\nu(x), \tag{19}$$

we have

$$(20)\quad \frac{d}{dx}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)=\frac{(-1)^n}{2}\int_0^\infty \sqrt{xy}Y_{\nu-1}(xy)y^{2n+1}g(y)\,dy$$
$$+\frac{(-1)^{n+1}}{2}\int_0^\infty \sqrt{xy}Y_{\nu+1}(xy)y^{2n+1}g(y)\,dy+\frac{(-1)^n}{2x}\int_0^\infty \sqrt{xy}Y_\nu(xy)y^{2n}g(y)\,dy.$$

The function $\sqrt{x}Y_\mu(x)$ is uniformly bounded on $[1,\infty)$, and is of the order $O\big(x^{1/2-|\mathcal{R}e\,\mu|}\big)$ on $(0,1)$. Therefore, for $x\in(0,1)$,

$$\left|\int_0^\infty \sqrt{xy}Y_\mu(xy)g(y)\,dy\right|\leqslant\left|\int_0^{1/x} \sqrt{xy}Y_\mu(xy)g(y)\,dy\right|+\left|\int_{1/x}^\infty \sqrt{xy}Y_\mu(xy)g(y)\,dy\right|$$
$$\leqslant Cx^{1/2-|\mathcal{R}e\,\mu|}\int_0^{1/x} y^{1/2-|\mathcal{R}e\,\mu|}\,|g(y)|\,dy+C\int_{1/x}^\infty |g(y)|\,dy$$
$$(21)\quad \leqslant Cx^{1/2-|\mathcal{R}e\,\mu|}\int_0^{\infty} y^{1/2-|\mathcal{R}e\,\mu|}\,|g(y)|\,dy+C\int_{0}^\infty |g(y)|\,dy.$$

Hence, in the neighbourhood of 0 we have

$$\frac{1}{x}\int_0^\infty \sqrt{xy}Y_\nu(xy)y^{2n}g(y)\,dy=O\big(x^{-1}\big),$$
$$\int_0^\infty \sqrt{xy}Y_{\nu-1}(xy)y^{2n+1}g(y)\,dy=O\Big(x^{\mathcal{R}e\,\nu-1/2}\Big),$$
$$(22)\quad \int_0^\infty \sqrt{xy}Y_{\nu+1}(xy)y^{2n+1}g(y)\,dy=O\Big(x^{-\mathcal{R}e\,\nu-1/2}\Big),\quad |\mathcal{R}e\,\nu|<1/2.$$

By combining (20) and (22), we obtain

$$x\frac{d}{dx}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)=O(1),\quad x\to 0;$$
$$(23)\quad n=0,1,\dots\,;\ |\mathcal{R}e\,\nu|<1/2.$$

(a-v) Let $|\mathcal{R}e\,\nu|<3/2$. For every $\varepsilon>0$ choose b so that the inequality (13) holds. Because $(xy)^{3/2}Y_\nu(xy)$, $|\mathcal{R}e\,\nu|<3/2$, is uniformly bounded for $x,y\in R_+$, $xy\leqslant 1$, then

$$(24)\quad \left|\int_0^{1/x}\sqrt{xy}Y_\nu(xy)y^{n+1}g(y)\,dy\right|\leqslant\frac{C}{x}\int_0^{1/x}y^n\,|g(y)|\,dy.$$

Hence,

$$(25)\quad \lim_{x\to\infty}\int_0^{1/x}\sqrt{xy}Y_\nu(xy)y^{n+1}g(y)\,dy=0,\quad |\mathcal{R}e\,\nu|<3/2.$$

Let

$$\Phi(x,y) = \begin{cases} \sqrt{xy}Y_\nu(xy)\,dy, & y > 1/x \\ 0, & y \leqslant 1/x. \end{cases} \tag{26}$$

Then $\Phi(x,y)$ is uniformly bounded. The integral

$$\int_{ax}^{bx} \sqrt{y}Y_\nu(y)\,dy, \quad |\mathcal{R}e\,\nu| < 3/2, \tag{27}$$

is uniformly bounded for all non-negative a, b and x such that $ax \geqslant 1$. Hence,

$$\int_a^b \Phi(x,y)\,dy = \frac{1}{x}\int_{max\{1,ax\}}^{bx} \sqrt{y}Y_\nu(y)\,dy, \quad |\mathcal{R}e\,\nu| < 3/2, \tag{28}$$

tends to 0 uniformly in a, b for $0 \leqslant a < b < \infty$ as x tends to infinity. Consequently, applying again the generalised Riemann-Lebesgue lemma [**8**] we get

$$\lim_{x\to\infty} \int_0^b \Phi(x,y)y^n g(y)\,dy = 0, \quad 0 < b < \infty, \tag{29}$$

This means that

$$\lim_{x\to\infty} \int_{1/x}^b \sqrt{xy}Y_\nu(xy)y^n g(y)\,dy = 0, \quad 0 < b < \infty, \ |\mathcal{R}e\,\nu| < 3/2. \tag{30}$$

Because ε can be taken arbitrarily small, from (13), (25) and (30) we obtain

$$\lim_{x\to\infty} \int_0^\infty \sqrt{xy}Y_\nu(xy)y^{n+1} g(y)\,dy = 0, \quad n = 0, 1, \dots, \ |\mathcal{R}e\,\nu| < 3/2. \tag{31}$$

If $|\mathcal{R}e\,\nu| < 1/2$, then $|\mathcal{R}e\,\nu \mp 1| < 3/2$. Hence,

$$\lim_{x\to\infty} \int_0^\infty \sqrt{xy}Y_{\nu-1}(xy)y^{2n+1} g(y)\,dy = 0,$$

$$\lim_{x\to\infty} \int_0^\infty \sqrt{xy}Y_{\nu+1}(xy)y^{2n+1} g(y)\,dy = 0, \quad |\mathcal{R}e\,\nu| < 1/2. \tag{32}$$

Applying now formulas (20), (31) and (32), we have

$$\lim_{x\to\infty} \frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n f(x) = 0, \quad n = 0, 1, \dots, \ |\mathcal{R}e\,\nu| < 1/2. \tag{33}$$

(a-vi) The special case $-1/2 < \mathcal{R}e\,\nu < 0$ has been proved in [**3**]. We give here a proof valid for all the range of ν. Integral (11) converges uniformly with respect to x

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on every compact subset of R_+. Therefore, one can interchange the order of integration in the following formula to obtain

$$\int_{1/N}^{N} x^{\nu-1/2}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\,dx$$

$$= (-1)^n \int_{1/N}^{N} x^{\nu-1/2}\int_0^\infty \sqrt{xy}Y_\nu(xy)y^{2n}g(y)\,dydx$$

$$(34)\qquad = (-1)^n \int_0^\infty y^{2n-\nu-1/2}g(y)\int_{y/N}^{yN} x^\nu Y_\nu(x)\,dx\,dy, \qquad 0<N<\infty.$$

The last inner integral in (34) is uniformly bounded for all nonnegative N and y, provided that $|\mathcal{R}e\,\nu|<1/2$. For $y^{2n-\nu-1/2}g(y)\in L_1(R_+)$ under the restriction $\mathcal{R}e\,\nu<0$, and $n\geqslant 1$ otherwise, one can apply the dominated convergence theorem to obtain

(35)

$$\lim_{N\to\infty}\int_{1/N}^{N} x^{\nu-1/2}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\,dx$$

$$= (-1)^n\int_0^\infty y^{2n-\nu-1/2}g(y)\int_0^\infty x^\nu Y_\nu(x)\,dx\,dy,\quad n=0,1,\dots;\ -1/2<\mathcal{R}e\,\nu<0,$$

$$n=1,2,\dots;\ 0\leqslant\mathcal{R}e\,\nu<1/2.$$

Applying now the formula [1]

$$\int_0^\infty x^\mu Y_\nu(x)\,dx=\frac{2^\mu}{\pi}\sin\frac{\pi}{2}(\mu-\nu)\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right),$$

$$(36)\qquad \mathcal{R}e\,(\mu+\nu)>-1,\ \ \mathcal{R}e\,\mu<1/2,$$

with $\mu=\nu$, we see that the inner integral on the right hand side of (35) equals 0. Hence,

$$\int_{\to 0}^{\to\infty} x^{\nu-1/2}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\,dx=0,\quad n=0,1,\dots,-1/2<\mathcal{R}e\,\nu<0,$$

$$(37)\qquad n=1,2,\dots,0\leqslant\mathcal{R}e\,\nu<1/2.$$

(b) Suppose now that f satisfies conditions (i)-(vi) of Theorem 1. Then $[d^2/dx^2+(1/x^2)((1/4)-\nu^2)]^n f(x)$, $n=0,1,\dots,$ belongs to $L_2(R_+)$.

(b-i) Let $-1/2<\mathcal{R}e\,\nu<0$ and $g_n(y)$ be the H-transforms $\mathbf{H}_\nu$, $-1/2<\mathcal{R}e\,\nu<0$, of $[d^2/dx^2+(1/x^2)((1/4)-\nu^2)]^n f(x)$, $n=0,1,\dots.$ Then

$$(38)\qquad g_n(y)=\int_0^\infty \sqrt{xy}\mathbf{H}_\nu(xy)\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\,dx,\quad n=0,1,2,\dots,$$

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where the integrals are understood in the $L_2(R_+)$ norm. Put

$$g_n^N(y) = \int_{1/N}^{N} \sqrt{xy}\mathbf{H}_\nu(xy)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n f(x)\,dx, \quad n = 0,1,2\ldots. \tag{39}$$

Then $g_n^N(y)$ tends to $g_n(y)$ in L_2 norm as $N \to \infty$. Let $n \geqslant 1$. Integrating (39) by parts twice, we obtain

$$\begin{aligned} g_n^N(y) = &\left\{\sqrt{xy}\mathbf{H}_\nu(xy)\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(x)\right\}\Bigg|_{x=1/N}^{x=N} \\ &- \left\{\frac{\partial}{\partial x}(\sqrt{xy}\mathbf{H}_\nu(xy))\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(x)\right\}\Bigg|_{x=1/N}^{x=N} \\ &+ \int_{1/N}^{N}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right](\sqrt{xy}\mathbf{H}_\nu(xy))\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(x)\,dx. \end{aligned} \tag{40}$$

Using formulas [1]

$$\begin{aligned} &\frac{\partial}{\partial x}(\sqrt{xy}\mathbf{H}_\nu(xy)) = (1/2 - \nu)\sqrt{\frac{y}{x}}\mathbf{H}_\nu(xy) + y\sqrt{xy}\mathbf{H}_{\nu-1}(xy), \\ &\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right](\sqrt{xy}\mathbf{H}_\nu(xy)) = \frac{2^{1-\nu}y^{\nu+3/2}}{\sqrt{\pi}\Gamma(\nu+1/2)}x^{\nu-1/2} - y^2\sqrt{xy}\mathbf{H}_\nu(xy), \end{aligned} \tag{41}$$

we have

(42)

$$g_n^N(y) = \sqrt{Ny}\mathbf{H}_\nu(Ny)\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(N)$$

$$- \sqrt{\frac{y}{N}}\mathbf{H}_\nu(y/N)\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(1/N) \tag{43}$$

$$+ \left(\nu - \frac{1}{2}\right)\sqrt{\frac{y}{N}}\mathbf{H}_\nu(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(N) \tag{44}$$

$$- y\sqrt{Ny}\mathbf{H}_{\nu-1}(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(N) \tag{45}$$

$$+ \left(\frac{1}{2} - \nu\right)\sqrt{Ny}\mathbf{H}_\nu(y/N)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(1/N) \tag{46}$$

$$(47)\qquad +y\sqrt{\frac{y}{N}}\mathbf{H}_{\nu-1}(y/N)\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1}f(1/N)$$

$$(48)\qquad -y^2\int_{1/N}^{N}\sqrt{xy}\mathbf{H}_{\nu}(xy)\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1}f(x)\,dx$$

$$(49)\qquad +\frac{2^{1-\nu}y^{\nu+3/2}}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_{1/N}^{N}x^{\nu-1/2}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1}f(x)\,dx.$$

Here $P(d/dx)\,f(N)$ means $P(d/dx)\,f(x)|_{x=N}$. As N tends to infinity, integral (49) vanishes because of property (vi). Applying the asymptotic formula for the Struve function [1]

$$(50)\qquad \mathbf{H}_{\nu}(y)=\begin{cases} O\big(y^{-1/2}\big), & y\to\infty,\quad \mathcal{R}e\,\nu<1/2,\\ O\big(y^{\mathcal{R}e\,\nu+1}\big), & y\to 0,\quad \forall\nu,\end{cases}$$

we obtain that $\sqrt{Ny}\mathbf{H}_{\nu}(Ny)$, $|\mathcal{R}e\,\nu|<1/2$, is uniformly bounded. The function $(d/dx)\left[d^2/dx^2+(1/x^2)\left((1/4)-\nu^2\right)\right]^{n-1}f(N)$ tends to 0 as N approaches infinity (property (v)), therefore, the expression on the right hand side of (42) tends to 0 as N approaches infinity. From (iv) we see that $(d/dx)\left[d^2/dx^2+(1/x^2)\left((1/4)-\nu^2\right)\right]^{n-1}$ $f(1/N)$ has order $O(N)$, whereas function $\sqrt{y/N}\mathbf{H}_{\nu}(y/N)$ has order $O\big(N^{-3/2-\nu}\big)$. Hence, expression (43) approaches 0 as N tends to infinity. Function $\sqrt{y/N}\mathbf{H}_{\nu}(Ny)$ has order $O\big(N^{-1}\big)$, whereas the expression $\left[d^2/dx^2+(1/x^2)\left((1/4)-\nu^2\right)\right]^{n-1}f(N)$ is $o(1)$ (property (iii)), therefore, expression (44) is $o(1)$. The function $y\sqrt{Ny}\mathbf{H}_{\nu-1}(Ny)$ is $O(1)$, hence, property (iii) shows that (45) is $o(1)$. Since $\sqrt{Ny}\mathbf{H}_{\nu}(y/N)$ has the order $O\big(N^{-1/2-\nu}\big)$, and $\left[d^2/dx^2+(1/x^2)\left((1/4)-\nu^2\right)\right]^{n-1}f(1/N)$ is $o(1)$ (property (iii)), expression (46) is also $o(1)$. The function $y\sqrt{y/N}\mathbf{H}_{\nu-1}\;(y/N)$ has the order $O\big(N^{-1/2-\nu}\big)$, hence, property (iii) shows that (47) is $o(1)$.

Therefore, the right hand side of (42), as well as all functions (43) – (49), except (48), vanish as N tends to infinity, whereas expression (48) converges to $-y^2g_{n-1}(y)$. Consequently, $g_n(y)=-y^2g_{n-1}(y)$, and therefore, $g_n(y)=\left(-y^2\right)^n g_0(y)$, $n=0,1,\dots$. Thus $g(y)=g_0(y)$ such that $y^{2n}g(y)\in L_2(R_+)$, $n=0,1,\dots$, is the H-transform $\mathbf{H}_{\nu}$ of the function $f(x)$. But the H-transform $\mathbf{H}_{\nu}$ is the inverse of the Y-transform Y_{ν} if $-1/2<\mathcal{R}e\;\nu<0$, so we obtain that and f is the Y-transform Y_{ν}, $-1/2<\mathcal{R}e\,\nu<0$, of a function g such that $y^n g(y)\in L_2(R_+)$, $n=0,1,\dots$.

(b-ii) Let now $0<\mathcal{R}e\;\nu<1/2$. The inverse of the Y-transform Y_{ν} in the range $0<\mathcal{R}e\,\nu<1$ has the form [3]

(51)

$$g(y)=y^{-\nu-1/2}\frac{d}{dy}y^{\nu+1/2}\int_0^{\infty}\sqrt{xy}\left[\mathbf{H}_{\nu+1}(xy)-\frac{(xy)^{\nu}}{2^{\nu}\sqrt{\pi}\Gamma(\nu+3/2)}\right]f(x)\,dx,\quad y\in R_+,$$

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that can be expressed in an equivalent form

$$g(y) = \lim_{N\to\infty} \int_{1/N}^{N} \left[\sqrt{xy}\mathbf{H}_\nu(xy) - \frac{(xy)^{\nu-1/2}}{2^{\nu-1}\sqrt{\pi}\Gamma(\nu+1/2)}\right] f(x)\,dx, \quad y \in R_+, \tag{52}$$

where the limit is understood in the $L_2(R_+)$ norm. Putting

$$g_n^N(y) = \int_{1/N}^{N} \left[\sqrt{xy}\mathbf{H}_\nu(xy) - \frac{(xy)^{\nu-1/2}}{2^{\nu-1}\sqrt{\pi}\Gamma(\nu+1/2)}\right] \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\,dx, \quad n = 0,1,2\dots, \tag{53}$$

we see that $g_n^N(y)$ tends to some functions $g_n(y)$ in the L_2 norm as $N \to \infty$. Let $n \geqslant 1$. Integrating (53) by parts twice and using formulae (40), (41) we obtain

$$g_n^N(y) = \sqrt{Ny}\mathbf{H}_\nu(Ny)\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(N) \tag{54}$$

$$-\sqrt{\frac{y}{N}}\mathbf{H}_\nu(y/N)\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(1/N) \tag{55}$$

$$+\left(\nu-\frac{1}{2}\right)\sqrt{\frac{y}{N}}\mathbf{H}_\nu(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(N) \tag{56}$$

$$-y\sqrt{Ny}\mathbf{H}_{\nu-1}(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(N) \tag{57}$$

$$+\left(\frac{1}{2}-\nu\right)\sqrt{Ny}\mathbf{H}_\nu(y/N)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(1/N) \tag{58}$$

$$+y\sqrt{\frac{y}{N}}\mathbf{H}_{\nu-1}(y/N)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(1/N) \tag{59}$$

$$-y^2\int_{1/N}^{N}\sqrt{xy}\left[\mathbf{H}_\nu(xy) - \frac{2^{1-\nu}(xy)^{\nu-1}}{\sqrt{\pi}\Gamma(\nu+1/2)}\right] \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(x)\,dx \tag{60}$$

$$-\frac{2^{1-\nu}y^{\nu-1/2}}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_{1/N}^{N} x^{\nu-1/2}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n} f(x)\,dx. \tag{61}$$

When N tends to infinity integral (61) vanishes because of property (vi) and $n \geqslant 1$. Reasoning the same as before, we can conclude that the right hand side of (54), as well as all functions (55)–(59), vanish as N tends to infinity, whereas the expression (60) converges to $-y^2 g_{n-1}(y)$. Consequently, $g_n(y) = -y^2 g_{n-1}(y)$, and therefore, $g_n(y) = \left(-y^2\right)^n g_0(y)$, $n = 0, 1, \ldots$. Thus $g(y) = g_0(y)$ such that $y^{2n} g(y) \in L_2(R_+)$, $n = 0, 1, \ldots$, is the transform (52) of function $f(x)$. But transform (52) is the inverse of the Y-transform Y_ν if $0 < \mathcal{R}e\ \nu < 1/2$, so we obtain that and f is the Y-transform Y_ν, $0 < \mathcal{R}e\ \nu < 1/2$, of a function g such that $y^n g(y) \in L_2(R_+)$, $n = 0, 1, \ldots$. Theorem 1 is thus proved. ∎

REMARK. The case $\mathcal{R}e\ \nu = 0$ has been excluded from Theorem 1. It was proved in [**3**] that in this case the range of the Y-transform in $L_2(R_+)$ is a proper subspace of $L_2(R_+)$.

3. Y-TRANSFORM OF SQUARE INTEGRABLE FUNCTIONS WITH COMPACT SUPPORTS

Now we describe the Y-transform of square integrable functions with compact supports (the Paley-Wiener theorem for the Y-transform).

THEOREM 2. *A function f is the Y-transform Y_ν, $0 < |\mathcal{R}e\ \nu| < 1/2$, of a square integrable function g with compact support on $[0, \infty)$ if and only if f satisfies conditions (i)-(vi) of Theorem 1 and moreover,*

$$\lim_{n\to\infty} \left\| \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right) \right]^n f(x) \right\|_2^{1/(2n)} = \sigma_g < \infty, \tag{62}$$

where

$$\sigma_g = \sup\{y : \ y \in \ supp\, g\}, \tag{63}$$

and the support of a function is the smallest closed set outside which the function vanishes almost everywhere [**12**].

PROOF: (a) Let $f(x)$ be the Y-transform of $g(y) \in L_2(R_+)$ and $\sigma_g < \infty$:

$$f(x) = \int_0^{\sigma_g} \sqrt{xy}\, Y_\nu(xy) g(y)\, dy, \quad 0 < |\mathcal{R}e\ \nu| < 1/2. \tag{64}$$

One can assume that $\sigma_g > 0$, otherwise it is trivial. Since $\sigma_g < \infty$ we have $y^n g(y) \in L_2(R_+)$ for all $n = 0, 1, 2, \ldots$. Therefore, f satisfies conditions (i)-(vi) of Theorem 1. Furthermore,

$$\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right) \right]^n f(x) = \int_0^{\sigma_g} \sqrt{xy}\, Y_\nu(xy) \left(-y^2\right)^n g(y)\, dy. \tag{65}$$

Consequently, applying the inequality (2) for the Y-transform (65), we obtain

$$\left\| \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right) \right]^n f(x) \right\|_2^2 \leqslant C \int_0^{\sigma_g} y^{4n} |g(y)|^2\, dy \leqslant C \sigma_g^{4n} \int_0^{\sigma_g} |g(y)|^2\, dy. \tag{66}$$

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Hence,

(67)
$$\overline{\lim_{n\to\infty}}\left\|\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\right\|_2^{1/(2n)} \leqslant \overline{\lim_{n\to\infty}} C^{1/(4n)}\sigma_g\left\{\int_0^{\sigma_g}|g(y)|^2\,dy\right\}^{1/(4n)} = \sigma_g.$$

On the other hand, since σ_g is the least upper bound of the support of g, for every ε, $0<\varepsilon<\sigma_g$, we have

(68)
$$\int_{\sigma_g-\varepsilon}^{\sigma_g}|g(y)|^2\,dy>0.$$

Consequently, using now inequality (3) for the Y-transform (65), we get

$$\underline{\lim_{n\to\infty}}\left\|\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\right\|_2^{1/(2n)} \geqslant \underline{\lim_{n\to\infty}} C^{-1/(4n)}\left\{\int_{\sigma_g-\varepsilon}^{\sigma_g} y^{4n}|g(y)|^2\,dy\right\}^{1/(4n)}$$

(69)
$$\geqslant(\sigma_g-\varepsilon)\underline{\lim_{n\to\infty}} C^{-1/(4n)}\left\{\int_{\sigma_g-\varepsilon}^{\sigma_g}|g(y)|^2\,dy\right\}^{1/(4n)} = \sigma_g-\varepsilon.$$

Because ε can be chosen arbitrarily small, from (69) and (67) we obtain (62).

(b) Suppose now that f satisfies the conditions of Theorem 1, and the limit in (62) exists and equals $\sigma<\infty$. Applying Theorem 1 we see that f is the Y-transform Y_ν of a function g with σ_g defined by (63) such that $y^n g(y)\in L_2(R_+)$, $n=0,1,\ldots$. We prove that $\sigma_g<\infty$ and moreover, $\sigma=\sigma_g$. Theorem 1 implies that (11) holds. Therefore, using inequalities (2) and (3) we obtain

(70)
$$C^{-1}\left\|y^{2n}g(y)\right\|_2 \leqslant \left\|\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\right\|_2 \leqslant C\left\|y^{2n}g(y)\right\|_2.$$

Hence,

$$\lim_{n\to\infty} C^{-1/(2n)}\left\|y^{2n}g(y)\right\|_2^{1/(2n)} \leqslant \lim_{n\to\infty}\left\|\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\right\|_2^{1/(2n)} = \sigma$$

(71)
$$\leqslant \lim_{n\to\infty} C^{1/(2n)}\left\|y^{2n}g(y)\right\|_2^{1/(2n)}.$$

Consequently,

(72)
$$\lim_{n\to\infty}\left\|y^{2n}g(y)\right\|_2^{1/(2n)}=\sigma.$$

Suppose that $\sigma_g>\sigma$. Then there exists a positive ε such that

(73)
$$\int_{\sigma+\varepsilon}^{\infty}|g(y)|^2\,dy>0.$$

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We have

$$\sigma = \lim_{n\to\infty} \left\|y^{2n} g(y)\right\|_2^{1/(2n)} \geqslant \underline{\lim}_{n\to\infty} \left\{ \int_{\sigma+\varepsilon}^{\infty} y^{4n} \left|g(y)\right|^2 dy \right\}^{1/(4n)}$$

$$\geqslant (\sigma+\varepsilon) \underline{\lim}_{n\to\infty} \left\{ \int_{\sigma+\varepsilon}^{\infty} \left|g(y)\right|^2 dy \right\}^{1/(4n)} = \sigma + \varepsilon. \tag{74}$$

This is impossible. Hence, $\sigma_g \leqslant \sigma$ and therefore, the function g has a compact support. Suppose now that $\sigma_g < \sigma$. Then there exists a positive ε such that

$$\int_{\sigma-\varepsilon}^{\infty} \left|g(y)\right|^2 dy = 0. \tag{75}$$

We have

$$\sigma = \lim_{n\to\infty} \left\|y^{2n} g(y)\right\|_2^{1/(2n)} \leqslant \overline{\lim}_{n\to\infty} \left\{ \int_{0}^{\sigma-\varepsilon} y^{4n} \left|g(y)\right|^2 dy \right\}^{1/(4n)}$$

$$\leqslant (\sigma-\varepsilon) \overline{\lim}_{n\to\infty} \left\{ \int_{0}^{\sigma-\varepsilon} \left|g(y)\right|^2 dy \right\}^{1/(4n)} = \sigma - \varepsilon. \tag{76}$$

This is also impossible. Hence, $\sigma_g \geqslant \sigma$, and consequently, $\sigma_g = \sigma < \infty$. Theorem 2 is thus proved. ∎

Remark. If a function f satisfies conditions of Theorem 1, then the limit (62) always exists. It equals to infinity if f is the Y-transform Y_ν of a function g with unbounded support.

4. Y-transform of analytic functions

We consider now the Y-transform Y_ν of functions analytic in some angle.

Theorem 3. *The Y-transform Y_ν, $-1 < \mathcal{R}e\,\nu < 1$, maps the space of all functions $g(z)$, regular in the angle $-\alpha < \arg z < \beta$, where $0 < \alpha, \beta \leqslant \pi$; of the order $O\left(|z|^{-a-\varepsilon}\right)$ for small z, and $O\left(|z|^{-b+\varepsilon}\right)$ for large z, where $a < 1/2 < b$, uniformly for any small positive ε in any angle interior to the above; and satisfying conditions*

$$\begin{aligned} &\int_0^\infty y^{\nu+2n+1/2} g(y)\, dy = 0, \quad n \in (-b/2 - \mathcal{R}e\,\nu/2 - 1/4, -a/2 - \mathcal{R}e\,\nu/2 - 1/4), \\ &\int_0^\infty y^{-\nu+2n+1/2} g(y)\, dy = 0, \quad n \in (-b/2 + \mathcal{R}e\,\nu/2 - 1/4, -a/2 + \mathcal{R}e\,\nu/2 - 1/4), \end{aligned} \tag{77}$$

for all nonnegative integers n, if there exists such n, one-to-one onto the space of all functions $f(z)$, regular in the angle $-\beta < \arg z < \alpha$, of the order $O\left(|z|^{1-b-\varepsilon}\right)$ for

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small z, and $O\left(|z|^{1-a+\varepsilon}\right)$ for large z, uniformly for any small positive ε in any angle interior to the above; and satisfying conditions

$$\int_0^\infty x^{\nu-2n-1/2} f(x)\,dx = 0, \quad n \in (a/2 + \mathcal{R}e\,\nu/2 - 1/4, b/2 + \mathcal{R}e\,\nu/2 - 1/4),$$

(78)

$$\int_0^\infty x^{\nu+2n+3/2} f(x)\,dx = 0, \quad n \in (-b/2 - \mathcal{R}e\,\nu/2 - 3/4, -a/2 - \mathcal{R}e\,\nu/2 - 3/4),$$

for all nonnegative integers n, if there exists such n. (For example, if $\mathcal{R}e\,\nu = 0$, then $n = 0$ always belongs to the interval $(a/2 - 1/4, b/2 - 1/4)$.)

PROOF: Let $g(z)$ satisfy the conditions of Theorem 3. Then the function $g(z)$ on R_+ belongs to $L_2(R_+)$ and its Mellin transform $g^*(s)$

$$g^*(s) = \int_0^\infty x^{s-1} g(x)\,dx \tag{79}$$

is an analytic function of s, regular for $a < \mathcal{R}e\,s < b$; and

$$g^*(s) = \begin{cases} O\left(e^{-(\beta-\varepsilon)\mathcal{I}m\,s}\right), & \mathcal{I}m\,s \to \infty \\ O\left(e^{(\alpha-\varepsilon)\mathcal{I}m\,s}\right), & \mathcal{I}m\,s \to -\infty \end{cases} \tag{80}$$

for every positive ε, uniformly in any strip interior to $a < \mathcal{R}e\,s < b$ (see [**8**]). Let $f(x)$ be the Y-transform Y_ν, $-1 < \mathcal{R}e\,\nu < 1$, of $g(y)$. Since $g(y)$ belongs to $L_2(R_+)$, the Parseval identity for the Y-transform Y_ν holds on the line $\mathcal{R}e\,s = 1/2$ [**6**]:

$$f^*(s) = 2^{s-1} \frac{\Gamma\left(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2}\right)\Gamma\left(\frac{1}{4} - \frac{\nu}{2} + \frac{s}{2}\right)}{\Gamma\left(-\frac{1}{4} - \frac{\nu}{2} + \frac{s}{2}\right)\Gamma\left(\frac{5}{4} + \frac{\nu}{2} - \frac{s}{2}\right)} g^*(1-s). \tag{81}$$

Because of (77) the function $g^*(1-s)$ equals 0 at the poles of function $\Gamma(1/4 + \nu/2 + s/2)$ $\Gamma(1/4 - \nu/2 + s/2)$ in the strip $1-b < \mathcal{R}e\,s < 1-a$, if there exists one. Hence, from (81) one can see that $f^*(s)$ is analytic in the strip $1-b < \mathcal{R}e\,s < 1-a$. Furthermore, since the function $2^{s-1/2}\Big(\Gamma(1/4+\nu/2+s/2)\Gamma(1/4-\nu/2+s/2)\Big)\Big/\Big(\Gamma(-1/4-\nu/2+s/2)$ $\Gamma(5/4+\nu/2-s/2)\Big)$ is uniformly bounded in any compact domain in the strip $1-b < \mathcal{R}e\,s < 1-a$, not containing the poles of function $\Gamma(1/4+\nu/2+s/2)\Gamma(1/4-\nu/2+s/2)$, and has at most only polynomial growth as $\mathcal{I}m\,s \to \pm\infty$, from (80) we see that function $f^*(s)$ decays exponentially

$$f^*(s) = \begin{cases} O\left(e^{(\beta-\varepsilon)\mathcal{I}m\,s}\right), & \mathcal{I}m\,s \to -\infty \\ O\left(e^{-(\alpha-\varepsilon)\mathcal{I}m\,s}\right), & \mathcal{I}m\,s \to \infty \end{cases} \tag{82}$$

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for every positive ε, uniformly in any strip interior to $1-b < \mathcal{R}e\ s < 1-a$. Hence, its inverse Mellin transform $f(z)$ is regular for $-\beta < \arg z < \alpha$, and of the order $O\left(|z|^{b-1-\varepsilon}\right)$ for small z, and $O\left(|z|^{a-1+\varepsilon}\right)$ for large z, uniformly in any angle interior to the above angle for any small positive ε [8]. Moreover, $f^*(s)$ has zeros at the poles of the function $\Gamma(-1/4-\nu/2+s/2)\Gamma(5/4+\nu/2-s/2)$ in the strip $1-b < \mathcal{R}e\ s < 1-a$, if there exists one. Hence (78) holds.

Conversely, let $f(z)$ satisfy the conditions of Theorem 3. Then $f(z)$ on R_+ belongs to $L_2(R_+)$ and its Mellin transform (79) $f^*(s)$ is analytic in the strip $1-b < \mathcal{R}e\ s < 1-a$ and satisfies (82). Furthermore, because of (78) the function $f^*(s)$ vanishes at the poles of the function $\Gamma(-1/4-\nu/2+s/2)\Gamma(5/4+\nu/2-s/2)$ in the strip $1-b < \mathcal{R}e\ s < 1-a$, if there exists one. Therefore, if we express $f^*(s)$ in the form (81), function $g^*(s)$ is analytic in the strip $a < \mathcal{R}e\ s < b$; and has the asymptotics (80) for every positive ε, uniformly in any strip interior to $a < \mathcal{R}e\ s < b$. Furthermore, $g^*(1-s)$ has zeros at the poles of the function $\Gamma(1/4+\nu/2+s/2)\Gamma(1/4-\nu/2+s/2)$ in the strip $1-b < \mathcal{R}e\ s < 1-a$. Consequently, the inverse Mellin transform $g(z)$ of $g^*(s)$ satisfies the conditions of Theorem 3 and f is the Y-transform of g. ☐

If in Theorem 3 we take $\alpha = \beta$ and $0 < a < \min\{|\nu|, |\nu+1|, |\nu-1|\}$, then in the strip $1/2-a < \mathcal{R}e\ s < 1/2+a$ there are no poles or zeros of the function $2^{s-1/2}\Big(\Gamma(1/4+\nu/2+s/2)\,\Gamma(1/4-\nu/2+s/2)\Big)\Big/\Big(\Gamma(-1/4-\nu/2+s/2)\,\Gamma(5/4+\nu/2-s/2)\Big)$, hence, we have

COROLLARY 1. *The Y-transform Y_ν, $0 < |\mathcal{R}e\ \nu| < 1$, is a bijection in the space of all functions, regular in the angle $|\arg z| < \alpha$, where $0 < \alpha \leqslant \pi$; of order $O\left(|z|^{a-1/2-\varepsilon}\right)$ for small z, and $O\left(|z|^{-a-1/2+\varepsilon}\right)$ for large z, uniformly for any small positive ε, $0 < \varepsilon < a$, in any angle interior to the above, where $0 < a < \min\{|\nu|, |\nu+1|, |\nu-1|\}$.*

5. Y-TRANSFORM IN SOME OTHER SPACES OF FUNCTIONS

In [9, 10] the Y-transform is proved to be a bijection in some spaces of functions $\mathcal{M}^{-1}_{c,\gamma}(L)$ introduced there. In this section the Y-transform in a space of functions including the spaces $\mathcal{M}^{-1}_{c,\gamma}(L)$ as special cases is considered.

Let Φ be any linear subspace of either $L_1(R)$ or $L_2(R)$ having properties:

(i) if $\phi(t) \in \Phi$ then $\phi(-t) \in \Phi$;

(ii) functions $\varphi(t) = 2^{it}\Gamma(1/2+\nu/2+it/2)\Gamma(1/2-\nu/2+it/2)\sin(\pi/2)(it-\nu), 0 < |\mathcal{R}e\ \nu| < 1$, and $\varphi^{-1}(t)$ are multipliers of Φ.

It is easy to see that $\varphi^{-1}(-t)$ is also a multiplier of Φ. The multipliers $\varphi(t)$ and $\varphi^{-1}(t)$ are infinitely differentiable and uniformly bounded on R, and their derivatives grow logarithmically. Therefore, many classical spaces on R are special cases of Φ (for example, any L_1 or L_2 space with L_∞-weights, the Schwartz space $\mathcal{S}(R)$, and the space of infinitely differentiable functions with compact support [12]). On R_+ we define $\mathcal{M}^{-1}(\Phi)$ to be the space of all functions g that can be represented in the form

$$g(x) = \int_{-\infty}^{\infty} \phi(t) x^{it-1/2}\,dt, \tag{83}$$

almost everywhere, where $\phi \in \Phi$ (if $\phi \notin L_1(R)$ the integral should be understood as the inverse Mellin transform in L_2 [8]). The spaces $\mathcal{M}^{-1}_{c,\gamma}(L)$ [10] as well as the space of functions considered in Corollary 1 are special cases of $\mathcal{M}^{-1}(\Phi)$.

Theorem 4. *The Y-transform Y_ν, $0 < |\mathcal{R}e\,\nu| < 1$, is a bijection in $\mathcal{M}^{-1}(\Phi)$.*

Proof: From (83) we see that if $g \in \mathcal{M}^{-1}(\Phi)$ then g can be expressed in the form of the inverse Mellin transform

$$g(x) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} g^*(s) x^{-s}\,ds, \tag{84}$$

where $g^*(1/2+it) \in \Phi$. Using formula (36) we obtain that the Mellin transform (79) of the function $k(x) = \sqrt{x}Y_\nu(x)$ is $k^*(s) = \varphi(i/2 - is)$. Applying the Parseval equation for the Mellin transform

$$\int_0^\infty k(xy)g(y)\,dy = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} k^*(s)g^*(1-s)x^{-s}\,ds, \quad 0 < |\mathcal{R}e\,\nu| < 1, \tag{85}$$

that has been proved for $g^*(1/2+it) \in L_2(R)$ in [8] and $g^*(1/2+it) \in L_1(R)$ in [9], we obtain

$$(Y_\nu g)(x) = \int_0^\infty \sqrt{xy}Y_\nu(xy)g(y)\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t)g^*(1/2-it)x^{-it-1/2}\,dt. \tag{86}$$

Since $\varphi(t)$ and $\varphi^{-1}(-t)$ are multipliers of Φ, then $\varphi(t)g^*(1/2-it)$ belongs to Φ if and only if $g^*(1/2+it)$ belongs to Φ. Therefore, $(Y_\nu g)(x) \in \mathcal{M}^{-1}(\Phi)$ if and only if $g \in \mathcal{M}^{-1}(\Phi)$. Theorem 4 is thus proved. ∎

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Department of Mathematics and Computer Science
Kuwait University
P.O. Box 5969
Safat 13060
Kuwait