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A VARIATION ON THE THEME OF NICOMACHUS

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Abstract

In this paper, we prove some conjectures of K. Stolarsky concerning the first and third moments of the Beatty sequences with the golden section and its square.

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1. Introduction

Nicomachus' theorem asserts that the sum of the first m cubes is the square of the mth triangular number,

$$1^3 + 2^3 + \cdots + m^3 = (1 + 2 + \cdots + m)^2. \tag{1.1}$$

(See [2].) With the notation

$$Q(\alpha, m) := \frac{\sum_{n=1}^{m} \lfloor \alpha n \rfloor^3}{(\sum_{n=1}^{m} \lfloor \alpha n \rfloor)^2}, \tag{1.2}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$, it implies that

$$\lim_{m\to\infty} Q(\alpha, m) = \alpha. \tag{1.3}$$

Here, $\lfloor x \rfloor$ is the integer part of the real number x. The limit in (1.3) follows from $\lfloor \alpha n \rfloor = \alpha n + O(1)$ and Nicomachus' theorem (1.1).

Recall that the Fibonacci and Lucas sequences, $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$, are given by $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ and the recurrence relations

$$F_{n+2} = F_{n+1} + F_n, \quad L_{n+2} = L_{n+1} + L_n$$

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for $n \geq 0$. In a personal communication to the second author, K. Stolarsky observed that the limit relation (1.3) can be 'quantified' for $\alpha = \phi$ and ϕ^2, where $\phi := \frac{1}{2}(1+\sqrt{5})$ is the golden mean, and a specific choice of m along the Fibonacci sequence. The corresponding result is Theorem 2.1 below. We complement it by a general analysis of moments of the Beatty sequences and give a solution to a related arithmetic question in Theorem 2.5.

2. Principal results

THEOREM 2.1. *For $k \geq 1$ an integer, define $m_k := F_k - 1$. Then*

$$Q(\phi^2, m_{2k}) - Q(\phi, m_{2k}) = \begin{cases} 1 - \dfrac{1}{(F_{k+1})^2 L_{k+2} L_{k-1}} & \text{if } k \text{ is even,} \\ 1 - \dfrac{1}{(L_{k+1})^2 F_{k+2} F_{k-1}} & \text{if } k \text{ is odd} \end{cases}$$

and

$$Q(\phi^2, m_{2k-1}) - Q(\phi, m_{2k-1}) = \begin{cases} 1 - \dfrac{F_{k-2}}{F_{k+1}(F_k)^2 (L_{k-1})^2} & \text{if } k \text{ is even,} \\ 1 - \dfrac{L_{k-2}}{L_{k+1}(L_k)^2 (F_{k-1})^2} & \text{if } k \text{ is odd.} \end{cases}$$

The theorem motivates our interest in the numerators and denominators of $Q(\phi, F_k - 1)$ and $Q(\phi^2, F_k - 1)$, which can be thought of as expressions of the form

$$A(k,s) := \sum_{n=1}^{F_k - 1} \lfloor \phi n \rfloor^s \quad \text{and} \quad A'(k,s) := \sum_{n=1}^{F_k - 1} \lfloor \phi^2 n \rfloor^s$$

for $k = 1, 2, \ldots$ and $s = 1, 3$. More generally, our analysis in Section 3 covers the sums

$$A(k,s,j) := \sum_{n=1}^{F_k - 1} n^j \lfloor \phi n \rfloor^s \quad \text{where } k = 1, 2, \ldots \text{ and } s, j = 0, 1, 2, \ldots. \tag{2.1}$$

Namely, we find a recurrence relation for $A(k, j, s)$ and deduce recursions from it for $A(k, s) = A(k, s, 0)$ and $A'(k, s)$. The strategy leads to the following expressions for the numerators and denominators in Theorem 2.1, which are given in Lemmas 2.2–2.4.

LEMMA 2.2. *Let $k \geq 1$ be an integer. Then*

$$\begin{aligned} A(k,1) &= \tfrac{1}{2}(F_{k+1} - 1)(F_k - 1), \\ A'(k,1) &= \tfrac{1}{2}(F_{k+2} - 1)(F_k - 1). \end{aligned} \tag{2.2}$$

LEMMA 2.3. *Let $k \geq 1$ be an integer. Then*

$$\begin{aligned} A(2k,3) &= \tfrac{1}{4}(F_{2k-1} - 1)(F_{2k+1} - 1)^2 (F_{2k+2} - 1), \\ A(2k-1,3) &= \tfrac{1}{4}(F_{2k-1} - 1)(F_{2k} - 1) \times \tfrac{1}{5}(L_{4k} - 3L_{2k+1} - L_{2k} + 3). \end{aligned}$$

LEMMA 2.4. *Let $k \geq 1$ be an integer. Then*

$$A'(2k,3) = \tfrac{1}{4}(F_{2k}-1)(F_{2k+2}-1) \times \tfrac{1}{5}(L_{4k+4} - 5L_{2k+3} + 13),$$
$$A'(2k-1,3) = \tfrac{1}{4}(F_{2k-1}-1)(F_{2k+1}-1) \times \tfrac{1}{5}(L_{4k+2} - 5L_{2k+2} + 7).$$

Finally, we present an arithmetic formula inspired by Stolarsky's original question.

THEOREM 2.5. *For $k \geq 1$,*

$$\mathrm{LCM}(A(2k,1), A'(2k,1)) = \begin{cases} \frac{1}{2} F_{k+1} F_k L_{k+2} L_{k+1} L_{k-1} & \text{if } 2 \mid k, \\ \frac{1}{2} F_{k+2} F_{k+1} F_{k-1} L_{k+1} L_k & \text{if } 2 \nmid k. \end{cases}$$

REMARK 2.6. Lemmas 2.2–2.4 indicate that the expression

$$Q(\phi^2, F_k - 1) - Q(\phi, F_k - 1)$$

is expressible as a fraction whose numerator and denominator are polynomials in Fibonacci and Lucas numbers with indices depending linearly on k according to the parity of k, yet the statement of Theorem 2.1 presents formulas for these quantities according to the congruence class of k modulo 4 rather than modulo 2. The discrepancy is related to different factorisations of the factors $F_n - 1$ that occur in the formulas for $A(k, j)$ and $A'(k, j)$ for $j \in \{1, 3\}$, since each of the factors $F_n - 1$ happens to be a product of a Fibonacci and a Lucas number according to the congruence class of n modulo 4 (see formulas (6.1)).

3. Recurrence relations for auxiliary sums

Here, we show how to compute the integer-part sums (2.1). This clearly covers the cases $A(k,s) = A(k,s,0)$. On using

$$\phi^2 = 1 + \phi,$$

which upon multiplication by the integer n and taking integer parts becomes

$$\lfloor \phi^2 n \rfloor = n + \lfloor \phi n \rfloor,$$

one also gets the explicit formulas

$$A'(k,s) = \sum_{i=0}^{s} \binom{s}{i} A(k, s-i, i).$$

Using the Binet formula

$$F_k = \frac{\phi^k - (-\phi^{-1})^k}{\sqrt{5}} \quad \text{for all } k \geq 0,$$

one easily proves that

$$\lfloor \phi F_k \rfloor = F_{k+1} - \epsilon_k \quad \text{where } \epsilon_k = \frac{1 + (-1)^k}{2}$$

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and

$$\lfloor \phi(F_k + n)\rfloor = F_{k+1} + \lfloor \phi n\rfloor \quad \text{for } 1 \le n \le F_{k-1} - 1$$

(see, for example, [1]). Thus,

$$\begin{aligned}
A(k+1, s, j) &= \sum_{n=1}^{F_k - 1} n^j \lfloor \phi n\rfloor^s + F_k^j \lfloor \phi F_k\rfloor^s + \sum_{n=F_k+1}^{F_{k+1}-1} n^j \lfloor \phi n\rfloor^s \\
&= A(k, s, j) + F_k^j (F_{k+1} - \epsilon_k)^s + \sum_{n=1}^{F_{k+1}-F_k-1} (F_k + n)^j \lfloor \phi(F_k + n)\rfloor^s \\
&= A(k, s, j) + F_k^j (F_{k+1} - \epsilon_k)^s + \sum_{n=1}^{F_{k-1}-1} (F_k + n)^j (F_{k+1} + \lfloor \phi n\rfloor)^s \\
&= A(k, s, j) + F_k^j \sum_{i=0}^{s} \binom{s}{i} F_{k+1}^i (-\epsilon_k)^{s-i} \\
&\quad + \sum_{n=1}^{F_{k-1}-1} \sum_{\ell=0}^{j} \binom{j}{\ell} F_k^\ell n^{j-\ell} \sum_{i=0}^{s} \binom{s}{i} F_{k+1}^i \lfloor \phi n\rfloor^{s-i} \\
&= A(k, s, j) + \sum_{i=0}^{s} \binom{s}{i} (-\epsilon_k)^{s-i} F_k^j F_{k+1}^i \\
&\quad + \sum_{\ell=0}^{j} \sum_{i=0}^{s} \binom{j}{\ell}\binom{s}{i} F_k^\ell F_{k+1}^i A(k-1, s-i, j-\ell).
\end{aligned}$$

The above reduction, the identity $A(k, 0, 0) = F_k - 1$ and induction on $k + j + s$ imply that

$$A(k, s, j) \in \operatorname{span}\{(\phi^i)^k,\ (-\phi^i)^k : |i| \le j + s + 1\};$$

in particular, for a fixed choice of s and j, the sequence $\{A(k, s, j)\}_{k\ge1}$ is linearly recurrent of order at most $4(s + j) + 6$. In the following section, we will use this observation about linear recurrency together with the following facts.

- If $\boldsymbol{u} = \{u_n\}_{n\ge0}$ is a linearly recurrent sequence whose roots are all simple in some set U, then, for fixed integers p and q, the sequence $\{u_{pn+q}\}_{n\ge0}$ is linearly recurrent with simple roots in $\{\alpha^p : \alpha \in U\}$.
- If $\boldsymbol{u} = \{u_n\}_{n\ge0}$ and $\boldsymbol{v} = \{v_n\}_{n\ge0}$ are linearly recurrent and their roots are all simple in some sets U and V, respectively, then $\boldsymbol{uv} = \{u_n v_n\}_{n\ge0}$ is linearly recurrent and its roots are all simple in $UV = \{\alpha\beta : \alpha \in U,\ \beta \in V\}$.

In this context, the roots of a linearly recurrent sequence are defined as the zeros of its characteristic polynomial, counted with their multiplicities. It then follows that, for a fixed s, each of the sequences $\{A(k, s)\}_{k\ge1}$ and $\{A'(k, s)\}_{k\ge1}$ is linearly recurrent of order at most $4s + 6$.

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4. The proofs of the lemmas

We first establish Lemma 2.2. By the argument in Section 3, both $A(k, 1)$ and $A'(k, 1)$ are linearly recurrent with simple roots in the set $\{\pm\phi^l : |l| \le 2\}$. The same is true for the right-hand sides in (2.2). Since the set of roots is contained in a set with 10 elements, it follows that the validity of (2.2) for $k = 1, \dots, 10$ implies that the relations hold for all $k \ge 1$.

Lemmas 2.3 and 2.4 are similar. The argument in Section 3 shows that the left-hand sides, $\{A(k, 3)\}_{k\ge1}$ and $\{A'(k, 3)\}_{k\ge1}$, are linearly recurrent with simple roots in $\{\pm\phi^l : |l| \le 4\}$. Splitting according to the parity of k, we deduce that $\{A(2k, 3)\}_{k\ge1}$, $\{A(2k - 1, 3)\}_{k\ge1}$, $\{A'(2k, 3)\}_{k\ge1}$ and $\{A'(2k - 1, 3)\}_{k\ge1}$ are linearly recurrent with simple roots in $\{\phi^{2l} : |l| \le 4\}$, a set with nine elements. The same is true about the right-hand sides in the lemmas. Thus, if the relations hold for $k = 1, \dots, 9$, then they hold for all $k \ge 1$.

A few words about the computation. For the identities presented in Lemmas 2.2–2.4, one can use brute force to compute $A(k, s)$ and $A'(k, s)$ for $s = 1, 3$ and $k = 2\ell + i$, where $i \in \{0, 1\}$ and ℓ is reasonably small (we dealt with $\ell \le 9$), with any computer algebra system. To check them up to larger values of k (around 100, say), the brute force strategy no longer works since the summation range up to $F_k - 1$ becomes too large. Instead one can use the recursion from Section 3 together with $A(k, 0, 0) = F_k - 1$ to find $A(k, 1, 0)$, $A(k, 2, 0)$ and $A(k, 3, 0)$ for all desired k and, similarly, $A(k, s, j)$ for small j, to evaluate $A'(k, s)$.

5. The proof of Theorem 2.1

Let us now address Theorem 2.1. When $k = 4\ell$, this can be rewritten as

$$\begin{aligned} &F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1}(A'(4\ell, 3)A(4\ell, 1)^2 - A(4\ell, 3)A'(4\ell, 1)^2) \\ &\quad = A(4\ell, 1)^2 A'(4\ell, 1)^2 (F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} - 1). \end{aligned} \tag{5.1}$$

Since $A(4\ell, s)$ and $A'(4\ell, s)$ are linearly recurrent (in ℓ) with roots contained in $\{\phi^{4l} : |l| \le s + 1\}$, and both the left-most factor in the left-hand side and the right-most factor in the right-hand side each have simple roots in $\{\phi^{4l} : |l| \le 2\}$, it follows that both the left-hand side and the right-hand side are linearly recurrent with simple roots contained in $\{\phi^{4l} : |l| \le 10\}$, a set with 21 elements. Thus, if the above formula holds for $\ell = 1, \dots, 21$, then it holds for all $\ell \ge 1$. A similar argument applies to the case when $k = 4\ell + i$ for $i \in \{1, 2, 3\}$. Hence, all claimed formulas hold provided they hold for all $k \le 100$, say.

Now we use the lemmas. For $k = 4\ell$, Lemmas 2.2–2.4 tell us that (5.1), after eliminating the common factor $(F_{4\ell} - 1)^2(F_{4\ell+1} - 1)^2(F_{4\ell+2} - 1)/16$, is equivalent to

$$\begin{aligned} &F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} \times (\tfrac{1}{5}(F_{4\ell} - 1)(L_{8\ell+4} - 5L_{4\ell+3} + 13) \\ &\qquad - (F_{4\ell+2} - 1)(F_{4\ell-1} - 1)(F_{4\ell+2} - 1)) \\ &\quad = (F_{4\ell} - 1)^2 (F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} - 1) \end{aligned}$$

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(and one can perform further reduction using (6.1)). It is sufficient to verify the resulting equality for $\ell = 1, \dots, 15$ and we have checked it for all $\ell = 1, \dots, 100$. The remaining cases for k modulo 4 are similar. We do not give further details here.

6. The proof of Theorem 2.5

This follows from Lemma 2.2, the classical formulas

$$\begin{aligned} F_{4\ell} - 1 &= F_{2\ell+1}L_{2\ell-1}, & F_{4\ell+1} - 1 &= F_{2\ell}L_{2\ell+1}, \\ F_{4\ell+2} - 1 &= F_{2\ell}L_{2\ell+2}, & F_{4\ell+3} - 1 &= F_{2\ell+2}L_{2\ell+1} \end{aligned} \tag{6.1}$$

as well as known facts about the greatest common divisor of Fibonacci and Lucas numbers with close arguments. For example, for $k = 2\ell$,

$$\begin{aligned} \mathrm{LCM}(2A(4\ell, 1), 2A'(4\ell, 1)) &= \mathrm{LCM}((F_{4\ell+1} - 1)(F_{4\ell} - 1), (F_{4\ell+2} - 1)(F_{4\ell} - 1)) \\ &= \mathrm{LCM}(F_{2\ell}L_{2\ell+1}, F_{2\ell}L_{2\ell+2})F_{2\ell+1}L_{2\ell-1} \\ &= F_{2\ell}L_{2\ell+1}L_{2\ell+2}F_{2\ell+1}L_{2\ell-1} \\ &= F_{k+1}F_kL_{k+2}L_{k+1}L_{k-1}, \end{aligned}$$

where we used the fact that $\gcd(L_{2\ell+1}, L_{2\ell+2}) = 1$. The case $k = 2\ell + 1$ is similar.

7. Further variations

First, we give an informal account of a more general result lurking, perhaps, behind the formulas in Theorem 2.1. Consider a homogeneous (rational) function $r(\boldsymbol{x}) = r(x_1, \dots, x_m)$ of degree 1, that is, satisfying

$$r(t\boldsymbol{x}) = tr(\boldsymbol{x}) \quad \text{for } t \in \mathbb{Q},$$

and an algebraic number α solving the equation

$$\sum_{k=0}^{m} c_k\alpha^k = 0, \tag{7.1}$$

where the c_k are integers. If $r(\boldsymbol{x})$ vanishes at a vector $\boldsymbol{x}^* = (x_1^*, \dots, x_m^*)$, then automatically

$$\sum_{k=0}^{m} c_k r(\alpha^k\boldsymbol{x}^*) = 0 \tag{7.2}$$

in view of the homogeneity of the function. We can then enquire whether equation (7.2) is 'approximately' true if $r(\boldsymbol{x}^*) = 0$ is 'approximately' true. In this note, we merely examined the golden ratio case in which (7.1) is $\alpha^2 - \alpha - 1 = 0$, while the choice

$$r(x_1, \dots, x_m) = \frac{\sum_{n=1}^m x_n^3}{(\sum_{n=1}^m x_n)^2}$$

for the rational function and $\boldsymbol{x}^* = (1, 2, \dots, m)$ for its exact solution originated from the Nicomachus identity.

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Notice that Nicomachus' theorem (1.1) is the first entry in the chain of identities

$$1^{2r-1} + 2^{2r-1} + \cdots + m^{2r-1} = P_r(1 + 2 + \cdots + m) \quad \text{for } r = 2, 3, \ldots,$$

where $P_r(x)$ are known as the Faulhaber polynomials. Our approach in this note gives a clear strategy to deal with the quantities that replace (1.2) in these settings.

Some further variations on the topic can be investigated in the q-direction, based on q-analogues of (1.1) (see [3]).

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